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The Attractor of the Scalar Reaction Diffusion Equation Is a Smooth Graph

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For the scalar reaction diffusion equation with Dirichlet boundary conditions, it is proved that its maximal compact attractor is the graph of a C^1 function from a subset with nonempty interior of a subspace of the state space the dimension of which is equal to the maximal Morse index of the equilibria of the equation.

KEY WORDS: Attractor; inertial manifold; zero number; reaction diffusion equation.
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1. INTRODUCTION

Consider the scalar reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t, 0) = u(t, 1) = 0. \tag{1.2}$$

We assume that f is C^2 and satisfies

$$\limsup_{|s| \to \infty} s^{-1} f(s) < \pi^2$$

and that all stationary solutions of (1.1) are hyperbolic. The [generic (Brunovský and Chow, 1984)] set of such f's we denote by $\mathscr{G}$.

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For $f \in \mathscr{G}$, (1.1), (1.2) can be considered as an abstract differential equation on the Hilbert space $X = L_2(0, 1)$ which generates a C^2 semiflow S_t on any of its dense subspaces X^α, $0 < \alpha \leqslant 1$ (Henry, 1981; Miklavčič, 1985), where X^α is the fractional space associated with the operator A given by $Au(x) = -u''(x)$ if defined and if $u(0) = u(1) = 0$. We note that $X^{1/2} = H_0^1$ and $X^1 = D(A) = H_0^1 \cap H^2$. The semiflow S_t is dissipative (i.e., there is a bounded set $B \subset X^\alpha$ such that each trajectory eventually enters B) and every trajectory has a compact closure (Hale *et al.*, 1984; Hale, 1987). The set E of equilibria is finite and contained in B.

By Hale *et al.* (1984) and (1987), S_t admits a maximal compact invariant set $\mathscr{A}$ which is given by

$$\mathscr{A} = \bigcup_{v \in E} W^u(v) \tag{1.3}$$

where $W^u(v)$ is the unstable manifold of v (Henry, 1981).

Brunovský and Fielder (1988, 1989) present a complete description of the connections between stationary solutions. The purpose of this paper is to prove additional properties of $\mathscr{A}$ announced by Brunovský (1989). In order to be able to formulate them, we introduce some notation.

Let $v \in E$. By $\lambda_0(v) < \lambda_1(v) < \lambda_2(v) < \cdots$ and $\phi_0(v)$, $\phi_1(v)$, $\phi_2(v)$, we denote, respectively, the eigenvalues and normalized (in X) eigenvectors of the linearization of (1.1), (1.2) at v which is the Sturm–Liouville problem

$$y'' + [f'(v(x)) + \lambda]\, y = 0, \qquad y(0) = y(1) = 0. \tag{1.4}$$

Further, for $0 \leqslant m \leqslant n$ we denote $X_m^n(v) = \operatorname{span}\{\phi_m(v),..., \phi_{n-1}(v)\}$, $X_n(v) = \operatorname{span}\{\phi_n,...\}$, $X_n^\alpha(v) := X_n(v) \cap X^\alpha$. The (Morse) instability index $i(v)$ of v is given by $\lambda_{i(v)-1} < 0 < \lambda_{i(v)}$ (note that since, by assumption, v is hyperbolic, $\lambda_n \neq 0$ for any $n \geqslant 0$).

The main result of this paper is the following

1.1. Theorem

Let $f \in \mathscr{G}$ and let $N := \max\{i(w)\colon w \in E\}$. Then, given $v \in E$, there exists an open subset U of $X_0^N(v)$ and a C^1 function $h\colon U \to X_N(v)$ such that $\mathscr{N} :=$ graph h contains $\mathscr{A}$ and is positively and locally negatively invariant.

This theorem extends a result due to Jolly (1989) by which, for special f [of the Chafee–Infante type (Henry, 1981)], $\mathscr{A}$ is the graph of a Lipschitz continuous map. Also, for $v \in E$ with $i(v) = N$ it answers positively the conjecture of Fusco [proved for finite dimensional approximations of S_t by Fusco (1987)] according to which $W^u(v)$ is a graph over a subset of

$X_0^{i(v)}(v)$. In fact [cf. Remark 3.4(1)], the same proof can be used to establish the correctness of Fusco's conjecture also for $v \in E$ with $i(v) < N$.

Theorem 1.1 is proved in Section 3. In Section 2 an invariant manifold theorem is established which is needed in the proof of Theorem 1.1. Some technical parts of the proofs of the results of Section 2 are presented separately in the Appendix.

2. LOCAL INVARIANT MANIFOLDS CONTAINING GIVEN TRAJECTORIES

In this section we consider an abstract differential equation,

$$dy/dt = Ay + \tilde{F}(y) \tag{2.1}$$

on $Y = \mathbb{R}^n$, where $\tilde{F}$ is C^1 on some neighborhood of 0 and satisfies

$$\tilde{F}(0) = 0, \qquad D\tilde{F}(0) = 0.$$

Equation (2.1) generates a local C^1 flow we denote by $\tilde{\varphi}_t$; by modifying $\tilde{\varphi}$ outside some neighborhood of 0 we can make $\tilde{\varphi}$ global.

We assume that the spectrum of A, $\sigma(A)$, is disjoint from the imaginary axis and the line $\operatorname{Re} \lambda = -\beta$. Then we have

$$\sigma(A) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$$

where

$$\Lambda_1 = \{\lambda \in \sigma(A) \colon \operatorname{Re} \lambda > 0\},$$

$$\Lambda_2 = \{\lambda \in \sigma(A) \colon 0 > \operatorname{Re} \lambda > -\beta\},$$

$$\Lambda_3 = \{\lambda \in \sigma(A) \colon \operatorname{Re} \lambda < -\beta\}.$$

By P_i and Y_i, $i = 1, 2, 3$, we denote the spectral projection corresponding to Λ_i and its image, respectively, and we write $A_i := A|_{Y_i}$. For $y \in Y$ we write $y_i = P_i y$, $\tilde{F}_i(y) = P_i \tilde{F}(y)$, $i = 1, 2, 3$. Adopting these notations we can write (2.1) equivalently as

$$\dot{y}_i + A_i y_i = \tilde{F}_i(y_1, y_2, y_3), \qquad i = 1, 2, 3. \tag{2.2}$$

It is well known that a scalar product $\langle \cdot, \cdot \rangle$ can be chosen in Y in such a way that the projections P_i are orthogonal and that for suitable $\delta > 0$, $0 < \gamma < \beta - \delta$, we have

$$\langle y_1, A_1 y_1 \rangle \geqslant \gamma |y_1|^2 \tag{2.3}$$

$$-(\beta - \delta) |y_2|^2 \leqslant \langle y_2, A_2 y_2 \rangle \leqslant -\gamma |y_2|^2 \tag{2.4}$$

$$\langle y_3, A_3 y_3 \rangle \leqslant -(\beta + \delta) |y_3|^2 \tag{2.5}$$

where the norm $|\cdot|$ is generated by this scalar product. This follows, e.g., from Palis and de Melo (1980, Corollary to Theorem 2.5, Chap. 2).

The formulation of the main proposition of this section as well as some of the arguments become more transparent after a coordinate change which places certain local invariant manifolds through 0 into coordinate planes. Those are

(i) the unstable manifold $W^u(0)$ which is C^1 and tangent to Y_1 at 0,

(ii) the stable manifold $W^s(0)$ which is C^1 and tangent to $Y_2 + Y_3$ at 0,

(iii) a locally invariant C^1 manifold V which is tangent to $Y_1 + Y_2$ at 0, and

(iv) the invariant manifold W which is tangent to Y_3 at 0.

While the existence of the unstable and stable manifolds is standard (Hartman, 1964; Palis and de Melo, 1980) and the existence of W is established, e.g., by Hartman (1964) (cf. also Brunovský and Fiedler, 1986), the existence of V does not seem to appear in this immediate form in the literature. After truncating the nonlinearity (in a way which is well known from the proofs of the center-unstable manifold theorems), it follows immediately from Chow and Lu, (1988). Alternatively, V can be obtained from general theorems establishing invariant manifolds for flows which admit splitting of the state variable into two components such that one component of the difference of two trajectories has a strictly larger exponential decay rate than the other one (Kurzweil, 1970).

We note that while $W^u(0)$, $W^s(0)$, and W are uniquely defined, V is not. Neverthless, it does have to contain $W^u(0)$. Unlike $W^u(0)$, $W^s(0)$, and W, in general, it may not be smoother than C^1 no matter what the order of smoothness of $\tilde{F}$ is.

Since the manifolds $W^u(0)$, $W^s(0)$, V, W are tangent to Y_1, $Y_2 + Y_3$, $Y_1 + Y_2$, Y_3, respectively, and since $W \subset W^s(0)$, $W^u(0) \subset V$, there exists a local C^1 coordinate transformation $x = \Phi(y)$ with $\Phi(0) = 0$, $D\Phi(0) = \mathrm{id}$, which places $W^u(0)$, $W^s(0)$, V, W into Y_1, $Y_2 + Y_3$, $Y_1 + Y_2$, Y_3, respectively. We work in this new coordinate system. Because of the lack of higher smoothness of Φ, some care is needed, however. Although Φ conjugates $\tilde{\varphi}_t$ with a C^1 flow φ_t in the x-space (by $\varphi_t = \Phi \circ \tilde{\varphi}_t \circ \Phi^{-1}$), the vector field $x \mapsto Ax + F(x)$, $F(x) = D\Phi(\Phi^{-1}(x))\, F(\Phi^{-1}(x))$ that generates φ_t may not be smoother than C^0 any more (Palis and de Melo, 1980). For this reason we have to avoid the differential equation in some of our arguments and work directly with the flow instead. This complication turns out to be minor and outweighed by better transparence of the statement and arguments in the new coordinates.

Since $D\Phi(0)$ is the identity we have

$$D\varphi_t(0) = D\tilde{\varphi}_t(0) = e^{At} \qquad \text{for all } t.$$

Hence, we have

$$\varphi_t(x) = e^{tA}x + R(t, x) \tag{2.6}$$

where R is C^1,

$$R(t, 0) = 0 \qquad \text{and} \qquad |D_x R(t, x)| \leqslant L(|x|) \tag{2.7}$$

for $0 \leqslant t \leqslant 1$ and $|x|$ sufficiently small, $L\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\lim_{\eta \to 0} L(\eta) = 0. \tag{2.8}$$

In addition, since φ_t leaves Y_1, $Y_1 + Y_2$, $Y_2 + Y_3$, and Y_3 invariant, we have

$$R_2(t, x_1, 0, 0) = 0, \qquad R_3(t, x_1, x_2, 0) = 0, \tag{2.9}$$

$$R_1(t, 0, x_2, x_3) = 0, \qquad R_2(t, 0, 0, x_3) = 0, \tag{2.10}$$

in some neighborhood of 0, where $R_j := P_j R$ for $j = 1, 2, 3$.

Of course, when addressing the differential equation

$$\dot{x} = Ax + F(x) \tag{2.11}$$

generating φ_t, we cannot assume F is C^1 any more but we still have

$$|F(x)| \leqslant L(|x|)\, |x|, \tag{2.12}$$

with L possibly larger but still satisfying (2.8).

