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Controllability and Linear Closed-loop Controls in Linear Periodic Systems*

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Consider a linear control system

$$\dot{x} = A(t)\,x + B(t)u \tag{1}$$

$x = (x_1, ..., x_n) \in R^n$, $u = (u_1, ..., u_m) \in R^m$, $A(t)$ and $B(t)$ being real continuous on $(-\infty, \infty)$ $n \times n$-matrices respectively.

The system (1) is called controllable on $(-\infty, \infty)$, if to any two points $x^1, x^2 \in R^n$ and any $t_0 \in (-\infty, \infty)$ there is a $t_1 > t_0$ and a measurable control function $u(t)$, $t \in [t_0, t_1]$ such that the solution $x(t)$ of (1), $x(t_0) = x^1$ under $u = u(t)$ satisfies $x(t_1) = x^2$ (cf. [1]).

It is a remarkable property of autonomous controllable systems ($A(t) = A$, $B(t) = B$; A, B constant) that to any prescribed spectrum Σ there is a closed-loop control $u = Qx$, (Q possibly complex) such that the spectrum of the system (1) with $u = Qx$, i.e. of the system

$$\dot{x} = (A + BQ)x$$

is Σ.

This fact has been known for a long time in the case of $u \in R^1$. For $u \in R^m$, $m > 1$ it was apparently first explicitly stated by Popov (cf. [2], [3]), who proved the equivalence of the above property to complete controllability (cf. also [5], where the problem is formulated in a somewhat different way).

Recently, Wonham [4] presented another proof of it. In addition to Popov, he has proved that if Σ contains with any complex number its conjugate with the same multiplicity, Q can be chosen real.

Let us note that a similar result can be obtained easily from the transformation of [6], (cf. Corollary 2), which is of a somewhat different kind than in [4] and [5].

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This paper is devoted to the proof of a similar property of controllable linear periodic systems, the spectrum of $A + BQ$ replaced by the characteristic multipliers of the system

$$\dot{x} = [A(t) + B(t)\,Q(t)]x. \tag{2}$$

Throughout this paper by a real-type (n-) spectrum Σ will always be meant a set of not necessarily distinct complex numbers $\sigma_1, \ldots, \sigma_n$, containing together with every complex number its complex conjugate with the same multiplicity. All other quantities occurring in this paper will be supposed to be real, unless stated otherwise.

Further, for any $r \times s$-matrix E denote $|E| = \sum_{i=1}^{r} \sum_{j=1}^{s} |e_{ij}|$, E' the transpose of E, vectors being regarded as one column matrices in this connection. By $Y(t, t_0)$ we shall denote the solution of the matrix equation

$$\dot{Y} = A(t)Y \tag{3}$$

with $Y(t_0, t_0) = I$, I being the unity matrix. $Y(t, 0)$ will be simply denoted by $Y(t)$.

If A, B are two matrices of $n \times n$ and $n \times m$ type respectively and the system $\dot{x} = Ax + Bu$ is controllable, we shall call $\langle A, B \rangle$ a controllable pair of matrices. It is well known (cf. [1]) that $\langle A, B \rangle$ is a controllable pair if and only if rank of the matrix $(B, AB, \ldots, A^{n-1}B)$ is n.

Before formulating the main theorem let us prove several auxiliary results, some of which are of interest by themselves.

PROPOSITION 1. *Let $\langle A, B \rangle$ be a controllable pair of matrices, $m \leqslant n$, and let B have rank m. Then, there are positive integers l_i, $i = 1, \ldots, m$ such that $\sum_{i=1}^{m} l_i = n$ and a nonsingular $n \times n$ matrix C such that $C^{-1}AC = D$, $C^{-1}B = G$, where*

$$D = \begin{pmatrix} D_{11}, \ldots, D_{1m} \\ \cdots\cdots\cdots \\ D_{m1}, \ldots, D_{mm} \end{pmatrix}, \qquad G = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix},$$

D_{ij} are $l_i \times l_j$,

$$D_{ij} = \begin{pmatrix} 0 & \ldots, & 0 \\ \cdots\cdots\cdots & & \\ 0 & \ldots, & 0 \\ \alpha_{i,k_{j-1}+1} & \ldots, & \alpha_{i,k_j} \end{pmatrix}$$

if $i \neq j$,

$$D_{ii} = \begin{pmatrix} 0 & , & 1 & ,\ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & , & 0 & ,\ldots, & 1 \\ \alpha_{i,k_{i-1}+1} & , & \alpha_{i,k_{i-1}+2} & ,\ldots, & \alpha_{i,k_i} \end{pmatrix},$$

$$k_i = \sum_{\nu=1}^{i} l_\nu ,$$

G_i *are* $l_i \times m$,

$$G_i = \begin{pmatrix} 0, & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \cdots & \cdots & \cdots & 0 \\ 0,\ldots, & 0, & 1, & \gamma_{i,i+1} ,\ldots, & \gamma_{im} \end{pmatrix}$$

(1 *is in the ith column*), $i, j = 1,\ldots, m$.
For the proof see [6].

COROLLARY 1. *If* $B = b$ *is* $n \times 1$, $\langle A, b\rangle$ *is a controllable pair, then there is a nonsingular matrix* C *such that* $C^{-1}AC = D$, $C^{-1}b = g$, *where*

$$D = \begin{pmatrix} 0, & 1, & 0,\ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0,\ldots, & 0, & 1 & \\ \alpha_1 ,\ldots, & \alpha_n & & \end{pmatrix}, \qquad g = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

COROLLARY 2. *Let* $\langle A, B\rangle$ *be a controllable pair of matrices. Then, to any (real type) spectrum* Σ *there is a complex (real) matrix* Q *such that* $A + BQ$ *has spectrum* Σ.

Proof. $\langle A, B\rangle$ being a controllable pair we can choose an $n \times \tilde{m}$ submatrix $\tilde{B}$ of $B(\tilde{m} \leqslant m)$, such that $\langle A, \tilde{B}\rangle$ is a controllable pair and $\tilde{B}$ has maximal rank (cf. [6], p. 771). Suppose $\tilde{B}$ consists of the first $\tilde{m}$ columns of B. By Proposition 1 we can find a matrix C such that the transformation $x = Cy$ transfers the system

$$\dot{x} = Ax + \tilde{B}u$$

to the system

$$\dot{y} = Dy + Gu$$

D and G being as in Proposition 1. Let $\lambda^n + \beta_n\lambda^{n-1} + \cdots + \beta_1$ be the polynomial with its set of roots equal to Σ. If Σ is real-type, β_i, $i = 1,\ldots, n$

are real. We define $u = Py$, where $p_{mj} = -\beta_j - \alpha_{mj}$ and recursively $p_{ij} = \delta_{k_i+1,j} - \alpha_{ij} - \sum_{\nu=i+1}^{m} \gamma_{i\nu} p_{\nu j}$, where δ_{ij} is the Kronecker's symbol.

