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CONVERGENCE IN GENERAL PERIODIC PARABOLIC EQUATIONS IN ONE SPACE DIMENSION

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IT HAS BEEN known for some time (see [8, 10, 14]) that scalar one-dimensional autonomous parabolic equations under separated boundary conditions have all bounded trajectories convergent. Recently, generalizations of this result to periodically timed-dependent equations have been established. Chen and Matano [5] have considered the equation

$$u_t = u_{xx} + f(t, u), \qquad t > 0, \qquad 0 < x < 1, \tag{1}$$

where f is of class C^2, $f(t + \tau, u) \equiv f(t, p)$ for some $\tau > 0$, under various types of boundary conditions (Dirichlet, Neumann, periodic). They have proved that any bounded solution of this boundary value problem converges to a τ-periodic solution of (1) and (2). In his thesis, Sandstede [13] extended this result by allowing f to depend on t, x, u and u_x. He proved the result for Dirichlet and, under some restrictions, for Neumann boundary conditions.

In this paper, we present a general convergence theorem with a simpler proof than the one in [13].

We consider a quasilinear parabolic equation

$$u_t = d(t, x, u, u_x)u_{xx} + f(t, x, u, u_x), \qquad t > 0, \qquad 0 < x < 1, \tag{2}$$

where $d, f \in C^2(\mathbb{R} \times [0, 1] \times \mathbb{R}^2, \mathbb{R})$, $d > 0$, are periodic in t with a common period $\tau > 0$. We consider either of the boundary conditions

$$u(t, i) = h_i(t), \qquad t > 0, \qquad i = 0, 1, \tag{3a}$$

$$u_x(t, i) = g_i(t, u(t, i)), \qquad t > 0, \qquad i = 0, 1. \tag{3b}$$

Here, $g_i(t, u)$ and $h_i(t)$, $i = 0, 1$, are C^2-functions, τ-periodic in t.

In the sequel the boundary conditions will be referred to as (3), assuming that only one of (3a), (3b) is chosen.

The problem (2), (3) is well posed on the Sobolev space $H^2 := H^2(0, 1)$ (see [1, 2]). For any $u_0(\cdot) \in H^2$ satisfying the compatibility conditions

$$u_0(i) = h_i(0), \qquad t > 0, \qquad i = 0, 1 \tag{4a}$$

or

$$u_{0x}(i) = g_i(0, u_0(i)), \qquad t > 0, \qquad i = 0, 1 \tag{4b}$$

[depending on whether we consider (3a) or (3b)], there exists a solution $u(t, \cdot)$ of (2), (3) with $u(0, \cdot) = u_0(\cdot)$. This solution is unique (up to the extension of the interval of existence) and depends continuously on u_0. Denoting the maximal interval of existence of $u(t, \cdot)$ by $[0, s_0)$, we have $s_0 = +\infty$ if $\|u(t, \cdot)\|_{H^2}$ stays bounded as $t \to s_0$. In the latter case, the set $\{u(t, \cdot): t > 0\}$ is relatively compact in H^2. By the regularity results of [1, 2], $u(t, x)$ is a classical solution (i.e. u_t, u_x, u_{xx} are continuous on $(0, s_0) \times [0, 1]$). Moreover, u_t has continuous derivative u_{tx}, hence (2) and the regularity of $d > 0$ and f imply that u_{xxx} is continuous.

The theorem we prove in this paper reads as follows.

THEOREM 1. Let $u(t, \cdot)$ be a bounded (in H^2) solution of (3), (4). Then there exists a τ-periodic solution $p(t, x)$ of (3), (4) such that

$$\lim_{t \to \infty} \|u(t, \cdot) - p(t, \cdot)\|_{H^2} = 0.$$

It will be useful to reformulate the conclusion of the theorem in terms of the Poincaré map T of the periodic problem (2), (3). By definition,

$$T(u_0) := u(\tau, \cdot)$$

if the solution $u(t, \cdot)$ with $u(0, \cdot) = u_0(\cdot)$ exists up to the time τ. The domain of definition of T is an open subset of the manifold

$$X := \{u_0 \in H^2: u_0 \text{ satisfies (4)}\}.$$

Clearly, $u(t, \cdot)$ is τ-periodic if and only if u_0 is a fixed point of T.

An obvious consequence of the conclusion of theorem 1 is that the sequence $T^n u_0 = u(n\tau, \cdot)$, $n = 0, 1, 2, \ldots,$ converges to $p_0 \in X$. By the continuous dependence of the solutions of (2), (3) on initial conditions, the opposite is also true: if $T^n u_0 \to p_0(\cdot)$ [in which case $p_0(\cdot)$ is a fixed point of T], then the solution $p(t, x)$ of (2), (3) with $p(0, x) \equiv p_0(x)$ satisfies the conclusion.

