JOURNAL OF DIFFERENTIAL EQUATIONS 38, 344–360 (1980)

Regular Synthesis for the Linear–Quadratic Optimal Control Problem with Linear Control Constraints

PAVOL BRUNOVSKÝ

Institute of Applied Mathematics and Computing, Comenius University, Mlynská dolina, 816 31 Bratislava, Czechoslovakia

Received October 31, 1979

1. INTRODUCTION

In [1] sufficient conditions for the existence of a regular optimal control synthesis for an abstract optimal control problem have been given. It has been indicated that the theorem is modelled after the linear–quadratic optimal control problem with linear control constraints. In this paper we prove a theorem which says that if a certain non-degeneracy (or, as we shall say, normality) condition is satisfied then the latter problem satisfies the hypotheses of the theorem of [1] and, therefore, it admits a regular synthesis of the optimal control. The proof includes a proof of a result of independent interest: under the normality assumption, every optimal control passes only finitely many times from one face of the control domain to another (or, as we shall say, has finitely many switching points). A general result of this kind has so far been known for the linear time-optimal control problem only [3, 4]. Recently, it has been extended to a certain class of nonlinear problems [5].

Although the paper is closely related to [1], most of its auxiliary results can be well understood without referring to the concepts of [1]. Therefore, the auxiliary results have been separated from the verification of the hypotheses of the abstract synthesis theorem of [1]. The former are contained in Sections 3, 4; the latter in Section 5. Section 2 contains the formulation of the optimal control problem as well as a summary of some of its known properties.

2. THE LINEAR–QUADRATIC OPTIMAL CONTROL PROBLEM WITH LINEAR CONTROL CONSTRAINTS

We consider the optimal control problem given by the linear system

$$\dot{x} = Ax + Bu \tag{1}$$

0022-0396/80/120344-17$02.00/0

with $x \in R^n$, $u \in R^m$, the performance index

$$J = \int_\tau^0 (x^*Qx + u^*Ru)\, dt \tag{2}$$

(asterisk standing for transpose), the control domain

$$U = \{u \in R^m \mid \langle c_i, u\rangle \leqslant d_i, i = 1,..., p\} \tag{3}$$

and the target point $x = 0$. We assume that Q, R are symmetric, $Q \geqslant 0$, $R > 0$, U is compact and contains the origin in its interior. Further, we assume that the time-optimal control problem for the system (1) and the control domain (3) is normal in the sense of [3, 4].

Note that U is a polytope but we do not express it (as usual in optimal control theory) as the convex hull of its vertices. Rather, we characterize it in the dual way, as a finite intersection of halfspaces. A sufficient condition for normality says that for no $0 \neq \psi \in R^n$ and no choice of $n - 1$ distinct linearly independent vectors $c_{i_1},..., c_{i_{n-1}}$, $\det(c_{i_1},..., c_{i_{n-1}}, e^{-tA^*}\psi)$ is identically zero.

By a control we understand any piecewise continuous function $u:[\tau, 0] \to U$. Given a point $(\tau, y) \in R^{n+1}$ and a control u on $[\tau, 0]$, by $x(t, \tau, y, u)$ we denote the solution of (1) with $u = u(t)$ such that $x(\tau, \tau, y, u) = y$. We say that the control u steers the system from y to 0 on $[\tau, 0]$ if $x(0, \tau, y, u) = 0$. We denote

$$J(\tau, u, y) = \int_\tau^0 [x^*(t, \tau, y, u)\, Qx(t, \tau, y, u) + u^*(t)\, Ru(t)]\, dt.$$

The control u on $[\tau, 0]$ is called optimal (for the initial state y) if it minimizes J among all controls steering the system from y to 0 on $[\tau, 0]$.

We recall some well known properties of the linear–quadratic optimal control problem that are valid under our assumptions, for which [4] is a good reference.

LQ1. The set $\mathscr{R}(\tau)$ of the points from which the system can be steered to 0 on $[\tau, 0]$ is convex closed and has a non-empty interior for any $\tau < 0$. Further, for any $\tau_1 < \tau_2 < 0$ we have $\mathscr{R}(\tau_2) \subset \operatorname{int}\mathscr{R}(\tau_1)$ (in the terminology of [3], $\mathscr{R}(-\tau)$ is expanding).

