SIAM J. CONTROL AND OPTIMIZATION
Vol. 21, No. 2, March 1983

0363-0129/83/2102-0008 $01.25/0

THE MATRIX RICCATI EQUATION AND THE NONCONTROLLABLE LINEAR-QUADRATIC PROBLEM WITH TERMINAL CONSTRAINTS*

PAVOL BRUNOVSKÝ† AND JOZEF KOMORNÍK‡

Abstract. It is proved that each positive semidefinite symmetric solution of the matrix Riccati equation corresponds to an optimal control problem with suitable terminal cost and constraints. The approximation scheme for the computation and characterization of the optimal cost and optimal controls of the problem with terminal constraints is extended to the noncontrollable case.

Key words. matrix Riccati, linear-quadratic, terminal constraints

Introduction. Consider the linear-quadratic optimal control problem on the interval $[s, T]$, $t_0 \leqq S \leqq T$, given by the equation

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

($x \in R^n$, $u \in R^r$), the initial state

$$x(s) = y, \tag{2}$$

the cost function

$$C_s^T(y, u) = \int_s^T c(t, x, u)\, dt + x'(T)Rx(T) \tag{3}$$

with $c(t, x, u) = x'Q(t)x + u'M(t)u$ and the terminal constraint

$$Dx(T) = 0, \tag{4}$$

D being $q \times n$, $q \leqq n$, with full rank, A, B, Q, M being continuous, Q, M symmetric, $Q \geqq 0$, $M > 0$ on $[t_0, T]$, $R \geqq 0$ symmetric.

Under the condition that the system with output $\xi = Dx$ is output controllable on $[s, T]$ for each $t_0 \leqq s < T$, we have shown in [1] that the minimal cost for this problem can be expressed by a solution of the corresponding matrix Riccati equation

$$\dot{W} + A'W + WA + Q - W'BM^{-1}B'W = 0 \tag{5}$$

(cf. also [2]) on $[t_0, T)$ that blows up for $t \nearrow T$. We have characterized this solution as a limit for $m \to \infty$ of solutions of (5) expressing the optimal cost of the corresponding unconstrained problem with cost

$$C_{s,m}^T(y, u) = C_s^T(y, u) + m\|Dx(T)\|^2 \tag{6}$$

containing a term penalizing the deviation of the response of u from the terminal subspace. Also, we have shown that the optimal control and optimal trajectory for the problem (1)–(4) are limits for $m \to \infty$ of those for the problems (1)–(3), (6).

This result can be put into an interesting context with the ideas of [3]. By associating with (5) a flow on the Grassmann manifold $GR(n)$ of n-dimensional subspaces of R^{2n}, we can prove an inverse theorem on the solutions of (5) (§ 3) and extend our results from [1] to noncontrollable problems and problems with constraints at several points (§ 4). In § 5 we show that the techniques of § 4 can be used to deal with the infinite interval problem in case the finiteness of cost is not assumed for all

* Received by the editors March 11, 1981, and in revised form October 7, 1981.
† Institute of Applied Mathematics, Comenius University, 84215 Bratislava, Czechoslovakia.
‡ Department of Probability and Statistics, Comenius University, 84215 Bratislava, Czechoslovakia.

points. Section 2 contains a summary of the items of [3] that are important for this paper.

All the necessary material about the Riccati matrix equation and the unconstrained linear-quadratic problem is summarized in [1].

2. The associated flow on *GR*(*n*). Denote

$$J=\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

where E_n is the $n\times n$ unity matrix. For $z_i=(x_i, p_i)\in R^n\times R^n$, $i=1,2$, denote

$$\omega(z_1, z_2)=z_1'Jz_2=x_1'p_2-x_2'p_1:$$

ω is a skew symmetric nondegenerate form on R^{2n}. An n-dimensional linear subspace L of R^{2n} is called Lagrangian, if the restriction of ω to L vanishes, i.e. $\omega(z_1, z_2)=0$ as soon as $z_1, z_2\in L$. We denote the set of Lagrangian subspaces of R^{2n} by $\mathscr{L}$.