Then,

$$D + GP = \begin{pmatrix} 0, 1, \ldots, 0 \\ \cdot \quad \cdot \\ 0, \ldots, 1 \\ -\beta_1, \ldots, -\beta_n \end{pmatrix}$$

and therefore its spectrum is Σ. Further, if we denote $\tilde{Q} = PC^{-1}$, then the matrix $A + \tilde{B}\tilde{Q} = C(D + GP)C^{-1}$ has also the spectrum Σ. Now, let Q be the matrix with the first $\tilde{m}$ rows equal to those of $\tilde{Q}$ and the remaining being zero. Then, $A + \tilde{B}\tilde{Q} = A + BQ$ and thus $A + BQ$ has spectrum Σ.

PROPOSITION 2. *If $\langle A, B\rangle$ is a controllable pair and B has maximal rank, then there is an $m \times n$-matrix Q such that $\langle A + BQ, b_m\rangle$ is controllable. If* $\det A > 0$, *Q can be chosen in such a way that*

$$\det\left(A + \sum_{\nu=i}^{m} b_\nu q_\nu'\right) > 0, \qquad i = 1, \ldots, m \tag{4}$$

where b_ν are the column vectors of B and q_ν are the row vectors of Q.

Proof. We can without loss of generality suppose that A, B are transformed to the special form of Proposition 1, i.e. $A = (D_{ij})$, $B = (G_i)$, $i, j = 1, \ldots, m$.

Suppose first $\det A > 0$. Denote $p_{mj} = -\alpha_{mj}$, $j = 1, \ldots, n$, $p_m = (p_{m1}, \ldots, p_{mn})'$. Then, the last row of $A + b_m p_m'$ is zero, and, consequently, $\det(A + b_m p_m') = 0$. Since $\det(A + b_m q_m') = \det A(1 + q_m' A^{-1} b_m)$ and $A^{-1}b_m \neq 0$, for any $\epsilon > 0$ we can find q_m such that $| q_m - p_m | < m^{-1} | B |^{-1}\epsilon$ and $\det(A + b_m q_m') > 0$. Further, we define recursively

$$p_{ij} = \delta_{k_i+1,j} - \alpha_{ij} - \sum_{\nu=i+1}^{m} \gamma_{i\nu} q_{\nu j}, \qquad p_i = (p_{i1}, \ldots, p_{in})'.$$

If

$$\det\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu' + b_i p_i'\right) \neq 0,$$

we define

$$q_{ij} = \operatorname{sign} \det\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu' + b_i p_i'\right) \delta_{k_i+1,j} - \alpha_{ij} - \sum_{\nu=i+1}^{m} \gamma_{i\nu} q_{\nu j}.$$

If

$$\det\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu' + b_i p_i'\right) = 0,$$

from

$$\det\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu' + b_i q_i'\right)$$

$$= \det\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu'\right) \cdot \left[1 + q_i'\left(A + \sum_{\nu=i+1}^{m} b_\nu q_\nu'\right)^{-1} b_i\right]$$

follows that there is a q_i such that $| p_i - q_i | < m^{-1} | B |^{-1}\epsilon$ and (4) is valid (recall that q_ν, $\nu = i + 1,..., m$ are already chosen so that $A + \sum_{\nu=i+1}^{m} b_\nu q_\nu'$ is nonsingular).

From the above construction, it follows that $A + BQ = T + S$, where

$$T = \begin{pmatrix} 0, \theta_1, 0,..., 0 \\ 0, 0, \theta_2,..., 0 \\ \cdots\cdots\cdots \\ 0,..., 0, \theta_{n-1} \\ 0,..., 0 \end{pmatrix},$$

$\theta_i = +1$ or -1 and $| S | < \epsilon$. Clearly the pair $\langle T, b_m \rangle$ is controllable. By ([7], Chapter 2, Theorem 1), for $\epsilon > 0$ sufficiently small $\langle A + BQ, b_m \rangle = \langle T + S, b_m \rangle$ is also controllable.

The assertion of the proposition for $\det A \leqslant 0$ can be obtained by a similar, rather simplified argument ($p_i = q_i$, $i = 1,..., m$).

PROPOSITION 3. *Let $A(t)$ and $B(t)$ be ω-periodic an integrable over $[0, \omega]$. Then, the system* (1) *is controllable if and only if the rows of the matrix function $Y^{-1}(s)\, B(s)$, $s \in [0, n\omega]$ are linearly independent.*

Proof. It is well known (cf. [1]) that (1) is controllable if and only if to every t_0 there is a t_1 such that the rows of the functional matrix $Y^{-1}(s)\, B(s)$ on $[t_0, t_1]$ are linearly independent. Clearly, this is true in the periodic case if and only if such a t_1 exists for every $t_0 = k\omega$, k integer. Further, we have $Y^{-1}(s, k\omega)\, B(s) = Y^{-1}(s - k\omega)\, B(s - k\omega) = Y^{-1}(t)\, B(t)$ for $s \in [k\omega, t_1]$ and $t \in [0, t_1 - k\omega]$. Consequently (1) is controllable if and only if the rows of $Y^{-1}(t)\, B(t)$ are linearly independent on $[0, t_1]$ for some $t_1 > 0$. It remains to prove that this is equivalent with linear independence of the rows of $Y^{-1}(t)\, B(t)$ on $[0, n\omega]$; the only nontrivial part of this statement is that the

linear independence of the rows of the matrix $Y^{-1}(t)\,B(t)$ on $[0, t_1]$ for $t_1 > n\omega$ implies their independence on $[0, n\omega]$. To prove this, note that from $c'Y^{-1}(t)\,B(t) \equiv 0$ on $[0, n\omega]$, $Y^{-1}(t)\,B(t) = Y^{-k}(\omega)\,Y^{-1}(t - k\omega)\,B(t - k\omega)$ and the fact that for $k \geqslant n$ the matrix $Y^{-k}(\omega)$ is a linear combination of the matrices $Y^{-i}(\omega)$, $i = 0, 1,..., n - 1$ follows $c'Y^{-1}(t)\,B(t) \equiv 0$ on any interval $[k\omega, (k + 1)\omega]$, $k \geqslant n\omega$.