LQ2. For every $\tau < 0$, $y \in \mathscr{R}(\tau)$ there exists a unique optimal control (we denote it by $u_{\tau,y}$). The optimal control $u_{\tau,y}$ satisfies the Pontrjagin maximum principle: there exists a non-zero solution $\psi(t) = (\psi^0, \eta(t))$ of the adjoint system

$$\begin{aligned} \dot\psi^0 &= 0, \\ \dot\eta &= -Qx(t)\,\psi^0 - A^*\eta \end{aligned} \tag{4}$$

such that $u_{\tau,y}$ is extremal (with respect to $\psi(t)$), i.e.,

$$L(\psi(t), u_{\tau,y}(t)) = \max_{u \in U} L(\psi(t), u) \tag{5}$$

where

$$L(\psi, u) = \psi^0 u^* R u + \langle \psi, Bu \rangle; \tag{6}$$

if $u(t)$ is extremal and steers y to 0 on $[\tau, 0]$ then $u(t) = u_{\tau,y}(t)$ for a.e. $t \in [\tau, 0]$ [4, Corollaries to Theorems 12 and 13, Chap. 3].

LQ3. For every $\tau < 0$, the boundary $\partial R(\tau)$ of $\mathrm{R}(\tau)$ coincides with the set of points which cannnot be steered to 0 in time $<\tau$. Also, $x \in \operatorname{int} \mathscr{R}(\tau)$ if and only if $(0, \eta) \notin E(\tau, y)$ for any $\eta \in R^n$, where $E(\tau, y)$ is the set of the initial values of the non-zero solutions of (4) with respect to which $u_{\tau,y}$ is extremal.

The first part of LQ3 is well known and follows e.g. from [3]. The second part follows from the maximum principle for the time-optimal control problem. Indeed, if $y \in \partial\mathscr{R}(\tau)$, then any control that steers y to 0 on $[\tau, 0]$ is time-optimal and, therefore, there exists a solution $\eta(t)$ of the adjoint equation

$$\dot{\eta} = -A^* \eta \tag{7}$$

for the time-optimal control problem such that

$$\langle \eta(t), B u_{\tau,y}(t) \rangle = \max_{u \in U} \langle \eta(t), Bu \rangle. \tag{8}$$

If we denote $\psi^0 = 0$, $\psi(t) = (0, \eta(t))$, then $\psi(t)$, $u_{\tau,y}(t)$ satisfy (4), (5) which implies $0, \eta(0)) \in E(\tau, y)$. Convversely, if $\psi(t) = (0, \eta(t))$ and $\psi(0) \in E(\tau, y)$, then $\psi(t), u_{\tau,y}(t)$ satisfy (7), (8). Since under our assumptions the maximum principle is a sufficient condition of optimality [3, Theorem 17.1], $u_{\tau,y}$ is time-optimal and, consequently, $y \in \partial\mathscr{R}(\tau)$.

Let us note that the assumption of compactness of U can be easily dropped and we have made it only for the sake of simplicity. Also, A, B, c_i, d_i can be allowed to vary analytically with time, but the normality conditions become more complicated.

3. The Solution of the Maximum Condition

To verify the hypotheses of the theorem of [1] we have to express the solution of the maximum condition (5) as a function of ψ. By R_0^{n+1} we denote the set of those $(\psi^0, \eta) \in R^{n+1}$ for which $\psi^0 < 0$.

Lemma 1. *Lor a given $\psi \in R_0^{n+1}$, there is a unique solution $w = w(\psi)$ of the equation*

$$L(\psi, w) = \max_{u \in U} L(\psi, u). \tag{9}$$

The function w is continuous in R_0^{n+1}.

Proof. The existence and uniqueness of w follow immediately from the compactness of U and the strict concavity of L in u respectively. For the proof of continuity assume ψ_k, $\psi \in R_0^{n+1}$, $\psi_k \to \psi$. Since U is compact, without loss of generality we may assume $w(\psi_k) \to w_0 \in U$. From the continuity of L in ψ and u we have for any $u \in U$

$$L(\psi, w_0) = \lim_{k\to\infty} L(\psi_k, w(\psi_k)) \leqslant \lim_{k\to\infty} L(\psi_k, u) = L(\psi, u).$$

So, w_0 is a solution of (9). From the unicity of its solutions it follows that $w_0 = w(\psi)$, which completes the proof.