A linear differential equation in R^{2n}

$$\dot z=H(t)z \tag{7}$$

is called Hamiltonian if ω is its integral, i.e. ω is constant along its solutions. This is equivalent to

$$H'J+JH=0. \tag{8}$$

By π_x we denote the natural projection of $R^{2n}=R^n\times R^n$ onto its first factor, and we denote

$$\mathscr{L}_0=\{L\in\mathscr{L}|\pi_x(L)=R^n\}.$$

We have $L\in\mathscr{L}_0$ if and only if there exists a symmetric $n\times n$ matrix W such that

$$L=\{(x, Wx)|x\in R^n\}.$$

The (time-dependent) flow of the equation (7) carries linear subspaces into linear subspaces of the same dimension and thus generates an associated (time-dependent) flow Φ on the Grassmann manifold $GR(n)$ of the subspaces of R^{2n} of dimension n. More precisely, if $L\in GR(n)$ and we denote by $\Phi_{t,s}(L)$ the linear subspace filled by the values at t of the solutions of (7) with values in L at time s, then there is a differential equation on $GR(n)$ such that $\Phi_{t,s}(L)$ is the value at time t of its solution having L as its value at time s. Since $GR(n)$ is compact, the solutions of this equation are defined for all $t\in R$. Since ω is an integral of (7), it is invariant under Φ, i.e. $L\in\mathscr{L}$ implies $\Phi_{t,s}(L)\in\mathscr{L}$ for all $t, s\in R$.

Consider now the flow Φ on $GR(n)$ associated with the differential equation

$$\dot x=Ax-BM^{-1}B'p, \qquad \dot p=-Qx-A'p \tag{9}$$

with A, B, M, Q coming from (1), (3). The matrix

$$H=\begin{pmatrix} A, & -BM^{-1}B'p \\ -Q & -A \end{pmatrix}$$

obviously satisfies (8), which means that (9) is Hamiltonian. If $L\in\mathscr{L}$ and $L(t)=\Phi_{t,s}(L)\in\mathscr{L}_0$ for all $t\in I=(t_1, t_2)$ and some $s\in I$, then there exists a matrix function $W(t)$, $t\in(t_1, t_2)$, such that $L(t)=\{(x, W(t)x)|x\in R^n\}$. This matrix satisfies (5).

Note that although $\lim_{t\to t_0} W(t)$ may not exist for $t_0=t_1$ or $t_0=t_2$, $L(t)=\Phi_{t,s}(L)$ can always be extended beyond I to all R.

3. The inverse theorem. Let A, B, Q, M be as in (1) (3).

THEOREM 1. *Let $W(t)$ be a positive semidefinite solution of the matrix Riccati equation (5) on $[t_0, T)$. Then there exist a $q \leqq n$, a $q \times n$ matrix D and a positive semidefinite symmetric matrix R such that $y'W(s)y$ is the optimal cost for the problem (1)–(4) for $t_0 \leqq s < T$.*

Proof. Denote $L(t) = \{(x, W(t)x) | x \in R^n\}$. If $\lim_{t \to T-} W(t)$ exists then we take $D = 0$, $R = \lim_{t \to T-} W(t)$; the statement of the theorem in this case is standard [4].

If $\lim_{t \to T-} W(t)$ does not exist, then $L(T) = \lim_{t \to T-} L(t) \notin \mathscr{L}_0$. Let $q = \operatorname{codim} \pi_x(L(T)) > 0$.

There exist $n \times n$ matrices S_1, S_2 such that rank

$$(S_1, S_2) = n \quad \text{and} \quad L(T) = \{(x, p) | S_1 x + S_2 p = 0\}.$$

If $x \in \pi_x(L(T))$ then there exists a p such that $S_2 p = -S_1 x$, i.e. $S_1 x \in \operatorname{Range} S_2$. The condition rank $(S_1, S_2) = n$ means

$$\text{Range } S_1 + \text{Range } S_2 = R^n, \tag{10}$$

from which it follows codim Range $S_2 = q$. Consequently, there exists a $q \times n$ matrix with full rank N such that $y \in \operatorname{Range} S_2$ if and only if $Ny = 0$. From (10) it also follows that if we denote $D = NS_1$, then rank D = rank $N = q$. Also, $x \in \pi_x(L(T))$ if and only if $Dx = 0$, i.e., $x \in \operatorname{Ker} D$.