COROLLARY 4. *Let $A(t), B(t)$ be continuous and ω-periodic. Then,* (1) *is controllable if and only if there are $r (1 \leqslant r \leqslant n)$ numbers*

$$0 \leqslant t_1 < \cdots < t_r < \omega \tag{5}$$

and integers $i_1,..., i_r$, $1 \leqslant i_j \leqslant m$, such that

(i) *The vectors $\tilde{b}_1 = Y^{-1}(t_1)\,b_{i_1}(t_1),..., \tilde{b}_r = Y^{-1}(t_r)\,b_{i_r}(t_r)$ are linearly independent (b_{i_j} denotes the i_jth column of B)*

(ii) *The pair of matrices $\langle Y(\omega), \tilde{B}\rangle$ is controllable, where $\tilde{B} = (\tilde{b}_1,..., \tilde{b}_r)$.*

Proof. Suppose (1) is controllable. From the set $\{Y^{-1}(t)\,b_i(t) \mid t \in [0, \omega], i = 1,..., m\}$ choose an arbitrary maximal set of linearly independent points $\tilde{b}_j = Y^{-1}(t_j)\,b_{i_j}(t_j)$, $0 \leqslant t_1 \leqslant \cdots \leqslant t_r \leqslant \omega$. Then (i) is satisfied and every point $Y^{-1}(t)\,b_i(t)$, $t \in [0, \omega]$ $i = 1,..., m$, is a linear combination of $\tilde{b}_j$, $j = 1,..., r$. Now, let $t \in [0, n\omega]$, $t = \tau + \mu\omega$, $\tau \in [0, \omega]$, $1 \leqslant i \leqslant m$. Then, there is an m-vector d such that $Y^{-1}(\tau)\,b_i(\tau) = \tilde{B}d$. We have

$$Y^{-1}(t)\,b_i(t) = Y^{-\mu}(\omega)\,Y^{-1}(\tau)\,b_i(\tau) = Y^{-\mu}(\omega)\,\tilde{B}d.$$

Consequently, the linear hull of the set of vectors $\{Y^{-1}(t)\,b_i(t) \mid t \in [0, \omega], i = 1,..., m\}$ is contained in the linear hull of the vectors $\{Y^{-\mu}(\omega)\,\tilde{b}_i \mid i = 1,..., r, \mu = 0,..., n - 1\}$. But the linear independence of the rows of the matrix $Y^{-1}(t)\,B(t)$ on $[0, n\omega]$ implies that the vectors $\{Y^{-1}(t)\,b_i(t) \mid t \in [0, n\omega], i = 1,..., m\}$ span R^n. Thus, the vectors $\{Y^{-\mu}(\omega)\tilde{b}_i \mid i = 1,..., r, \mu = 0,..., n - 1\}$ span R^n, or, equivalently, rank $(\tilde{B}, Y^{-1}(\omega)\,\tilde{B},..., Y^{-n+1}(\omega)\tilde{B}) = n$. Since $Y(\omega)$ is nonsingular, this is equivalent with rank $(\tilde{B}, Y(\omega)\,\tilde{B},..., Y^{n-1}(\omega)\tilde{B}) = n$.

The numbers $t_1,..., t_r$ are not necessarily distinct, but since $Y^{-1}(t)\,B(t)$ are continuous, a sufficiently small change of the numbers t_i will not affect the rank of the matrices $\tilde{B}$ and $(\tilde{B}, Y(\omega)\,\tilde{B},..., Y^{n-1}(\omega)\tilde{B})$. Therefore, by a small change of the numbers t_i we can achieve that both (5) and (i), (ii) will be valid.

In the other direction, the corollary is obvious.

Remark 1. Since $Y(\omega)$ is nonsingular, $\langle Y(\omega), \tilde{B}\rangle$ is a controllable pair if and only if $\langle Y(\omega), Y(\omega)\tilde{B}\rangle$ is controllable.

PROPOSITION 4. *Let $A(t), A_k(t)$ be ω-periodic integrable matrices such that $A_k(t) \to A(t)$ for $k \to \infty$ in $L_1(0, \omega)$. Then, $Y_k(\omega) \to Y(\omega)$ for $k \to \infty$, where $Y_k(t)$ is the solution of $\dot{Y} = A_k(t)Y$ with $Y_k(0) = I$.*

Proof. We have

$$| Y_k(t)| \leqslant n + \int_0^t | A_k(s)| \, | Y_k(s)| \, ds.$$

By Gronwall's inequality

$$| Y_k(t)| \leqslant n \exp \int_0^t | A_k(s)| \, ds \leqslant n \exp \int_0^\omega | A_k(s)| \, ds \quad \text{for} \quad t \in [0, \omega].$$

Since $A_k(t)$ converge in $L_1(0, \omega)$, $\int_0^\omega | A_k(s)| \, ds$ is bounded; hence, $| Y_k(t)|$ are equibounded on $[0, \omega]$ say $| Y_k(t)| \leqslant \kappa$. Further, we have

$$\begin{aligned} | Y_k(t) - Y(t)| &\leqslant \int_0^t | A_k(s) \, Y_k(s) - A(s) \, Y(s)| \, ds \\ &\leqslant \int_0^\omega | Y_k(s)| \, | A_k(s) - A(s)| \, ds \\ &\quad + \int_0^t | A(s)| \cdot | Y_k(s) - Y(s)| \, ds \\ &\leqslant \kappa \int_0^\omega | A_k(s) - A(s)| \, ds + \int_0^t | A(s)| \, | Y_k(s) - Y(s)| \, ds. \end{aligned}$$

Applying Gronwall's inequality, we obtain $|Y_k(t) - Y(t)| \leqslant \kappa \int_0^\omega |A_k(s) - A(s)| \, ds \cdot \exp\{\int_0^t | A(s)| \, ds\}$ which completes the proof.

THEOREM. *Let $A(t)$, $B(t)$ be ω-periodic and C^1 in t and let* (1) *be controllable. Then,*

(i) *To any real-type spectrum $\sum = \{\sigma_1, ..., \sigma_n\}$ such that $\sigma_i \neq 0$, $i = 1, ..., n$ and $\prod_{i=1}^n \sigma_i > 0$ there is an ω-periodic $m \times n$ matrix $Q(t)$ such that the characteristic multipliers of* (2) *are equal to σ_i.*

(ii) *To any real-type spectrum $\sum = \{\sigma_1, ..., \sigma_n\}$ such that $\sigma_i \neq 0$, $i = 1, ..., n$ there is a 2ω-periodic $m \times n$ matrix $Q(t)$ such that the characteristic multipliers of* (2) (*considered as 2ω-periodic system*) *are equal to σ_i^2.*

Moreover, both (i) *and* (ii) *are sufficient for complete controllability of* (1).

