Nonlinear Analysis, Theory, Methods & Applications, Vol. 10, No. 2, pp. 179–193, 1986.
Printed in Great Britain.

0362–546X/86 \$3.00 + .00

NUMBERS OF ZEROS ON INVARIANT MANIFOLDS IN REACTION–DIFFUSION EQUATIONS

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(*Received* 10 *November* 1984; *received for publication* 9 *April* 1985)

Key words and phrases: Lap number, zero number, invariant manifolds, reaction–diffusion equation.

INTRODUCTION

CONSIDER the one-dimensional reaction–diffusion equation

$$u_t = u_{xx} + f(x, u), t > 0, 0 < x < 1 \tag{0.1}$$

with the Dirichlet boundary conditions

$$u(t, 0) = u(t, 1) = 0, \tag{0.2}$$

where f: $[0, 1] \times \mathbb{R} \to \mathbb{R}$ is $BC^1 \cap C^\kappa$, $\kappa > 1$. The equations (0.1), (0.2) can be viewed as a particular case of the abstract equation

$$\mathrm{d}u/\mathrm{d}t + Au = f(u) \tag{0.3}$$

in a Banach space X, the basic theory of which is developed in [5]. For (0.1), (0.2), $X = L^2[0, 1]$, A is the closure of the operator defined by $Av = -v''$ for $v \in C^2[0, 1]$, $v(0) = v(1) = 0$, F: $L^2[0, 1] \to L^2[0, 1]$ is given by $F(v)(x) = f(x, v(x))$. We frequently work in the Hilbert space $X^1 = \mathfrak{D}(A) = H_0^1([0, 1]) \cap H^2([0, 1])$ with F: $X^1 \to X^1$ also being C_κ. Let $|\cdot|$ denote the norm on X^1.

Applying the results of [5] one obtains that (0.1), (0.2) generates a local semiflow S on X^1. The semiflow S is a continuous map of an open neighbourhood U of $\{0\} \times X^1$ in $\mathbb{R}^+ \times X^1$ into X^1 defined by

$$S_t(v)(x) = u(t, x) \quad \text{for} \quad (t, v) \in U,$$

where u is the solution of (0.1), (0.2), satisfying

$$u(0, x) = v(x) \quad \text{for} \quad 0 < x < 1. \tag{0.4}$$

It has the properties $S_0(v) = v$, $S_{t+s}(v) = S_t \circ S_s(v)$ as long as (s, v) and $(t, S_s(v))$ are in U [5]. In order not to obscure the formulations by technicalities we shall assume that S is a global semiflow, i.e. $U = \mathbb{R}^+ \times X^1$. This is by no means an essential restriction; sufficient conditions can be found in [5, Chapter 3].

The critical points of S are the stationary solutions of (0.1), (0.2), i.e. the solutions of the equation

$$v'' + f(x, v) = 0, \; v(0) = v(1) = 0. \tag{0.5}$$

The qualitative properties of S near a critical point v are determined by the linearization of (0.1), (0.2) at v which is the equation

$$y_t = y_{xx} + f_u(x, v(x))y \tag{0.6}$$

$$y(t, 0) = y(t, 1) = 0. \tag{0.7}$$

The solution v is called hyperbolic if 0 is not an eigenvalue of the operator $L = A - F'(v)$, i.e. (0.6), (0.7) do not admit a nontrivial stationary solution y.

An important information about the global structure of the semiflow of (0.1), (0.2) is given by the orbit connections of different stationary solutions [2, 3, 5]. By a connecting orbit of the stationary solutions v_1, v_2 we understand a solution u of (0.1), (0.2) which exists for all $t \in (-\infty, \infty)$ and satisfies

$$\lim_{t \to -\infty} u(t, x) = v_1(x), \; \lim_{t \to \infty} u(t, x) = v_2(x)$$

in $H^2[0, 1]$. In the terminology of [5], $u(t, \cdot)$ has to be in the stable manifold of v_2 and the unstable manifold of v_1, provided v_1, v_2 are hyperbolic.

In this paper we obtain estimates on the number of zeros (or, more precisely, the zero number defined below) of $u(t, \cdot)$-v_1 and $u(t, \cdot)$-v_2. This information can be used to conclude existence and nonexistence of connections. Our approach provides an alternative to the (slightly different) zero number of u_t, which was used by Hale and Nascimento [3] to solve the connection problem for f of the Chafee–Infante type (see e.g. [5, Section 5.3]).

For any continuous function ϕ: $[0, 1] \to \mathbb{R}$ we define the *zero number* $z(\phi)$ as follows. Let $n \geqq 0$ be the maximal element of $\mathbb{N}_0 \cup \{\infty\}$ such that there is a strictly increasing sequence $0 \leqq x_0 < x_1 < \ldots < x_n \leqq 1$ with $\phi(x_j)$ of alternating signs:

$$\phi(x_j) \cdot \phi(x_{j+1}) < 0 \quad \text{for } 0 \leqq j < n.$$

If n is finite let $z(\phi) := n$, and $z(\phi) := \infty$ otherwise. Note that we put $z(0) := 0$.

As a first example consider the linearized equation (0.6), (0.7). The operator $L = A - F'(v)$ has eigenvalues $\lambda_0 < \lambda_1 < \ldots$ with eigenfunctions $\phi_0, \phi_1, \ldots$. By Sturm–Liouville theory $z(\phi_k) = k$ and indeed it is a classical result (see [0, p. 549]) that for $0 \leqq i < j < \infty$

$$i \leqq z(\phi) \leqq j, \tag{0.8}$$

whenever ϕ is a (nontrivial) linear combination of $\phi_i, \ldots, \phi_j$. As a trivial illustration of our approach we prove estimate (0.8) in corollary 1.2, using the dynamic equation (0.6), (0.7).

All our results depend on a basic observation, lemma 1.1, going back to Redheffer, Walter [8] and, more recently, Matano [6]. According to lemma 1.1,

$$z(u(t, \cdot)) \text{ is nonincreasing}$$

as a function of time t along solutions of equation (0.1), (0.2) provided that f satisfies the condition.

$$f(x, 0) = 0 \quad \text{for } 0 < x < 1. \tag{0.9}$$

The proof is elementary and relies on the maximum principle for parabolic equations. For the convenience of the reader we present it in detail below.

In the nonlinear case, let v be a hyperbolic stationary solution of (0.1), (0.2). Then the eigenvalues λ_j of the linearized equation with corresponding eigenfunctions ϕ_j satisfy $\lambda_0 < \ldots < \lambda_{n-1} < 0 < \lambda_n < \ldots$ for some $n \geqq 0$. Further by [5, theorems 5.2.1, 6.1.9] there exist immersed invariant C^K-manifolds W^u and $W^s \subset X^1$ of the flow S through $v = 0$ with the properties:

(i) for $w \in W^u$ (resp. W^s) the solution $u(t, \cdot) = S(t)w$ exists for all real t and satisfies $\lim u(t, \cdot) = v$ as $t \to -\infty$, (resp. $t \to +\infty$);

(ii) the tangent space of W^u (resp. W^s) at v is spanned by the ϕ_k with $k < n$ (resp. $k \geqq n$).

