NUMBERS OF ZEROS ON INVARIANT MANIFOLDS IN REACTION–DIFFUSION EQUATIONS

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INTRODUCTION

Consider the one-dimensional reaction–diffusion equation

\[ u_t = u_{xx} + f(x, u), \quad t > 0, \quad 0 < x < 1 \]  

(0.1)

with the Dirichlet boundary conditions

\[ u(t, 0) = u(t, 1) = 0, \]  

(0.2)

where \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is \( BC^1 \cap C^k, \ k > 1. \) The equations (0.1), (0.2) can be viewed as a particular case of the abstract equation

\[ \frac{du}{dt} + Au = f(u) \]  

(0.3)

in a Banach space \( X, \) the basic theory of which is developed in [5]. For (0.1), (0.2), \( X = L^2[0, 1], \) \( A \) is the closure of the operator defined by \( Av = -v'' \) for \( v \in C^2[0, 1], \) \( v(0) = v(1) = 0, \) \( F: L^2[0, 1] \to L^2[0, 1] \) is given by \( F(v)(x) = f(x, v(x)). \) We frequently work in the Hilbert space \( X^1 = \mathcal{D}(A) = H^1_0([0, 1]) \cap H^2([0, 1]) \) with \( F: X^1 \to X^1 \) also being \( C_k. \) Let \( |\cdot| \) denote the norm on \( X^1. \)

Applying the results of [5] one obtains that (0.1), (0.2) generates a local semiflow \( S \) on \( X^1. \) The semiflow \( S \) is a continuous map of an open neighbourhood \( U \) of \( \{0\} \times X^1 \) in \( \mathbb{R}^+ \times X^1 \) into \( X^1 \) defined by

\[ S_t(v)(x) = u(t, x) \quad \text{for} \quad (t, v) \in U, \]

where \( u \) is the solution of (0.1), (0.2), satisfying

\[ u(0, x) = v(x) \quad \text{for} \quad 0 < x < 1. \]  

(0.4)

It has the properties \( S_0(v) = v, \) \( S_{t+s}(v) = S_t \circ S_s(v) \) as long as \( (s, v) \) and \( (t, S_t(v)) \) are in \( U \) [5]. In order not to obscure the formulations by technicalities we shall assume that \( S \) is a global semiflow, i.e. \( U = \mathbb{R}^+ \times X^1. \) This is by no means an essential restriction; sufficient conditions can be found in [5, Chapter 3].

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The critical points of \( S \) are the stationary solutions of (0.1), (0.2), i.e. the solutions of the equation

\[
v'' + f(x, v) = 0, \quad v(0) = v(1) = 0.
\] (0.5)

The qualitative properties of \( S \) near a critical point \( v \) are determined by the linearization of (0.1), (0.2) at \( v \) which is the equation

\[
y_t = y_{xx} + f_u(x, v(x)) y
\] (0.6)
\[
y(t, 0) = y(t, 1) = 0.
\] (0.7)

The solution \( v \) is called hyperbolic if 0 is not an eigenvalue of the operator \( L = A - F'(v) \), i.e. (0.6), (0.7) do not admit a nontrivial stationary solution \( y \).

An important information about the global structure of the semiflow of (0.1), (0.2) is given by the orbit connections of different stationary solutions [2, 3, 5]. By a connecting orbit of the stationary solutions \( v_1, v_2 \) we understand a solution \( u \) of (0.1), (0.2) which exists for all \( t \in (-\infty, \infty) \) and satisfies

\[
\lim_{t \to -\infty} u(t, x) = v_1(x), \quad \lim_{t \to \infty} u(t, x) = v_2(x)
\]
in \( H^2[0, 1] \). In the terminology of [5], \( u(t, \cdot) \) has to be in the stable manifold of \( v_2 \) and the unstable manifold of \( v_1 \), provided \( v_1, v_2 \) are hyperbolic.

In this paper we obtain estimates on the number of zeros (or, more precisely, the zero number defined below) of \( u(t, \cdot) \)-\( v_1 \) and \( u(t, \cdot) \)-\( v_2 \). This information can be used to conclude existence and nonexistence of connections. Our approach provides an alternative to the (slightly different) zero number of \( u_0 \), which was used by Hale and Nascimento [3] to solve the connection problem for \( f \) of the Chafee–Infante type (see e.g. [5, Section 5.3]).

For any continuous function \( \varphi : [0, 1] \to \mathbb{R} \) we define the zero number \( z(\varphi) \) as follows. Let \( n \geq 0 \) be the maximal element of \( \mathbb{N}_0 \cup \{\infty\} \) such that there is a strictly increasing sequence \( 0 \leq x_0 < x_1 < \ldots < x_n \leq 1 \) with \( \varphi(x_j) \) of alternating signs:

\[
\varphi(x_j) \cdot \varphi(x_{j+1}) < 0 \quad \text{for } 0 \leq j < n.
\]

If \( n \) is finite let \( z(\varphi) := n \), and \( z(\varphi) := \infty \) otherwise. Note that we put \( z(0) := 0 \).

As a first example consider the linearized equation (0.6), (0.7). The operator \( L = A - F'(v) \) has eigenvalues \( \lambda_0 < \lambda_1 < \ldots \) with eigenfunctions \( \varphi_0, \varphi_1, \ldots \). By Sturm–Liouville theory \( z(\varphi_k) = k \) and indeed it is a classical result (see [0, p. 549]) that for \( 0 \leq i < j < \infty \)

\[
i \leq z(\varphi) \leq j,
\] (0.8)

whenever \( \varphi \) is a (nontrivial) linear combination of \( \varphi_0, \ldots, \varphi_j \). As a trivial illustration of our approach we prove estimate (0.8) in corollary 1.2, using the dynamic equation (0.6), (0.7).

All our results depend on a basic observation, lemma 1.1, going back to Redheffer, Walter [8] and, more recently, Matano [6]. According to lemma 1.1,

\[
z(u(t, \cdot)) \text{ is nonincreasing}
\]
as a function of time \( t \) along solutions of equation (0.1), (0.2) provided that \( f \) satisfies the condition.

\[
f(x, 0) = 0 \quad \text{for } 0 < x < 1.
\] (0.9)
The proof is elementary and relies on the maximum principle for parabolic equations. For the convenience of the reader we present it in detail below.