Proof. Suppose first that (1) is not controllable. Then, there is at least one nonzero n-vector c such that

$$c' Y^{-1}(t) \, B(t) = 0 \qquad \text{for all} \qquad t. \tag{6}$$

The set of all c satisfying (6) is a linear subspace of R^n invariant under the action of $Y(\omega)'$, since

$$(Y(\omega)'c)' Y^{-1}(t) \, B(t) = c' \cdot Y^{-1}(t - \omega) \, B(t - \omega) = 0 \qquad \text{for all} \qquad t.$$

Therefore, it contains at least one eigenvector of $Y(\omega)'$, i.e. there is a vector c_0 satisfying (6) such that $Y(\omega)'c_0 = \lambda c_0$. Now, let $Q(t)$ be any periodic $m \times n$ matrix and $X(t)$ be the fundamental matrix of (2) with $X(0) = I$. Using the variation of constants formula we obtain

$$\begin{aligned} c_0' X(\omega) &= c_0' Y(\omega) + \int_0^\omega Y^{-1}(t)\, B(t) Q(t)\, X(t)\, dt \\ &= \lambda c_0' + \lambda \int_0^\omega c_0' Y^{-1}(t)\, B(t) Q(t)\, X(t)\, dt = \lambda c_0' \end{aligned}$$

Thus, λ is an eigenvalue of $X(\omega)'$ (and, thus, of $X(\omega)$) for any Q.

Now, let (1) be controllable. Choose the numbers $0 \leqslant t_1 < \cdots < t_r < \omega$ and $\{i_1, \ldots, i_n\}$ and define $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_r)$ as in Corollary 4.

The proof will be accomplished in several steps which we shall number for better orientation.

1°. For an arbitrary $r \times n$ matrix $Q = (q_1, \ldots, q_r)'$ and

$$0 < h \leqslant h_0 = \min_{i=1,\ldots,r} \{t_i - t_{i-1}, \omega - t_r\}$$

denote

$$Q_h(t) = \begin{cases} \dfrac{1}{h} Q^{(j)} & \text{for} \quad t \in [t_j + \nu\omega, t_j + \nu\omega + h],\ \nu \text{ integer} \\ 0 \text{ elsewhere} \end{cases}$$

where $Q^{(j)}$ is the matrix with i_jth row q_j and the remaining rows equal to zero.

Denote $X_{Q,h}(t, \tau)$ the solution of the matrix equation

$$\dot{X} = (A(t) + B(t) Q_h(t))x$$

with $X_{Q,h}(\tau, \tau) = I$. We prove that for $t \in [0, 1]$

$$X_{Q,h}(t_j + ht, t_j) = e^{t b_{i_j}(t_j) q_j'} + O(h) \tag{7}$$

locally uniformly in Q, which implies

$$\begin{aligned} X_{Q,h}(\omega, 0) &= Y(\omega, t_r)\, e^{b_{i_r}(t_r) q_r'}\, Y(t_r, t_{r-1})\, e^{b_{i_{r-1}}(t_{r-1}) q_{r-1}'} \cdots e^{b_{i_1} q_1'}\, Y(t_1) + O(h) \\ &= Y(\omega) \prod_{j=1}^{r} e^{\tilde{b}_j \tilde{q}_j'} + O(h), \end{aligned} \tag{8}$$

locally uniformly in Q, where $\tilde{q}_j = Y(t_j)'\, q_j$. This allows us to define

$$X_{Q,0}(\omega, 0) = \lim_{h \to 0} X_{Q,h}(\omega, 0) = Y(\omega) \prod_{j=1}^{r} e^{\tilde{b}_j \tilde{q}_j'}.$$

For $t \in [0, 1]$, $0 < h \leqslant h_0$ we have

$$| X_{Q,h}(t_j + ht, t_j)| \leqslant n + \int_0^t (\alpha h + \beta | q_j |)| X_{Q,h}(t_j + hs, t_j)| \, ds$$

where

$$\alpha = \max_{t \in [0, h_0]} | A(t_j + t)|, \qquad \beta = \max_{t \in [0, h_0]} | b_{i_j}(t_j + t)|$$

and, consequently, by Gronwall's inequality

$$| X_{Q,h}(t_j + ht, t_j)| \leqslant n e^{\alpha h t} \cdot e^{\beta t |q_j|} \leqslant \kappa e^{\beta |q_j|} \quad \text{for} \quad t \in [0, 1] \tag{9}$$

where κ is a constant independent of h, Q for h sufficiently small.

Further, we have for $t \in [0, 1]$, $h \in [0, h_0]$

$$\begin{aligned}
&| X_{Q,h}(t_j + ht, t_j) - e^{t b_{i_j}(t_j) q'_j} | \\
&\quad \leqslant \int_0^t | [hA(t_j + hs) + b_{i_j}(t_j + hs) \, q'_j] \, X_{Q,h}(t_j + hs, t_j) \\
&\qquad - b_{i_j}(t_j) \, q'_j e^{s b_{i_j}(t_j) q'_j} | \, ds \\
&\quad \leqslant \int_0^t | \, b_{i_j}(t_j)\| \, q_j \| \, X_{Q,h}(t_j + hs, t_j) \\
&\qquad - e^{b_{i_j}(t_j) q'_j} | \, ds + \int_0^t h \, | \, A(t_j + hs)\| \, X_{Q,h}(t_j + hs, t_j)| \, ds \\
&\qquad + \int_0^t | \, b_{i_j}(t_j + hs) - b_{i_j}(t_j)\| \, q_j \| \, X_{Q,h}(t_j + hs, t_j)| \, ds.
\end{aligned} \tag{10}$$

According to (9),

$$\int_0^t h \, | \, A(t_j + hs)\| \, X_{Q,h}(t_j + hs, t_j)| \, ds \leqslant | \, h \, | \, \kappa \alpha e^{\beta |q_j|} \tag{11}$$

$$\int_0^t | \, b_{i_j}(t_j + hs) - b_{i_j}(t_j \| \, q_j \| \, X_{Q,h}(t_j + hs, t_j)| \, ds \leqslant | \, h \, \| \, q_j \, | \cdot \kappa e^{\beta |q_j|} \cdot \beta_1 \tag{12}$$

where

$$\beta_1 = \max_{t \in [t_j, t_j + h_0]} | \, \dot{b}_{i_j}(t)|$$

From (10), (11), (12) it follows

$$\begin{aligned}
&| X_{Q,h}(t_j + ht, t_j) - e^{t b_{i_j}(t_j) q'_j} | \\
&\qquad \leqslant h \cdot \gamma_1(q) + \gamma_2(q) \int_0^t | \, X_{Q,h}(t_j + hs, t_j) - e^{b_{i_j}(t_j) q'_j} | \, ds
\end{aligned}$$

for $t \in [0, 1]$, where $\gamma_1(q), \gamma_2(q)$ do not depend on h for $h \in [0, h_0]$ and are locally bounded in q. Using Gronwall's inequality, we obtain

$$| X_{Q,h}(t_j + ht, t_j) - e^{tb_{ij}(t_j)q'_j} | \leqslant h\gamma_1(q)\, e^{\gamma_2(q)} \qquad \text{for} \quad t \in [0, 1]$$

which proves (7).