Let K be any $n \times n$ matrix, the restriction of which to Range S_2 is a right inverse of S_2, i.e. we have $S_2 K S_2 = S_2$. Then, $(x, p) \in L(T)$ if and only if $x \in \operatorname{Ker} D$ and

$$p + KS_1 x \in \operatorname{Ker} S_2. \tag{11}$$

Denote $R_0 = -KS_1$.

Since $L(T) \in \mathscr{L}$, for any $p_1, p_2 \in \operatorname{Ker} S_2$, $x_1, x_2 \in \operatorname{Ker} D$ we have

$$(R_0 x_1 + p_1)' x_2 - (R_0 x_2 + p_2)' x_1 = 0. \tag{12}$$

Choosing $x_1 = 0$ and using (12) we obtain $p'x = 0$ for any $p \in \operatorname{Ker} S_2$, $x \in \operatorname{Ker} D$. However, $p'x = 0$ for all $x \in \operatorname{Ker} D$ is equivalent to $p \in \operatorname{Range} D'$, so $\operatorname{Ker} S_2 \subset \operatorname{Range} D'$. Since rank $D = q$ = codim Range S_2 = dim Ker S_2, we have

$$\operatorname{Ker} S_2 = \operatorname{Range} D'. \tag{13}$$

Choosing $p_1 = p_2 = 0$ in (12) we have $x_1' R_0' x_2 = x_1' R_0 x_2$ for all $x_1, x_2 \in \operatorname{Ker} D$. Also, if we take any $x \in \operatorname{Ker} D$, then $(x, R_0 x) \in L(T)$. Since $L(T) = \lim_{t \to T-} L(t)$ (in $GR(n)$), there exists a sequence of points $t_i \nearrow T$, $(x_i, p_i) = (x_i, W(t_i)x_i) \in L(t_i)$, $(x_i, p_i) \to (x, R_0 x)$. Since $W(t)$ is positive semidefinite, for each $t < T$, we have $x' R_0 x = \lim_{i \to \infty} x_i' W(t_i) x_i \geqq 0$.

Denote $R_0^1 = PR_0$, $R_0^2 = (E - P)R_0$, where P is the orthogonal projection of R^n onto Ker D. For any $x_1, x_2 \in \operatorname{Ker} D$ we have

$$x_1' R_0 x_2 = x_1' R_0^1 x_2$$

and, consequently,

$$x_1' R_0^1 x_1 \geqq 0, \qquad x_1' R_0^1 x_2 = x_2' R_0^1 x_1.$$

Thus, the restriction of R_0^1 to Ker D is symmetric and positive semidefinite. Obviously, we can find an R symmetric and positive semidefinite on all R^n such that

$$R|_{\operatorname{Ker} D} = R_0^1|_{\operatorname{Ker} D}. \tag{14}$$

By (13), for $x \in \operatorname{Ker} D$, (11) is equivalent to

$$p - R_0 x = p - R_0^1 x - R_0^2 x \in \operatorname{Range} D'.$$

Since $R_0^2 x$ is orthogonal to $\operatorname{Ker} D$, we have $R_0^2 x \in \operatorname{Range} D'$, which means that (11) is equivalent to

(15) $$p - Rx \in \operatorname{Range} D'$$

for all $x \in \operatorname{Ker} D$. By [5], [2], (14) is the transversality condition for the solution of the adjoint equation of the problem (1)–(4). Since $R \geqq 0$, $M(t) > 0$ and $Q(t) \geqq 0$ for $t \in [t_0, T]$, if $(x(t), p(t))$ is a solution of (9) with $Dx(T) = 0$ and $p(T)$ satisfying (15), then $x(t)$, $t \in [s, T]$ is an optimal trajectory for the problem (1), (3), (4) with initial state $x(s)$, the corresponding optimal control being generated by the feedback law

(16) $$u(t) = -M^{-1}(t)B(t)p(t) = -M^{-1}(t)B(t)W(t)x(t)$$

for $s \leqq t < T$. Since $\pi_x(L(S)) = R^n$, the points $x(s)$ obtained in this way for all possible choices of $x(T)$ and $p(T)$ fill up all R^n.