In the nonlinear case, let $v$ be a hyperbolic stationary solution of (0.1), (0.2). Then the eigenvalues $\lambda_j$ of the linearized equation with corresponding eigenfunctions $\phi_j$ satisfy $\lambda_0 < \ldots < \lambda_{n-1} < 0 < \lambda_n < \ldots$ for some $n \geq 0$. Further by [5, theorems 5.2.1, 6.1.9] there exist immersed invariant $C^K$-manifolds $W^u$ and $W^s \subset X^1$ of the flow $S$ through $v = 0$ with the properties:

(i) for $w \in W^u$ (resp. $W^s$) the solution $u(t, \cdot) = S(t)w$ exists for all real $t$ and satisfies $\lim u(t, \cdot) = v$ as $t \to -\infty$, (resp. $t \to +\infty$);
(ii) the tangent space of $W^u$ (resp. $W^s$) at $v$ is spanned by the $\phi_k$ with $k < n$ (resp. $k \geq n$).

$W^u$ is called the unstable manifold and $W^s$ the stable manifold of $v$.

Our main result, given in Sections 2 and 3, states that

\[ z(w - v) < \dim W^u \text{ for } w \in W^u \]  
(0.10)

(theorem 2.1) and

\[ z(w - v) \geq \dim W^u \text{ for } w \in W^u \setminus \{v\} \]  
(0.11)

(theorem 3.2). Note that these estimates are suggested by the respective tangent spaces of $W^u$ and $W^s$, together with the Sturm–Liouville estimate (0.8).

The crucial observation of our proof is that for $v \equiv 0$:

\[ \lim_{t \to -\infty} \frac{u(t)}{|u(t)|} = \phi_k \]  
(0.12)

— for $t \to -\infty$ on $W^u$ and some $k < n$
— for $t \to +\infty$ on $W^s$ and some $k \geq n$, provided that $z(u(t, \cdot))$ is eventually finite.

Actually it is quite simple to prove (0.12) on $W^u$, as we will indicate at the end of Section 2. However, analysis on the infinite dimensional stable manifold $W^s$ is quite delicate and we need detailed information on the fine structure of $W^s$ before we can prove (0.12). For illustration we pursue an analogous approach to $W^u$ in Section 1, as a preparation to the stable manifold case.

1. COUNTING ZEROS

In the introduction we defined the zero number $z(\phi)$ of a continuous real function $\phi$ as the maximal number of sign changes of $\phi$. In this section we show that $z$ decreases along solutions $u(t, \cdot)$ of the parabolic equation (0.1) with Dirichlet boundary conditions, assuming that

\[ f(x, 0) = 0 \text{ for all } x \in I, \]  
(1.1)

$I := [0, 1]$. This result is essentially in [8, corollary 3] who consider $f = f(t, x, u, u_x)$ independent of $u$. Similarly, Matano [6] investigates the lap number of $\phi$, which is the zero number of $\phi_x$ and was called “maximum order of a saw in $\phi$” by Redheffer and Walter [8].

Note that by definition the function

\[ z: C^0(I) \to \mathbb{N} \cup \{\infty\} \]

is lower semicontinuous. Further, $z$ is constant in a $C^1$-neighbourhood of any $C^1$-function $\phi$ with only simple zeros. These trivial facts will become important later on.
The parabolic equation (0.1), (0.2) generates a semiflow \( S(t)u_0 = u(t) = u(t, \cdot) \) on \( X^1 \subset H^1_0 \subset C^0(I) \), thus \( z(u(t)) \) is well defined along solutions.

**Lemma 1.1.** [6, 8]. Let \( f(x, 0) = 0 \) for all \( x \in I \). Then the zero number \( z(u(t, \cdot)) \) is nonincreasing as a function of \( t \) along solutions \( u(\cdot, \cdot) \) of (0.1), (0.2).

**Proof.** With \( a(t, x) := \left( f(x, u(t, x)) / (u(t, x)) \right) \) we write (0.1) as

\[
 u_t = u_{xx} + au, \tag{1.2}
\]

where \( a \) is \( C^0 \). We apply the maximum principle to (1.2) to prove: if \( x_1', x_2' \in I \) are such that \( u(t, x_1') < 0 < u(t, x_2') \) then there exist continuous paths \( \gamma_i \) in \( I \times [0, t] \) connecting \( (t, x_1') \) to a point \( (0, x_i) \), such that \( u < 0 \) (resp. \( u > 0 \)) along \( \gamma_1 \) (resp. \( \gamma_2 \)). To see that assume \( 0 < x_1' < x_2' < 1 \), the case \( x_1' > x_2' \) is analogous. Let \( D_i \) be the path connected component of \( (t, x') \) in the relatively open set

\[
 K_i := \{ (\tau, \xi) \in [0, t] \times I \mid (-1)^i u(\tau, \xi) > 0 \}.
\]

We claim that we can find elements \( (0, x_i) \in D_i \). Otherwise, e.g. \( D_2 \) is contained in the strip \( (0, t] \times I \). Replacing \( u \) by \( we^{wt} \) does not change \( d_2 \) and allows us to assume \( a < 0 \), hence \( Au := u_{xx} - u_t \equiv 0 \) on \( D_2 \). Let \( M := \max u > 0 \) and choose a point \( (i, \hat{x}) \in D_2 \) with minimal \( i \) such that \( u(i, \hat{x}) = M \). From \( M > 0 \) we conclude \( (i, \hat{x}) \in D_2 \), hence \( i > 0 \). This implies a contradiction to the strong maximum principle: let \( E := D_2 \) and apply [7, III.2, lemma 3] to conclude \( u < M \) on \( D_2 \cap \{ (i) \times I \} \), contradicting \( (i, \hat{x}) \in D_2 \cap \{ (i) \times I \} \). Therefore there are points \( (0, x_i) \in D_i \).

Invoking the Jordan curve theorem completes the proof. \( \blacksquare \)

As a trivial but illustrative application, we prove estimate (0.8) for finite linear combinations

\[
 \phi^0 = \sum_{k=i}^j \alpha_k \cdot \phi_k \tag{1.3}
\]

of Sturm–Liouville eigenfunctions \( \phi_k \) for the potential \( a(x) := f_u(x, v(x)) \). We use the flow (1.2), defining a solution \( \phi(t, \cdot) \) with initial condition \( \phi(0, \cdot) = \phi^0 \) and Dirichlet conditions.