2°. Let p be any vector such that $\det(I + \tilde{b}_j p') > 0$. Then, there is a real vector q such that

$$Y^{-1}(t_j)\, e^{b_{ij}(t_j)q'}\, Y(t_j) = I + \tilde{b}_j p' \tag{13}$$

or, equivalently,

$$e^{\tilde{b}_j \tilde{q}'} = I + \tilde{b}_j p'.$$

where $\tilde{q} = Y(t_j)'q$.

We have

$$e^{\tilde{b}_j \tilde{q}'} = I + \tilde{b}_j \tilde{q}' \frac{e^{\tilde{q}'\tilde{b}_j} - 1}{\tilde{q}'\tilde{b}_j} \tag{14}$$

(Here and further we understand $(e^\xi - 1)/\xi = 1$ if $\xi = 0$).

Using (14), (13) can be rewritten as

$$\tilde{b}_j \tilde{q}' \frac{e^{\tilde{q}'\tilde{b}_j} - 1}{\tilde{q}'\tilde{b}_j} = \tilde{b}_j p' \tag{15}$$

Denote $z_1, \ldots, z_{n-1}$ arbitrary n vectors such that $z_1, \ldots, z_{n-1}$, $\tilde{b}_j$ form a basis in R^n. Since $1 + p'\tilde{b}_j = \det(I + \tilde{b}_j p') > 0$, we can define

$$\tilde{q}'\tilde{b}_j = \ln(1 + p'\tilde{b}_j) \tag{16}$$

$$\tilde{q}' z_\nu = \frac{p' \ln(1 + p'\tilde{b}_j)}{p'\tilde{b}_j} \cdot z_\nu, \qquad \nu = 1, \ldots, n - 1 \tag{17}$$

(again, $\xi^{-1} \ln(1 + \xi) = 1$ if $\xi = 0$). Since z_ν, $\nu = 1, \ldots, n - 1$ and $\tilde{b}_j$ form a basis, $\tilde{q}'$ is uniquely determined by (16), (17). From (16) follows

$$e^{\tilde{q}'\tilde{b}_j} - 1 = p'\tilde{b}_j \tag{18}$$

or, equivalently,

$$\tilde{q}' \frac{e^{\tilde{q}'\tilde{b}_j} - 1}{\tilde{q}'\tilde{b}_j} \tilde{b}_j = p'\tilde{b}_j\,. \tag{19}$$

From (16), (17), (18) follows

$$\tilde{q}' z_\nu = \frac{\tilde{q}' \tilde{b}_j p'}{e^{\tilde{q}' \tilde{b}_j} - 1} \cdot z_\nu$$

or, equivalently,

$$\tilde{q}' \frac{e^{\tilde{q}' \tilde{b}_j} - 1}{\tilde{q}' \tilde{b}_j} z_\nu = p' z_\nu . \tag{20}$$

Since z_ν, $\nu = 1,..., n-1$ and $\tilde{b}_j$ form a basis, from (19) and (20) follows

$$\tilde{q}' \cdot \frac{e^{\tilde{q}' \tilde{b}_j} - 1}{\tilde{q}' \tilde{b}_j} = p'. \tag{21}$$

Multiplying this equation from the left by $\tilde{b}_j$ we obtain (15).

As a consequence of this we obtain that to any set of r vectors $p_1 ,..., p_r$ such that $\det(I + \tilde{b}_j p_j') > 0$ we can find a matrix Q such that

$$X_{Q,0}(\omega, 0) = Y(\omega) \prod_{j=1}^{r} (I + \tilde{b}_j p_j'). \tag{22}$$

3°. To any $r \times n$-matrix V, $V = (v_1 ,..., v_r)'$ having the property

$$\det \left(I + \sum_{\nu=j}^{r} \tilde{b}_\nu v_\nu' \right) > 0 \quad j = 1,..., r \tag{23}$$

there are vectors $p_1 ,..., p_r$ such that

$$\prod_{j=1}^{r} (I + \tilde{b}_j p_j') = I + \sum_{j=1}^{r} \tilde{b}_j v_j' , \det(I + \tilde{b}_j p_j') > 0. \tag{24}$$

It is easy to verify that under our assumptions the vectors

$$p_j = \left(I + \sum_{\nu=j+1}^{r} \tilde{b}_\nu v_\nu' \right)^{-1'} v_j$$

solve equation (24) and we have

$$\det(I + \tilde{b}_j p_j') = \det \left[I + \tilde{b}_j v_j' \left(I + \sum_{\nu=j+1}^{r} \tilde{b}_\nu v_\nu' \right)^{-1} \right]$$

$$= \det \left(I + \sum_{\nu=j+1}^{r} \tilde{b}_\nu v_\nu' \right)^{-1} \cdot \det \left(I + \sum_{\nu=j}^{r} \tilde{b}_\nu v_\nu' \right) > 0.$$

Combining (22) and (24) we obtain that to any $r \times n$ matrix V such that (23) is satisfied there is a matrix Q such that

$$X_{Q,0}(\omega, 0) = Y(\omega) + \sum_{j=1}^{r} Y(\omega)\, \tilde{b}_j v_j' \tag{25}$$

4°. By Liouville's theorem $\det Y(\omega) = \exp \int_0^\omega \operatorname{tr} A(t)\, dt > 0$. Therefore, according to Proposition 2 and Remark 1 there is an $r \times n$ matrix V such that $\langle Y(\omega)(I + \tilde{B}V), Y(\omega)\, \tilde{b}_r \rangle$ is controllable and $\det(Y(\omega) + \sum_{\nu=j}^{r} Y(\omega)\, \tilde{b}_\nu v_\nu') > 0$, $j = 1, \ldots, r$, which is equivalent with (23). From ([7], Chap. 2 Theorem 11) again follows that there is an $\epsilon > 0$ such that if $| Z - Y(\omega)(I + \tilde{B}V)| < \epsilon$, $| b - \tilde{b}_r | < \epsilon$ then $\langle Z, Y(\omega) b \rangle$ is also a controllable pair.