W^u is called the unstable manifold and W^s the stable manifold of v.

Our main result, given in Sections 2 and 3, states that

$$z(w - v) < \dim W^u \quad \text{for} \quad w \in W^u \tag{0.10}$$

(theorem 2.1) and

$$z(w - v) \geqq \dim W^u \quad \text{for} \quad w \in W^s \backslash \{v\} \tag{0.11}$$

(theorem 3.2). Note that these estimates are suggested by the respective tangent spaces of W^u and W^s, together with the Sturm–Liouville estimate (0.8).

The crucial observation of our proof is that for $v \equiv 0$:

$$\lim \frac{u(t)}{|u(t)|} = \phi_k \tag{0.12}$$

—for $t \to -\infty$ on W^u and some $k < n$

—for $t \to +\infty$ on W^s and some $k \geqq n$, provided that $z(u(t, \cdot))$ is eventually finite.

Actually it is quite simple to prove (0.12) on W^u, as we will indicate at the end of Section 2. However, analysis on the infinite dimensional stable manifold W^s is quite delicate and we need detailed information on the fine structure of W^s before we can prove (0.12). For illustration we pursue an analogous approach to W^u in Section 1, as a preparation to the stable manifold case.

1. COUNTING ZEROS

In the introduction we defined the zero number $z(\phi)$ of a continuous real function ϕ as the maximal number of sign changes of ϕ. In this section we show that z decreases along solutions $u(t, \cdot)$ of the parabolic equation (0.1) with Dirichlet boundary conditions, assuming that

$$f(x, 0) = 0 \quad \text{for all} \quad x \in I, \tag{1.1}$$

$I := [0, 1]$. This result is essentially in [8, corollary 3] who consider $f = f(t, x, u_x, u_{xx})$ independent of u. Similarly, Matano [6] investigates the lap number of ϕ, which is the zero number of ϕ_x and was called "maximum order of a saw in ϕ" by Redheffer and Walter [8].

Note that by definition the function

$$z\colon C^0(I) \to \mathbb{N} \cup \{\infty\}$$

is lower semicontinuous. Further, z is constant in a C^1-neighbourhood of any C^1-function ϕ with only simple zeros. These trivial facts will become important later on.

The parabolic equation (0.1), (0.2) generates a semiflow $S(t)u_0 = u(t) = u(t, \cdot)$ on $X^1 \subset H_0^1 \subset C^0(I)$, thus $z(u(t))$ is well defined along solutions.

LEMMA 1.1. [6, 8]. Let $f(x, 0) = 0$ for all $x \in I$. Then the zero number $z(u(t, \cdot))$ is nonincreasing as a function of t along solutions $u(t, \cdot)$ of (0.1), (0.2).

Proof. With $a(t, x) := (f(x, u(t, x)))/(u(t, x))$ we write (0.1) as

$$u_t = u_{xx} + au, \tag{1.2}$$

where a is C^0. We apply the maximum principle to (1.2) to prove: if $x'_1, x'_2 \in I$ are such that $u(t, x'_1) < 0 < u(t, x'_2)$ then there exist continuous paths γ_i in $I \times [0, t]$ connecting (t, x'_i) to a point $(0, x_i)$, such that $u < 0$ (resp. $u > 0$) along γ_1 (resp. γ_2). To see that assume $0 < x'_1 < x'_2 < 1$, the case $x'_1 > x'_2$ is analogous. Let D_i be the path connected component of (t, x'_i) in the relatively open set

$$K_i := \{(\tau, \xi) \in [0, t] \times I | (-1)^i u(\tau, \xi) > 0\}.$$

We claim that we can find elements $(0, x_i) \in D_i$. Otherwise, e.g. D_2 is contained in the strip $(0, t] \times I$. Replacing u by $ue^{\alpha t}$ does not change d_2 and allows us to assume $a < 0$, hence $Au := u_{xx} - u_t \geqq 0$ on $\bar{D}_2$. Let $M := \max_{\bar{D}_2} u > 0$ and choose a point $(\bar{t}, \bar{x}) \in \bar{D}_2$ with minimal $\bar{t}$ such that $u(\bar{t}, \bar{x}) = M$. From $M > 0$ we conclude $(\bar{t}, \bar{x}) \in D_2$, hence $\bar{t} > 0$. This implies a contradiction to the strong maximum principle: let $E := D_2$ and apply [7, III.2, lemma 3] to conclude $u < M$ on $d_2 \cap (\{\bar{t}\} \times I)$, contradicting $(\bar{t}, \bar{x}) \in D_2 \cap (\{\bar{t}\} \times I)$. Therefore there are points $(0, x_i) \in D_i$. Invoking the Jordan curve theorem comletes the proof. ■

As a trivial but illustrative application, we prove estimate (0.8) for finite linear combinations

$$\phi^0 = \sum_{k=i}^{j} \alpha_k \cdot \phi_k \tag{1.3}$$

of Sturm–Liouville eigenfunctions ϕ_k for the potential $a(x) := f_u(x, v(x))$. We use the flow (1.2), defining a solution $\phi(t, \cdot)$ with initial condition $\phi(0, \cdot) = \phi^0$ and Dirichlet conditions.

COROLLARY 1.2. If the Sturm–Liouville potential a is continuous, $0 \leqq i < j < \infty$ and $\phi^0 \not\equiv 0$, then

$$i \leqq z(\phi^0) \leqq j$$

Proof. We use the explicit representation

$$\phi(t, \cdot) = \sum_{k=i}^{j} \alpha_k e^{\lambda_k t} \phi_k \tag{1.4}$$

of the solution $\phi(t, \cdot)$, $t \in \mathbb{R}$ of (1.2) through ϕ^0. From (1.4), $\phi^0 \not\equiv 0$ it is immediate that there exist integers $k^{\pm} \in \{i, i+1, \ldots, j\}$ such that

$$\lim_{t \to \pm\infty} \phi(t)/|\phi(t)| = \text{sign}(\alpha_{k^{\pm}}) \phi_{k^{\pm}}$$

in the C^1-topology (normalizing $|\phi_k| = 1$), because the λ_k are pairwise disjoint. The ϕ_k have only simple zeros, hence z is constant in a C^1-neighbourhood of ϕ_k. By monotonicity of z along solutions of (1.2) (lemma 1.1) we conclude for $T>0$ sufficiently large

$$i \leqq k^+ = z(\phi(T,\cdot)/|\phi(T,\cdot)|) = z(\phi(T,\cdot)) \leqq z(\phi^0) \leqq z(\phi(-T,\cdot)) = k^- \leqq j$$

and the proof is complete. ■

Note that the corollary holds even if $j=\infty$.