We shall henceforth understand that among the inequalities defining U there are no redundant ones, i.e., for every $1 \leqslant i \leqslant p$ there exists a $u \in R^m$ such that $\langle c_i, u\rangle > d_i$ and $\langle c_j, u\rangle \leqslant d_j$ for $j \neq i$. The control domain U is a finite disjoint union of its (open) faces of different dimensions which can be expressed by the formula

$$U_I = \{u \in R^m \mid \langle c_i, u\rangle = d_i \text{ for } i \in I, \langle c_i, u\rangle < d_i \text{ for } i \notin I\},$$

for I running through all subsets of the set $\{1,..., p\}$ of cardinality $\leqslant m$ such that c_i, $i \in I$ are linearly independent. We shall call such sets I admissible.

The sets U_I are obviously relatively open. Since we have no redundant constraints and int $U \neq \varnothing$, $U_I = \varnothing$ if the vectors c_i, $i \in I$ are linearly dependent. For $I = \varnothing$, U_I is the interior of U.

Denote

$$W_I = \{\psi \in R_0^{n+1} \mid w(\psi) \in U_I\}.$$

In a series of lemmas we prove that the sets W_I, completed by the point 0, are polyhedral cones and we find the explicit expressions for the function w and the linear inequalites by which the sets W_I are defined.

LEMMA 2. *The family* $\mathscr{U} = \{U_I \mid I \text{ admissible}\}$ *is a finite stratification of* U *by relatively open convex sets*; $U_I \subset \overline{U}_J$ *if and only if* $J \subset I$.

Since we shall not need the fact that $\mathscr{U}$ is a stratification, we do not introduce its precise definition, for which the reader is referred to [1]. The verification of the lemma is straightforward.

Given I admissible, we denote by

$$P_I = \{u \in R^m \mid \langle c_i, u\rangle = d_i \text{ for } i \in I\}$$

the affine hull of the face U_I. For $\psi \in R_0^{n+1}$ we define $w_I(\psi) \in P_I$ by

$$L(\psi, w_I(\psi)) = \max_{u\in P_I} L(\psi, u).$$

Since L is strictly concave in u and tends to $-\infty$ for $|u| \to \infty$ for any fixed $\psi \in R_0^{n+1}$, $w_I(\psi)$ is well defined. This follows also from the following lemma which gives an explicit expression for w_I.

LEMMA 3. *Let* $\psi = (\psi^0, \eta) \in R_0^{n+1}$, $I = \{i_1, ..., i_r\}$ *admissible. Then,*

$$\begin{aligned} w_I(\psi) &= (2\psi^0 R)^{-1}[CK^{-1}(2\psi^0 d + C^* R^{-1} B^* \eta) - B^* \eta] \\ &= R^{-1}[CK^{-1}d + \tfrac{1}{2}(CK^{-1}C^*R^{-1}B^* - B^*)(\psi^0)^{-1}\eta], \end{aligned} \tag{10}$$

where $C = (c_{i_1}, ..., c_{i_r})$, $d = (d_{i_1}, ..., d_{i_r})^*$, $K = C^* R^{-1} C > 0$.

Proof. By the Lagrange multiplier rule there are constants λ_i, $i \in I$ which, together with $u = w_I(\psi)$, satisfy the system of equations

$$(\partial L/\partial u)(\psi, u) - \sum_{i \in I} \lambda_i (\partial/\partial u)\langle c_i, u\rangle = 0, \tag{11}$$

$$\langle c_i, u\rangle = d_i, \qquad i \in I. \tag{12}$$

Substituting for L and computing the derivatives in (11) we obtain

$$2\psi^0 Ru + B^*\eta - \sum_{i \in I} \lambda_i c_i = 0$$

from which we obtain

$$u = (2\psi^0 R)^{-1}(C\lambda - B^*\eta), \tag{13}$$

where $\lambda = (\lambda_{i_1}, ..., \lambda_{i_r})^*$. Substituing (13) into (12) we obtain

$$\langle c_i, (2\psi^0 R)^{-1}(C\lambda - B^*\eta)\rangle = d_i, \qquad i \in I,$$

or

$$(2\psi^0)^{-1}(c_i^* R^{-1} C\lambda - c_i^* R^{-1} B^* \eta) = d_i, \qquad i \in I,$$

which is the same as

$$(2\psi^0)^{-1} K\lambda = d + (2\psi^0)^{-1} C^* R^{-1} B^* \eta.$$

From this expression we can eliminate λ:

$$\lambda = K^{-1}(2\psi^0 d + C^* R^{-1} B^* \eta).$$

Substituting for λ into (13) we obtain (10).