**Corollary 1.2.** If the Sturm–Liouville potential \( a \) is continuous, \( 0 \leq i < j < \infty \) and \( \phi^0 \neq 0 \), then

\[
 i \leq z(\phi^0) \leq j
\]

**Proof.** We use the explicit representation

\[
 \phi(t, \cdot) = \sum_{k=i}^j \alpha_k e^{\lambda_k t} \phi_k \tag{1.4}
\]

of the solution \( \phi(t, \cdot) \), \( t \in \mathbb{R} \) of (1.2) through \( \phi^0 \). From (1.4), \( \phi^0 \neq 0 \) it is immediate that there exist integers \( k^\pm \in \{ i, i+1, \ldots, j \} \) such that

\[
 \lim_{t \to \pm \infty} \phi(t)/|\phi(t)| = \text{sign}(\alpha_k \pm) \phi_k \pm
\]

in the \( C^1 \)-topology (normalizing \( |\phi_k| = 1 \)), because the \( \lambda_k \) are pairwise disjoint. The \( \phi_k \) have
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only simple zeros, hence \( z \) is constant in a \( C^1 \)-neighbourhood of \( \phi_k \). By monotonicity of \( z \) along solutions of (1.2) (lemma 1.1) we conclude for \( T > 0 \) sufficiently large

\[
i \leq k^+ = z(\phi(T, \cdot)/|\phi(T, \cdot)|) = z(\phi(T, \cdot)) \leq z(\phi^0) \leq z(\phi(-T, \cdot)) = k^- \leq j
\]

and the proof is complete.  

Note that the corollary holds even if \( j = \infty \).

2. ZEROS ON THE UNSTABLE MANIFOLD

In this section we prove that for any element \( w \) of an \( n \)-dimensional unstable manifold of \( v \) there are less than \( n \) zeros of \( w - v \). On our way we investigate the fine structure of the unstable manifold. Finally we relate \( z(w - v) \) to the number of zeros of \( v \).

Let \( v \) be a hyperbolic stationary solution of (0.1), (0.2) with eigenvalues \( \lambda_0 < \ldots < \lambda_{n-1} < 0 < \lambda_n < \ldots \) of the linearization (0.6), (0.7) and eigenfunctions \( \phi_k \). By \( E^s, E^u, E^s \oplus E^u = I \) we denote the complementary projections of \( X \) onto the stable and unstable spaces of the linearization \( L = A - F'(v) \) at \( v \), and by \( E_k, k = 0, \ldots, n-1 \), \( E_0 \oplus E_1 + \ldots \oplus E_{n-1} = E^u \) the projections onto the subspaces spanned by \( \phi_k \).

**Theorem 2.1.** Let \( v \) be a hyperbolic stationary solution as above. Then there exists an increasing sequence \( W_0 \subset \ldots \subset W_{n-1} = W^u \) of invariant \( C^\infty \)-submanifolds of the unstable manifold \( W^u \) through \( v \) such that

(i) \( \dim W_k = k + 1 \), and the tangent space to \( W_k \) at \( v \) is spanned by \( \phi_0, \ldots, \phi_k \);

(ii) for any \( w \in W_k \setminus W_{k-1} \)

\[
\lim_{t \to -\infty} (S_t(w) - v)/|S_t(w) - v| = \pm \phi_k
\]

where the flow \( S_t \) for \( t \leq 0 \) is defined by \( S_{-t}(S_t(w)) = w \) on \( W^u \);

(iii) for \( w \in W_k \setminus W_{k-1} \) and \( t \) near \( -\infty \) the zero number \( z \) satisfies

\[
z(S_t(w) - v) = k;
\]

and \( S_t(x) - v \) has precisely \( k \) simple zeros in (0,1);

(iv) for \( w \in W_k \setminus W_{k-1} \), we obtain

\[
z(w - v) \leq k,
\]

and consequently for all \( w \in W^u \)

\[
z(w - v) < \dim W^u.
\]

Note that by [5, Section 7.3], \( S_t \) is well defined for \( t \leq 0 \) on \( W^u \).

At the end of this section we outline a simple idea for the proof of theorem 2.1 which uses finite dimensionality of \( W^u \). Another idea which also works for the infinite dimensional stable manifold (see Section 3) can be illustrated in the case \( \dim W^u = 2 \). The linearization of the flow on \( W^u \) near \( v \) looks like Fig. 1, where \( \phi_0, \phi_1 \) are represented by the coordinate vectors. All integral curves \( \gamma(t) = \alpha_0(t)\phi_0 + \alpha_1(t)\phi_1 \) which are not identically zero have the property \( \alpha_0(t)\alpha_1^{-1}(t) \to 0 \) for \( t \to -\infty \) except of two which have \( \alpha_1(t) = 0 \). Qualitatively, this picture is not destroyed by nonlinearities. The exceptional trajectories become \( W_0 \) in the notation of the
A trajectory $\gamma$ on $W^u$ satisfies
$$\gamma(t) = v + \alpha_0(t)\phi_0 + \alpha_1(t)\phi_1 + O(|\alpha_0(t)| + |\alpha_1(t)|) \quad \text{for } t \to -\infty.$$ 

The exceptional ones satisfy in addition $\alpha_1(t) = o(\alpha_0(t))$, all the others $\alpha_0(t) = o(\alpha_i(t))$ for $t \to -\infty$. Consequently, $\alpha_0^{-1}(t)(\gamma(t) - v)$ mimicks $\phi_0$ in the first case while $\alpha_1^{-1}(t)(\gamma(t) - v)$ mimicks $\phi_1$ in the second case for $t$ near $-\infty$. In particular, it will have the same zero number as $\phi_0$, $\phi_1$ respectively. We employ lemma 1.1 to conclude that $(\gamma(t) - v)$ does not increase with $t$, hence
$$z(\gamma(0)) \leq \max(z(\phi_0), z(\phi_1)) = 1.$$ 

To carry out the idea in detail we need the following.