2. ZEROS ON THE UNSTABLE MANIFOLD

In this section we prove that for any element w of an n-dimensional unstable manifold of v there are less than n zeros of w–v. On our way we investigate the fine structure of the unstable manifold. Finally we relate $z(w-v)$ to the number of zeros of v_x.

Let v be a hyperbolic stationary solution of (0.1), (0.2) with eigenvalues $\lambda_0 < \ldots < \lambda_{n-1} < 0 < \lambda_n < \ldots$ of the linearization (0.6), (0.7) and eigenfunctions ϕ_k. By E^s, E^u, $E^s \oplus E^u = I$ we denote the complementary projections of X onto the stable and unstable spaces of the linearization $L = A - F'(v)$ at v, and by E_k, $k = 0, \ldots, n-1$, $E_0 \oplus E_1 + \ldots \oplus E_{n-1} = E^u$ the projections onto the subspaces spanned by ϕ_k.

THEOREM 2.1. Let v be a hyperbolic stationary solution as above. Then there exists an increasing sequence $W_0 \subset \ldots \subset W_{n-1} = W^u$ of invariant C^κ-submanifolds of the unstable manifold W^u through v such that

(i) $\dim W_k = k+1$, and the tangent space to W_k at v is spanned by $\phi_0, \ldots, \phi_k$;
(ii) for any $w \in W_k \setminus W_{k-1}$

$$\lim_{t\to -\infty} (S_t(w) - v)/|S_t(w) - v| = \pm\, \phi_k \tag{2.1}$$

where the flow S_t for $t<0$ is defined by $S_{-t}(S_t(w)) = w$ on W^u;
(iii) for $w \in W_k \setminus W_{k-1}$ and t near $-\infty$ the zero number z satisfies

$$z(S_t(w) - v) = k;$$

and $S_t(x) - v$ has precisely k simple zeros in (0,1);
(iv) for $w \in W_k \setminus W_{k-1}$, we obtain

$$z(w-v) \leqq k,$$

and consequently for all $w \in W^u$

$$z(w-v) < \dim W^u.$$

Note that by [5, Section 7.3], S_t is well defined for $t<0$ on W^u.

At the end of this section we outline a simple idea for the proof of theorem 2.1 which uses finite dimensionality of W^u. Another idea which also works for the infinite dimensional stable manifold (see Section 3) can be illustrated in the case $\dim W^u = 2$. The linearization of the flow on W^u near v looks like Fig. 1, where ϕ_0, ϕ_1 are represented by the coordinate vectors. All integral curves $\gamma(t) = \alpha_0(t)\phi_0 + \alpha_1(t)\phi_1$ which are not identically zero have the property $\alpha_0(t)\alpha_1^{-1}(t) \to 0$ for $t \to -\infty$ except of two which have $\alpha_1(t) = 0$. Qualitatively, this picture is not destroyed by nonlinearities. The exceptional trajectories become W_0 in the notation of the

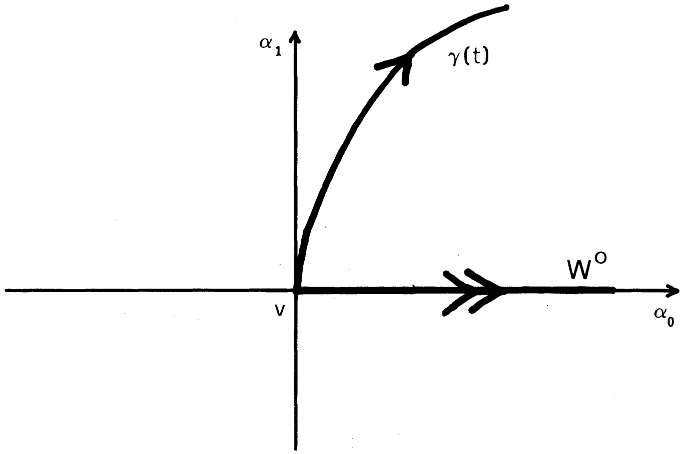

Fig. 1. The strongly unstable manifold W^0 and a general trajectory γ outside W^0.

theorem. A trajectory γ on W^u satisfies

$$\gamma(t) = v + \alpha_0(t)\phi_0 + \alpha_1(t)\phi_1 + O(|\alpha_0(t)| + |\alpha_1(t)|) \quad \text{for } t \to -\infty.$$

The exceptional ones satisfy in addition $\alpha_1(t) = o(\alpha_0(t))$, all the others $\alpha_0(t) = o(\alpha_1(t))$ for $t \to -\infty$. Consequently, $\alpha_0^{-1}(t)(\gamma(t) - v)$ mimicks ϕ_0 in the first case while $\alpha_1^{-1}(t)(\gamma(t) - v)$ mimicks ϕ_1 in the second case for t near $-\infty$. In particular, it will have the same zero number as ϕ_0, ϕ_1 respectively. We employ lemma 1.1 to conclude that $(\gamma(t) - v)$ does not increase with t, hence

$$z(\gamma(0)) \leqq \max(z(\phi_0), z(\phi_1)) = 1.$$

To carry out the idea in detail we need the following.

LEMMA 2.2. Consider a differential equation on a neighbourhood U of the origin in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}) \tag{2.2}$$

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) \tag{2.3}$$

($\mathbf{x} \in \mathbb{R}^p$, $\mathbf{y} \in \mathbb{R}^q$). Assume that all eigenvalues of $\mathbf{A}$ ($\mathbf{B}$) have negative real parts $\leqq a_0$ ($\geqq b_0$, respectively), where $a_0 < b_0 < 0$, $\mathbf{f}$, $\mathbf{g}$ are C^k, $k > 0$ and satisfy

$$\lim_{(\mathbf{x},\mathbf{y})\to 0} \mathbf{f}(\mathbf{x}, \mathbf{y})\, |(\mathbf{x}, \mathbf{y})|^{-1} = \mathbf{0}, \quad \lim_{(\mathbf{x},\mathbf{y})\to 0} \mathbf{g}(\mathbf{x}, \mathbf{y})\, |(\mathbf{x}, \mathbf{y})|^{-1} = \mathbf{0}.$$

Then, there exists a positively invariant neighbourhood Ω of $\mathbf{0}$ and a p-dimensional C^k submanifold W of Ω through $(\mathbf{0}, \mathbf{0})$ tangent to the subspace $\mathbf{y} = \mathbf{0}$ at $(\mathbf{0}, \mathbf{0})$ such that each solution $(\mathbf{x}(t), \mathbf{y}(t))$ of (2.2), (2.3) with $(\mathbf{x}(0), \mathbf{y}(0)) \in \Omega \backslash W$ satisfies

$$\lim_{t\to\infty} |\mathbf{y}(t)|^{-1}\, \mathbf{x}(t) = \mathbf{0}. \tag{2.4}$$

Proof. For the finite dimensional case considered here, it is easy to prove (2.4) directly from (2.2), (2.3), choosing suitable scalar products on $\mathbb{R}^p$, $\mathbb{R}^q$ and deriving a differential inequality for $\eta(t) := |\mathbf{x}(t)|^2/|\mathbf{y}(t)|^2$. However, we give a different proof which carries over without change to an infinite dimensional situation occurring in the stable manifold (see lemma 3.1 and its proof in the appendix).