5°. From 2°, 3°, 4° follows that there is a matrix Q such that

$$X_{Q,0}(\omega, 0) = Y(\omega)(I + \tilde{B}V)$$

and that for sufficiently small $h > 0$,

$$| X_{Q,h}(\omega, 0) - X_{Q,0}(\omega, 0)| < \tfrac{1}{2}\epsilon. \tag{26}$$

Since $b_{i_r}(t)$ and $Y(\omega, t)$ are continuous, we can choose $h > 0$ so small that $| Y(\omega, t_r + h)\, b_{i_r}(t_r + h) - Y(\omega, t_r)\, \tilde{b}_r | < \epsilon$. Denote

$$R_h^\delta(t) = \begin{cases} h^{-1} Q^{(j)} \xi_\delta[h^{-1}(t - t_j)] & \text{for} \quad t \in [t_j + \nu\omega, t_j + \nu\omega + h], \nu \text{ integer} \\ 0 \text{ elsewhere} \end{cases}$$

where $\xi_\delta(t) = 0$ for $t = 0, 1$, $\xi_\delta(t) = 1$ for $t \in [\delta, 1 - \delta]$ $0 \leqslant \xi_\delta(t) \leqslant 1$ for $t \in [0, 1]$ and $\xi_\delta(t)$ is C^1 on $[0, 1]$. Clearly $A(t) + B(t)\, R_h^\delta(t)$ is ω-periodic, C^1 and $A(t) + B(t)\, R_h^\delta(t) \to A(t) + B(t) Q_h(t)$ for $\delta \to 0$ in $L_1(0, \omega)$. Thus, if we denote $W_h^\delta(t)$ the solution of the matrix equation

$$\dot{x} = (A(t) + B(t)\, R_h^\delta(t))x, \tag{27}$$

with $W_h^\delta(0) = I$, we have by Proposition 4

$$| W_h^\delta(\omega) - X_{Q,h}(\omega, 0)| < \tfrac{1}{2}\epsilon \tag{28}$$

for sufficiently small $\delta > 0$. Combining (26), and (28) we obtain

$$| W_h^\delta(\omega) - Y(\omega)(I + \tilde{B}V)| < \epsilon$$

Hence, by 4°, $\langle W_h^\delta(\omega), Y(\omega, t_r + h)\, b_{i_r}(t_r + h) \rangle$ is controllable. Since $A(t) + B(t)$, $R_h^\delta(t) = A(t)$ for $t \in (t_r + h, \omega)$, $Y(\omega, t_r + h) = W_h^\delta(\omega)$, $W_h^\delta(t_r + h)^{-1}$ and, consequently, $\langle W_h^\delta(\omega), W_h^\delta(\omega)\, W_h^\delta(t_r + h)^{-1}\, b_{i_r}(t_r + h) \rangle$ is controllable.

6°. By the above procedure we have reduced our problem to the case of $A(t)$, $B(t)$ in (1) being C^1 and the pair $\langle Y(\omega), Y(\omega, t_1)\, b_1(t_1)\rangle$ being controllable for some $t_1 \in [0, \omega)$, since the system

$$\dot{x} = \tilde{A}(t)x + B(t)u \tag{29}$$

with $\tilde{A}(t) = A(t) + B(t)\, R_h^{\delta}(t)$ and suitably re-ordered columns of B satisfies the above properties. If the matrix $\tilde{Q}(t)$ solves our problem for the system (29), then the matrix $R_h^{\delta}(t) + \tilde{Q}(t) = Q(t)$ solves the problem for the original system (1).

7°. Let us hence suppose that $A(t)$, $B(t)$ are C^1 and $\langle Y(\omega), Y(\omega)\tilde{b}\rangle$, $\tilde{b} = Y^{-1}(t_1)\, b_1(t_1)$ is controllable. Then, there is a nonsingular matrix C such that $D = C^{-1}Y(\omega)C$, $g = C^{-1}Y(\omega)\tilde{b}$ have the special form of Corollary 1. It is easy to verify that the linear change of variables $x = Cy$ transforms $Y(\omega)$ into D, $Y(\omega)\, \tilde{b}_1$ into g without changing the characteristic multipliers of the system so that we can without loss of generality assume that $Y(\omega)$, $Y(\omega)\tilde{b}$ have already this special form of D and g of Corollary 1.

Now, choose an arbitrary spectrum containing no zero element. Choose p according to Corollary 2 in such a way that the spectrum of

$$Y(\omega) + Y(\omega)\, \tilde{b}p' = Y(\omega)[I + \tilde{b}p']$$

is Σ. If $\sigma_1 \cdots \sigma_n > 0$, then $\det(I + \tilde{b}p') > 0$ and according to 2°, there is a vector q^0 such that

$$X_{q^0,0}(\omega, 0) = Y(\omega)[I + \tilde{b}p']$$

(where $X_{q,h}(t, \tau)$ stands now for $X_{Q,h}(t, \tau)$ with $Q = (q, 0, \ldots, 0)'$). If $\sigma_1 \cdots \sigma_n < 0$, then certainly $(\sigma_1 \cdots \sigma_n)^2 > 0$ and we can apply our argument for (1) considered as a 2ω-periodic system.

8°. The proof will be complete if we show that there is an $h > 0$ and a vector q such that $X_{q,h}(\omega, 0)$ is similar to $X_{q^0,0}(\omega, 0)$. This will be proved by an implicit function argument, for which we need first to prove the continuity of

$$X_{q,h}(\omega, 0), \quad \frac{\partial}{\partial q_i} X_{q,h}(\omega, 0) \qquad \text{and} \qquad \frac{\partial}{\partial h} X_{q,h}(\omega, 0)$$

in q and h in the right (in h) neighborhood of the point $(q^0, 0)$. Since

$$X_{q,h}(\omega, 0) = Y(\omega, t_1 + h) \cdot X_{q,h}(t_1 + h, t_1)\, Y(t_1)$$

it is obvious that if we denote $X_{q,0}(t_1 + 0, t_1) = e^{b_1(t_1)q'}$, it is sufficient to prove the continuous differentiability of $X_{q,h}(t_1 + h, t_1)$. For the sake of simplicity we shall use further the notation $X_{q,h}(t_1 + t, t_1) = Z_{q,h}(t)$, $b_1(t_1 + t) = b(t)$, $b(0) = b$, $X_{q,0}(t_1 + 0, t_1) = Z_{q,0}$.