**Lemma 2.2.** Consider a differential equation on a neighbourhood $U$ of the origin in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ defined by
$$\dot{x} = Ax + f(x, y) \quad (2.2)$$
$$\dot{y} = By + g(x, y) \quad (2.3)$$

$(x \in \mathbb{R}^p, y \in \mathbb{R}^q)$. Assume that all eigenvalues of $A$ ($B$) have negative real parts $\leq a_0$ ($\leq b_0$, respectively), where $a_0 < b_0 < 0$, $f$, $g$ are $C^k$, $k > 0$ and satisfy
$$\lim_{(x, y) \to 0} f(x, y) |(x, y)|^{-1} = 0, \lim_{(x, y) \to 0} g(x, y) |(x, y)|^{-1} = 0.$$ 

Then, there exists a positively invariant neighbourhood $\Omega$ of 0 and a $p$-dimensional $C^k$ submanifold $W$ of $\Omega$ through $(0, 0)$ tangent to the subspace $y = 0$ at $(0, 0)$ such that each solution $(x(t), y(t))$ of (2.2), (2.3) with $(x(0), y(0)) \in \Omega \setminus W$ satisfies
$$\lim_{t \to \infty} |x(t)|^{-1} x(t) = 0. \quad (2.4)$$
Proof. For the finite dimensional case considered here, it is easy to prove (2.4) directly from (2.2), (2.3), choosing suitable scalar products on $\mathbb{R}^p$, $\mathbb{R}^q$ and deriving a differential inequality for $\eta(t) := |x(t)|^2/|y(t)|^2$. However, we give a different proof which carries over without change to an infinite dimensional situation occurring in the stable manifold (see lemma 3.1 and its proof in the appendix).

The existence of $\Omega$ and an invariant manifold $W$ tangent to the subspace $y = 0$ at $(0,0)$ follows from [4, lemma 4.1 and corollary 5.1, chapter IX]. If $\Omega$ is chosen sufficiently small, $W$ can be represented as the graph of a $C^k$ function $h$ from some neighbourhood of $0$ in the $x$-space into $\mathbb{R}^q$ with $h'(0) = 0$. It follows from [4] that if one introduces in $\Omega$ new coordinates $u = x$, $v = y - h(x)$ then the $(u, v)$-representation $\Phi: (u, v) \rightarrow (u_1, v_1)$ of the time one map of (2.2), (2.3) satisfies

$$u_1 = \tilde{A}u + U(u, v)$$
$$v_1 = \tilde{B}v + V(u, v)$$

with $U$, $V$ having similar properties as $f$, $g$ in (2.2), (2.3) and, in addition, $V(u, 0) = 0$. The time one map of a differential equation maps initial values of its solutions into their values at time one.

By choosing suitable norms $|\cdot|$ in the $u$-, $v$-spaces we can assume

$$|Au| < (a + \theta)|u|$$
$$|Bv| > (b - \theta)|v|$$

where $a := \exp a_0$, $b := \exp b_0$ and $0 < \theta < (b - a)/2$, $\theta < b$. Also, there is a positive function $\kappa(\rho)$ on some right neighbourhood of zero such that $\kappa(\rho) \rightarrow 0$ for $\rho \rightarrow 0$ and

$$|U(u, v)| < \kappa(\rho)(|u| + |v|), |V(u, v)| < \kappa(\rho)|v|$$

if $|u| + |v| < \rho$.

Let now $(u, v) \in \Omega$ and let $\Omega$ be so small that $|u_1| < |u|, |v_1| < |v|$. Then, we have

$$\frac{|u_1|}{|v_1|} < \frac{(a + \theta)|u| + \kappa(\rho)(|u| + |v|)}{(b - \theta)|v| - \kappa(\rho)|v|} = \frac{a + \theta + \kappa(\rho)}{b - \theta - \kappa(\rho)} \frac{|u|}{|v|} + \frac{\kappa(\rho)}{b - \theta - \kappa(\rho)}. \quad (2.7)$$

Let

$$\alpha \in \left(\frac{a + \theta}{b - \theta}, 1\right), \beta(\rho) := \frac{\kappa(\rho)}{b - \theta - \kappa(\rho)}.$$ 

We have $\lim_{\rho \rightarrow 0} \beta(\rho) = 0$ and there exists a $\rho_0 > 0$ such that $a + \theta + \kappa(\rho) < \alpha(b - \theta - \kappa(\rho))$ for any $\rho < \rho_0$. From (2.7) we have for $\varepsilon \in (0, 1 - \alpha)$

$$\frac{|u_1|}{|v_1|} < (\alpha + \varepsilon) \frac{|u|}{|v|} \text{ as soon as } \frac{|u|}{|v|} > \frac{\beta(\rho)}{\varepsilon}. \quad (2.8)$$

Choose any $(u_0, v_0) \in \Omega$ with $v_0 \neq 0$ and any $\gamma > 0$. We prove that there exists an $N > 0$ such that $|u_k| < \gamma |v_k|$ for all $k > N$ where $(u_k, v_k) = \Phi^k(u_0, v_0)$. Indeed, assume the contrary. Since $(u_k, v_k) \rightarrow 0$, there exists an $N_0$ such that $|u_k| + |v_k| < \rho_1 \leq \rho_0$ for all $k \geq N_0$, where $\beta(\rho_1)/\varepsilon < \gamma$. From (2.8) it follows that

$$|u_{k+1}| < \gamma |v_{k+1}| \text{ as soon as } |u_k| + |v_k| < \rho_1 \text{ and } |u_k| < \gamma |v_k|. \quad (2.9)$$
If \(|u_k| > \gamma |v_k|\) for \(k \geq N_0\) then by (2.8) also
\[
\gamma < \frac{|u_k|}{|v_k|} < (\alpha + \varepsilon)^{k-N_0} \frac{|u_{N_0}|}{|v_{N_0}|} \quad \text{for } k \geq N_0
\]
which is impossible. Thus, there exists an \(N \geq N_0\) for which \(|u_N| < \gamma |v_N|\). By (2.9), we have \(|u_k| < \gamma |v_k|\) for all \(k > N\). Since \(\gamma\) was arbitrary, \(\lim_{k \to \infty} |v_k|^{-1}u_k = 0\).