We have

$$\begin{aligned} x'(s)W(s)x(s) = p(s)x(s) &= -\int_s^T \frac{d}{dt}(p(t)x(t))\,dt + p'(T)x(T) \\ &= x'(T)Rx(T) - \int_s^T [\dot{p}'(t)x(t) + p'(t)\dot{x}(t)]\,dt \\ &= x'(T)Rx(T) + \int_s^T [x'(t)Q(t)x(t) + u'(t)M(t)u(t)]\,dt. \end{aligned}$$

Since $x(t)$, $u(t)$ are the optimal trajectory and control, respectively, this completes the proof.

4. The noncontrollable problem. In this section we consider the problem (1)–(4), but unlike in [1], [2], we shall not assume that the system (1) with output $\xi = Dx$ is output controllable. It is obvious that the set of points that can be controlled to the terminal set $Dx(T) = 0$ on $[s, T]$ is a linear subspace of R^n, but for a nonautonomous problem it is moving with s in general, and it is not entirely obvious how to characterize it.

The following theorem gives two characterizations of this subspace—one in terms of the flow on $GR(n)$, the other in terms of the approximation scheme of [1]. Also, it shows that for this approximation scheme to work, the output controllability assumption is not essential.

As in [1], we denote by $\mathcal{U}_s^T(y)$ the set of controls steering the system from the point y to the terminal set (4) on $[s, T]$ and by W_m the solution of (5) satisfying the terminal condition $W_m(T) = R + mD'D$. Note that $y'W_m(s)y$ is the minimal value of the cost for the unconstrained problem (1)–(3), (6). The optimal control $u_m(t)$ for this problem is given by the optimal feedback law

(17) $$u = -M^{-1}(t)B'(t)W_m(t)x,$$

i.e., we have $u_m(t) = -M^{-1}(t)B'(t)W_m(t)x_m(t)$, where $x_m(t)$ is the solution of the equation

$$\dot{x} = (A - BM^{-1}B'W_m)x$$

with $x_m(s) = y$.

Denote

$$U(s)=\{y|\mathcal{U}_s^T(y)\neq\varnothing\},$$

$$V(s)=\{y|\limsup_{m\to\infty} y'W_m(s)y<\infty\},$$

$$L(s)=\Phi_{s,T}(\{(x,p)|Dx=0, p-Rx\in \text{Range } D'\}).$$

THEOREM 2. *For all* $s\in[t_0, T)$,

$$U(s)=V(s)=\pi_x(L(s)). \tag{18}$$

For $y\in U(s)$, *the optimal control* $u_0(t)$ *for the problem* (1)–(4) *is given by*

$$u_0(t)=\lim_{m\to\infty} u_m(t)=-\lim_{m\to\infty} M^{-1}(t)B'(t)W_m(t)x_m(t),$$

and the optimal value of the cost is given by

$$\min_{u\in\mathcal{U}_s^T(t)} C_s^T(y,u)=C_s^T(y,u_0)=\lim_{m\to\infty} y'W_m(s)y. \tag{19}$$

Proof. First, we prove $V(s)\subset U(s)$. From (9) we obtain by simple calculation for any k, m, s fixed, $y=x_i(s)$ and $p_i(t)=W_i(t)x_i(t)$, $i=k, m$,

$$\begin{aligned}
&\int_s^T [(x_m(t)-x_k(t))'Q(t)(x_m(t)-x_k(t))+(u_m(t)-u_k(t))'M(t)(u_m(t)-u_k(t))]\,dt\\
&\quad=-\int_s^T \frac{d}{dt}[(p_m(t)-p_k(t))'(x_m(t)-x_k(t))]\,dt\\
&\quad=-(p_m(T)-p_k(T))'(x_m(T)-x_k(T))\\
&\quad=-(x_m(T)-x_k(T))'(W_m(T)x_m(T)-W_k(T)x_k(T))\\
&\quad=-(x_m(T)-x_k(T))'(R+kD'D)(x_m(T)-x_k(T))\\
&\qquad-(x_m(T)-x_k(T))'(m-k)D'Dx_m(T)\\
&\quad=-(x_m(T)-x_k(T))'(R+kD'D)(x_m(T)-x_k(T))\\
&\qquad-(m-k)x_m'(T)D'Dx_m(T)+x_k'(T)(W_m(T)-W_k(T))x_m(T).
\end{aligned} \tag{20}$$