Denote

$$\begin{aligned}
\Gamma(\eta) &:= \{x\colon |x_2| = \eta,\ |x_1| < \eta\},\\
\hat{\Gamma}(\eta) &:= \{x \in \Gamma(\eta)\colon |x_3| \leqslant \eta\},\\
\Gamma_{12}(\eta) &:= (P_1 + P_2)\, \Gamma(\eta),\\
\Omega(\eta) &= \{x\colon |x_1| < \eta,\ |x_2| < \eta\}.
\end{aligned}$$

For a given subset Σ of Y denote

$$\begin{aligned}
\Phi(\Sigma) &:= \{\varphi_t(x)\colon t > 0,\ x \in \Sigma\},\\
\Phi_\eta(\Sigma) &:= \{\varphi_t(x)\colon x \in \Sigma,\ t > 0,\ \varphi_s(x) \in \Omega(\eta) \text{ for } 0 < s \leqslant t\}.
\end{aligned}$$

Below we frequently deal with manifolds which are positively invariant and locally negatively invariant. We call them briefly PLN-invariant.

2.1. Proposition

Let $\mathscr{R}$ be a PLN-invariant manifold for φ_t of dimension $dim(Y_1 + Y_2)$. Assume the following for some $\eta > 0$ sufficiently small:

(i) $U := (P_1 + P_2)\mathscr{R} \cap \Gamma(\eta)$ *is an open subset of* $\Gamma_{12}(\eta)$, $U \cap Y_2 \neq \emptyset$.

(ii) *There is an open neighborhood B of $\bar{U}$ in $Y_1 + Y_2$ and a C^1 function $\sigma: B \to Y_3$ such that $\Sigma := \mathscr{R} \cap \Gamma(\eta) = \text{graph } \sigma|_U$ and*

$$|\sigma(x_1, x_2)| \leqslant \eta, \tag{2.13}$$

$$|\sigma(x_1, x_2) - \sigma(x'_1, x'_2)| \leqslant |x_1 - x'_1| + |x_2 - x'_2|, \tag{2.14}$$

for (x_1, x_2), $(x'_1, x'_2) \in B$.

(iii) $\mathscr{R} \cap \Omega(\eta) = \Phi_\eta(\Sigma)$.

(iv) $\Phi(\Gamma(\eta_1) \cap \mathscr{R}) = \Phi(\Gamma(\eta_1)) \cap \mathscr{R}$ *for* $0 < \eta_1 \leqslant \eta$.

(v) $\mathscr{R} \cap W^u(0) = \emptyset$.

Then, $\mathscr{R}$ extends to a C^1 PLN-invariant manifold $\mathscr{M} = \mathscr{R} \cup \Phi(D)$ containing $W^u(0)$, where D is a locally invariant open disk of dimension $dim(Y_1 + Y_2)$ such that $0 \in D$.

The proof requires several preparatory lemmas. Observe that because of Lemma A.1 (ii) we have $|P_3 x| < 2\eta$ provided $x \in \Phi_\eta(\Sigma)$ and $\eta > 0$ is sufficiently small.

2.2. Lemma

For $\eta > 0$ sufficiently small let $U \subseteq \Gamma_{12}(\eta)$ and let B be an open neighborhood of $\bar{U}$ in $Y_1 + Y_2$. Let $\sigma: B \to X_3$ be C^1 and satisfy (2.13), (2.14) for each (x_1, x_2), $(x'_1, x'_2) \in B$. Then, $(P_1 + P_2)\,\Phi_\eta(\Sigma)$, where $\Sigma := \sigma(U)$, is an open subset of $Y_1 + Y_2$ and there exists a C^1 function $s: (P_1 + P_2)\,\Phi_\eta(\Sigma) \to X_3$ such that

(i) $\Phi_\eta(\Sigma) = \text{graph } s$;

(ii) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|s(x_1, x_2)| \leqslant \varepsilon\,|x_2|, \tag{2.15}$$

$$|s'(x_1, x_2)| < \varepsilon, \tag{2.16}$$

$$|s(x_1, x_2) - s(x'_1, x'_2)| < \varepsilon(|x'_1 - x'_1| + |x_2 - x'_2|), \tag{2.17}$$

provided $|x_2|$, $|x'_2| < \delta$;

(iii) s extends to a C^1 function in a neighborhood of each point $(x_1, x_2) \in (P_1 + P_2)\,\overline{\Phi_\eta(\Sigma)} \setminus Y_1$.

Proof

By (2.13), (2.14), and (2.11), for $x \in \bar{\Sigma}$ we have

$$\langle x_2, A_2 x_2 + F_2(x) \rangle < -\gamma\, |x_2|^2 + |x_2|\, |F_2(x)| \leqslant (-\gamma + L(\eta))\eta^2 < 0 \qquad (2.18)$$

provided η is so small that $L(\eta) < \gamma$. The inequality (2.18) means that the tangent vector to the trajectory of $x \in \Sigma$ at x is not contained in $T_x\Sigma$. Therefore, $\Phi(\Sigma)$ and $\Phi_\eta(\Sigma)$ [as an open subset of $\Phi(\Sigma)$] are C^1 submanifolds of Y and

$$\dim \Phi_\eta(\Sigma) = \dim \Phi(\Sigma) = \dim(Y_1 + Y_2). \qquad (2.19)$$

We prove that for $\eta > 0$ sufficiently small, $\Phi_\eta(\Sigma)$ is a graph over its $(P_1 + P_2)$-projection, i.e., that for any $(x_1, x_2) \in (P_1 + P_2)\,\Phi_\eta(\Sigma)$, there exists a unique $x_3 \in Y_3$ such that $(x_1, x_2, x_3) \in \Phi_\eta(\Sigma)$.

The proof is indirect. Assume that for some (x_1, x_2) there exist x_3, x_3' such that both $x := (x_1, x_2, x_3) \in \Phi_\eta(\Sigma)$ and $x' := (x_1, x_2, x_3') \in \Phi_\eta(\Sigma)$. Then there are $\xi \neq \xi' \in \Sigma$ and $t, \tau \geqslant 0$ such that $x = \varphi_{t+\tau}(\xi)$ and $x' = \varphi_t(\xi')$. By Lemma A.5, for sufficiently small $\eta > 0$,

$$|P_3(\varphi_{\tau+s}(z) - \varphi_s(z')|\; |(P_1 + P_2)(\varphi_{\tau+s}(z) - \varphi_s(z')|^{-1}$$

is bounded for $s \geqslant 0$ by some constant $c > 0$. For $s = t$ we obtain

$$0 \neq |x_3 - x_3'| \leqslant c(|x_1 - x_1'| + |x_2 - x_2'|) = 0, \qquad (2.20)$$

with $x_1 := x_1'$, $x_2 := x_2'$, a contradiction.

For $(x_1, x_2) \in (P_1 + P_2)\,\Phi_\eta(\Sigma)$ we can now define $s(x_1, x_2)$ as the unique x_3 such that $(x_1, x_2, x_3) \in \Phi_\eta(\Sigma)$. By the implicit function theorem, s is C^1 if and only if $(P_1 + P_2)|_{\Phi_\eta(\Sigma)}$ is a local diffeomorphism at each $x \in \Phi_\eta(\Sigma)$. Because of (2.19) this is equivalent to

$$(P_1 + P_2)\, y \neq 0 \qquad \text{for any} \quad 0 \neq y \in T_x \Phi_\eta(\Sigma).$$

This, however, follows immediately from (2.20) if we let $x_1' \to x_1$, $x_2' \to x_2$.

To prove (ii) we first show that there exists a function $T\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $T(\delta) \to \infty$ for $\delta \to 0$ and $t \geqslant T(\delta)$ as soon as $x \in \Phi_\eta(\Sigma)$, $|x_2| < \delta$, $x \in \varphi_t(\Sigma)$.

Indeed, assume that this is not the case. Then there exists a sequence of points $x^k \in \Phi_\eta(\Sigma)$ such that $x_2^k \to 0$ and $x^k = \varphi_{t_k}(\xi^k)$, $t_k \to t^* < \infty$, $\xi^k \to \xi^* \in \bar{\Sigma}$. By continuity we have $P_2(\varphi_{t^*}(x^*) = 0$, which contradicts Corollary A.2.

Define $k(t)$ and $\Delta(t)$ by (A.6) and (A.35), respectively. From Lemma A.1 (iv), and Lemma A.5 it follows, respectively, that there exists a $T > 0$ such that

$$k(t) < \varepsilon, \qquad \Delta(t) < \varepsilon \qquad \text{for} \quad t \geqslant T. \tag{2.21}$$

Let $\delta > 0$ be so small that $T(\delta) \geqslant T$. Then (2.21) means that (2.15) and (2.17) are satisfied for $|x_2|$, $|x_2'| < \delta$; (2.16) follows immediately from (2.17) for $|x_j - x_j'| \to 0$ for $j = 1, 2$. This completes the proof of (ii).

To prove (iii) we first show

$$\overline{\Phi_\eta(\Sigma)} \backslash Y_1 \subseteq \Phi_{2\eta}(\bar{\Sigma}). \tag{2.22}$$

Indeed, let $\{x^n\} \to x$, $x^n = \varphi_{t_n}(\xi^n)$, $\xi^n \in \Sigma$, and $x \notin Y_1$, i.e., $x_2 \neq 0$. Then $\{t_n\}$ is bounded by Lemma A.1 (iii) and, therefore, we may assume $\xi^n \to \xi \in \bar{\Sigma}$, $t_n \to t < \infty$. By continuity of φ_t we have $\varphi_t(\xi) = x$, $|(P_1 + P_2)\, \varphi_s(\xi)| < 2\eta$ for $0 < s \leqslant t$, hence $x \in \Phi_{2\eta}(\bar{\Sigma})$.

Because of (2.20), to obtain an extension of s to a neighborhood of $(P_1 + P_2)x$, we repeat its construction with U replaced by some neighborhood of $\bar{U}$ in the sphere $|x_2| = \eta$ and $\Omega(\eta)$ replaced by $\{x\colon |x_1| < 2\eta,\ |x_2| < \eta\}$. ∎

Let now U, η, σ, and s be as in Lemma 2.2. Extend s to $\overline{(P_1 + P_2)\, \Phi_\eta(\Sigma)} \cap Y_1$ by defining $s(x_1, 0) = 0$ for $(x_1, 0) \in \overline{(P_1 + P_2)\, \Phi_\eta(\Sigma)}$. By Lemma 2.2 (iii), at each point $(x_1, x_2) \in \overline{(P_1 + P_2)\, \Phi_\eta(\Sigma)} \backslash Y_1$, s is a restriction of a C^1 function defined in some neighborhood of (x_1, x_2). Therefore, s satisfies the hypotheses of the Whitney C^1 extension theorem (Abraham and Robbin, 1967) at each such (x_1, x_2). The estimates (2.15)–(2.17) of Lemma 2.2 mean that these hypotheses are satisfied at points of $\overline{(P_1 + P_2)\, \Phi_\eta(\Sigma)} \cap Y_1$ as well with 0 as the candidate for $s'(x_1, 0)$. Applying the Whitney extension theorem we obtain the following.