The existence of Ω and an invariant manifold W tangent to the subspace $\mathbf{y} = \mathbf{0}$ at $(\mathbf{0}, \mathbf{0})$ follows from [4, lemma 4.1 and corollary 5.1, chapter IX]. If Ω is chosen sufficiently small, W can be represented as the graph of a C^k function $\mathbf{h}$ from some neighbourhood of $\mathbf{0}$ in the x-space into $\mathbb{R}^q$ with $\mathbf{h}'(\mathbf{0}) = \mathbf{0}$. It follows from [4] that if one introduces in Ω new coordinates $\mathbf{u} = \mathbf{x}$, $\mathbf{v} = \mathbf{y} - \mathbf{h}(\mathbf{x})$ then the $(\mathbf{u}, \mathbf{v})$-representation $\mathbf{\Phi}: (\mathbf{u}, \mathbf{v}) \to (\mathbf{u}_1, \mathbf{v}_1)$ of the time one map of (2.2), (2.3) satisfies

$$\mathbf{u}_1 = \bar{\mathbf{A}}\mathbf{u} + \mathbf{U}(\mathbf{u}, \mathbf{v}) \tag{2.5}$$

$$\mathbf{v}_1 = \bar{\mathbf{B}}\mathbf{v} + \mathbf{V}(\mathbf{u}, \mathbf{v}) \tag{2.6}$$

with $\mathbf{U}$, $\mathbf{V}$ having similar properties as $\mathbf{f}$, $\mathbf{g}$ in (2.2), (2.3) and, in addition, $\mathbf{V}(\mathbf{u}, \mathbf{0}) = \mathbf{0}$. The time one map of a differential equation maps initial values of its solutions into their values at time one.

By choosing suitable norms $|\cdot|$ in the $\mathbf{u}$-, $\mathbf{v}$-spaces we can assume

$$|\mathbf{A}\mathbf{u}| < (a + \theta)\,|\mathbf{u}|$$

$$|\mathbf{B}\mathbf{v}| > (b - \theta)\,|\mathbf{v}|$$

where $a := \exp a_0$, $b := \exp b_0$ and $0 < \theta < (b - a)/2$, $\theta < b$. Also, there is a positive function $\kappa(\rho)$ on some right neighbourhood of zero such that $\kappa(\rho) \to 0$ for $\rho \to 0$ and

$$|\mathbf{U}(\mathbf{u}, \mathbf{v})| < \kappa(\rho)\,(|\mathbf{u}| + |\mathbf{v}|),\ |\mathbf{V}(\mathbf{u}, \mathbf{v})| < \kappa(\rho)|\mathbf{v}|$$

if $|\mathbf{u}| + |\mathbf{v}| < \rho$.

Let now $(\mathbf{u}, \mathbf{v}) \in \Omega$ and let Ω be so small that $|\mathbf{u}_1| < |\mathbf{u}|$, $|\mathbf{v}_1| < |\mathbf{v}|$. Then, we have

$$\frac{|\mathbf{u}_1|}{|\mathbf{v}_1|} < \frac{(a + \theta)\,|\mathbf{u}| + \kappa(\rho)\,(|\mathbf{u}| + |\mathbf{v}|)}{(b - \theta)\,|\mathbf{v}| - \kappa(\rho)\,|\mathbf{v}|} = \frac{a + \theta + \kappa(\rho)}{b - \theta - \kappa(\rho)}\frac{|\mathbf{u}|}{|\mathbf{v}|} + \frac{\kappa(\rho)}{b - \theta - \kappa(\rho)}. \tag{2.7}$$

Let

$$\alpha \in \left(\frac{a + \theta}{b - \theta}, 1\right) \quad, \beta(\rho) := \frac{\kappa(\rho)}{b - \theta - \kappa(\rho)}.$$

We have $\lim_{\rho \to 0} \beta(\rho) = 0$ and there exists a $\rho_0 > 0$ such that $a + \theta + \kappa(\rho) < \alpha(b - \theta - \kappa(\rho))$ for any $\rho < \rho_0$. From (2.7) we have for $\varepsilon \in (0, 1 - \alpha)$

$$\frac{|\mathbf{u}_1|}{|\mathbf{v}_1|} < (\alpha + \varepsilon)\frac{|\mathbf{u}|}{|\mathbf{v}|} \quad \text{as soon as} \frac{|\mathbf{u}|}{|\mathbf{v}|} > \frac{\beta(\rho)}{\varepsilon}. \tag{2.8}$$

Choose any $(\mathbf{u}_0, \mathbf{v}_0) \in \Omega$ with $\mathbf{v}_0 \neq 0$ and any $\gamma > 0$. We prove that there exists an $N > 0$ such that $|\mathbf{u}_k| < \gamma|\mathbf{v}_k|$ for all $k > N$ where $(\mathbf{u}_k, \mathbf{v}_k) = \mathbf{\Phi}^k(\mathbf{u}_0, \mathbf{v}_0)$. Indeed, assume the contrary. Since $(\mathbf{u}_k, \mathbf{v}_k) \to \mathbf{0}$, there exists an N_0 such that $|\mathbf{u}_k| + |\mathbf{v}_k| < \rho_1 \leqq \rho_0$ for all $k \geqq N_0$, where $\beta(\rho_1)/\varepsilon < \gamma$. From (2.8) it follows that

$$|\mathbf{u}_{k+1}| < \gamma|\mathbf{v}_{k+1}| \quad \text{as soon as} \quad |\mathbf{u}_k| + |\mathbf{v}_k| < \rho_1 \quad \text{and} \quad |\mathbf{u}_k| < \gamma|\mathbf{v}_k| \tag{2.9}$$

If $|\mathbf{u}_k| > \gamma|\mathbf{v}_k|$ for $k \geqq N_0$ then by (2.8) also

$$\gamma < \frac{|\mathbf{u}_k|}{|\mathbf{v}_k|} < (\alpha + \varepsilon)^{k-N_0} \frac{|\mathbf{u}_{N_0}|}{|\mathbf{v}_{N_0}|} \quad \text{for } k \geqq N_0$$

which is impossible. Thus, there exists an $N \geqq N_0$ for which $|\mathbf{u}_N| < \gamma|\mathbf{v}_N|$. By (2.9), we have $|\mathbf{u}_k| < \gamma|\mathbf{v}_k|$ for all $k > N$. Since γ was arbitrary, $\lim_{k\to\infty} |\mathbf{v}_k|^{-1}\mathbf{u}_k = 0$.