Using the invariance of ω, we have

$$\begin{aligned}
x_k'(T)(W_m(T)-W_k(T))x_m(T)&=p_m'(T)x_k(T)-p_k'(T)x_m(T)\\
&=p_m'(s)x_k(s)-p_k'(s)x_m(s)\\
&=y'(W_m(s)-W_k(s))y.
\end{aligned}$$

Denote $\delta_{k,m}(s)=y'(W_m(s)-W_k(s))y$.

$$\bar\delta_k(s)=\lim_{m\to\infty}\delta_{k,m}(s);$$

$$0\leqq\delta_{k,m}(s)\leqq\bar\delta_k(s)<\infty,\quad \lim_{k\to\infty}\bar\delta_k(s)=0\quad \text{for } k<m, y\in V(s). \tag{21}$$

From (20) it follows that

$$(22)\qquad \begin{aligned}\delta_{k,m}(s)=\int_s^T &[(x_m(t)-x_k(t))'Q(t)(x_m(t)-x_k(t))\\ &+(u_m(t)-u_k(t))'M(t)(u_m(t)-u_k(t))]\,dt\\ &+(x_m(T)-x_k(T))'(R+kD'D)(x_m(T)-x_k(T))\\ &+(m-k)x'_m(T)D'Dx_m(T).\end{aligned}$$

Since all the right-hand side terms are nonnegative, we have

$$(23)\qquad \begin{aligned}&(m-k)\|Dx_m(T)\|^2=(m-k)x'_m(T)D'Dx_m(T)=\delta_{k,m}(s)\leqq\bar{\delta}_k(s),\\ &0\leqq\|Dx_m(T)\|^2\leqq\frac{1}{m-k}\bar{\delta}_k(s),\end{aligned}$$

and, by (21),

$$\lim_{m\to\infty}\|Dx_m(T)\|=0.$$

Also, from (22) it follows that

$$\sup_m\int_s^T(u_m(t)-u_k(t))'M(t)(u_m(t)-u_k(t))\,dt\leqq\bar{\delta}_k(s).$$

Since $M(t)$ is continuous and positive definite on $[s, T]$, it is uniformly positive definite on $[s, T]$. From this and (21) it follows that $\{u_m\}$ is a Cauchy sequence in $L_2(s, T)$ and therefore has a limit $u_0(t)$ in $L_2(s, T)$. From the representation of $x_m(t)$ by the variation of constant formula it follows immediately that $\{x_m\}$ converges uniformly to the response $x_0(t)$ of $u_0(t)$ satisfying $x_0(s)=y$.

By (23), we have

$$Dx_0(T)=0.$$

This proves $V(s)\subset U(s)$ and also the second equality of (19). To prove the first equality (having as its consequence the optimality of u_0) we note that for each $u\in\mathcal{U}_s^T(y)$ we have

$$C_s^T(y,u)=C_{s,m}^T(y,u)\geqq\min C_{s,m}^T(y,u)=y'W_m(s)y.$$

This also proves $U(s)\subset V(s)$. To complete the proof of the theorem it remains to prove the second equality of (18).

If $y\in\pi_x(L(s))$ then there exists a solution $(x(t), p(t))$ of (9) with $Dx(T)=0$ such that $x(s)=y$. The function $x(t)$ is a response of the control $u(t)=-M^{-1}(t)B'(t)p(t)$ which means $u\in\mathcal{U}_s^T(y)$. Consequently, $\mathcal{U}_s^T(y)\neq\varnothing$ and $y\in U(s)$.

On the other hand, if $y\in U(s)$, then by [5], there exists an optimal control u_0 in $\mathcal{U}_s^T(y)$, the response $x_0(t)$ of which, together with a suitable function $p(t)$, satisfies (9). In addition, $p(T)$ satisfies the transversality condition (15). This proves $y\in\pi_x(L(s))$.

Remark 1. Since $u_m\to u_0$ in $L_2(s, T)$ we have

$$(24)\qquad \lim_{m\to\infty}C_s^T(y,u_m)=C_s^T(y,u_0).$$

On the other hand, we have

$$(25)\qquad \begin{aligned}C_s^T(y,u_0)&=\lim_{m\to\infty}y'W_m(s)y=\lim_{m\to\infty}C_{s,m}^T(y,u_m)\\ &=\lim_{m\to\infty}C_s^T(y,u_m)+m\|Dx_m(T)\|^2.\end{aligned}$$

From (24), (25) we obtain

$$\lim_{m\to\infty} m\|Dx_m(T)\|^2 = 0,$$

or

$$\|Dx_m(T)\|^2 = o(m^{-1/2}).$$

This gives an estimate for the deviation of the endpoint of the optimal trajectory of the approximate unconstrained problem from the terminal set.