The proof of the theorem is (of course) based on the properties of the zero number of solutions of a linearization of (2), (3). In fact, the quasilinearity of the equation is of no relevance. The arguments apply to any equation

$$u_t = F(t, x, u, u_x, u_{xx}), \qquad t > 0, \qquad 0 < x < 1, \tag{5}$$

with $F(t, x, u, p, q) \in C^2$, $F_q \in C^2$, F periodic in t, and $F_q \geq \beta > 0$ everywhere, provided the basic theory (existence, uniqueness, continuous dependence) and sufficient regularity (in general, continuity of u_{xxxx} is needed) is available. The reader interested in fully nonlinear equations is referred to [6, 7] and the references therein, where the basic properties are studied. Note that in the fully nonlinear case, compactness of the closure of a trajectory is not assured by its boundedness (so in formulations of convergence results, compactness must be assumed).

In order to prepare the proof of theorem 1, we now state a lemma which appears to be crucial.

Consider the linear equation

$$v_t = a(t, x)v_{xx} + b(t, x)v_x + c(t, x)v, \qquad t > 0, \qquad 0 < x < 1, \tag{6}$$

with the boundary conditions

$$v(t, i) = 0, \qquad i = 0, 1, \qquad t > 0, \tag{7a}$$

or

$$v_x(t, i) = \alpha_i(t)v(t, i), \qquad i = 0, 1, \qquad t > 0. \tag{7b}$$

LEMMA 1. Assume that a_t, a_x, a_{xx}, b_t, b_x and c are continuous on $[0, \infty) \times [0, 1]$, $a > 0$ everywhere, and that α_i, $i = 0, 1$, are bounded C^1 functions on $[0, \infty)$. Let $v(t, x) \not\equiv 0$ be a classical solution of (6), (7). Then there exists a t^* such that for any $t > t^*$ we have

$$v_x(t, 0) \neq 0, \text{ in the case of (8a) and}$$

$$v(t, 0) \neq 0, \text{ in the case of (8b).}$$

In the case of Dirichlet boundary conditions this lemma follows directly from the results of [3]. Indeed, by [3, theorem C], for $t > 0$, the "zero number"

$$z(v(t, \cdot)) := \sup\{k \in \mathbb{Z}: \text{there exist } 0 < x_1 < x_2 < \cdots < x_k < 1,$$
$$\text{such that } v(t, x_j)v(t, x_{j+1}) < 0, \text{ for } j = 1, 2, \ldots, k-1\}$$

is finite and nonincreasing in t. Moreover, $z(v(t, \cdot))$ drops at any t such that $v(t, \cdot)$ has a multiple zero in $[0, 1]$. Since the integer value $z(v(t, \cdot)) \geq 0$ can drop only a finite number of times, there exists a t^* such that for $t > t^*$, $v(t, \cdot)$ has only simple zeros. In particular, (7a) implies that $u_x(t, 0) \neq 0$ for $t > t^*$.

In the case of the boundary condition (7b), lemma 1 is not so immediate. For the reader's convenience, its proof is included at the end of the paper.

The application of lemma 1 in the proof of theorem 1 is based upon the following observation: if u_1, u_2 are two solutions of (2), (3) then the difference $v := u_1 - u_2$ is a classical solution of (6), (7) with a, b and c defined by

$$a(t, x) = d(t, x, u_1, u_{1x}),$$
$$b(t, x) = \{f(t, x, u_1, u_{1x}) - f(t, x, u_1, u_{2x}) + (d(t, x, u_1, u_{1x}) - d(t, x, u_1, u_{2x}))u_{2xx}\}(u_{1x} - u_{2x})^{-1},$$
$$c(t, x) = \{f(t, x, u_1, u_{2x}) - f(t, x, u_2, u_{2x}) + (d(t, x, u_1, u_{2x}) - d(t, x, u_2, u_{2x}))u_{2xx}\}(u_1 - u_2)^{-1},$$

for $u_1 \neq u_2$, $u_{1x} \neq u_{2x}$, and extended continuously to the set where $u_1 = u_2$ or $u_{1x} = u_{2x}$. [For brevity we have omitted the argument (t, x).] In the case of Neumann boundary conditions [(3b) for u_1, u_2 and (7b) for v] we have

$$\alpha_i(t) = (g_i(t, u_1(t, i)) - g_i(t, u_2(t, i)))(u_1(t, i) - u_2(t, i))^{-1}.$$

By hypotheses and by the regularity properties of the solutions of (2), (3), the functions a, b, c and α_i satisfy the regularity assumptions of lemma 1. Moreover, α_1, α_2 are bounded if u_1, u_2 are.

We now prove theorem 1.