For the differential equation (2.2), (2.3) this means that if $(\mathbf{x}(t), \mathbf{y}(t))$ is its solution with $(\mathbf{x}(0), \mathbf{y}(0)) \in \Omega\backslash W$ (or, equivalently, $\mathbf{y}(0) \neq \mathbf{h}(\mathbf{x}(0))$), then

$$\lim_{\substack{k\to\infty \\ k \text{ integer}}} \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|} = 0. \tag{2.10}$$

We have

$$\begin{aligned} \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k)|} &= \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(k))|} \frac{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|}{|\mathbf{y}(k)|} \\ &\leqq \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|} \left(1 + \frac{|\mathbf{h}(\mathbf{x}(k))|}{|\mathbf{y}(k)|}\right) \\ &\leqq \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|} \left(1 + \frac{|\mathbf{h}(\mathbf{x}(k))|}{|\mathbf{x}(k)|} \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k)|}\right), \end{aligned} \tag{2.11}$$

or,

$$\frac{|\mathbf{x}(k)|}{|\mathbf{y}(k)|} \left(1 - \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|} \frac{|\mathbf{h}(\mathbf{x}(k))|}{|\mathbf{x}(k)|}\right) \leqq \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k) - \mathbf{h}(\mathbf{x}(k))|}.$$

Since $\mathbf{h}(\mathbf{x}) = o(|\mathbf{x}|)$, from (2.10), (2.11) we obtain

$$\lim_{\substack{k\to\infty \\ k \text{ integer}}} \frac{|\mathbf{x}(k)|}{|\mathbf{y}(k)|} = 0. \tag{2.12}$$

Let now $k \leqq t < k + 1$. By standard Gronwall estimates and the variation of constants formula we obtain:

$$|(\mathbf{x}(t), \mathbf{y}(t))| \leqq C|(\mathbf{x}(k), \mathbf{y}(k))| =: \rho \qquad \text{for all } k \text{ and } t \in [k, k+1)$$

with some $C \geqq 1$. Again by Gronwall and variation of constants we obtain

$$|\mathbf{x}(t)| \leqq C_1(|\mathbf{x}(k)| + \hat{\kappa}(\rho) \cdot (|\mathbf{x}(k)| + |\mathbf{y}(k)|))$$

$$|\mathbf{y}(t)| \geqq C_2(|\mathbf{y}(k)| - \hat{\kappa}(\rho) \cdot (|\mathbf{x}(k)| + |\mathbf{y}(k)|))$$

for all $k \in \mathbb{N}$, $t \in [k, k+1)$, suitable constants C_1, $C_2 > 0$ and a function $\hat{\kappa}(\rho)$ satisfying

$$\lim_{\rho\to 0} \hat{\kappa}(\rho) = 0$$

Thus we have (for all $k \in \mathbb{N}$, $t \in [k, k+1)$)

$$\frac{|\mathbf{x}(t)|}{|\mathbf{y}(t)|} \leqq \frac{C_1}{C_2} \cdot \frac{|\mathbf{x}(k)| \cdot |\mathbf{y}(k)|^{-1} \cdot (1 + \hat{\kappa}(\rho)) + \hat{\kappa}(\rho)}{1 - \hat{\kappa}(\rho)(1 + |\mathbf{x}(k)| \cdot |\mathbf{y}(k)|^{-1})}$$

and (2.12) readily implies (2.4), completing the proof of the lemma. ■

Proof of theorem 2.1. A neighbourhood V of v in W^u can be considered as an open subset Ω of $\mathbb{R}^n$, the coordinates $\mathbf{z} = (z_0, \ldots, z_{n-1})$ chosen in such a way that $z_k(w) = E_k(w - v)$ for $w \in W^u$ near v. Then, locally at v, the restriction of (0.1), (0.2) to W^u has the form

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{q}(\mathbf{z}), \tag{2.13}$$

where $\mathbf{C} = \operatorname{diag}\{-\lambda_0, \ldots, -\lambda_{n-1}\}$, q is C^κ and $\mathbf{q}'(\mathbf{0}) = \mathbf{0}$.

Consider the associated system

$$\mathrm{d}\mathbf{z}/\mathrm{d}\tau = -\mathbf{C}\mathbf{z} - \mathbf{q}(\mathbf{z})$$

which is obtained from (2.13) by time reversal $\tau = -t$. This system satisfies the assumptions of lemma 2.2 with $\mathbf{x} = (z_0, \ldots, z_{n-2})$, $\mathbf{y} = z_{n-1}$. We denote by $\tilde{W}_{n-2}$ the submanifold W the existence of which is asserted in lemma 2.2. It is given by an equation

$$z_{n-1} = h_{n-2}(z_0, \ldots, z_{n-2}), \qquad \mathbf{z} \in \Omega$$

where h_{n-2} is C^κ and satisfies

$$h_{n-2}(\mathbf{0}) = 0. \tag{2.14}$$

By lemma 2.2, if $\mathbf{z}(t)$ is a solution of (2.13) with

$$\mathbf{z}(0) \in \Omega, z_{n-1}(0) \neq h_{n-2}(z_0, \ldots, z_{n-2}), \tag{2.15}$$

then

$$\lim_{t \to -\infty} |z^{n-1}(t)|^{-1} |(z_0(t), \ldots, z_{n-2}(t))| = 0. \tag{2.16}$$

From (2.14) it follows that

$$\lim_{t \to -\infty} |z_{n-1}(t)|\, |(z_0(t), \ldots, z_{n-2}(t)|^{-1} = 0 \tag{2.17}$$

if (2.15) does not hold. Since W^u is tangent to the unstable space of L, from (2.16), (2.17) it follows respectively

$$\lim_{t \to -\infty} |E_{n-1}(S_t(w) - v)|^{-1}\, |(I - E_{n-1})(S_t(w) - v)| = 0 \tag{2.18}$$

for $w \in V \backslash \tilde{W}_{n-2}$ and

$$\lim_{t \to -\infty} |(E_{n-1} + E^s)\,(S_t(w) - v)|\, |(I - E_{n-1} - E^s)(S_t(w) - v)|^{-1} = 0 \tag{2.19}$$

for $w \in \tilde{W}_{n-2}$. We define $W_{n-2} = \{S_t(\tilde{W}_{n-2}) | t \geqq 0\}$. By [5, theorem 6.1.9], W_{n-2} is an invariant submanifold of W^u. The properties (2.18), (2.19) obviously extend to $w \in W^u \backslash W_{n-2}$, W_{n-2}, respectively.