2.3. Corollary

Let the assumptions of Lemma 2.2 be satisfied. Then s extends to a C^1 function $\tilde{s}$ on $Y_1 + Y_2$ such that $\tilde{s}(x_1, 0) = 0$, $\tilde{s}'(x_1, 0) = 0$ if $(x_1, 0) \in \overline{(P_1 + P_2)\, \Phi_\eta(\Sigma)} \cap Y_1$.

Combining Lemma 2.2 and Corollary 2.3 we obtain the following.

2.4. Lemma

Let Σ, η, σ, U, B, and $\tilde{s}$ be as in Lemma 2.2 and Corollary 2.3. Assume $(Y_2+Y_3)\cap U\neq\varnothing$. Then there exists a $0<\eta_1\leqslant\eta$ such that $\sigma_1:=\tilde{s}|_{\Gamma_{12}(\eta_1)}$ extends to a C^1 function $s_1\colon Y_1+Y_2\to U_3$ such that

$$\operatorname{graph} s_1\cap\Omega(\eta_1)=\Phi_{\eta_1}(\operatorname{graph}\sigma_1)\cup(Y_1\cap\Omega(\eta_1)$$

is a locally invariant manifold of φ_t containing $(\Phi_\eta(\Sigma)\cup Y_1)\cap\Omega(\eta_1)$.

Proof

Since locally $Y_2+Y_3=W^s(0)$, for $x\in U\cap(Y_2+Y_3)\neq 0$ we have $\lim_{t\to\infty}\varphi_t(x)=0$, which implies $0\in\overline{\Phi_\eta(\Sigma)}$. By Corollary 2.3 we have $\tilde{s}(0,0)=0$, $\tilde{s}'(0,0)=0$. Thus, for $\eta_1\leqslant\eta$ sufficiently small the function $\sigma_1:=\tilde{s}|_{(P_1+P_2)\Gamma(\eta_1)}$ admits a C^1 extension to a neighborhood of $(P_1+P_2)\,\Gamma(\eta_1)$ satisfying (2.13), (2.14) with η replaced by η_1 and $U:=(P_1+P_2)\,\Gamma(\eta_1)$. Also, by its definition, σ_1 admits a C^1 extension to a neighborhood of $\bar{U}$. Applying Lemma 2.2 (i) to σ_1 instead of σ allows us to define s_1: $(P_1+P_2)\,\Phi_\eta(\Sigma_1)\to Y_3$ by graph $s_1:=\Phi_{\eta_1}(\Sigma_1)$, where $\Sigma_1:=\operatorname{graph}\sigma_1$. Extend s_1 to $Y_1\cap\Omega_1(\eta_1)$ by defining $s(x_1,0):=0$. Near any point of $\overline{(P_1+P_2)\,\Phi_{\eta_1}(\Sigma_1)}\backslash Y_1$, s_1 is a restriction of a C^1 function by Lemma 2.2 (iii) (applied to σ_1), while at any point of $Y_1\cap\Omega(\eta_1)$, s_1 satisfies the assumptions of the Whitney C^1 extension theorem because of Lemma 2.2 (ii) (applied to σ_1). Therefore, s_1 extends to a C^1 function on Y_1+Y_2.

Trivially, both $\Phi_{\eta_1}(\Sigma_1)$ and $Y_1\cap\Omega(\eta_1)$ are locally invariant and their union contains $(\Phi_\eta(\Sigma)\cap\Phi_{\eta_1}(\Sigma_1))\cup(Y_1\cap\Omega(\eta_1))$. Thus, all that remains to be proved is

$$(P_1+P_2)\,\Phi_{\eta_1}(\Sigma_1)=(P_1+P_2)\,\Omega(\eta_1)\backslash Y_1. \tag{2.23}$$

Since the restriction of P_1+P_2 to $\Phi_{\eta_1}(\Sigma_1)$ is a local isomorphism, $(P_1+P_2)\,\Phi_{\eta_1}(\Sigma_1)$ is open in $(P_1+P_2)\,\Omega(\eta_1)\backslash Y_1$. To prove (2.23) we show: that it is also closed in $(P_1+P_2)\,\Omega(\eta_1)$.

Let $(x_1,x_2)\in(P_1+P_2)\,\Omega(\eta_1)$ and $x_2\neq 0$. Assume that there are sequences $\xi_k\in\Sigma_1$, $\xi_k\to\xi\in\bar{\Sigma}_1$, $t_k\geqslant 0$ such that $\varphi_{t_k}(\xi_k)\in\Phi_{\eta_1}(\Sigma_1)$ and $(P_1+P_2)\,\varphi_{t_k}(\xi_k)\to(x_1,x_2)$.

Since $x_2\neq 0$, from Lemma A.1 (iii) it follows that $\{t_k\}$ is bounded. Therefore, we may assume that $t_k\to t^*\geqslant 0$. By continuity we have $(x_1,x_2)=(P_1+P_2)\,\varphi_{t^*}(\xi)\in\overline{\Phi_{\eta_1}(\Sigma_1)}$, hence $\varphi_{t^*}(\xi)\in\Phi_{\eta_1}(\Sigma_1)$ by Lemma A.1 (ii). This completes the proof. ▮

Proof of Proposition 2.1

All the hypotheses of Lemmas 2.2 and 2.4 and Corollary 2.3 being met, define s_1 as in Lemma 2.4. Denote

$$\mathscr{M} := \mathscr{R} \cup \Phi(\text{graph } s_1 \cap \Omega(\eta_1)). \tag{2.24}$$

The PLN-invariance of $\mathscr{M}$ follows immediately from its definition and the PLN-invariance of $\mathscr{R}$. By Lemma 2.4 we have

$$\Phi(\text{graph } s_1 \cap \Omega(\eta_1)) \supseteq \Phi(Y_1 \cap \Omega(\eta_1)) = W^u(0).$$

It remains to be proved that $\mathscr{M}$ is a manifold.

Since graph $\sigma_1 \subset \Gamma(\eta_1)$, applying consequently Lemma 2.4, hypotheses (v), (iv), (iii), and Lemma 2.4 again we obtain

$$\begin{aligned}
&\mathscr{R} \cap \Phi(\text{graph } s_1 \cap \Omega(\eta_1)) \\
&= \mathscr{R} \cap \Phi(\Phi_{\eta_1}(\Sigma_1)) = \mathscr{R} \cap \Phi(\Gamma(\eta_1)) \cap \Phi(\Sigma_1) \\
&= \Phi(\mathscr{R} \cap \Gamma(\eta_1)) \cap \Phi(\Sigma_1) = \Phi(\mathscr{R} \cap \Omega(\eta) \cap \Gamma(\eta_1)) \cap \Phi(\Sigma_1) \\
&= \Phi(\Phi_\eta(\Sigma) \cap \Gamma(\eta_1)) \cap \Phi(\Sigma_1) = \Phi(\Phi_\eta(\Sigma) \cap \Gamma(\eta_1)).
\end{aligned}$$

By (2.18), the trajectories of φ_t cross $\Gamma(\eta_1)$ transversally, hence $\Phi(\Phi_\eta(\Sigma) \cap \Gamma(\eta_1)) = \mathscr{R} \cap \Phi(\text{graph } s_1 \cap \Omega(\eta_1))$ is a submanifold of Y of dimension $\dim Y_1 + \dim Y_2$. Since both $\mathscr{R}$ and $\Phi(\text{graph } s_1 \cap \Omega(\eta_1))$ are submanifolds of Y of the same dimension, so is $\mathscr{M} = \Phi(\text{graph } s_1 \cap \Omega(\eta_1)) \cup \mathscr{R}$. ∎

2.5. Remark

When introducing the invariant manifold V which is tangent to $Y_1 + Y_2$ at 0, we have mentioned that it is not unique. Proposition 2.1 gives a method to construct additional manifolds tangent to $Y_1 + Y_2$ at 0 provided one of such manifolds is known (in our case the latter is represented by the manifold which we have placed to the $Y_1 + Y_2$-plane by our coordinate transform). If one takes $\eta > 0$ sufficiently small, defines $U := \Gamma(\eta) \cap (Y_1 + Y_2)$ in Proposition 2.1 (and Lemma 2.4), and chooses a function $\sigma: U \to Y_3$ satisfying the estimates (2.13) and (2.14), then there is a unique invariant manifold tangent to $Y_1 + Y_2$ and containing graph σ. Note that the right-hand sides of (2.13) and (2.14) can be replaced by $p\eta$ and q, respectively, $p > 0$ and $q > 0$ arbitrary.

3. PROOF OF THEOREM 1.1

Recall that we denote

$$N = \max_{w \in E} i(w); \tag{3.1}$$

by $|\cdot|$ we denote the norm of X. Also, recall the notation introduced before Theorem 1.1. We start our proof by preparatory lemmas. Their proofs are heavily dependent on the papers (Brunovský and Fiedler, 1986, 1988, 1989). Therefore, we have to introduce some notation used there.

By the zero number of a continuous function $v \not\equiv 0$ on $[0, 1]$, denoted by $z(v)$, we understand the number of its strict sign changes (Brunovský *et al.*, 1986, 1988, 1989). We denote

$$Z_n := \{v \in E: z(v) = n \text{ or } v \equiv 0\}$$

if $f(0) = 0$, n is even and $i(0) = n$ or $n+1$ and

$$Z_n := \{v \in E: z(v) = n\}$$

otherwise (Brunovský and Fiedler, 1989, pp. 6, 11). Further, for an interval $I \subseteq \mathbb{R}$, we denote

$$EI = \{v \in E: v'(0) \in I\}.$$

As Brunovský and Fiedler, (1989), we order the elements of E by their initial slope $v'(0)$ and we use freely the terminology above, below, maximal, neighbor, etc., relatively to this ordering.

3.1. Lemma

For all $v_1 \neq v_2 \in E$ one has $z(v_1 - v_2) < N$.

Proof

Without loss of generality assume

$$v_1'(0) > 0, \qquad |v_2'(0)| \leqslant v_1'(0), \tag{3.2}$$

[if $v_1'(0) < 0$ and $|v_2'(0)| \leqslant v_1'(0)$, replace $f(u)$ by $-f(-u)$]. Then by Brunovský and Fiedler (1989, Lemma 4.2) we have

$$z(v_1 - v_2) = z(v_1). \tag{3.3}$$

Further, we have

$$N - 1 \leqslant \max_{w \in E} z(w) \leqslant N. \tag{3.4}$$

This follows from Brunovský and Fiedler (1988, Lemma 5.1) or Brunovský and Fiedler (1989, Lemma 2.1), according to which for $0 \not\equiv w \in E$ we have

$$i(w) \in \{z(w), z(w)+1\} \tag{3.5}$$

Suppose now $z(v_1 - v_2) \geqslant N$. Then, from (3.3) and (3.4) it follows that

$$z(v_1) = z(v_1 - v_2) = N. \tag{3.6}$$

From (3.1) and (3.5) it follows that

$$i(v_1) = N. \tag{3.7}$$

We complete the proof by showing that the existence of v_1, v_2 such that (3.2), (3.6), and (3.7) hold simultaneously is contradictory.