COROLLARY 1. *The function* $\psi^0 w$ *is globally Lipschitz continuous on* R_0^{n+1}.

This follows immediately from the continuity of w on R_0^{n+1}, the linearity of the functions $\psi^0 w_I$ and the formula

$$w(\psi) = \max\{w_I(\psi) \mid I \text{ admissible}\}.$$

LEMMA 4. *We have $W_I = X_I \backslash Y_I$, where X_I is the set of those ψ for which $w_I(\psi) \in U_I$ and Y_I is the set of those ψ for which*

$$\sup_{u \in U_J} L(\psi, u) > \sup_{u \in U_I} L(\psi, u) \tag{14}$$

for some $J \subset I$.

Proof. If $w_I(\psi) \notin U_I$ then $L(\psi, w_I(\psi)) > L(\psi, u)$ and the function $\varphi(\mu) = L(\psi, (1-\mu)\, u + \mu w_I(\psi))$ is increasing for each $u \in U_I$. Let $\mu_0 = \sup\{\mu \mid (1-\mu)\, u + \mu w_I(\psi) \in U_I\}$. Since U is closed and U_I is open in U, we have $(1-\mu_0)\, u + \mu_0 w_I(\psi) \in U \backslash U_I$ but, since φ is increasing, $\varphi(\mu_0) > L(\psi, u)$. Since $u \in U_I$ was arbitrary, $w(\psi) \notin U_I$ and, consequently, $\psi \notin W_I$. This proves $W_I \subset X_I$. The fact that W_I is contained in the complement of Y_I follows directly from the definition of Y_I. Thus, $W_I \subset X_I \backslash Y_I$.

To prove the converse inclusion, assume that it is not valid. Then, there exists a $\psi \in X_I \backslash Y_I$ such that $w(\psi) \in U_J$ for some $J \neq I$, $J \not\subset I$. Since $\psi \in X_I$, $w_I(\psi) \in U_I$. Denote $z = \frac{1}{2}(w_I(\psi) + w(\psi))$. Since $L(\psi, w_I(\psi)) < L(\psi, w(\psi))$ and L is concave in u, we have

$$L(\psi, z) > L(\psi, w_I(\psi)). \tag{15}$$

However, if $j \in J \backslash I$, then $\langle c_j, w(\psi)\rangle = d_j > \langle c_j, w_I(\psi)\rangle$ from which it follows that $\langle c_j, z\rangle < d_j$. Consequently, $z \in U_K$ for some $K \subset I$, so (15) contradicts $\psi \notin Y_I$. This completes the proof.

LEMMA 5. *For every $I \in \{1,..., p\}$ admissible we have $W_I = X_I \cap Z_I$, where X_I is defined in Lemma 4 and*

$$Z_I = \bigcap_{\substack{J \supset I \\ \operatorname{card} J + \operatorname{card} I - 1}} \{\psi \in R_0^{n+1} \mid \langle c_i, w_J(\psi)\rangle \geqslant d_i \text{ for } i \in I \backslash J\}.$$

For the proof of this lemma we shall need two further lemmas.

LEMMA 6. *Let $c_1, ..., c_p \in R^n$ be linearly independent and let $x_0, ..., x_p \in R^n$, $d_1, ..., d_p \in R$ be such that $\langle c_i, x_i\rangle > d_i$, $\langle c_i, x_j\rangle = d_i$, $\langle c_i, x_0\rangle \leqslant d_i$, $i, j = 1,..., p$, $i \neq j$. Then, there exists a point in the convex hull of the points $x_0, ..., x_p$ such that*

$$\langle c_i, x\rangle = d_i \qquad \textit{for} \quad i = 1,..., p. \tag{16}$$

Furthermore, x can be expressed as a convex combination $x = \lambda_0 x_0 + \cdots + \lambda_p x_p$ with $\lambda_0 > 0$.