On W_{n-2}, the differential equation is again of form (2.13) with $\mathbf{C} = \operatorname{diag}\{-\lambda_0, \ldots, -\lambda_{n-2}\}$. Applying lemma 2.2 to the equation on W_{n-2} we obtain an $(n-2)$-dimensional submanifold $\tilde{W}_{n-3}$ of $\tilde{W}_{n-2}$ represented by

$$z_{n-2} = h_{n-3}(z_0, \ldots, z_{n-3})$$

with

$$h_{n-3}(\mathbf{0}) = 0 \tag{2.20}$$

such that

$$\lim_{t\to-\infty} |z_{n-2}(t)|^{-1} \, |(z_0(t), \ldots, z_{n-3}(t))| = 0 \tag{2.21}$$

for all solutions $z(t)$ with $z_{n-2}(0) \neq h_{n-2}(z_0, \ldots, z_{n-3})$. Again, we extend $\tilde{W}_{n-3}$ to an invariant submanifold of W_{n-2} by $W_{n-3} = \{S_t(\tilde{W}_{n-3}),\ t \geqq 0\}$. From (2.19) and (2.21) it follows that

$$\lim_{t\to-\infty} |E_{n-2}(S_t(w)-v)|^{-1} \cdot |(I-E_{n-2})(S_t(w)-v)| = 0$$

for $w \in W_{n-2} \backslash W_{n-3}$ while for $w \in W_{n-3}$ it follows from (2.19) and (2.20) that

$$\lim_{t\to-\infty} \left|\sum_{k=0}^{n-3} E_k(S_t(w)-v)\right|^{-1} \left|\left(I - \sum_{k=0}^{n-3} E_k(S_t(w)-v)\right|\right. = 0.$$

In this way we may proceed further and after $n-1$ steps obtains all the $(k+1)$-dimensional manifolds W_k such that for $w \in W_k \backslash W_{k-1}$ we have

$$\lim_{t\to-\infty} |(I-E_k)(S_t(w)-v)|/|E_k(S_t(w)-v)| = 0.$$

This in turn implies for $w \in W_k \backslash W_{k-1}$ that

$$\lim_{t\to-\infty} (S_t(w)-v)/|S_t(w)-v| = \pm\,\phi_k. \tag{2.1}$$

Recall that the above limit is considered in $X^1 \subset C^1(I)$, and ϕ_k has only simple zeros with $z(\phi_k) = k$. By our remark preceding lemma 1.1 this implies

$$z(S_t(w)-v) = k$$

for t near $-\infty$.

Now we invoke lemma 1.1 for $z(u(t))$, $u(t) := S_t(w) - v$. Note that u satisfies an equation

$$u_t = u_{xx} + \hat{f}(x,u),$$
$$u(t,0) = u(t,1) = 0,$$

where $\hat{f}(x,u) := f(x, u+v(x)) - f(x, v(x))$. Hence $\hat{f}(x,0) = 0$ and lemma 1.1 implies for t near $-\infty$

$$z(w-v) = z(u(0)) \leqq z(u(t)) = z(S_t(w)-v) = k.$$

This completes the proof of theorem 2.1. ■

From our theorem we deduce a relation between the number of changes of monotonicity of a hyperbolic stationary solution v (some "lap-number", cf. [6]) and the zero number $z(w-v)$ on the unstable manifold of v.

COROLLARY 2.3. Let v be a stationary hyperbolic solution of (0.1), (0.2), $v_x \not\equiv 0$, and let $w \in W^u$ be in its unstable manifold. Then

$$z(w - v) < z(v_x).$$

Proof. Due to theorem 2.1 it suffices to prove that $n := \dim W^u \leqq z(v_x)$.
The function $y := v_x$ solves the linearized equation

$$y_{xx} + f_u(x, v(x))y = 0.$$

On the other hand, the eigenfunction ϕ_{n-1} has $n+1$ zeros on the closed interval $[0, 1]$. By the comparison theorem, between any two consecutive zeros of ϕ_{n-1} there has to be a zero of v_x. By $v_x \not\equiv 0$, all zeros of v_x are simple. This implies $z(v_x) \geqq n$ and the proof is complete. ■

We outline an alternate proof of theorem 2.1, (iv) which works only for W^u, as far as we know. Consider any trajectory $u(t)$ on $W^u \backslash \{v\}$ and let $y(t) := u(t)/|u(t)|$ be its projection onto the unit sphere. Then obviously

$$\lim_{t \to -\infty} E^s y(t) = 0.$$

Since W^u is finite dimensional, we may thus pick a sequence $t_k \to -\infty$ such that

$$\phi := \lim_{t_k \to -\infty} y(t_k) \tag{2.22}$$

exists in $X^1 \subset C^1(I)$. But ϕ is in the unstable eigenspace of v, hence Section 1 implies for t_k near $-\infty$

$$z(w - v) = z(u(0)) \leqslant z(u(t_k)) = z(y(t_k)) = z(\phi) < n = \dim W^u,$$

without any intermediate construction of W_k.

3. ZEROS ON THE STABLE MANIFOLD

We turn to investigate the zero number $z(w - v)$ on the stable manifold W^s of the hyperbolic stationary solution v of (0.1), (0.2), keeping the assumptions and notations of Section 2 in effect.

Similarly to the unstable case we need the following lemma on the fine structure of W^s.

LEMMA 3.1. Assuming hyperbolicity of v above and $f \in C^\kappa$, $\kappa \geqq 2$, there exists a decreasing sequence $W^s = W_n \supset W_{n+1} \supset \dots$ of invariant C^κ-submanifolds of the stable manifold W^s through v such that

(i) the tangent space to W_k at v is spanned by $\phi_k, \phi_{k+1}, \dots$

(ii) for any $w \in W_k \backslash W_{k+1}$

$$\lim_{t \to \infty} (S_t(w) - v)/|S_t(w) - v| = \pm\, \phi_k. \tag{3.1}$$

We defer the proof of this lemma to the appendix.

As an immediate consequence of lemma 3.1 we can conclude for $w \in W_k \backslash W_{k+1}$, $k \geqq n$, that $u(t) := S_t(w) - v$ satisfies

$$z(w - v) \geqq \lim_{t\to\infty} z(u(t)) = \lim_{t\to\infty} z(u(t)/|u(t)|) \geqq z\left(\lim_{t\to\infty} u(t)/|u(t)|\right) \tag{3.2}$$
$$= z(\pm\phi_k) = k,$$

by lower semicontinuity of z and monotonicity of z (lemma 1.1). However, this does not imply $z \geqq n$ on all of W^s, if for example

$$\bigcap_{k\geqq n} W_k \neq \{v\}.$$

To remedy this point we use the following alternative which is proved in [1]:

(i) either $z(u(t))$ stays infinite for all $t \geqq 0$;

(ii) or $z(u(t_0)) < \infty$ for some $t_0 \geqq 0$, and $u(t)$ has only simple zeros for an open dense set of $t \in [t_0, \infty)$.