For the differential equation (2.2), (2.3) this means that if \((x(t), y(t))\) is its solution with \((x(0), y(0)) \in \Omega \setminus W\) (or, equivalently, \(y(0) \neq h(x(0))\)), then
\[
\lim_{k \to \infty} \frac{|x(k)|}{|y(k) - h(x(k))|} = 0. \tag{2.10}
\]

We have
\[
\frac{|x(k)|}{|y(k)|} = \frac{|x(k)|}{|y(k) - h(x(k))|} \frac{|y(k) - h(x(k))|}{|y(k)|} \leq \frac{|x(k)|}{|y(k) - h(x(k))|} \left(1 + \frac{|h(x(k))|}{|y(k)|}\right)
\]
\[
\leq \frac{|x(k)|}{|y(k) - h(x(k))|} \left(1 + \frac{|h(x(k))| |x(k)|}{|y(k)| |y(k)|}\right), \tag{2.11}
\]
or,
\[
\frac{|x(k)|}{|y(k)|} \left(1 - \frac{|x(k)|}{|y(k) - h(x(k))|} \frac{|h(x(k))|}{|x(k)|}\right) \leq \frac{|x(k)|}{|y(k) - h(x(k))|}.
\]

Since \(h(x) = o(|x|)\), from (2.10), (2.11) we obtain
\[
\lim_{k \to \infty} \frac{|x(k)|}{|y(k)|} = 0. \tag{2.12}
\]

Let now \(k \leq t < k + 1\). By standard Gronwall estimates and the variation of constants formula we obtain:
\[
|(x(t), y(t))| \leq C |(x(k), y(k))| : \rho \quad \text{for all } k \text{ and } t \in [k, k + 1]
\]
with some \(C \geq 1\). Again by Gronwall and variation of constants we obtain
\[
|x(t)| \leq C_1 (|x(k)| + \tilde{k}(\rho) \cdot (|x(k)| + |y(k)|))
\]
\[
|y(t)| \leq C_2 (|y(k)| - \tilde{k}(\rho) \cdot (|x(k)| + |y(k)|))
\]
for all \(k \in \mathbb{N}, t \in [k, k + 1]\), suitable constants \(C_1, C_2 > 0\) and a function \(\tilde{k}(\rho)\) satisfying
\[
\lim_{\rho \to 0} \tilde{k}(\rho) = 0
\]
Thus we have (for all $k \in \mathbb{N}$, $t \in [k, k + 1)$)
\[
\frac{|x(t)|}{|y(t)|} \leq \frac{C_1}{C_2} \cdot \frac{|x(k)| \cdot |y(k)|^{-1} \cdot (1 + \kappa(k) + \kappa(p))}{1 - \kappa(p)(1 + |x(k)| \cdot |y(k)|^{-1})}
\]
and (2.12) readily implies (2.4), completing the proof of the lemma. □

**Proof of theorem 2.1.** A neighbourhood $V$ of $v$ in $W^u$ can be considered as an open subset $\Omega$ of $\mathbb{R}^n$, the coordinates $z = (z_0, \ldots, z_{n-1})$ chosen in such a way that $z_k(w) = E_k(w - v)$ for $w \in W^u$ near $v$. Then, locally at $v$, the restriction of (0.1), (0.2) to $W^u$ has the form
\[
\dot{z} = Cz + q(z),
\]
where $C = \text{diag}\{-\lambda_0, \ldots, -\lambda_{n-1}\}$, $q$ is $C^\infty$ and $q'(0) = 0$.

Consider the associated system
\[
\frac{dz}{d\tau} = -Cz - q(z)
\]
which is obtained from (2.13) by time reversal $\tau = -t$. This system satisfies the assumptions of lemma 2.2 with $x = (z_0, \ldots, z_{n-2})$, $y = z_{n-1}$. We denote by $W_{n-2}$ the submanifold $W$ the existence of which is asserted in lemma 2.2. It is given by an equation
\[
z_{n-1} = h_{n-2}(z_0, \ldots, z_{n-2}), \quad z \in \Omega
\]
where $h_{n-2}$ is $C^\infty$ and satisfies
\[
h_{n-2}(0) = 0. \quad (2.14)
\]

By lemma 2.2, if $z(t)$ is a solution of (2.13) with
\[
z(0) \in \Omega, z_{n-1}(0) \neq h_{n-2}(z_0, \ldots, z_{n-2}), \quad (2.15)
\]
then
\[
\lim_{t \to -\infty} |z_{n-1}(t)|^{-1} |(z_0(t), \ldots, z_{n-2}(t))| = 0. \quad (2.16)
\]
From (2.14) it follows that
\[
\lim_{t \to -\infty} |z_{n-1}(t)||z_0(t), \ldots, z_{n-2}(t)|^{-1} = 0 \quad (2.17)
\]
if (2.15) does not hold. Since $W^u$ is tangent to the unstable space of $L$, from (2.16), (2.17) it follows respectively
\[
\lim_{t \to -\infty} |E_{n-1}(S_t(w) - v)|^{-1} |(I - E_{n-1})(S_t(w) - v)| = 0 \quad (2.18)
\]
for $w \in V \setminus W_{n-2}$ and
\[
\lim_{t \to -\infty} |(E_{n-1} + E^\nu)(S_t(w) - v)| |(I - E_{n-1} - E^\nu)(S_t(w) - v)|^{-1} = 0 \quad (2.19)
\]
for $w \in W_{n-2}$. We define $W_{n-2} = \{S_t(W_{n-2})| t \geq 0\}$. By [5, theorem 6.1.9], $W_{n-2}$ is an invariant submanifold of $W^u$. The properties (2.18), (2.19) obviously extend to $w \in W^u \setminus W_{n-2}$, $W_{n-2}$, respectively.
On $W_{n-2}$, the differential equation is again of form (2.13) with $C = \text{diag} \{-\lambda_0, \ldots, -\lambda_{n-2}\}$. Applying lemma 2.2 to the equation on $W_{n-2}$ we obtain an $(n-2)$-dimensional submanifold $\bar{W}_{n-3}$ of $\bar{W}_{n-2}$ represented by

$$z_{n-2} = h_{n-3}(z_0, \ldots, z_{n-3})$$

with

$$h_{n-3}(0) = 0$$

(2.20)

such that

$$\lim_{t \to -\infty} |z_{n-2}(t)|^{-1} |(z_0(t), \ldots, z_{n-3}(t))| = 0$$

(2.21)

for all solutions $z(t)$ with $z_{n-2}(0) \neq h_{n-2}(z_0, \ldots, z_{n-3})$. Again, we extend $\bar{W}_{n-3}$ to an invariant submanifold of $W_{n-2}$ by $W_{n-3} = \{ S_t(\bar{W}_{n-3}), t \geq 0 \}$. From (2.19) and (2.21) it follows that

$$\lim_{t \to -\infty} |E_{n-2}(S_t(w) - v)|^{-1} \cdot \|(I - E_{n-2})(S_t(w) - v)\| = 0$$

for $w \in W_{n-2}\setminus W_{n-3}$ while for $w \in W_{n-3}$ it follows from (2.19) and (2.20) that

$$\lim_{t \to -\infty} \left| \sum_{k=0}^{n-3} E_k(S_t(w) - v) \right|^{-1} \left| (I - \sum_{k=0}^{n-3} E_k(S_t(w) - v)) \right| = 0.$$ 