Let $u(t, x)$ be a bounded solution of (2), (3). As mentioned above, it suffices to prove that the sequence $T^n u(0, \cdot) = u(n\tau, \cdot)$, $n = 0, 1, \ldots$, is convergent. We first prove that the real sequence

$$\eta_n := (1 - \delta)u(n\tau, 0) + \delta u_x(n\tau, 0) \tag{8}$$

is convergent. Here $\delta = 1$ in the case of (3a) and $\delta = 0$ in the case of (3b).

Consider the function

$$v(t, x) := u(t + \tau, x) - u(t, x).$$

Due to periodicity, $u(t + \tau, x)$ satisfies (2), (3) [as does $u(t, x)$], hence $v(t, x)$ is a classical solution of some linear problem (6), (7). By lemma 1, unless $v \equiv 0$ (in which case u is τ-periodic and the assertion is trivial), there exists a t^* such that the function

$$t \mapsto (1 - \delta)v(t, 0) + \delta v_x(t, 0)$$

is of constant nonzero sign in (t^*, ∞).

Observe that

$$\eta_{n-1} - \eta_n = (1 - \delta)v(n\tau, 0) + \delta v_x(n\tau, 0).$$

Thus for $n > t^*\tau^{-1}$, η_n is a monotone sequence. Since η_n is bounded [because $u(t, \cdot)$ is bounded in H^2 and, thus, in C^1], it is convergent.

Denote

$$\eta_\infty := \lim_{n \to \infty} \eta_n. \tag{9}$$

We now prove that the ω-limit set $\omega(u)$, defined as the set of all accumulation points of $u(n\tau, \cdot)$ as $n \to \infty$, consists of a single point, i.e. $u(n\tau, \cdot)$ is convergent (recall that this sequence is relatively compact in the submanifold $X \subset H^2$). In the proof we use the obvious fact that

$$(1 - \delta)w(0) + \delta w_x(0) = \eta_\infty \tag{10}$$

for any $w(\cdot) \in \omega(u)$ [see (8), (9)].

Let $p_0(\cdot)$, $q_0(\cdot) \in \omega(u)$, and let $p(t, x)$, $q(t, x)$ be the solutions of (2), (3) with $p(0, x) \equiv p_0(x)$, $q(0, x) \equiv q_0(x)$. In order to prove that $p_0 = q_0$, we apply lemma 1 again, this time with the function

$$v(t, x) := p(t, x) - q(t, x).$$

By continuity of the Poincaré map T, we have $p(n\tau, \cdot) = T^n p_0 \in \omega(u)$ and, similarly, $q(n\tau, \cdot) \in \omega(u)$ for all n. Therefore, by (10),

$$(1 - \delta)p(n\tau, 0) + \delta p_x(n\tau, 0) \equiv \eta_\infty \equiv (1 - \delta)q(n\tau, 0) + \delta q_x(n\tau, 0).$$

Hence,

$$\delta v(n\tau, 0) + (1 - \delta)v_x(n\tau, 0) = 0 \qquad \text{for } n = 1, 2, \ldots.$$

By lemma 1, this is possible only if $v \equiv 0$. This shows that $p = q$ and completes the proof of the theorem.

We now prove lemma 1 for the boundary condition (7b). To this end we employ the following lemma which can be proved by adapting standard maximum principle arguments [4, 9, 11] in a straightforward way.

Lemma 2. Let the functions a, b and c be defined on $D := [t_1, t_2] \times [x_1, x_2]$ with a_t, a_x, a_{xx}, b_t continuous and $a > 0$. Let $v(t, x)$ be a classical solution of (7) on D and for both $i = 0$ and $i = 1$ let one of the following conditions be satisfied:

(a) $u_x(t, x_i) \equiv 0$ for all $t \in [t_1, t_2]$,
(b) $u(t, x_i) \neq 0$ for any $t \in [t_1, t_2]$.

Then, $z_{[x_1,x_2]}(u(t, \cdot))$ is a nonincreasing function of t.

Here $z_{[x_1,x_2]}$ stands for the "zero number" on the interval $[x_1, x_2]$ which is defined similarly to the zero number on $[0, 1]$.

Proof of lemma 1 *for* (8b). Let $v \not\equiv 0$ be a classical solution of (6), (7b) on $Q := (0, \infty) \times [0, 1]$. First we show that $z(v(t, \cdot)) < \infty$ for some $t_0 > 0$. This, in conjunction with the nonincrease of $z(v(t, \cdot))$ will imply that $z(v(t, \cdot))$ is constant at some interval (t^*, ∞). We then conclude the proof by showing that $v(t, 0) \neq 0$ for $t > t^*$.