We start the proof by showing

$$E(0, v_1'(0)) = \varnothing. \tag{3.8}$$

First, we prove

$$E(0, v_1'(0)) \cap Z_N = \varnothing. \tag{3.9}$$

Suppose that this is not true and denote $\bar{w}$ the maximal element of $E(0, v_1'(0)) \cap Z_N$. Then, $\bar{w}$ is the neighbor of v_1 in Z_N and $v_1'(0)\, \bar{w}'(0) > 0$. Thus, $i(\bar{w}) \neq i(v_1)$ by Brunovský and Fiedler (1988, Lemma 2.2). Since $i(\bar{w}) \in \{z(\bar{w}),\ z(\bar{w})+1\} = \{N, N+1\}$, by (3.4) we have $i(\bar{w}) > N$, which contradicts (3.1).

Knowing (3.9), from Brunovský and Fiedler [1989, Lemma 2.2(i)], we conclude $Z_k \cap E(0, v_1'(0)) = \varnothing$ also for $k < N$. Since $Z_k \cap E(0, v_1'(0) = \varnothing$ for $k > N$ by (3.4), this proves (3.8).

From (3.8) it follows $v_2 \in E[-v_1'(0), 0]$. In order to show that this is impossible we distinguish two cases:

(a) N odd, (b) N even.

In case (a) it follows from (3.2) that $v_1(x) := v_1(1-x)$ is the maximal element of $E(-\infty, 0) \cap Z_N$. Indeed, since $\hat{v}_1'(0) = -v_1'(0)$, if $w \in E(-v_1'(0), 0) \cap Z_N$, then $w \in E(0, v_1'(0))$, which contradicts (3.8) (cf. also Brunovský and Fiedler (1989, Lemma 2.5).

From Brunovský and Fiedler (1989, Lemma 2.4) it follows that $E(v_1'(0),\ v_1'(0)) \neq \varnothing$. Since $i(v_1) = i(v_1) = z(v_1) = z(v_1)$ by symmetry, we have $[E(-v_1'(0), 0) \cup (0, v_1'(0))] \cap Z_k = \varnothing$ for $k < N$ by Brunovský and Fiedler [1989, Lemma 2.6($i_\pm$)]. Therefore, $v_2'(0) = 0$. By Brunovský and Fiedler (1989, Lemma 2.3), this is possible only if $f(0) = 0$ and $v_2 \equiv 0$ and can be excluded by the perturbation argument used at the end of the proof of Brunovský and Fiedler (1986, Theorem 1.5). In case (b), $-v_1'(0) \leqslant v_2'(0) \leqslant 0$ and $E(0, v_1'(0)) = \varnothing$ implies that the maximal element w of $E(-\infty, v_1'(0))$ satisfies $-v_1'(0) \leqslant w'(0) \leqslant 0$. If $w'(0) < 0$, by Brunovský and Fiedler (1989, Lemma 2.4) and (3.7) we have $z(w) = N-1$. Since $N-1$ is odd, $\hat{w}(x) := w(1-x) \in E$, $0 < w'(0) < v_1'(0)$, a contradiction to (3.8). Therefore, $w(0) = 0$ which can again be excluded by the perturbation argument mentioned in case (a). ▌

3.2. Lemma

For every $u_1 \neq u_2 \in \mathscr{A}$ *one has* $z(u_1 - u_2) < N$.

Proof

From (1.3) it follows that $u_1 \in W^u(v_1)$, $u_2 \in W^u(v_2)$ for some $v_1, v_2 \in E$. We distinguish two cases:

$$\text{(a)} \quad v_1 \neq v_2, \qquad \text{(b)} \quad v_1 = v_2.$$

Case (a)

Since $S_t(u_j) \to v_j$, $j = 1, 2$ and $t \to -\infty$, in $H_2 \cap H_1^0$ and since (by Brunovský and Fiedler, 1989, Lemma 3.2) $v_1 - v_2$ has simple zeros, for t near $-\infty$ we have $z(S_t(u_1) - S_t(u_2)) = z(v_1 - v_2)$. Since by Lemma 3.1 $z(v_1 - v_2) < N$ and since $z(S_t(u_1) - S_t(u_2))$ does not increase with time (Brunovský and Fiedler, 1989), $z(u_1 - u_2) < N$.

Case (b)

Denote $v := v_1 = v_2$ and $y(t, x) := S_t(u_1)(x) - S_t(u_2)(x)$. The function $y(t, x)$ solves the linear equation

$$y_t = y_{xx} + a(t, x)\, y \tag{3.10}$$

with the boundary conditions

$$y(t, 0) = y(t, 1) = 0,$$

where

$$a(t, x) = \int_0^1 f'((1-\vartheta)\, S_t(u_2)(x) + \vartheta S_t(u_1))\, d\vartheta.$$

We have

$$\lim_{t \to -\infty} y(t) = 0 \qquad \text{and} \qquad \lim_{t \to -\infty} a(t, x) = f'(v(x)) \tag{3.11}$$

uniformly in x.

Since $y(t) \not\equiv 0$, by Henry (1985), it follows from (3.11) that

$$\lim_{t \to -\infty} y(t) \, |y(t)|^{-1} = \pm\phi_j(v)$$

[$\phi_j(v)$ defined in Section 1] for some $0 \leqslant j < N$. This implies $z(y(t)) = z(\phi_j) = j < N$ for t near $-\infty$ and, since $z(y(t))$ does not increase with t, also $z(y(0)) = z(u_1 - u_2) < N$. ▮

3.3. Proposition

For any $v \in E$, $\mathscr{A}$ is the graph of a function h: $P_0^N(v)\mathscr{A} \to X_N(v)$.

Proof

The statement of the lemma is equivalent to: $u = u'$ whenever $u, u' \in \mathscr{A}$ and $u - u' \in X_N(v)$. Since by Atkinson (1964, Exercise 2, p. 549), $z(u-u') \geqslant N$ if $0 \neq u - u' \in X_N(v)$, we have $u = u'$ by Lemma 3.2. ▮

3.4. Remarks

(1) The argument used to prove Proposition 3.3 remains valid if X_0^N, X_N are replaced by any two subspaces Y, Z such that $\dim Y = N$, $Y \oplus Z = X$, $Y \cap Z = \{0\}$ and

$$z(u) \geqslant N \qquad \text{for all} \quad u \in Z. \tag{3.12}$$

In particular, by Atkinson (1964, Exercise 2, p. 549), (3.12) holds if Z is the subspace spanned by all the eigenfunctions except of the first N ones of any Sturm–Liouville problem and Y is any complement of Z.

(2) By a straightforward modification of the proofs of Lemma 3.2 and Proposition 3.3, one can prove that for any $v \in E$, $i(v) = n$, $W^u(v)$ is a graph of a function h: $P_0^n(v)\, W^u(v) \to X_n(v)$. This property of $W^u(v)$ has been conjectured and proved for finite dimensional approximations of (1.1), (1.2) by Fusco (1987).

3.5. Lemma

For each $v \in E$ there exists a $q > 0$ such that for any two points $w_1, w_2 \in \mathscr{A}$, one has

$$|P_N(v)(w_1 - w_2)| \leqslant q \, |P_0^N(v)(w_1 - w_2)|. \tag{3.13}$$

In other words, the function h of Lemma 3.3 is globally Lipschitz.

Proof

It follows from Chow and Lu (1988) and Foias *et al.* (1986) that for a given $v \in E$, there exists an $M > N$ and a C^1 M-dimensional PLN-invariant submanifold $\mathcal{M}$ of X (called inertial manifold) such that $\mathcal{A} \subset \mathcal{M}$. For any chosen $v \in E$, the manifold $\mathcal{M}$ is a graph of a globally Lipschitz C^1 function $g: U \to X_M(v)$, where U is an open subset of $X_0^M(v)$.

Assume that q does not exist. Then, since $\mathcal{A}$ and the unit sphere in $X_0^M(v)$ are compact, there exist sequences $\{w_1^k\}$, $\{w_2^k\}$ such that $w_j^k \to w_j^*$, $w_j^* \in \mathcal{A}$ for $j = 1, 2$, $P_0^M(w_1^k - w_2^k) \, |P_0^M(w_1^k - w_2^k)|^{-1} \to y$, $|y| = 1$ and $|P_N(x_1^k - w_2^k)| \, |P_0^N(w_1^k - w_2^k)|^{-1} \geqslant k$ (here and below in this proof we drop the argument v at the projection operators and their images).

Note that

$$\begin{aligned} P_N(w_1^k - w_2^k) &= P_N^M(w_1^k - w_2^k) + P_M(w_1^k - w_2^k) \\ &= P_N^M(w_1^k - w_2^k) + g(P_0^M w_1^k) - g(P_0^M w_2^k), \end{aligned}$$

hence

$$|P_N(w_1^k - w_2^k)| \leqslant |P_N^M(w_1^k - w_2^k)| + 1 \, |P_0^M(w_1^k - w_2^k)|,$$

where 1 is the Lipschitz constant of g. Thus,

$$\begin{aligned} \frac{|P_N^M(w_1^k - w_2^k)|}{|P_0^N(w_1^k - w_2^k)|} &\geqslant \frac{|P_N(w_1^k - w_2^k)|}{|P_0^N(w_1^k - w_2^k)|} - 1 \, \frac{|P_0^M(w_1^k - w_2^k)}{|P_0^N(w_1^k - w_2^k)|} \\ &\geqslant k - 1 - 1 \, \frac{|P_N^M(w_1^k - w_2^k)|}{|P_0^N(w_1^k - w_2^k)|}, \end{aligned}$$

or

$$\frac{|P_N^M(w_1^k - w_2^k|}{|P_0^N(w_1^k - w_2^k)|} \geqslant \frac{1}{1+1} (k-1) \to \infty \qquad \text{for} \quad k \to \infty.$$

Consequently,

$$|P_0^N y| = \lim_{k \to \infty} \frac{|P_0^N(w_1^k - w_2^k)|}{|P_0^M(w_1^k - w_2^k)|} \leqslant \lim_{k \to \infty} \frac{|P_0^N(w_1^k - w_2^k)|}{|P_N^M(w_1^k - w_2^k)|} = 0$$

which means $y \in X_N^M$.

We have

$$\begin{aligned} &|P_0^M(w_1^k - w_2^k)|^{-1}(w_1^k - w_2^k) \\ &\quad = |P_0^M(w_1^k - w_2^k)|^{-1} [P_0^M(w_1^k - w_2^k) + g(P_0^M(w_1^k)) - g(P_0^M(w_2^k)] \\ &\quad = \left[I + \int_0^1 g'(P_0^M((1-\theta) w_1^k + \theta w_2^k) \, d\theta) \, y \right] |P_0^M(w_1^k - w_2^k)|^{-1} P_0^M(w_1^k - w_2^k) \\ &\quad = \left[I + \int_0^1 g'(P_0^M((1-\theta) w_1^* + \theta w_2^*) \, d\theta) \right] y + \psi_k \end{aligned}$$

where $\lim_{k \to \infty} \psi_k = 0$.

We have $y \in X_N^M(v)$, $\int_0^1 g'(P_0^M(1-\theta)w_1^* + \theta w_2^*)\,d\theta y \in X_M(v)$, hence

$$P_0^N \left[I + \int_0^1 g'(P_0^M((1-\theta)w_1^* + \theta w_2^*)\,d\theta \right] y = 0,$$

and therefore,

$$z\left(\left[I + \int_0^1 g'(P_0^M((1-\theta)w_1^* + \theta w_2^*)\,d\theta \right] y \right) \geqslant N$$

by Atkinson (1964, Exercise 2, p. 549).