Proof. We prove this lemma by induction in p. For $p = 1$ the lemma is obvious. Given $p > 1$, assume that it is valid for $p - 1$. Then, by the induction hypothesis there exist numbers $\mu_0, \ldots, \mu_{p-1}$ such that $0 \leqslant \mu_j \leqslant 1$, $\mu_0 > 0$, $\mu_0 + \cdots + \mu_{p-1} = 1$ and the point $y = \mu_0 x_0 + \cdots + \mu_{p-1} x_{p-1}$ satisfies $\langle c_i, y\rangle = d_i$ for $i = 1, \ldots, p-1$. Obviously, it satisfies also $\langle c_p, y\rangle \leqslant d_p$. Since the lemma is valid for $p = 1$, there exists a $\lambda > 0$ such that the point $x = \lambda y + (1 - \lambda) x_p$ satisfies (16) for $i = p$. Since $\langle c_i, y\rangle = d_i$ and $\langle c_i, x_p\rangle = d_i$ for all $1 \leqslant i < p$, (16) is satisfied also for $1 \leqslant i \leqslant p$. We have

$$x = \lambda\mu_0 x_0 + \cdots + \lambda\mu_{p-1} x_{p-1} + (1 - \lambda) x_p ,$$

$\lambda\mu_0 > 0$, which proves the lemma.

LEMMA 7. *Let $f\colon R^n \to R$ be differentiable and strictly concave and let $x \in R^n$, $S \subset R^n$, $x \notin S$ be such that for each $y \in S$ there are vectors x_i, $i = 1, \ldots, p$ and non-negative constants $\lambda_1, \ldots, \lambda_p$ such that $y = x + \lambda_1 x_1 + \cdots + \lambda_p x_p$ and the functions $\varphi_i(t) = f(x + tx_i)$ are strictly decreasing for ea·h $i = 1, \ldots, p$. Then, $f(y) < f(x)$ for each $y \in S$.*

Proof. Let $y \in S$. We have $0 \geqslant (d\varphi_i/dt)(0) = df(x)\, x_i$. Because of the strict concavity of f we have

$$f(y) - f(x) < df(x)(y - x) = df(x) \sum_{i=1}^{p} \lambda_i x_i = \sum_{i=1}^{p} \lambda_i\, df(x)\, x_i \leqslant 0$$

which proves the lemma.

Proof of Lemma 5. First we prove $W_I \subset X_I \cap Z_I$.

Let $\psi \in W_I$. Then, by Lemma 4, $\psi \in X_I \backslash Y_I$. Assume that for some $J \subset I$, $\operatorname{card} J = \operatorname{card} I - 1$, $i \in I \backslash J$ we have $\langle c_i, w_J(\psi)\rangle < d_i$. Since $U_I \subset P_J$ and $w_J(\psi) \notin U_I$, for each $u \in U_I$ we have $L(\psi, u) < L(\psi, w_J(\psi))$ and, consequently, $L(\psi, g(\lambda)) > L(\psi, u)$ for $0 < \lambda \leqslant 1$, where $g(\lambda) = \lambda w_J(\psi) + (1 - \lambda) u$. For $\lambda > 0$ sufficiently small we have $g(\lambda) \in U_J$. Thus, for each $u \in U_I$ there exists a $v \in U_J$ such that $L(\psi, v) > L(\psi, u)$ which implies (14) and, consequently, $\psi \in Y_I$, contrary to our assumption.

Now, we prove that if $\psi \in X_I \cap Z_I$ then $\psi \notin Y_I$ which will complete the proof of this lemma.

Let $\mathscr{J}$ be the family of the sets $J \subset I$ satisfying $\operatorname{card} J = \operatorname{card} I - 1$. We have to distinguish two cases: $w_J(\psi) \in U_I$ (and, consequently, $w_J(\psi) = w_I(\psi)$) for all $J \in J$ and $w_J(\psi) \notin U_I$ for at least one $J \in \mathscr{J}$.