In this way we may proceed further and after $n-1$ steps obtains all the $(k+1)$-dimensional manifolds $W_k$ such that for $w \in W_k \setminus W_{k-1}$ we have

$$\lim_{t \to -\infty} \| (I - E_k)(S_t(w) - v) \| / \| E_k(S_t(w) - v) \| = 0.$$ 

This in turn implies for $w \in W_k \setminus W_{k-1}$ that

$$\lim_{t \to -\infty} (S_t(w) - v) / \| S_t(w) - v \| = \pm \phi_k.$$ (2.1)

Recall that the above limit is considered in $X^1 \subset C^1(I)$, and $\phi_k$ has only simple zeros with $z(\phi_k) = k$. By our remark preceding lemma 1.1 this implies

$$z(S_t(w) - v) = k$$

for $t$ near $-\infty$.

Now we invoke lemma 1.1 for $z(u(t))$, $u(t) := S_t(w) - v$. Note that $u$ satisfies an equation

$$u_t = u_{xx} + \hat{f}(x, u),$$

$$u(t, 0) = u(t, 1) = 0,$$

where $\hat{f}(x, u) := f(x, u + \nu(x)) - f(x, \nu(x))$. Hence $\hat{f}(x, 0) = 0$ and lemma 1.1 implies for $t$ near $-\infty$

$$z(w - v) = z(u(0)) \equiv z(u(t)) = z(S_t(w) - v) = k.$$ 

This completes the proof of theorem 2.1. ■

From our theorem we deduce a relation between the number of changes of monotonicity of a hyperbolic stationary solution $v$ (some "lap-number", cf. [6]) and the zero number $z(w - v)$ on the unstable manifold of $v$. 

COROLLARY 2.3. Let \( v \) be a stationary hyperbolic solution of (0.1), (0.2), \( v_x \neq 0 \), and let \( w \in W^u \) be in its unstable manifold. Then
\[
z(w - v) < z(v_x).
\]

Proof. Due to theorem 2.1 it suffices to prove that \( n := \dim W^u \leq z(v_x) \).

The function \( y := v_x \) solves the linearized equation
\[
y_{xx} + f_u(x, v(x))y = 0.
\]

On the other hand, the eigenfunction \( \phi_{n-1} \) has \( n + 1 \) zeros on the closed interval \([0, 1]\). By the comparison theorem, between any two consecutive zeros of \( \phi_{n-1} \) there has to be a zero of \( v_x \). By \( v_x \neq 0 \), all zeros of \( v_x \) are simple. This implies \( z(v_x) \geq n \) and the proof is complete. \( \blacksquare \)

We outline an alternate proof of theorem 2.1, (iv) which works only for \( W^u \), as far as we know. Consider any trajectory \( u(t) \) on \( W^u \setminus \{v\} \) and let \( y(t) := u(t)/|u(t)| \) be its projection onto the unit sphere. Then obviously
\[
\lim_{t \to -\infty} E^s y(t) = 0.
\]

Since \( W^u \) is finite dimensional, we may thus pick a sequence \( t_k \to -\infty \) such that
\[
\phi := \lim_{t_k \to -\infty} y(t_k) \tag{2.22}
\]
exists in \( X^1 \subset C^1(I) \). But \( \phi \) is in the unstable eigenspace of \( v \), hence Section 1 implies for \( t_k \) near \( -\infty \)
\[
z(w - v) = z(u(0)) \leq z(u(t_k)) = z(y(t_k)) = z(\phi) < n = \dim W^u,
\]
without any intermediate construction of \( W_k \).

3. ZEROS ON THE STABLE MANIFOLD

We turn to investigate the zero number \( z(w - v) \) on the stable manifold \( W^s \) of the hyperbolic stationary solution \( v \) of (0.1), (0.2), keeping the assumptions and notations of Section 2 in effect.

Similarly to the unstable case we need the following lemma on the fine structure of \( W^s \).

LEMMA 3.1. Assuming hyperbolicity of \( v \) above and \( f \in C^x, \kappa \geq 2 \), there exists a decreasing sequence \( W^s = W_n \supset W_{n+1} \supset \ldots \) of invariant \( C^x \)-submanifolds of the stable manifold \( W^s \) through \( v \) such that
(i) the tangent space to \( W_k \) at \( v \) is spanned by \( \phi_k, \phi_{k+1}, \ldots \)
(ii) for any \( w \in W_k \setminus W_{k+1} \)
\[
\lim_{t \to \infty} (S_t(w) - v)/|S_t(w) - v| = \pm \phi_k. \tag{3.1}
\]

We defer the proof of this lemma to the appendix.
As an immediate consequence of lemma 3.1 we can conclude for \( w \in W_k \setminus W_{k+1}, k \geq n \), that
\[
\lim_{t \to \infty} z(u(t)) = \lim_{t \to \infty} z(u(t)/|u(t)|) \geq z \left( \lim_{t \to \infty} u(t)/|u(t)| \right)
\]
by lower semicontinuity of \( z \) and monotonicity of \( z \) (lemma 1.1). However, this does not imply \( z \equiv n \) on all of \( W^n \), if for example
\[
\bigcap_{k \geq n} W_k \neq \{v\}.
\]
To remedy this point we use the following alternative which is proved in [1]:
(i) either \( z(u(t)) \) stays infinite for all \( t \geq 0 \);
(ii) or \( z(u(t_0)) < \infty \) for some \( t_0 \geq 0 \), and \( u(t) \) has only simple zeros for an open dense set of \( t \in [t_0, \infty) \).
Using this fact, we will conclude below that
\[
\bigcap_{k \geq n} W_k \subset \{w \mid z(w - v) = \infty\} \cup \{v\}.
\]
**Theorem 3.2.** Let \( v \) be a hyperbolic stationary solution of (0.1), (0.2) as above. Then for \( w \in W_k \subseteq W^n, w \neq v \) we obtain
\[
z(w - v) \geq k
\]
and in particular for all \( w \in W^n \setminus \{v\} \)
\[
z(w - v) \geq \dim W^w.
\]
**Proof.** With the preceding remarks it is sufficient to prove for \( w \neq v \)
\[
z(w - v) \geq k \quad \text{for all } w \in W_{k+1}, \quad k \geq n.
\]
Obviously we may assume that \( z(w - v) < \infty \). Then, by [1, theorem], there exists a \( t \geq 0 \) such that \( u(t, \cdot) = S_t(w) - v \) has only simple zeros. Because \( W_{k+1} \) has codimension 1 in \( W_k \) we may then choose \( \tilde{u} \in W_k \setminus W_{k+1} \) such that
\[
z(\tilde{u}) = z(u(t))
\]
(just choosing \( \|u - u(t)\|_{C^1(\Omega)} \) small enough). But by the remarks above
\[
z(\tilde{u}) \geq k,
\]
thus monotonicity of \( z \) (lemma 1.1) yields
\[
z(w - v) = z(u(0)) \geq z(u(t)) = z(\tilde{u}) \geq k
\]
and we are done. \( \blacksquare \)