The zero number is lower semicontinuous on X, hence for sufficiently large k we have

$$z(w_1^k - w_2^k) \geqslant N.$$

This contradicts Lemma 3.2. ∎

2.6. Corollary

For each $v \in E$ there exists a $q > 0$ such that

$$|P_N(v)\,y| \leqslant q\,|P_0^N(v)\,y|$$

for any $y \in T_u W^u(w)$, $u \in W^u(w)$, $w \in E$.

Indeed, since $W^u(w) \subset \mathscr{A}$, we have for any C^1 curve $\gamma: [0, \varepsilon_0) \to W^u(w)$, $\gamma(0) = u$, $\gamma'(0) = y \in T_u W^u(w)$

$$|P_N(y)| = \lim_{\varepsilon \to 0} 1/\varepsilon\,|P_N(\gamma(\varepsilon) - \gamma(0)|$$

$$\leqslant q \lim_{\varepsilon \to 0} 1/\varepsilon\,|P_0^N(\gamma(\varepsilon) - \gamma(0)| = q\,|P_0^N(v)\,y|.$$

Proof of Theorem 1.1

By Proposition 3.3, for any chosen $v \in E$ the set $\mathscr{A}$ is a graph of a function $h: P_0^N(v)\mathscr{A} \to X_N$. Also, from Corollary 3.6 it follows that h is C^1 on each $P_0^N(W^u(w))$, $w \in E$. It remains to be proved that there exists a PLN-invariant C^1 manifold $\mathscr{N}$ of dimension N containing $\mathscr{A}$. Since $\mathscr{A}$ is a compact attractor, it is obvious that $\mathscr{N}$ can be restricted in such a way that it will preserve its invariance properties, contain $\mathscr{A}$, and be a graph of a C^1 extension of h.

The manifold $\mathscr{N}$ will be constructed by induction. Let us order the equilibria into a sequenbce $w_1, \ldots, w_r$ in such a way that $i(w_j) \geqslant i(w_k)$ if $j < k$.

We construct a sequence $\{\mathcal{N}_j\}$ of C^1 locally invariant manifolds of dimension N such that $\mathcal{N}_{j+1}$ extends an open submanifold of $\mathcal{N}_j$ and

$$\mathcal{A}_j := \bigcup_{v \leqslant j} W^u(w_v) \subset \mathcal{N}_j.$$

Then, $\mathcal{N}_r$ will be a locally invariant manifold of dimension N containing A.

Denote

$$j_n := \max\{j : i(w_j) \geqslant n\}.$$

We define

$$\mathcal{N}_{j_N} := \bigcup_{1 \leqslant j \leqslant j_N} W^u(w_j).$$

For a given $j > j_N$ denote $n := i(w_j)$ and assume that $\mathcal{N}_{j-1}$ has been constructed to contain $\bigcup_{1 \leqslant v \leqslant j-1} W^u(w_v)$. To complete the induction step we extend an open submanifold of $\mathcal{N}_{j-1}$ containing $\mathcal{A}_{j-1}$ to a PLN-invariant manifold $\mathcal{N}_j$ containing $W^u(w_j)$.

To this end we employ Proposition 2.1. First we note that the inertial manifold theorem of Chow and Lu (1988) and Foias *et al.* (1986) allows us to reduce the extension step to one for finite dimensional systems. As mentioned in the proof of Lemma 3.5, [8, 9, 20] provide for an $M < N$-dimensional PLN-invariant manifold $\mathcal{M}$ which can be expressed by $\mathcal{M} :=$ graph g, where $g: Q \to X_M(w_j)$ is C^1 and Q is an open subset of $X_0^M(w_j)$ containing $P_0^M(w_j)(\mathcal{A})$. The semiflow induces a local flow φ_t on Q by

$$\varphi_t(u) = P_0^M(w_j)\, S_t(u + g(u)).$$

To simplify the formulations we extend φ_t to a global flow on $X_0^M(w_j)$ by modifying it outside some neighborhood of $P_0^M(w_j)\mathcal{A}$ if necessary.

Since $\mathcal{A} \subset \mathcal{M}$, in particular, we have $W^u(w_v) \subset \mathcal{M}$ for all v. Therefore, we may add $\mathcal{N}_{j-1} \subset \mathcal{M}$ to our induction hypotheses. In addition, we assume that

$$\mathcal{N}_{j-1} = \bigcup_{v < j} \Phi(D_v) \tag{3.14}$$

where D_v is a locally invariant open disk of dimension N containing a neighborhood of w_v in $W^u(w_v)$. From the construction of $\mathcal{N}_j$ it is seen immediately that it also lies in $\mathcal{M}$ and satisfies (3.14) with $j-1$ replaced by j.

We now introduce some notation to match that of Proposition 2.1. First, we note that the spectrum Λ of the operator L of the linearized problem (1.4) for $v := w_j$ [defined by $(Ly)(x) = y''(x) + f'(v(x))\, y(x)$ for $y \in H_0^1 \cap H^2$] admits a partition $\Lambda \cup \Lambda_4$, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, where $\Lambda_1 := \{-\lambda_0(w_j),..., -\lambda_{n-1}(w_j)\}$, $\Lambda_2 := \{-\lambda_n(w_j),..., -\lambda_{N-1}(w_j)\}$, $\Lambda_3 := \{-\lambda_N(w_j),..., -\lambda_{M-1}(w_j)\}$, and $\Lambda_4 := \{\lambda_M, ...\}$. The corresponding splitting of $Y := X_0^M(w_j)$ is $Y = Y_1 \oplus Y_2 \oplus Y_3$, where $Y_1 := X_0^n(w_j)$, $Y_2 := X_n^N(w_j)$, and $Y_3 := X_N^N(w_j)$. Then for $A_i := L|_{\Lambda_i}$ (2.3)–(2.5) are satisfied with

$$\gamma < \min\{|\lambda_{n-1}(w_j)|, |\lambda_n(w_j)|\},$$
$$\beta = 1/2(\lambda_{N-1}(w_j) + \lambda_N(w_j)),$$
$$0 < \delta < 1/2(\lambda_N(w_j) - \lambda_{N-1}(w_j)).$$

As in Section 2 we introduce coordinates $x = (x_1, x_2, x_3)$ in such a way that $x(w_j) = 0$ and the manifolds $W^u(w_j)$, $W^s(w_j)$, V, and W (the latter two introduced in Section 2) locally at 0 coincide with Y_1, $Y_2 + Y_3$, $Y_1 + Y_2$, and Y_3; by P_i we denote the orthogonal projection $Y \to Y_j$, $j = 1, 2, 3$. We do not distinguish between $\mathcal{N}_{j-1}$, $\mathcal{A}_{j-1}$, φ_t, etc., and their representations in the x-coordinates. Then, φ_t is generated by the differential equation (2.11) and satisfies (2.6), with R satisfying (2.7)–(2.10). It is then sufficient to construct $\mathcal{N}_j$ as a submanifold of the x-space.

As the manifold $\mathcal{R}$ of Proposition 2.1 we take a suitable restriction of $\mathcal{N}_{j-1}$ of the form (3.14) (with D_v possibly replaced by their open subdisks) which contains $\mathcal{A}_{j-1}$. Below, we prove that $\mathcal{M}$ can be chosen to satisfy the hypotheses of Proposition 2.1. The PLN-invariant manifold $\mathbb{R}$ which is provided by Proposition 2.1 is of the form (3.14) and contains both $\mathcal{R}$ and $W^u(0)$. Trivially, $\mathcal{M} \subseteq \mathcal{M}$. Therefore, we can take it for $\mathcal{N}_j$. This completes the induction step and, thus, also the proof of the theorem.

It remains to be verified that the requirements of Proposition 2.1 can be met by a suitable choice of $\mathcal{R}$.

By Lemma 3.5 we have $\mathcal{A} = \operatorname{graph} h$, where h: $(P_1 + P_2)\mathcal{A} \to Y_3$ is C^1 and satisfies

$$|h(x_1, x_2) - h(x_1', x_2')| \leqslant q(|x_1 - x_1'| + |x_2 - x_2'|)$$

for some $q > 0$ and any (x_1, x_2), $(x_1', x_2') \in (P_1 + P_2)\mathcal{A}$; by rescaling Y_3 we can achieve $q \leqslant 1/4$. Since $\mathcal{A}$ is compact it follows that there is a neighborhood C of $\mathcal{A}$ such that

$$|x_3 - x_3'| \leqslant (1/2)(|x_1 - x_1'| + |x_2 - x_2'|) \tag{3.15}$$

for any $x, x' \in (\mathcal{N}_{j-1} \cup \mathcal{A}) \cap C$.

The set $\mathcal{A}$, being the maximal compact attractor, is Lyapunov stable, i.e., for any neighborhood Q of $\mathcal{A}$ there is a neighborhood R of $\mathcal{A}$ such that $\Phi(R) \subseteq Q$. In particular, by possibly restricting the disks D_v we can make $\tilde{\mathscr{R}} := \bigcup_{v<j} \Phi(D_v)$ to satisfy $\mathrm{cl}\, \tilde{\mathscr{R}} \subset C$. Then (3.15) holds for all $x, x' \in \mathscr{R} \cup \mathscr{A}$.

By Henry [1985, Properties (5) and (1), p. 191], $0 \in \mathrm{cl}\, W^u(w_v)$ for some v implies $W^u(w_v) \cap W^s(0) \neq 0$, the intersection being transversal by Henry (1985, Theorem 7). Since $W^s(0)$ coincides with Y_2 locally at 0, for sufficiently small $\eta > 0$ we have

$$W^u(w_v) \cap \Gamma(\eta) \cap Y_2 \neq \varnothing \qquad \text{if} \quad 0 \in \mathrm{cl}\, W^u(w_v), \tag{3.16}$$

$$W^u(w_v) \cap \overline{Q(\eta)} = \varnothing \qquad \text{if} \quad 0 \notin \mathrm{cl}\, W^u(w_v). \tag{3.17}$$

Since $i(w_v) \geqslant i(0)$ for $v \leqslant j$, from (1, Theorem 7) and (3.16), it follows that

$$\overline{Q(\eta)} \cap (\mathscr{A} - \mathscr{A}_{j-1}) \neq \varnothing \tag{3.18}$$

for $\eta > 0$ small.

By Henry (1985, Property (5, p. 191), $w_v \notin W^u(w_j)$ for $v < j$. Therefore, by possibly restricting D_v we can achieve

$$\bigcup_{v<j} D_v \cap (\overline{Q(\eta)} \cup W^u(w_j)) = \varnothing \tag{3.19}$$

for $\eta > 0$ sufficiently small.