Remark 2. From $\pi_x(L(s)) = U(s)$ it follows that the dimension of $\pi_x(L(s))$ cannot decrease with s decreasing. From [6] it follows that for A, B analytic it is constant for $s < T$ and equal to the dimension of the space $\operatorname{Ker} D + C$, where $C = \operatorname{span}\{b_i(T), (\mathscr{A}b_i)(T), \cdots, (\mathscr{A}^{n-1}b_i)(T) | i = 1, \cdots, n\}$, where b_i are the column vectors of B and $\mathscr{A}f(t) = \dot{f}(t) - Af(t)$ for a differentiable function f on $[t_0, T]$.

Theorem 2 allows us to deal with the problem (1)–(3) with additional constraints and costs at intermediate points of the interval. We shall restrict ourselves to the case of one intermediate point, the extension to the case of a higher number of points being straightforward.

Let $T_1 \in (t_0, T)$, $q_1 \leqq n$ and let $R_1 \geqq 0$, D_1 be $n \times n$ symmetric and $q_1 \times n$ with full rank, respectively. Consider the problem given by the system (1), the initial point (2), the cost function

$$\tilde{C}_s^T(y, u) = C_s^T(y, u) + x'(T_1)R_1x(T_1), \tag{26}$$

the constraints (4) and

$$D_1x(T_1) = 0. \tag{27}$$

Of course, for $s \in (T_1, T]$ the problem coincides with the problem (1)–(4).

Let $U(t)$, $W_m(t)$ be defined as in Theorem 2. It is obvious that the optimal control for the problem (1), (2), (26), (4), (27) for $s = T_1$ will be a concatenation of the optimal control on $[s, T_1]$ for the problem (1), (2), the cost function

$$\int_s^{T_1} c(t, x, u)\, dt + x'(T_1)R_1x(T_1) + \lim_{M\to\infty} x'(T_1)W_m(T_1)x(T_1)$$

and the linear constraint

$$x(T_1) \in U(T_1) \cap \operatorname{Ker} D_1,$$

and the optimal control for the problem (1), (3), (4), with initial point $x(t_1)$ on $[t_1, T]$.

5. The infinite interval. Consider the unconstrained problem (1), (3) with $R = 0$ and denote W^T the corresponding solution of (5), which is the solution satisfying $W^T(T) = 0$. For fixed s, y, denote u^T, x^T the optimal control and trajectory respectively. In [1], we have shown that $\lim_{T\to\infty} W^T(s)$ exists and represents the optimal cost for the infinite interval problem, provided for each s, y there exists a u such that $C_s^\infty(y, u) = \lim_{T\to\infty} C_s^T(y, u) < \infty$. Like Theorem 2, the following theorem deals with problems not satisfying this condition.

By $U^\infty(s)$ we denote the set of those $y \in R^n$ for which there is a control u on $[s, \infty)$ such that $C_s^T(y, u) < \infty$. Further, we denote

$$V^\infty(s) = \left\{ y \,\middle|\, \lim_{T\to\infty} y'W^T(s)y < \infty \right\}.$$

By $L_M^2(s,\infty)$ we denote the space of functions $u:[s,\infty)\to R^r$ which are square integrable with weight $M(t)$, i.e., $\int_s^\infty u'(t)M(t)u(t)\,dt<\infty$. $L_M^2(s,\infty)$ is a Banach space.

THEOREM 3. *We have $U^\infty(s)=V^\infty(s)$ for every $s\geqq t_0$. For $y\in U^\infty(s)$ we have*

$$\min_u C_s^\infty(y,u)=\lim_{T\to\infty} y'W^T(s)y.$$

The optimal control $u^\infty(t)$ and trajectory $x^\infty(t)$ are given by

$$u^\infty(t)=\lim_{T\to\infty} u^T(t)\qquad (in\ L_M^2(s,\infty)),\tag{28}$$

$$x^\infty(t)=\lim_{T\to\infty} x^T(t)\tag{29}$$

(uniformly on each finite interval).