From the definition of $Z_{q,h}(h)$ it is evident that $Z_{q,h}(h)$ is continuous in q, h for $h > 0$. From (7) it follows that it is continuous in q for $h = 0$. Therefore, the continuity of $Z_{q,h}(h)$ in q, h for $h \geqslant 0$ follows from the local uniformity of $O(h)$ in (7).

$(\partial/\partial q_i)\, Z_{q,h}(t)$ is the solution of the equation

$$\dot{Z} = [A(t_1 + t) + h^{-1}b(t)\, q']\, Z + h^{-1}b(t)\, e_i' Z_{q,h}(t) \tag{30}$$

with $Z(0) = 0$, where e_i is the vector with ith component 1 and the remaining 0. From (30) follows

$$\begin{aligned}\frac{\partial Z_{q,h}(h)}{\partial q_i} &= h^{-1} \int_0^h Z_{q,h}(h)\, Z_{q,h}^{-1}(s)\, b(s)\, e_i' Z_{q,h}(s)\, ds \\ &= \int_0^1 Z_{q,h}(h)\, Z_{q,h}^{-1}(hs)\, b(hs)\, e_i' \cdot Z_{q,h}(hs)\, ds\end{aligned}$$

and by (7),

$$\begin{aligned}\lim_{h\to 0} \frac{\partial Z_{q,h}(h)}{\partial q_i} &= \int_0^1 e^{(1-s)bq'} b e_i' e^{sbq'}\, ds = b e_i' \frac{e^{q'b} - 1}{q'b} \\ &\quad + bq' b_i \frac{-e^{q'b} + 1 + q'b e^{q'b}}{(q'b)^2} \\ &= \frac{\partial}{\partial q_i}\left(I + bq' \frac{e^{q'b} - 1}{q'b}\right) = \frac{\partial}{\partial q_i} e^{bq'} = \frac{\partial}{\partial q_i} Z_{q,0}\,, \end{aligned} \tag{31}$$

(where b_i stands for the ith coordinate of b) if $q'b \neq 0$; the validity of (31) can be similarly verified if $q'b = 0$.

Since the convergence in (31) is locally uniform in q, the continuity of $(\partial/\partial q_i)\, Z_{q,h}(h)$ is proved.

Now, consider for $h, k < h_0$:

$$\begin{aligned}\frac{d}{dt}[Z_{q,h}(ht) - Z_{q,k}(kt)] &= [hA(t_1 + ht) + b(ht)\, q']\, Z_{q,h}(ht) \\ &\quad - [kA(t_1 + kt) + b(kt)\, q']\, Z_{q,k}(kt) \\ &= [hA(t_1 + ht) + b(ht)\, q'][Z_{q,h}(ht) - Z_{q,k}(kt)] \\ &\quad + [hA(t_1 + ht) - kA(t_1 + kt) + (b(ht) \\ &\quad - b(kt))\, q']\, Z_{q,k}(kt). \end{aligned} \tag{32}$$

Denote $[hA(t_1 + ht) - kA(t_1 + kt) + (b(ht) - b(kt))\, q'] = \Gamma(h, k, t)$. Using the variation of constants formula, we obtain from (32)

$$Z_{q,h}(h) - Z_{q,k}(k) = \int_0^1 Z_{q,h}(h)\, Z_{q,h}^{-1}(ht)\, \Gamma(h, k, t)\, Z_{q,k}(kt)\, dt. \tag{33}$$

We have

$$\begin{aligned} \Gamma(h, k, t) &= (h-k)\,A(t_1+ht) + k[A(t_1+ht) - A(t_1+kt)] \\ &\quad + [b(ht) - b(kt)]\,q' \\ &= (h-k)\,A(t_1+ht) + k[\dot{A}(t_1+ht)(h-k)t] \\ &\quad + \dot{b}(ht)(h-k)\,t\cdot q' + \omega(h, h-k), \end{aligned} \tag{34}$$

where $\omega(h, h-k) = o(h-k)$ uniformly in $0 < h < h_0$ and locally uniformly in q. From (33) and (34) we conclude

$$\begin{aligned} \frac{d}{dh} Z_{q,h}(h) &= \lim_{k\to h} (h-k)^{-1}\,[Z_{q,h}(h) - Z_{q,k}(k)] \\ &= \int_0^1 Z_{q,h}(h)\,Z_{q,h}^{-1}(ht)[A(t_1+ht) + ht\dot{A}(t_1+ht) \\ &\quad + t\dot{b}(ht)\,q']\,Z_{q,h}(ht)\,dt. \end{aligned}$$

For $h \to 0$ we obtain

$$\lim_{h\to 0} \frac{d}{dh} Z_{q,h}(h) = \int_0^1 e^{(1-t)bq'}[A(t_1) + t\dot{b}(0)\,q']\,e^{tbq'}\,dt \tag{35}$$

locally uniformly in q. Since the function on the right-hand side of (35) is continuous in q, the continuity of $(d/dh)\,Z_{q,h}(h)$ is established.

9°. We construct a nonsingular $n \times n$-matrix S such that

$$X_{q,h}(\omega, 0)\cdot S = SX_{q^0,0}(\omega, 0) \tag{36}$$

for $h > 0$ sufficiently small and appropriate q. Further, we shall simply denote $X_{q,h}(\omega, 0)$ by $X_{q,h}$.

Denote s_i, $i = 1,\ldots, n$ the columns of S and choose $s_n = e_n$. Then, taking into account that by 7°

$$X_{q^0,0} = \begin{pmatrix} 0, 1,\ldots, 0 \\ \cdots\cdots\cdots \\ 0,\ldots, 1 \\ -\beta_1,\ldots, -\beta_n \end{pmatrix}$$

we see that (36) is equivalent with the set of equations

$$\begin{aligned} X_{q,h}s_1 &= -\beta_1 e_n \\ X_{q,h}s_2 &= s_1 - \beta_2 e_n \\ &\cdots\cdots\cdots\cdots \\ X_{q,h}e_n &= s_{n-1} - \beta_n e_n\,. \end{aligned}$$

or, equivalently,

$$\begin{aligned} s_{n-1} &= (X_{q,h} + \beta_n)\, e_n \\ s_{n-2} &= X_{q,h} s_{n-1} + \beta_{n-1} e_n \\ &\cdots\cdots\cdots\cdots \\ 0 &= X_{q,h} s_1 + \beta_1 e_n . \end{aligned} \tag{37}$$