Let $\eta > 0$ be so small that (3.17) and (3.19) hold. To complete the proof we distinguish two cases:

$$\text{(a)} \quad 0 \in \mathrm{cl}\, \mathscr{A}_{j-1}, \qquad \text{(b)} \; 0 \notin \mathrm{cl}\, \mathscr{A}_{j-1}.$$

Case (a)

By (3.17) we have $\Gamma(\eta) \cap \mathscr{R} \cap Y_2 \neq \varnothing$; because of (3.16), (3.17), by possibly restricting D_v further, we can achieve that

$$\Phi(D_v) \cap \mathrm{cl}\, \Omega(\eta) = \varnothing \qquad \text{if} \quad 0 \notin \mathrm{cl}\, W^u(w_v) \tag{3.20}$$

provided $\eta > 0$ is sufficiently small. Then since (3.15) holds for all $x, x' \in \tilde{\mathscr{R}} \cup \mathscr{A}$, and since $0 \in \mathscr{A}$, we have

$$|x_3| = |x_3 - 0| \leqslant (1/2)(|x_2| + |x_3|) \qquad \text{for} \quad x \in \mathrm{cl}\, Q(\eta) \cap \tilde{\mathscr{R}}. \tag{3.21}$$

Take open subdisks G_v of D_v containing w_v such that $\bar{G}_v \subset D_v$, $v < j$, and denote $\mathscr{R} := \bigcup_{v<j} \Phi(G_v)$, $U := (P_1 + P_2)\mathscr{R} \cap \Gamma(\eta)$. Then there is a neighborhood B of $\bar{U}$ in $Y_1 + Y_2$ such that $B \subseteq (P_1 + P)\mathscr{R}$ and, therefore, we can define $\sigma := h|_B$. By (3.15) and (3.21), $\mathscr{R}$, U, B, and σ satisfy hypotheses (i) and (ii) of Proposition 2.1; by (3.19), hypothesis (v) is satisfied as well.

To verify hypothesis (iv) we first note that from $\Phi(\mathscr{R})=\mathscr{R}$ it follows

$$\Phi(\Gamma(\eta_1)\cap\mathscr{R})\subseteq\Phi(\Gamma(\eta_1))\cap\mathscr{R}. \tag{3.22}$$

The opposite inclusion follows from the fact that S_t and, consequently, also φ, is gradient-like, i.e., there is a scalar function V on Y which decreases strictly along nonconstant trajectories (Henry, 1981).

Indeed, let $x\in\mathscr{R}$, $x=\varphi_t(\xi)$, $\xi\in\Gamma(\eta_1)$, $t\geqslant 0$. By definition of $\mathscr{R}$ we have $x=\varphi_{t'}(\xi')$ for some $t'\geqslant 0$, $\xi'\in D_v$, $v<j$; by (3.17) we have

$$W^u(w_v)\cap W^s(0)\neq 0. \tag{3.23}$$

Since V decreases along nonconstant trajectories, from (3.22) it follows $V(0)<V(w_v)$. If $\eta>0$, D_v are chosen sufficiently small it follows that $V(x')<V(x'')$ for all $x'\in\Gamma(\eta)$, $x''\in D_v$, hence $V(\xi)\leqslant V(\xi')$, from which it follows that $t'-t\geqslant 0$. This means $\xi\in\Phi(\mathscr{R})\cap\Gamma(\eta_1)=\mathscr{R}\cap\Gamma(\eta_1)$ and $x=\varphi_t(\xi)\in\Phi(\mathscr{R}\cap\Gamma(\eta_1))$. This completes the verification of hypothesis (iv).

It remains to verify hypothesis (iii). From (3.22) and the definition of Σ it follows that

$$\mathscr{R}\cap\Omega(\eta)=\Phi_\eta(\Sigma)\cup\Phi_\eta(\mathscr{R}\cap\Delta(\eta)), \tag{3.24}$$

where $\Delta(\eta):=\{x\colon |x_1|=\eta,\ |x_2|\leqslant\eta,\ |x_3|\leqslant\eta\}$. For $x\in\Delta(\eta)$, $x(t):=\varphi_t(x)$ we have

$$\begin{aligned}(1/2)\,d/dt\,|x_1|^2|_{t=0}&=\langle A_1x_1\cdot x_1\rangle+\langle F_1(x),x_1\rangle\\ &\geqslant[\gamma\eta^2-L(\eta)]\eta>0,\end{aligned}$$

provided $\eta>0$ is sufficiently small. This proves $\Phi_\eta(\Delta(\eta))=\varnothing$ and, by (3.17), verifies hypothesis (iii).

Case (b)

By assumption it follows from (3.17) and (3.18) that we can restrict the disks D_v in such a way that, for sufficiently small $\eta>0$, we have $\overline{Q(\eta)}\cap\tilde{\mathscr{R}}=\varnothing$. For such $\eta>0$ we can chose $U:=\Gamma(\eta)$, $s\colon\Gamma_{12}(\eta)\to Y_3$ arbitrarily satisfying (2.13), (2.14) and $\mathscr{R}:=\tilde{\mathscr{R}}\cup\varphi_{(-\varepsilon,\infty)}(\text{graph }\sigma)$ for some $\varepsilon>0$ sufficiently small. This choice of $\mathscr{R}$ obviously satisfies the hypotheses of Proposition 2.1 ▌

APPENDIX

In this Appendix we prove several technical lemmas which are needed in Section 2.

We consider the differential equation

$$\dot{x}_i = A_i x_i + F_i(x_i), \qquad i := 1, 2, 3 \tag{A.1}$$

on $Y = Y_1 + Y_2 + Y_3$ from Section 2 in the transformed coordinates. That is, we assume that A_i satisfy (2.3)–(2.5), F_i are continuous and satisfy (2.6), and (A.1) generates a unique flow φ_t which can be represented by

$$\varphi_t(x) = e^{tA}x + R(t, x)$$

with $x := (x_1, x_2, x_3)$ and $R := (R_1, R_2, R_3)$ being C^1 and satisfying (2.9) and (2.10). As in Section 2, by P_j and R_j we denote the orthogonal projection $Y \to Y_j$ and $P_j R$, respectively, $j = 1, 2, 3$.

Note that (2.3)–(2.5) imply

$$|e^{-A_1 t}| \leqslant e^{-\gamma t}, \tag{A.2}$$

$$|e^{A_2 t}| \leqslant e^{-\gamma t}, \qquad |e^{-A_2 t}| \leqslant e^{(\beta - \delta) t}, \tag{A.3}$$

$$|e^{A_3 t}| \leqslant e^{-(\beta + \delta) t} \tag{A.4}$$

for $t \geqslant 0$, respectively.

Recall the definitions of $\Gamma(\eta)$, $\Omega(\eta)$, and $\Gamma(\eta)$ from Section 2 and denote

$$\hat{\Omega}(\eta) := \{x \in \Omega(\eta) : |x_3| \leqslant \eta\}.$$

For given $\eta > 0$ define

$$p(t) = \sup\{|(P_2 + P_3)\,\varphi_t(x)| \; x \in \hat{\Omega}(\eta) \cup \hat{\Gamma}(\eta),\ \varphi_s(x) \in \Omega(\eta) \text{ for } 0 < s \leqslant t\}, \tag{A.5}$$

$$k(t) = \sup\left\{\frac{|P_3 \varphi_t(x)|}{|P_2 \varphi_t(x)|} : x \in \Gamma(\eta),\ \varphi_s(x) \in \Omega(\eta) \text{ for } 0 < s \leqslant t\right\}. \tag{A.6}$$

A.1. Lemma

For $\eta > 0$ sufficiently small we have the following.

(*i*) *If $x \in \hat{\Gamma}(\eta) \cap \hat{\Omega}(\eta)$ and $\varphi_s(x) \in \Omega(\eta)$ for $0 < s \leqslant t$, then*

$$|P_3 x(t)| \leqslant 2\eta. \tag{A.7}$$

(*ii*) *If $x \in \operatorname{cl} \hat{\Gamma}(\eta)$, $\varphi_s(x) \in \Omega(\eta)$ for $0 < s < t$ and $|P_1 \varphi_t(x)| = \eta$, then $|P_1 x(t + \tau)| > \eta$ for $\tau > 0$ sufficiently small.*

(*iii*) $\lim_{t \to \infty} p(t) = 0$.

(*iv*) $|k(t)| \leqslant 2$ *for $t \geqslant 0$ and*

$$\lim_{t \to \infty} k(t) = 0. \tag{A.8}$$

To simplify the formulations in the proofs in this Appendix, once we consider $\varphi_t(x)$ for some $t>0$ we automatically will assume that $\varphi_s(x)\in \mathrm{cl}\,\Omega(\eta)$ for $0<s\leqslant t$ without explicitly saying so. In other words, we restrict φ_t to the (local) flow in $\mathrm{cl}\,\Omega(\eta)$. Once we prove Lemma A.1(i) it allows us, in addition, to restrict φ_t to $\hat{\Omega}(\eta)\cup\hat{\Gamma}(\eta)$. This does not concern the formulations of the results, which are given in full.

Proof

(i) For $x\in\hat{\Gamma}(\eta)\cup\hat{\Omega}(\eta)$, η sufficiently small and $0\leqslant t\leqslant 1$ it follows from (2.9) and (2.10) that

$$|R_2(t, x_1, x_2, x_3)|\leqslant L(\eta)(|x_2|+|x_3|), \tag{A.9}$$

$$|R_1(t, x_1, x_2, x_3)|\leqslant L(\eta)|x_1|, \tag{A.10}$$

$$|R_3(t, x_1, x_2, x_3)|\leqslant L(\eta)|x_3|. \tag{A.11}$$

Denote $x(t):=\varphi_t(x)$. If $|x_3(t)|=\eta$, from (2.5) and (2.12), it follows that

$$\begin{aligned}(1/2)d|x_3(t)|^2/dt &= \langle x_3(t), A_3x_3(t)+F_3(x(t))\rangle\\ &\leqslant -(\beta+\delta)\eta^2+L(\eta)\eta^2<0\end{aligned} \tag{A.12}$$

provided η is so small that $L(\eta)<\beta+\delta$. This proves (A.7).

(ii) If η is so small that (i) holds, by (2.3) and (2.11) we have

$$1/2\, d\,|x_1(t)|^2/dt=\langle x_1(t), Ax_1(t)+F_1(x(t))\rangle\geqslant\gamma\eta^2-\eta^2L(\eta)>0,$$

provided $\eta>0$ is so small that $L(\eta)<\gamma$. This proves (ii).

(iii) Let $\eta>0$ be so small that (i) holds. Then $|x_j(t)|\leqslant 2\eta$ for $j=1, 2, 3$, and by (A.9) we have

$$|x_2(t+1)|\leqslant(e^{-\gamma}+L(\eta))\,|x_2(t)|+L(\eta)\,|x_3(t)|,$$
$$|x_3(t+1)|\leqslant(e^{-(\beta+\delta)}+L(\eta))\,|x_3(t)|,$$

hence

$$|x_2(t+1)|+|x_3(t+1)|\leqslant(e^{-\gamma}+2L(\eta))(|x_2(t)|+|x_3(t)|). \tag{A.13}$$

Let $\eta>0$ be so small that $a:=e^{-\gamma}+2L(\eta)<1$. Applying (A.13) to $t=0, 1,\ldots, n-1$, we obtain

$$|x_2(n)|+|x_3(n)|\leqslant 2a^n\eta.$$

If $n := [t]$, the integer part of t, we have

$$\begin{aligned}|x_2(t)| + |x_3(t)| &\leqslant e^{-\gamma(t-n)}\,|x_2(n)| + L(\eta)(|x_2(n)| + |x_3(n)|)\\ &\quad + e^{-(\beta+\delta)(t-n)}\,|x_3(n)| + L(\eta)\,|x_3(n)|\\ &\leqslant (1+2L(2\eta))(|x_2(n)| + |x_3(n)|) \leqslant 2(1+2L(\eta))a^n\eta\end{aligned}$$

Since the left-hand side of the inequality depends only on η, this proves (iii).