Let us note that in (28), (29) we understand u^T, x^T to be extended to $[s,\infty)$ by having value 0 for $t>T$.

Proof. Let $y\in V^\infty(s)$, $T_2=T_1\geqq s$. Denote $W^{T_i}=W_i$, $x^{T_i}=x_i$, $u^{T_i}=u_i$, $i=1,2$. By computations similar to those leading to (20) we obtain

$$\begin{aligned}y'(W_1(s)-W_2(s))y=\int_s^T &[(x_1(t)-x_2(t))'Q(t)(x_1(t)-x_2(t))\\&+(u_1(t)-u_2(t))'M(t)(u_1(t)-u_2(t))]\,dt\\&+(x_1(T_1)-x_2(T_1))W_1(T_1)(x_1(T_1)-x_2(T_1))\\&+x_2(T_1)(W_2(T_1)-W_1(T_1))x_2(T_1)\\&=\int_s^{T_1}[(x_1(t)-x_2(t))'Q(t)(x_1(t)-x_2(t))+(u_1(t)\\&\qquad-u_2(t))'M(t)(u_1(t)-u_2(t))]\,dt\\&+\int_{T_1}^{T_2}[x_2(t)'Q(t)x_2(t)+u_2'(t)M(t)u_2(2)]\,dt\\&\geqq\int_s^{T_1}(u_1(t)-u_2(t))'M(t)(u_1(t)-u_2(t))\,dt.\end{aligned}\tag{30}$$

From the estimate (30) it follows that the family of functions $\{u_T|T\geqq s\}$ is a Cauchy family in $L_M^2(s,\infty)$. Since $L_M^2(s,\infty)$ is complete, it has a limit $u\in L_M^2(s,\infty)$. From the variation of constants formula it follows immediately that the response x^∞ of u^∞ is a pointwise limit of the functions x^T, the convergence being uniform on each finite subinterval of $[s,\infty)$.

For every fixed $T_0\geqq s$ we have

$$C_s^{T_0}(y,u^\infty)=\lim_{T\to\infty} C_s^{T_0}(y,u^T)=\lim_{T\to\infty} C_s^T(y,u^T)=\lim_{T\to\infty} y'W^T(s)y,$$

from which it follows that $C_s^\infty(y,u^\infty)$ is finite and, thus, that $V^\infty(s)\subset U^\infty(s)$. On the other hand, we have for any control u,

$$C_s^{T_0}(y,u)\geqq y'W^{T_0}(s)y.\tag{31}$$

From (30), (31) it follows that

$$C_s^\infty(y,u)\geqq C_s^\infty(y,u^\infty)=\lim_{T\to\infty} y'W^T(s)y,$$

which implies that u^∞ is optimal.

The inclusion $U^{\infty}(s) \subset V^{\infty}(s)$ follows immediately from (31).

Note added in proof. There is an overlap of our § 4 and the paper of G. Chen and W. Mills, *Finite elements and terminal penalization for quadratic cost optimal control problems governed by ordinary differential equations*, this Journal, 19 (1981), pp. 744–764. In particular, the essential part of Theorem 3 or our paper is contained in Theorem 2.2 of the quoted paper.

REFERENCES

[1] P. BRUNOVSKÝ AND J. KOMORNÍK, *The Riccati equation solution of the linear-quadratic problem with constrained terminal state*, IEEE Trans. Automat. Control, AC-26 (1981), pp. 398–402.

[2] B. FRIEDLAND, *On solutions of the Riccati equation in optimization problems*, IEEE Trans. Automat. Control, AC-12 (1967), pp. 303–304.

[3] R. HERMANN, *Cartanian Geometry, Nonlinear Waves and Control Theory*, Part A, Math. Science Press, Brookline, MA, 1979.

[4] V. KUČERA, *A review of the matrix Riccati equation*, Kybernetika (Prague), 9 (1973), pp. 42–61.

[5] E. B. LEE AND L. MARKUS, *Foundations of Optimal Control Theory*, John Wiley, New York, 1967.

[6] H. J. SUSSMANN AND V. JURDJEVIC, *Controllability of non-linear systems*, J. Differential Equations, 12 (1972), pp. 95–116.