Using this fact, we will conclude below that

$$\bigcap_{k\geqq n} W_k \subset \{w | z(w - v) = \infty\} \cup \{v\}.$$

THEOREM 3.2. Let v be a hyperbolic stationary solution of (0.1), (0.2) as above. Then for $w \in W_k \subseteq W^s$, $w \neq v$ we obtain

$$z(w - v) \geqq k$$

and in particular for all $w \in W^s \backslash \{v\}$

$$z(w - v) \geqq \dim W^u.$$

Proof. With the preceding remarks it is sufficient to prove for $w \neq v$

$$z(w - v) \geqq k \qquad \text{for all } w \in W_{k+1}, \qquad k \geqq n.$$

Obviously we may assume that $z(w - v) < \infty$. Then, by [1, theorem], there exists a $t \geqq 0$ such that $u(t, \cdot) = S_t(w) - v$ has only simple zeros. Because W_{k+1} has codimension 1 in W_k we may then choose $\tilde{u} \in W_k \backslash W_{k+1}$ such that

$$z(\tilde{u}) = z(u(t))$$

(just choosing $\|u - u(t)\|_{C^1(I)}$ small enough). But by the remarks above

$$z(\tilde{u}) \geqq k,$$

thus monotonicity of z (lemma 1.1) yields

$$z(w - v) = z(u(0)) \geqq z(u(t)) = z(\tilde{u}) \geqq k$$

and we are done. ∎

4. APPENDIX

We give a proof of the fine structure of the stable manifold claimed in lemma 3.1. To this end we first construct an invariant manifold corresponding to a line, splitting the spectrum of the linearization. We use a general analytic semigroup setting

$$\frac{du}{dt} + Au = f(u) \tag{4.1}$$

in a Banach space X with norm $|\cdot|$, where A is sectorial linear $X \to X$; f: $U \to X$ is C^κ, where U is a neighborhood of 0 in X^α, $\kappa \geqq 1$, $0 \leqq \alpha < 1$; $f(0) = 0$.

Let $L := A - f'(0)$ have spectrum $\sigma(L)$. By $u(t; u_0)$ we denote the solution of (4.1) with initial data $u(0; u_0) = u_0 \in X^\alpha$.

The following lemma is well known in the finite dimensional case. It replaces [4, lemma 5.1 and corollary 5.1, chapter IX] in the proof of the infinite dimensional version of lemma 2.2. Its proof is modelled in close analogy to [5, theorem 5.2.1]. Nevertheless, for the convenience of the reader we give a detailed proof.

LEMMA 4.1. Assume $\gamma > 0$ is such that $\sigma(L) = \sigma_1 \cup \sigma_2$, $\sigma_1 = \sigma(L) \cap \{\mathrm{Re}\,\lambda < \gamma\}$, $\sigma_2 = \sigma(L) \cap \{\mathrm{Re}\,\lambda > \gamma\}$ is a decomposition of $\sigma(L)$ into spectral sets. Let $X = X_1 \oplus X_2$ be the decomposition of X corresponding to the decomposition of $\sigma(L)$ and let E_1 and E_2 be the spectral projections onto X_1 and X_2 respectively, $E_1 \oplus E_2 = I$.

Then there exist $\rho > 0$, $M > 0$ and a local invariant C^κ submanifold S of the ball $\{|u|_\alpha < \rho/2M\}$ such that:
(i) S is C^κ diffeomorphic under $E_2|_S$ to an open neighborhood of 0 in $X_2^\alpha := X_2 \cap X^\alpha$;
(ii) S is tangent to X_2^α at 0;
(iii) if $|E_2 u(0)|_\alpha < \rho/2M$ and $|u(t)|_\alpha e^{\gamma t} < \rho$ for all $t \geqq 0$ then $u(0) \in S$;
(iv) if $u(0) \in S$ then

$$\sup_{t \geqq 0} |u(t)|_\alpha\, e^{\gamma t} < \infty.$$

Proof. Without loss of generality assume $\sigma(A) \subset \{\mathrm{Re}\,\lambda > 0\}$. By L_1, L_2 denote the restrictions of L to X_1, X_2 respectively, let $T_i(t) := \exp(-L_i t)$ be the semigroup on X_i generated by L_i and $u_i := E_i u$ the X_i-component of u. Note that $\dim X_1 < \infty$, L_1 is bounded and there exist $0 < \beta < \gamma < \delta$ such that

$$|T_1(t)| \leqq M e^{-\beta t}, \; |A^\alpha T_1(t)| \leqq M e^{-\beta t} \qquad \text{for } t \leqq 0,$$
$$|A^\alpha T_2(t) E_2 A^{-\alpha}| \leqq M e^{-\delta t}, \; |A^\alpha T_2(t)| \leqq M t^{-\alpha} e^{-\delta t} \qquad \text{for } t \geqq 0. \tag{4.2}$$

Write $g(u) := f(u) - f'(0)u$ with components $g_i := E_i g$. Then there exists a positive function k on $(0, \rho_0)$, $\rho_0 > 0$ such that $k(\rho) \to 0$ for $\rho \to 0$ and

$$|g(u^1) - g(u^2)| \leqq k(\rho)\, |u^1 - u^2|_\alpha$$

as soon as $|u^j|_\alpha < \rho$, $j = 1, 2$. By [5, lemma 3.3.2], $u(t)$ solves (4.1) iff $u(t)$ solves the variation of constants version of (4.1)

$$\begin{aligned} u_1(t) &= T_1(t) u_1(0) + \int_0^t T_1(t-s) g_1(u(s))\, ds \\ u_2(t) &= T_2(t) u_1(0) + \int_0^t T_2(t-s) g_2(u(s))\, ds. \end{aligned} \tag{4.1$'$}$$

Assuming that the solution $u(t)$ satisfies

$$|u(t)|_\alpha\, e^{\gamma t} \text{ is bounded as } t \to \infty, \tag{4.3}$$

we conclude that for $t \to \infty$

$$|T_1(-t) u_1(t)|_\alpha \leqq M e^{\beta t} |u_1(t)|_\alpha \to 0$$

which implies

$$u_1(0) = -\int_0^\infty T_1(-s) g_1(u(s))\, ds,$$

and, again by (4.1)$'$, we obtain

$$u(t) = T_2(t) a + \int_0^t T_2(t-s) g_2(u(s))\, ds - \int_t^\infty T_1(t-s) g_1(u(s))\, ds \tag{4.4}$$

where $a := E_2 u(0) \in X_2$.