(iv) Let $\eta > 0$ be so small that (i) holds. If $x_2(t) \neq 0$, denote $\chi(t) = |x_3(t)|\,|x_2(t)|^{-1}$. If $\chi(t) \leqslant 1$, we have from (A.9) and (A.11)

$$\begin{aligned}\chi(t+1) - \frac{|x_3(t+1)|}{|x_2(t+1)|} &\leqslant \frac{[e^{-(\beta+\delta)} + L(\eta)]\,|x_3(t)|}{e^{-(\beta-\delta)}|x_2(t)| - L(\eta)[|x_2(t)| + |x_3(t)|]},\\ &\leqslant \frac{e^{-(\beta+\delta)} + L(\eta)}{e^{-(\beta-\delta)} - L(\eta)(1+\chi(t))}\,\chi(t).\end{aligned}$$

Let $\eta > 0$ be so small that $b := [e^{-(\beta+\delta)} + L(\eta)][e^{-(\beta-\delta)} - 2L(\eta)]^{-1} < 1$. Then, from $\chi(0) \leqslant 1$ we obtain by induction $\chi(n) \leqslant 1$ for $t \geqslant n \geqslant 0$ integer and, in turn, also $\chi(n) \leqslant b^n$.

Let now $n = [t]$. We have

$$\begin{aligned}|\chi(t)| = |x_3(t)|\,|x_2(t)|^{-1} &\leqslant [e^{-(\beta+\delta)(t-n)} + L(\eta)]\,|x_3(n)|\\ &\quad \times [e^{-(\beta-\delta)(t-n)}|x_2(n)| - L(\eta)(|x_2(n)| + |x_3(n)|)]^{-1}\\ &\leqslant (1+L(\eta))(1-2L(\eta))^{-1}\chi(n) \leqslant (1+L(\eta))(1-2L(\eta))^{-1}b^n.\end{aligned}$$

Since b and the right-hand side of the inequality depend on η only, this proves (iv). ∎

From the local invariance of $Y_1 \cap \Omega(\eta_1)$ and Lemma A.1 (iv) we obtain the following.

A.2. Corollary

For sufficiently small $\eta > 0$, *one has* $|P_2\varphi_t(x)| \neq 0$ *provided* $x \in \hat{\Gamma}(\eta)$ *and* $\varphi_s(x) \in Q(\eta)$ *for* $0 < s \leqslant t$.

In most of the arguments below there is no need to consider the components x_1 and x_2 separately. Therefore, in order to shorten the formulas we frequently aggregate them into one component $x_{12} := x_1 + x_2$. Correspondingly we write $Y_{12} := Y_1 + Y_2$, $A_{12} := A_1 + A_2$, $P_{12} := P_1 + P_2$, etc.

For fixed x, x' denote $y(t) := \varphi_t(x') - \varphi_t(x)$. From (2.6) it follows that

$$y_i(s+\tau) = e^{\tau A_i}\,y_i(s) + b_{i,12}(\tau, s)\,y_{12}(s) + b_{i,3}(\tau, s)\,y_3(s) \qquad \text{(A.14)}$$

for $0 \leqslant s \leqslant t$, $0 \leqslant \tau \leqslant \min\{1, t-s\}$, and $i = 12, 3$, where

$$b_{i,j}(\tau, s) = \int_0^1 D_{x_j} R_i(\tau, (1-\vartheta)\,\varphi_s(x) + \vartheta\varphi_s(x'))\, d\vartheta.$$

In the lemma below we consider $y(t)$ satisfying (A.14) with $b_{i,j}(\tau, t)$ such that

$$|b_{i,j}(\tau, t)| \leqslant L \qquad \text{for} \quad i, j = 12, 3, \qquad \tau \geqslant 0, \qquad \text{and some } L > 0, \tag{A.15}$$

$$|b_{3,12}(\tau, t)| \leqslant \rho(t), \tag{A.16}$$

where ρ satisfies

$$\lim_{t \to \infty} \rho(t) = 0. \tag{A.17}$$

A.3. Lemma

Let $q > 0$ be given. Let $y(t) = (y_{12}(t), y_3(t))$ satisfy (A.14) with $b_{i,j}$ satisfying (A.15) and (A.16), ρ satisfying (A.17). Then for sufficiently small $L > 0$ there exists a positive function r: $[0, \infty) \to [0, \infty)$ depending on ρ only and satisfying

$$\lim_{t \to \infty} r(t) = 0 \tag{A.18}$$

such that if

$$|y_3(0)| \leqslant q\, |y_{12}(0)| \tag{A.19}$$

then

$$|y_3(t)| \leqslant r(t)\, |y_{12}(t)|. \tag{A.20}$$

Proof

Let $y(0)$ satisfy (A.19) and $y_{12}(0) \neq 0$. Denote

$$\lambda(t) := |y_3(t)|\, |y_{12}(t)|^{-1}.$$

If $\lambda(n) \leqslant q$, we have

$$\lambda(n+1) \leqslant \frac{(|e^{A_3}| + L)\, |y_3(n)| + \min\{\rho(n), L\}\, |y_{12}(n)|}{(|e^{-A_{12}}|^{-1} - L(1+q))\, |y_{12}(n)|},$$

$$\leqslant \chi\lambda(n) + \sigma_n,$$

where

$$\chi = \frac{e^{-(\beta+\delta)} + L}{e^{-(\beta-\delta)} - (1+q)L} = \frac{e^{-2\delta} + Le^{\beta-\delta}}{1-(1+q)\,Le^{\beta-\delta}}.$$

$$\sigma_n = \frac{\min\{\rho_n, L\}}{e^{-(\beta-\delta)} - L(1+q)}.$$

Let $L > 0$ be so small that

$$\chi < 1 \tag{A.21}$$

and

$$\chi q + \sigma_n < q. \tag{A.22}$$

From (A.22) it follows $\lambda(n) \leqslant q$ by induction, and from (A.21) we obtain

$$\lambda(n) < \chi^n q + \sum_{j=0}^{n-1} \chi^{n-1-j} \sigma_j = \chi^n q + K \sum_{j=0}^{n-1} \chi^{n-1-j} \min\{\rho_j, L\}$$

where $K = [e^{-(\beta-\delta)} - L(1+q)]^{-1}$.

Let $\varepsilon > 0$. Choose $N = N(\varepsilon)$ so large that

$$(1-\chi)^{-1}\rho_n \leqslant \varepsilon/3K \quad \text{for} \quad n \geqslant N, \qquad \chi^{2N} q < \varepsilon/3, \qquad \chi^N < (NLK)^{-1}\varepsilon/3.$$

Then, we have for $n \geqslant 2N$

$$\lambda(n) \leqslant \chi^{2N} q + \chi^N K \sum_{j=0}^{N-1} \chi^{N-1-j} L + K \sum_{j=N}^{n} \chi^{n-1-j} \rho_j \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 < \varepsilon. \tag{A.23}$$

Let now $n \leqslant t < n+1$. We have

$$\lambda(t) \leqslant \frac{(|e^{A_3(t-n)}| + L)\,|y_3(n)| + \rho_n\,|y_{12}(n)|}{[|e^{-A_{12}(t-n)}|^{-1} - (1+q)L]\,|y_{12}(n)|},$$

$$\leqslant M_2 \lambda(n) + M_1 \rho(n), \tag{A.24}$$

where $M_1 := [e^{-(\beta-\delta)L} - L(q+1)]^{-1}$ and $M_2 := (1+L)M_1$.

Let $\{\varepsilon_n\}$ be any sequence of positive reals satisfying $\varepsilon_n \to 0$ for $n \to \infty$. Define

$$r(t) := M_2 \varepsilon_n + M_1 \rho([t]) \qquad \text{for} \quad 2N(\varepsilon_n) \leqslant t < 2N(\varepsilon_{n+1}).$$

Then r depends on ρ only and satisfies (A.18); from (A.23) and (A.24) we obtain (A.20). ∎

A.4. Lemma

There exists a $q>0$, *such that if* η *is sufficiently small,* $x, x' \in \hat{\Gamma}(\eta)$, *and* $|x_3 - x_3'| \leqslant |x_{12} - x_{12}'|$, *then*

$$|P_3(\varphi_t(x) - x')| \leqslant q\,|(P_1 + P_2)(\varphi_t(x) - x')| \tag{A.25}$$

whenever $t \geqslant 0$ *and* $\varphi_s(x) \in \Omega(\eta)$ *for* $0 < s \leqslant t$.

Proof

As usual we write $x(t) := \varphi_t(x)$ and assume that $\eta > 0$ is so small that Lemma A.1(i) holds true. We split the proof into three cases:

(a) $t \leqslant \tau$, $|x_{12} - x_{12}'| \leqslant \chi_1 \eta$,

(b) $t \leqslant \tau$, $|x_{12} - x_{12}'| > \chi_1 \eta$,

(c) $t > \tau$,

χ_1 and τ to be determined later. In each of the three cases we prove separately that for sufficiently small $\eta > 0$, a $q > 0$ satisfying the requirements of the lemma can be found.

Case (a)

We have

$$\frac{|x_3(t) - x_3'|^2}{|x_{12}(t) - x_{12}'|^2} \leqslant \frac{2[|x_3(t) - x_3|^2 + |x_3 - x_3'|^2]}{|x_{12}(t) - x_{12}|^2 + |x_{12} - x_{12}'|^2 - 2\langle x_{12}(t) - x_{12}, x_{12} - x_{12}' \rangle}$$

We find $\chi_1 > 0$ and $0 < \tau \leqslant 1$, for which there exists a constant $\lambda < 1$ such that

$$\langle x_{12}(t) - x_{12}, x_{12} - x_{12}' \rangle \leqslant \lambda |x_{12}(t) - x_{12}|\,|x_{12} - x_{12}'| \tag{A.26}$$

provided $t \leqslant \tau$, $|x_{12} - x_{12}'| \leqslant \chi_1 \eta$ and $\eta > 0$ is sufficiently small.