4. Appendix

We give a proof of the fine structure of the stable manifold claimed in lemma 3.1. To this end we first construct an invariant manifold corresponding to a line, splitting the spectrum of the linearization. We use a general analytic semigroup setting.
in a Banach space $X$ with norm $\| \cdot \|$, where $A$ is sectorial linear $X \to X$; $f: U \to X$ is $C^r$, where $U$ is a neighborhood of 0 in $X^k$, $k \geq 1$, $0 \leq \alpha < 1$; $f(0) = 0$.

Let $L := A - f'(0)$ have spectrum $\sigma(L)$. By $u(t; u_0)$ we denote the solution of (4.1) with initial data $u(0; u_0) = u_0 \in X^k$.

The following lemma is well known in the finite dimensional case. It replaces [4, lemma 5.1 and corollary 5.1, chapter IX] in the proof of the infinite dimensional version of lemma 2.2. Its proof is modelled in close analogy to [5, theorem 5.2.1]. Nevertheless, for the convenience of the reader we give a detailed proof.

**Lemma 4.1.** Assume $\gamma > 0$ is such that $\sigma(L) = \sigma_1 \cup \sigma_2$, $\sigma_1 = \sigma(L) \cap \{\text{Re} \lambda < \gamma\}$, $\sigma_2 = \sigma(L) \cap \{\text{Re} \lambda > \gamma\}$ is a decomposition of $\sigma(L)$ into spectral sets. Let $X = X_1 \oplus X_2$ be the decomposition of $X$ corresponding to the decomposition of $\sigma(L)$ and let $E_1$ and $E_2$ be the spectral projections onto $X_1$ and $X_2$ respectively. Then there exist $p > 0$, $M > 0$ and a local invariant $C^\infty$ submanifold $S$ of the ball $\{|u|_a < \rho/2M\}$ such that:

(i) $S$ is $C^\infty$ diffeomorphic under $E_2|S$ to an open neighborhood of 0 in $X_1 := X_1 \cap X^k$;

(ii) $S$ is tangent to $X_1$ at 0;

(iii) if $|E_1 u(0)|_a < \rho/2M$ and $|u(t)|_a e^{\gamma t} < \rho$ for all $t \geq 0$ then $u(0) \in S$;

(iv) if $u(0) \in S$ then

\[
\sup_{t \geq 0} |u(t)|_a e^{\gamma t} < \infty.
\]

**Proof.** Without loss of generality assume $\sigma(A) \subset \{\text{Re} \lambda > 0\}$. By $L_1$, $L_2$ denote the restrictions of $L$ to $X_1$, $X_2$ respectively, let $T_i(t) := \exp(-tL_i)$ be the semigroup on $X_i$ generated by $L_i$ and $u_i = E_i u$ the $X_i$-component of $u$. Note that dim $X_1 < \infty$, $L_1$ is bounded and there exist $0 < \beta < \gamma < \delta$ such that

\[
|T_1(t)| \leq Me^{-\beta t}, \quad |A^\alpha T_1(t)| \leq Me^{-\beta t} \quad \text{for } t \leq 0,
\]

\[
|A^\alpha T_2(t)E_2 A^{-\alpha}| \leq Me^{-\delta t}, \quad |A^\alpha T_2(t)| \leq Mt^{-\alpha} e^{-\delta t} \quad \text{for } t \geq 0.
\]

Write $g(u) := f(u) - f'(0)u$ with components $g_i := E_i g$. Then there exists a positive function $k$ on $(0, \rho_0)$, $\rho_0 > 0$ such that $k(\rho) \to 0$ for $\rho \to 0$ and

\[
|g(u^1) - g(u^2)| \leq k(\rho) |u^1 - u^2|_a
\]

as soon as $|u_i|_a < \rho$, $j = 1, 2$. By [5, lemma 3.3.2], $u(t)$ solves (4.1) iff $u(t)$ solves the variation of constants version of (4.1)

\[
u_i(t) = T_i(t)u_i(0) + \int_0^t T_i(t - s)g_1(u(s)) \, ds
\]

(4.1)'

Assuming that the solution $u(t)$ satisfies

\[
|u(t)|_a e^{\gamma t} \text{ is bounded as } t \to \infty,
\]

we conclude that for $t \to \infty$

\[
|T_1(-t)u_1(t)|_a \leq Me^{\beta t} |u_1(t)|_a \to 0
\]

which implies

\[
u_1(0) = - \int_0^\infty T_1(-s)g_1(u(s)) \, ds,
\]

and, again by (4.1)', we obtain

\[
u(t) = T_2(t)a + \int_0^t T_2(t - s)g_2(u(s)) \, ds - \int_t^\infty T_1(t - s)g_1(u(s)) \, ds
\]

(4.4)

where $a := E_2 u(0) \in X_2$.