This set of equations is equivalent with

$$s_i = \left[X_{q,h}^{n-i} + \sum_{j=0}^{n-i-1} X_{q,h}^{j} \beta_{i+j+1} \right] e_n , \qquad i = 1,\ldots, n-1 \tag{38}$$

$$0 = \left[X_{q,h}^{n} + \sum_{j=0}^{n-1} X_{q,h}^{j} \beta_{1+j} \right] e_n . \tag{39}$$

Denote $\phi(q, h) = [X_{q,h}^n + \sum_{j=0}^{n-1} \beta_{1+j} X_{q,h}^j]\, e_n$. We have $\phi(q^0, 0) = 0$, because the square bracket in (39) is the characteristic polynomial of $X_{q^0,0}$. Since $X_{q,h}$ is a continuously differentiable function of q, h, so is ϕ. Therefore, by the implicit function theorem, if we prove that $(\partial/\partial q)(\phi(q, 0)|_{q=q_0}$ is non-singular, it follows that there is a continuous function $q(h)$ for h sufficiently small such that $\phi(q(h), h) = 0$ and $q(0) = q^0$. By 2^0 and 7^0 we have

$$X_{q,0} = \begin{pmatrix} 0, 1,\ldots, 0 \\ \cdots\cdots\cdots\cdots \\ 0,\ldots, 1 \\ \alpha_1 + p_1 ,\ldots, \alpha_n + p_n \end{pmatrix}$$

where $p = (p_1 ,\ldots, p_n) = q(e^{\tilde{b}'\tilde{q}} - 1)/\tilde{b}'\tilde{q}$ and $\tilde{q} = Y(t_1)'q$ (cf. (21)). From this it follows that

$$X_{q,0}^j e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \alpha_n + p_n \\ \alpha_{n-1} + p_{n-1} + f_{j,j+2}(p_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n-j+1} + p_{n-j+1} + f_{j,n}(p_n ,\ldots, p_{n-j+2}) \end{pmatrix}$$

where 1 is in the n-jth row and f_{jv} are polynomials. Consequently,

$$\phi(q, 0) = \begin{pmatrix} \alpha_n + p_n \\ \alpha_{n-1} + p_{n-1} + f_2(p_n) \\ \cdots\cdots\cdots\cdots \\ \alpha_1 + p_1 + f_n(p_n ,\ldots, p_2) \end{pmatrix} \tag{40}$$

where $f_2 ,\ldots, f_n$ are polynomials.

We have

$$\frac{\partial \phi(q, 0)}{\partial q} = \frac{\partial \phi(q, 0)}{\partial p} \cdot \frac{\partial p}{\partial \tilde{q}} \cdot \frac{\partial \tilde{q}}{\partial q}.$$

From (40) follows that $(\partial \phi(q, 0))/\partial p$ is triangular with ones in the diagonal and, consequently, nonsingular.

Further, if we denote $\hat{p} = Y^{-1}(\omega)'p$, $\hat{q} = Y^{-1}(\omega)'\tilde{q}$, we have

$$\hat{p} = \frac{e^{(Y(\omega)\tilde{b})'\hat{q}} - 1}{(Y(\omega)\tilde{b})'\hat{q}} \cdot \hat{q} = \frac{e^{\hat{q}_n} - 1}{\hat{q}_n} \cdot \hat{q}$$

so that

$$\frac{\partial \hat{p}_i}{\partial \hat{q}_j} = \frac{e^{\hat{q}_n} - 1}{\hat{q}_n} \delta_{ij} \quad \text{if} \quad j \neq n, \qquad \frac{\partial \hat{p}_n}{\partial \hat{q}_n} = \frac{\partial}{\partial \hat{q}_n}(e^{\hat{q}_n} - 1) \neq 0$$

which proves that $\partial \hat{p}/\partial \hat{q}$ is nonsingular. Consequently,

$$\partial p/\partial \tilde{q} = Y(\omega)'(\partial \hat{p}/\partial \hat{q})\, Y^{-1}(\omega)'$$

is nonsingular. Since $\partial \tilde{q}/\partial q = Y(t_1)'$ is also nonsingular, we have proved that $(\partial \phi/\partial q)(q^0, 0)$ is nonsingular.

Thus, for any q^0 we can find an $h > 0$ and q such that (39) and, consequently (36) is satisfied and q is arbitrarily close to q^0. From (37) it follows that S is a continuous function of q and h. But for $q = q^0$ and $h = 0$, $S = I$ so that for $h > 0$ sufficiently small S will be nonsingular. This completes the proof.

Remark 2. If we allow $Q(t)$ to be complex, then the characteristic multipliers can be shifted to any nonzero numbers $\sigma_1, \ldots, \sigma_n$ by closed loop control $u = Q(t)x$.

Remark 3. If $A(t), B(t)$ are only continuous, the theorem is still valid in a weaker form: namely, if (1) is controllable and Σ is a spectrum such that $\sigma_1, \ldots, \sigma_n \geqslant 0$, then to any $\epsilon > 0$ there is a matrix $Q(t)$ such that the characteristic multipliers σ_i' of the system (7) satisfy $|\sigma_i' - \sigma_i| < \epsilon$. (The case (ii) of the theorem can be changed in a similar manner). Also the sufficiency part remains valid.

This can be seen from the fact that (7) can still be proved in a weaker form

$$X_{Q,h}(\omega, 0) = Y(\omega) \cdot \prod_{i=1}^{r} e^{\tilde{b}_i \hat{q}_i'} + \theta(h) = \psi_{Q,0}(\omega, 0) + \theta(h) \qquad (41)$$

where $\lim_{h\to 0} \theta(h) = 0$ locally uniformly in q. Thus, the steps 2°–7° of the proof can be repeated without change and we can find the Q^0 such that $X_{Q^0,0}(\omega, 0)$ has spectrum Σ, or an arbitrary close spectrum to Σ, if Σ contains zero elements. Since the spectrum of a matrix is a continuous function of its entries the statements follow from (41).

Remark 4. The matrix, which we have constructed to solve our problem, is discontinuous. This is not essential and it can be verified that there is a C^∞-matrix $Q(t)$ which solves the problem. For this purpose, let us first note that the function $\xi_\delta(t)$ of 5° can be chosen C^∞ with all required properties preserved. Choosing such a function the proof can be carried out essentially in the same way (with some calculations, of course, more complicated) with $Q(t)$ replaced by

$$\hat{Q}_h(t) = \begin{cases} h^{-1}Q^{(j)}\xi_\delta[h^{-1}(t - t_j)] & \text{in } [t_j + \nu\omega, t_j + \nu\omega + h], \nu \text{ integer} \\ 0 & \text{elsewhere.} \end{cases}$$

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