To see that $z(v(t, \cdot))$ is finite for $t > 0$, we use the following simple observation: arbitrarily near 0 there exists an open interval $U \subseteq (0, \infty)$ such that one of the following three alternatives holds:

(i) $v(t, i) \equiv 0$ for any $t \in U$, $i = 0, 1$;
(ii) $v(t, i) \neq 0$ for any $t \in U$, $i = 0, 1$;
(iii) $v(t, 0) \equiv 0$ and $v(t, 1) \neq 0$ for any $t \in U$;
(iv) $v(t, 0) \neq 0$ and $v(t, 1) \equiv 0$ for any $t \in U$.

In the case of alternatives (i) and (ii), theorems C and D of [3] apply respectively to the solution $v(t, x)$ on $U \times [0, 1]$. By these theorems, $z(v(t, \cdot)) < \infty$ for $t \in U$. In the case of alternatives (iii) and (iv), the proofs of the above theorems have to be combined (cf. [3, pp. 81, 82]) to conclude the result.

In order to be able to apply lemma 2 in our next argument we "transform" the boundary conditions. To this end, consider the function

$$w(t, x) = \begin{cases} v(t, x)(1 + \xi(x)\alpha_0(t))^{-1} & \text{for } x \in [0, 1/2],\ t > 0, \\ v(t, x)(1 + \xi(x)\alpha_1(t))^{-1} & \text{for } x \in [1/2, 1],\ t > 0, \end{cases} \tag{11}$$

where $\xi(x)$ is a smooth function on $[0, 1]$ satisfying the following conditions:

$$\inf\{\xi(x)\alpha_i(t)\colon x \in [0, 1],\ t > 0,\ i \in \{0, 1\}\} > -1,$$

$$\xi \equiv 0 \text{ in a neighbourhood of } x = 1/2,$$

$$\xi(i) = 0,\ \xi_x(i) = 1 \text{ for } i := 0, 1.$$

It is easy to see that such a function ξ exists and that w is a classical solution of a linear equation

$$w_t = \tilde{a}w_{xx} + \tilde{b}w_x + \tilde{c}w,$$

where $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ have the same regularity as a, b and c. Moreover, w satisfies

$$w_x(t, 0) = w_x(t, 1) = 0.$$

Thus, the transformation (11) leads to a boundary value problem (6), (7b) with

$$\alpha_1(t) \equiv 0, \qquad i := 0, 1. \tag{12}$$

Since v and w have the same zeros, this transformation shows that without loss of generality we may proceed in the proof for $v(t, x)$, assuming (12).

By lemma 2, $z(v(t, \cdot))$ is nonincreasing. Since $z(v(t, \cdot))$ is finite for $t > 0$, there is a t^* such that for $t > t^*$ we have

$$z(v(t, \cdot)) \equiv \text{const.}$$

We prove that $v(t, 0) \neq 0$ for $t > t^*$. Suppose the opposite holds, i.e. $v(t_1, 0) = 0$ for some $t_1 > t^*$. We show that this leads to a contradiction.

Since $z(v(t_1, \cdot)) < \infty$ and $v(t_1, \cdot) \not\equiv 0$ on $[0, 1]$ (otherwise $v(t, x) \equiv 0$ by the maximum principle), there exists an $x_0 \in (0, 1)$ such that

$$v(t_1, x_0) \neq 0 \qquad \text{and} \qquad v(t_1, x)v(t_1, x_0) \geq 0 \qquad \text{for } x \in [0, x_0]. \tag{13}$$

Assume, e.g. $v(t_1, x_0) > 0$ [the case $v(t_1, x_0) < 0$ is analogous]. Choose t_2, t_3 satisfying $t^* < t_2 < t_1 < t_3$ such that

$$v(t, x_0) > 0 \qquad \text{for } t \in [t_2, t_3].$$

By lemma 2, the functions

$$z_{[0,x_0]}(v(t, \cdot)), \qquad z_{[x_0,1]}(v(t, \cdot))$$

are both nonincreasing. Since we obviously have

$$z(v(t, \cdot)) = z_{[0,x_0]}(v(t, \cdot)) + z_{[x_0,1]}(v(t, \cdot)) \qquad \text{for } t \in (t_2, t_3)$$

and $z(v(t, \cdot))$ is constant for $t > t_2 > t^*$, $z_{[0,x_0]}(v(t, \cdot))$ must be constant as well. Hence, by (13),

$$z_{[0,x_0]}(v(t, \cdot)) \equiv 0 \qquad \text{for } t \in (t_2, t_3).$$

Consequently,

$$v(t, x) \geq 0 \qquad \text{on } Q_0 := [t_2, t_3] \times [0, x_0].$$

We see that 0 is the minimum of v in Q_0 and it is achieved at the boundary point (t_1, x_0). Now, the Neumann condition $u_x(t_1, x_0) = 0$ contradicts the Hopf boundary principle [12].

This contradiction shows that $v(t, 0) \neq 0$ for $t > t^*$ and lemma 1 is proved.

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