We show that for $\rho > 0$ sufficiently small integral equation (4.4) has a unique solution $u_\rho(t)$ satisfying $|u_\rho(t)|_a e^{\gamma t} < \rho$ provided $|a|_a < \rho/2M$. 
Let \( R_p \) be the set of continuous functions \( u: [0, \infty) \to X^2 \) such that

\[
\|u(\cdot)\| := \sup_{t \geq 0} |u(t)|_a e^{\eta t} \leq \rho
\]

is finite. The set \( R_p \) endowed with the metric generated by \( \| \cdot \| \) is a complete metric space. We claim that for \( \rho \) small enough and \( |a|_a < \rho / 2M, a \in X^2 \) the map \( F_a \) defined by

\[
F_a(u(\cdot)) (t) := T_2(t)a + \int_0^t T_2(t-s)g_2(u(s)) \, ds - \int_t^\infty T_1(t-s)g_1(u(s)) \, ds
\]

is a contraction \( R_p \to R_p \). Indeed

\[
\|F_a(u(\cdot))\| \leq \sup_{t \geq 0} e^{\eta t} |T_2(t)a|_a + \sup_{t \geq 0} \int_0^t e^{\eta t} |A^s T_2(t-s)| \cdot |g_2(u(s))| \, ds
\]

\[
+ \sup_{t \geq 0} \int_t^\infty e^{\eta t} |A^s T_1(t-s)| \cdot |g_1(u(s))| \, ds
\]

\[
\leq M |a|_a + |E_2| \sup_{t \geq 0} \int_0^t e^{\eta t} M(t-s)^{-\alpha} e^{-\delta(t-s)} k(\rho)|u(s)|_a \, ds
\]

\[
+ |E_1| \sup_{t \geq 0} \int_t^\infty e^{\eta t} M e^{-\delta(t-s)} k(\rho)|u(s)|_a \, ds
\]

\[
\leq M |a|_a + |E_2| M \cdot k(\rho) \int_0^\infty r^{-\alpha} e^{\eta r - \delta r} \, dr \cdot \|u(\cdot)\|_a
\]

\[
+ |E_1| M k(\rho) \int_0^\infty e^{(\delta - \eta)r} \, dr \cdot \|u(\cdot)\|_a
\]

\[
\leq M \cdot |a|_a + M k(\rho) \cdot C \|u(\cdot)\|_a,
\]

with some constant \( C \) independent of \( \rho \). Thus, if \( |a|_a < \rho / 2M \) and \( \rho > 0 \) is small enough that \( k(\rho) \cdot C < \rho / 2M \), then \( F_a \) maps \( R_p \) into \( R_p \). Also, repeating the same steps as in (4.5) we find

\[
\|F_a(u_1(\cdot)) - F_a(u_2(\cdot))\| \leq \|u_1(\cdot) - u_2(\cdot)\|
\]

as soon as \( \|u(\cdot)\| \leq \rho, j = 1, 2 \), so \( F_a \) is a contraction in \( R_p \). Consequently, \( F_a \) has a unique fixed point \( u(\cdot) \in R_p \) which solves (4.4).

The map \((u(\cdot), a) \to F_a(u(\cdot))\) is \( C^1 \) on \( R_p \times ([|a|_a < \rho / 2M] \cap X^2) \). Indeed, the map is linear in \( a \) and estimating as in (4.5) one obtains

\[
\sup_{t \geq 0} e^{\eta t} |e^{-\delta \cdot \epsilon} (F_a(u(\cdot)) + e \nu(\cdot))(t) - F_a(u(\cdot))(t))
\]

\[
- \int_0^t T_2(t-s)g_2(u(s))\nu(s) \, ds + \int_t^\infty T_1(t-s)g_1(u(s))\nu(s) \, ds|_a \to 0 \quad \text{for } \epsilon \to 0.
\]

Therefore

\[
(v(\cdot), b) \to T_2(t)b + \int_0^t T_2(t-s)g_2(u(s))\nu(s) \, ds - \int_t^\infty T_1(t-s)g_1(u(s))\nu(s) \, ds
\]

(4.6)

is the Gâteaux differential of the map \((u(\cdot), a) \to F_a(u(\cdot))\). Since the map (4.6) is continuous in \((v(\cdot), b)\), the differential is Fréchet and \((u(\cdot), a) \to F_a(u(\cdot))\) is \( C^1 \). To obtain \( C^* \) we iterate the arguments above.

By [5, 1.2.6] the fixed point \( u_a(\cdot) \) of \( F_a \) is a \( C^* \)-function of \( a \) in \([|a|_a < \rho / 2M] \cap X^2\). Consequently the map \( h: [|a|_a < \rho / 2M] \cap X^2 \to X_a \) defined by

\[
h(a) := u_a(0) = a - \int_0^\infty T_1(-s)g_1(u_a(s)) \, ds
\]

is \( C^* \) and, since \( E_2 h(a) = E_2 a = a \), has a \( C^* \) inverse on its image \( S \). Thus,

\[
h: [|a|_a < \rho / 2M] \cap X^2 \to X_a
\]

is a \( C^* \)-diffeomorphism. This proves (i) and, using \( g_1(0) = 0 \), as a direct consequence (ii). By definition of \( R_p \), (iv) holds.

By construction and (4.4), \( S \) is invariant with respect to the semiflow (4.1). If \( |E_2 u(0)|_a < \rho / 2M \) and \( |u(t)|_a e^{\eta t} < \rho \),
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for all \( t \geq 0 \), then we have shown that \( u(\cdot) \) satisfies (4.4). Since \( u(\cdot) \in R_p \) and \( u(t) = u_a(t) \) with \( a := E_3(0) \), \( u(0) \in S \). Thus (iii) holds and the proof is complete. •

Proof of lemma 3.1. Existence of the manifolds \( W_k \) as claimed in lemma 3.1 follows from lemma 4.1, with \( \bar{\lambda}_{k-1} < \gamma < \bar{\lambda}_k \).

Using existence of the manifolds \( W_k \), we apply the proof of lemma 2.2 successively for each \( k \) on a neighborhood \( U \) of \( v := 0 \) (w.l.o.g.) in \( W_k \), with coordinates \( y = E_k u \) and \( x = \sum_{j > k} E_j u \) as in the notation of Section 2. Note that the proof of lemma 2.2 carries over to analytic semigroups without the assumption that \( x \) is finite dimensional. Now lemma 2.2, together with \( u(t) = S(t) \rightarrow 0 \) and lemma 4.1, (ii) imply

\[
\pm \phi_k = \lim_{t \to +\infty} \frac{\sum_{j \geq k} E_j u(t)}{|E_k u|} = \lim_{t \to +\infty} \frac{\sum_{j \geq k} E_j u(t)}{|\sum_{j \geq k} E_j u(t)|} = \lim_{t \to +\infty} \frac{u(t)}{|u(t)|}
\]

and the proof is complete. •

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