We show that for $\rho > 0$ sufficiently small integral equation (4.4) has a unique solution $u_a(t)$ satisfying $|u_a(t)|_\alpha e^{\gamma t} < \rho$ provided $|a|_\alpha < \rho/2M$.

Let R_ρ be the set of continuous functions u: $[0,\infty)\to X^\alpha$ such that

$$\|u(\cdot)\| := \sup_{t\geqq 0} |u(t)|_\alpha \, e^{\gamma t} \leqq \rho$$

is finite. The set R_ρ endowed with the metric generated by $\|\cdot\|$ is a complete metric space. We claim that for ρ small enough and $\|a\|_\alpha < \rho/2M$, $a\in X_2^\alpha$ the map F_a defined by

$$F_a(u(\cdot))\,(t) := T_2(t)a + \int_0^t T_2(t-s)g_2(u(s))\,ds - \int_t^\infty T_1(t-s)g_1(u(s))\,ds$$

is a contraction $R_\rho\to R_\rho$. Indeed

$$\begin{aligned}
\|F_a(u(\cdot))\| &\leqq \sup_{t\geqq 0} e^{\gamma t}|T_2(t)a|_\alpha + \sup_{t\geqq 0}\int_0^t e^{\gamma t}|A^\alpha T_2(t-s)|\cdot|g_2(u(s))|\,ds\\
&+ \sup_{t\geqq 0}\int_t^\infty e^{\gamma t}|A^\alpha T_1(t-s)|\cdot|g_1(u(s))|\,ds\\
&\leqq M|a|_\alpha + |E_2|\sup_{t\geqq 0}\int_0^t e^{\gamma t}M(t-s)^{-\alpha}e^{-\delta(t-s)}k(\rho)|u(s)|_\alpha\,ds\\
&+ |E_1|\sup_{t\geqq 0}\int_t^\infty e^{\gamma t}Me^{-\beta(t-s)}k(\rho)|u(s)|_\alpha\,ds\\
&\leqq M|a|_\alpha + |E_2|M\cdot k(\rho)\int_0^\infty t^{-\alpha}e^{(\gamma-\delta)t}\,dt\cdot\|u(\cdot)\|\\
&+ |E_1|Mk(\rho)\int_0^\infty e^{(\beta-\gamma)t}\,dt\cdot\|u(\cdot)\|\\
&\leqq M\cdot|a|_\alpha + Mk(\rho)\cdot C\|u(\cdot)\|,
\end{aligned} \tag{4.5}$$

with some constant C independent of ρ. Thus, if $|a|_\alpha < \rho/2M$ and $\rho>0$ is small enough that $k(\rho)\cdot C < \rho/2M$, then F_a maps R_ρ into R_ρ. Also, repeating the same steps as in (4.5) we find

$$\|F_a(u^1(\cdot)) - F_a(u^2(\cdot))\| \leqq \tfrac{1}{2}\|u^1(\cdot) - u^2(\cdot)\|$$

as soon as $\|u^j(\cdot)\|\leqq\rho$, $j=1,2$, so F_a is a contraction in R_ρ. Consequently, F_a has a unique fixed point $u(\cdot)\in R_\rho$ which solves (4.4).

The map $(u(\cdot),a)\to F_a(u(\cdot))$ is C^κ on $R_\rho\times(\{|a|_\alpha<\rho/2M\}\cap X_2^\alpha)$. Indeed, the map is linear in a and estimating as in (4.5) one obtains

$$\sup_{t\geqq 0} e^{\gamma t}\Big|\varepsilon^{-1}(F_a(u(\cdot)+\varepsilon v(\cdot))(t) - F_a(u(\cdot))(t)) - \int_0^t T_2(t-s)g_2'(u(s))v(s)\,ds + \int_t^\infty T_1(t-s)g_1'(u(s))v(s)\,ds\Big|_\alpha \to 0 \qquad \text{for } \varepsilon\to 0.$$

Therefore

$$(v(\cdot),b)\to T_2(t)b + \int_0^t T_2(t-s)g_2'(u(s))v(s)\,ds - \int_t^\infty T_1(t-s)g_1'(u(s))v(s)\,ds \tag{4.6}$$

is the Gâteaux differential of the map $(u(\cdot),a)\to F_a(u(\cdot))$. Since the map (4.6) is continuous in $(v(\cdot),b)$, the differential is Fréchet and $(u(\cdot),a)\to F_a(u(\cdot))$ is C^1. To obtain C^κ we iterate the arguments above.

By [5, 1.2.6] the fixed point $u_a(\cdot)$ of F_a is a C^κ-function of a in $\{|a|_\alpha<\rho/2M\}\cap X_2^\alpha$. Consequently the map h: $\{|a|_\alpha<\rho/2M\}\cap X_2^\alpha\to X_\alpha$ defined by

$$h(a) := u_a(0) = a - \int_0^\infty T_1(-s)g_1(u_a(s))\,ds$$

is C^κ and, since $E_2h(a) = E_2a = a$, has a C^κ inverse on its image S. Thus,

$$h\colon \{|a|_\alpha<\rho/2M\}\cap X_2^\alpha\to X_\alpha$$

is a C^κ-diffeomorphism. This proves (i) and, using $g_1'(0)=0$, as a direct consequence (ii). By definition of R_ρ, (iv) holds.

By construction and (4.4), S is invariant with respect to the semiflow (4.1). If $|E_2u(0)|_\alpha<\rho/2M$ and $|u(t)|_\alpha e^{\gamma t}<\rho$

for all $t \geqq 0$, then we have shown that $u(\cdot)$ satisfies (4.4). Since $u(\cdot) \in R_\rho$ and $u(t) = u_a(t)$ with $a := E_2(0$ $u(0) \in S$. Thus (iii) holds and the proof is complete. ■

Proof of lemma 3.1. Existence of the manifolds W_k as claimed in lemma 3.1 follows from lemma 4.1, c with $\lambda_{k-1} < \gamma < \lambda_k$.

Using existence of the manifolds W_k, we apply the proof of lemma 2.2 successively for each k on a neighborhood U of $v := 0$ (w.l.o.g.) in W_k, with coordinates $y = E_k u$ and $x = \sum_{j>k} E_j u$ as in the notation of Section 2. Note that the proof of lemma 2.2 carries over to analytic semigroups without the assumption that x is finite dimensional. Now lemma 2.2, together with $u(t) = S_t(w) \to 0$ and lemma 4.1, (ii) imply

$$\pm\, \phi_k = \lim_{t\to\infty} \frac{\sum_{j\geqq k} E_j u(t)}{|E_k u|} = \lim_{t\to\infty} \frac{\sum_{j\geqq k} E_j u(t)}{\left|\sum_{j\geqq k} E_j u(t)\right|} = \lim_{t\to\infty} \frac{u(t)}{|u(t)|}$$

and the proof is complete. ■

Acknowledgements—The authors are indebted to W. Alt for helpful criticism.

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