Suppose for a moment that (A.26) holds. Since

$$|x_{12}(t) - x_{12}|\,|x_{12} - x_{12}'| \leqslant 1/2[|x_{12}(t) - x_{12}|^2 + |x_{12} - x_{12}'|^2],$$

we then have

$$\begin{aligned}
\frac{|x_3(t) - x_3'|^2}{|x_{12}(t) - x_{12}'|^2} &\leqslant \frac{2[|x_3(t) - x_3|^2 + |x_3 - x_3'|^2]}{(1-\lambda)[|x_{12}(t) - x_{12}|^2 + |x_{12} - x_{12}'|^2]} \\
&\leqslant \frac{2}{(1-\lambda)} \left[\frac{|x_3(t) - x_3|^2}{|x_{12}(t) - x_{12}|^2} + \frac{|x_3 - x_3'|^2}{|x_{12} - x_{12}'|^2} \right] \\
&\leqslant \frac{2}{1-\lambda} \left[1 + \frac{|x_3(t) - x_3|^2}{|x_{12}(t) - x_{12}|^2} \right].
\end{aligned}$$

For $t \leqslant 1$ we have by (2.6)

$$|x_3(t)-x_3| \leqslant |x(t)-x| \leqslant \int_0^t |Ax(x)+F(x(s))|\, ds \leqslant K_1 t\eta, \qquad \text{(A.27)}$$

where $K_1 := 2(|A| + L(\eta))$. Further, using the variation of constants formula we obtain

$$\begin{aligned} |x_{12}(t)-x_{12}| &\geqslant |x_2(t)-x_2| \geqslant |x_2| - |x_2(t)| \\ &\geqslant \eta - e^{-\gamma t}\eta - 2te^{|A_3|}\, L(\eta)\eta \geqslant K_2 t\eta \end{aligned}$$

with $K_2 = 1 - e^{-\gamma} - 2e^{|A_3|} L(\eta)$. Hence, if $\tau \leqslant 1$ and if $\eta > 0$ is so small that $e^{-\gamma} - 2e^{|A_3|} L(\eta) < 1$, for $0 \leqslant t \leqslant \tau$ we have

$$\frac{|x_3(t)-x_3'|^2}{|x_{12}(t)-x_{12}'|^2} \leqslant \frac{2}{1-\lambda}\left[1+\frac{K_1}{K_2}\right],$$

i.e., (A.25) is satisfied with $q = (1-\lambda)^{-1}(1+K_1^2/K_2^2)$.

To complete Case (a) it remains to find $1 \geqslant \tau > 0$ and $\chi_1 > 0$ such that (A.26) is satisfied for $0 \leqslant t \leqslant \tau$ and $|x_{12}-x_{12}'| \leqslant \chi_1 \eta$ for $\eta > 0$ sufficiently small.

Let π_η be the orthogonal projection of the neighborhood of the point x_{12} in $P_{12}\Gamma(\eta)$ into $T_{x_{12}}P_{12}\Gamma(\eta)$. Denote

$$f := \pi_\eta(x_{12}-x_{12}')\, |x_{12}-x_{12}'|^{-1}.$$

Then there exists a function ω: $\mathbb{R}^+ \to \mathbb{R}^+$ (independent of η) such that $\omega(t) \to 0$ for $t \to 0$ and

$$|x_{12}-x_{12}'-f|x_{12}-x_{12}'|| \leqslant \omega(\eta^{-1}|x_{12}-x_{12}'|)\, |x_{12}-x_{12}'|.$$

We have

$$\begin{aligned} \langle x_{12}(t)-x_{12}, x_{12}-x_{12}'\rangle &= |x_{12}-x_{12}'|\, \langle x_{12}(t)-x_{12}, f\rangle \\ &\quad + \langle x_{12}(t)-x_{12}, x_{12}-x_{12}'-f\, |x_{12}-x_{12}'|\rangle, \end{aligned}$$

hence

$$\begin{aligned} &\langle x_{12}(t)-x_{12}, x_{12}-x_{12}'\rangle - |x_{12}-x_{12}'|\, \langle x_{12}(t)-x_{12}, f\rangle \\ &\qquad \leqslant \omega(\eta^{-1}|x_{12}-x_{12}'|)\, |x_{12}-x_{12}'|\, |x_{12}(t)-x_{12}|. \end{aligned} \qquad \text{(A.28)}$$

We now estimate $\langle x_{12}(t)-x_{12}, f\rangle$. Since $\langle x_2, f\rangle = 0$, $\langle x_1, x_2\rangle = 0$, we have

$$\begin{aligned} \langle x_{12}(t)-x_{12}, f\rangle &= |x_{12}(t)-x_{12}|^2 - \langle x_{12}-x_{12}(t), |x_2|^{-1}x_2\rangle^2 \\ &= |x_{12}(t)-x_{12}|^2 - \eta^{-2}\langle x_2(t)-x_2, x_2\rangle^2. \end{aligned} \qquad \text{(A.29)}$$

Further, for $0 \leqslant t \leqslant 1$, by (2.4), (2.6), and (A.27) we have

$$\langle x_2(t) - x_2, x_2 \rangle = \left\langle \int_0^t [A_2 x_2(s) - F_2(x(s))], x_2 \right\rangle ds$$

$$\geqslant t \langle A_2 x_2, x_2 \rangle - \eta \int_0^t [|A_2|\,|x_2(s) - x_2| + 2L(2\eta)]\, ds$$

$$\geqslant t[\gamma - |A_2|\, t K_1 - L(\eta)]\eta^2.$$

Hence, for $\tau \geqslant (1/4)|A_2|^{-1} K_1^{-1}\gamma$ and $L(\eta) < \gamma/4$ we have

$$\langle x_2(t) - x_2, x_2 \rangle \geqslant t(\gamma/2)\eta^2. \tag{A.30}$$

Substituting (A.30) into (A.29) we obtain

$$\langle x_{12}(t) - x_{12}, f \rangle^2 \leqslant |x_{12}(t) - x_{12}|^2 - t^2\gamma^2/4.$$

On the other hand, from (A.27) we obtain for $0 \leqslant t \leqslant 1$

$$t \geqslant (K_1 \eta)^{-1} |x(t) - x| \geqslant (K_1 \eta)^{-1} |x_{12}(t) - x_{12}|,$$

which implies

$$\langle x_{12}(t) - x_{12}, f \rangle^2 \leqslant \lambda_1\, |x_{12}(t) - x_{12}|^2 \tag{A.31}$$

with $\lambda_1 := 1 - (1/4)\, K_1^{-2}\gamma^2 < 1$. From (A.28) and (A.31) it follows that

$$\langle x_{12}(t) - x_{12}, x_{12} - x'_{12} \rangle \leqslant \lambda\, |x_{12} - x'_{12}|\, |x_{12}(t) - x_{12}|$$

where $\lambda = \lambda_1 + \omega(\eta^{-1}|x_{12} - x'_{12}|)$. If χ_1 is chosen so small that $\omega(\chi_1) < 1 - \lambda_1$, we have $\lambda < 1$.

Case (b)

If $t < \tau$ and $|x_{12} - x'_{12}| \geqslant \chi_1 \eta$ we have

$$\frac{|x_3(t) - x'_3|}{|x_{12}(t) - x'_{12}|} \leqslant \frac{2\eta}{\chi_1 \eta - |x_{12}(t) - x_{12}|}.$$

Further, by (A.27) we have

$$|x_{12}(t) - x_{12}| \leqslant K_1 \eta t.$$

If $\tau \leqslant (1/2)\, \chi_1 K_1^{-1}$, then

$$|x_3(t) - x'_3| \leqslant 4\chi_1^{-1}\, |x_{12}(t) - x'_{12}|,$$

i.e., (A.25) holds with $q := 4\chi_1^{-1}$.

Case (c)

Let $\tau \leqslant 1$ be given. For $t \geqslant 0$, $0 \leqslant s \leqslant \tau$ we obtain from (A.3) and the variation of constants formula

$$|x_2(t+s)| = \left| e^{A_2 s} x_2(t) + \int_0^s e^{A_2(s-\sigma)} F_2(x(t+\sigma))\, d\sigma \right|$$
$$\leqslant e^{-\gamma s} |x_2(t)| + 2se^{|A_2|} L(\eta)\eta. \tag{A.32}$$

In particular, for $t = \tau$ we have

$$|x_2(t)| \leqslant \chi_2 \eta \tag{A.33}$$

where $\chi_2 := e^{-\gamma\tau} + \tau e^{|A_2|} L(\eta) < 1$ for $\eta > 0$ sufficiently small.

Assume that (A.33) holds for $t = k\tau$. We prove that for $\eta > 0$ sufficiently small (A.33) extends to all $t \in [k\tau, (k+1)\tau]$.

From (A.32) and (A.33) applied to $t := k\tau$ we obtain for $0 \leqslant s \leqslant \tau$

$$x_2(k\tau + s) \leqslant e^{-\gamma s} \chi_2 \eta + se^{|A_2|} L(\eta)\eta \leqslant \chi_2 \eta,$$

provided $e^{-\gamma s}\chi_2 + se^{|A_2|}L(\eta) \leqslant \chi_2$ for $0 \leqslant s \leqslant \tau$, i.e., if $(1 - e^{-\gamma s})\chi_2 - se^{|A_2|}L(\eta) \geqslant 0$. Since $1 - e^{-\gamma s}$ is convex, this is true if $\eta > 0$ is chosen so small that $1 - e^{-\gamma\tau} \geqslant \tau e^{|A_2|} L(\eta)$.

By induction we obtain $|x_2(t)| \leqslant \chi_2 \eta$ for $t \geqslant \tau$. Hence, for $t \geqslant \tau$ we have

$$\frac{|x_3(t) - x_3'|}{|x_{12}(t) - x_{12}'|} \leqslant \frac{|x_3(t)| + |x_3'|}{|x_2'| - |x_2(t)|} \leqslant 2(1 - \chi_2)^{-1}.$$

Hence, (A.25) holds with $q := 2(1 - \chi_2)$. ∎

A.5. Lemma

For $\eta > 0$ sufficiently small we have

$$\lim_{t \to \infty} \Delta(t) = 0, \tag{A.34}$$

where

$$\Delta(t) := \sup \left\{ \frac{|P_3(S_{t+\tau}(x) - S_t(x'))|}{|(P_1 + P_2)(S_{t+\tau}(x) - S_t(x')|} : \tau \geqslant 0;\ x, x' \in \hat{\Gamma}(\eta), \right.$$

$$\varphi_s(x) \in \Omega(\eta) \quad \textit{for} \quad 0 < s \leqslant t + \tau, \qquad \varphi_\sigma(x') \in \Omega(\eta) \quad \textit{for} \quad 0 < \sigma \leqslant t,$$

$$\left. |x_3 - x_3'| \leqslant |x_{12} - x_{12}'| \right\}. \tag{A.35}$$

Proof

First, we note that by Lemma A.4, for sufficiently small $\eta > 0$ there exists a $q > 0$ such that

$$|P_3(\varphi_\tau(x) - x')| \leqslant q |(P_1 + P_2)(\varphi_\tau(x) - x')|$$

provided $x, x' \in \hat{\Gamma}(\eta)$, and $|x_3 - x_3'| \leqslant |x_{12} - x_{12}'|$.

The assumptions of Lemma A.3 are satisfied for

$$y(t) := \varphi_{t+\tau}(x) - \varphi_t(x'),$$

with $L := L(\eta)$ and

$$\rho(t) = \sup\{|D_{x_{12}} R_3(s, (1-\vartheta)\, \varphi_{t+\tau}(x) + \vartheta \varphi_t(x'))| : t \geqslant 0, \qquad 0 \leqslant s \leqslant 1, \\ 0 \leqslant \vartheta \leqslant 1, \qquad x, x' \in \operatorname{cl} \hat{\Omega}(\eta)\}.$$

To check that $\rho(t)$ satisfies (A.17), note that from (2.9) it follows that $D_{x_{12}} R_3(s, x_{12}, 0) = 0$ and, since R is C^1, $D_{x_{12}} R_3(s, x) \to 0$ for $x_3 \to 0$ uniformly for $0 \leqslant s \leqslant 1$ and $x \in \hat{\Omega}(\eta)$. Hence, (A.17) follows from Lemma A.1(iii). Now (A.34) is an immediate consequence of Lemma A.3. ∎

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