In the first case let $y \in K \subset I$, $K \neq I$. The set $U_K - w_I(\psi)$ (algebraic minus) is obviously contained in the convex cone spanned by the sets $U_J - w_I(\psi)$,

$J \in \mathscr{J}$. Therefore, there exist vectors $x_J \in U_J - w_I(\psi)$ and non-negative constants λ_J, $J \in \mathscr{J}$, such that

$$y = w_I(\psi) + \sum_{J \in \mathscr{J}} \lambda_J x_J .$$

The application of Lemma 7 yields $L(\psi, y) < L(\psi, w_I(\psi))$. Since $y \in K \subset I$ was arbitrary, this means $\psi \notin Y_I$.

In the second case denote $J_1, ..., J_r$ those elements of $\mathscr{J}$ for which

$$\langle c_{j_\nu}, w_{J_\nu}(\psi)\rangle > d_{j_\nu} \qquad \text{for} \quad j_\nu \in I \backslash J_\nu .$$

By assumption, $r > 0$. Since $\psi \in X_I$, we have $w_I(\psi) \in U_I$. Since $U_I \subset P_{J_\nu}$ and $w_{J_\nu}(\psi) \notin U_I$, we have $L(\psi, w_{J_\nu}(\psi)) > L(\psi, w_I(\psi))$.

Assume $\psi \in Y_I$. Then, there exist a $K \subset I$ and a point $u_0 \in U_K$ such that $L(\psi, u_0) > L(\psi, w_I(\psi))$. Applying Lemma 6 to the points $u_0, u_\nu = w_{J_\nu}(\psi)$, $\nu = 1, ..., r$ we obtain that there exists a point $u = \lambda_0 u_0 + \lambda_1 w_{J_1}(\psi) + \cdots + \lambda_r w_{J_r}(\psi)$, $\lambda_1 + \cdots + \lambda_r = 1$, $\lambda_0 > 0$, $\lambda_\nu \geqslant 0$ for $\nu > 0$ such that

$$\langle c_{j_\nu}, u\rangle = d_{j_\nu}, \qquad \nu = 1, ..., r.$$

Obviously, we have also $\langle c_i, u\rangle \leqslant d_i$ for $i \in I \backslash \{j_1, ..., j_r\}$ since these inequalities are satisfied by $w_{J_\nu}(\psi)$, $\nu = 1, ..., r$ as well as u_0. We have

$$L(\psi, u) \geqslant \lambda_0 L(\psi, u_0) + \sum_{i=1}^{r} \lambda_\nu L(\psi, w_{J_\nu}(\psi)) > L(\psi, w_I(\psi)).$$

From this inequality it follows that $u \notin P_I$. Consequently, there exists an $i_0 \in I \backslash \{j_1, ..., j_r\}$ such that $\langle c_{i_0}, u\rangle < d_{i_0}$. The point $u - w_I(\psi)$ is obviously contained in the convex cone spanned by the sets $U_J - w_I(\psi)$, $J \in \mathscr{J} \backslash \{J_1, ..., J_r\}$. Therefore, we can apply the same argument as in the first case to complete the proof.

As a consequence of Lemmas 3, 5 we obtain

THEOREM 1. *For every $I \subset \{1, ..., p\}$ admissible and $\psi \in W_I$ we have*

$$w_I(\psi) = R^{-1}[C_I K_I^{-1} d_I + \tfrac{1}{2}(C_I K_I^{-1} C_I^* R^{-1} B^* - B^*)(\psi^0)^{-1} \eta],$$

where $\psi = (\psi^0, \eta)$ and

$$\begin{aligned} W_I = \{\psi \in R_0^{n+1} \mid\ & (c_i^* R^{-1} C_I K_I^{-1} d_i - d_I)\, \psi^0 \\ & + \tfrac{1}{2} c_i^* R^{-1}(C_I K_I^{-1} C_I^* R^{-1} B^* - B^*)\eta < 0 \\ & \textit{for all } i \notin I \textit{ and } (c_j^* R^{-1} C_J K_J^{-1} d_J - d_j)\, \psi^0 \\ & + \tfrac{1}{2} c_j^* R^{-1}(C_J K_J^{-1} C_J^* R^{-1} B^* - B^*)\eta \geqslant 0 \\ & \textit{for all } J \subset I, \text{ card } J = \text{card } I - 1, j \in I \backslash J\} \end{aligned} \tag{17}$$

(the capital subscript attached to C, K, d indicates the index set to which these quantities defined in Lemma 3 refer).

COROLLARY 2. *The collection of sets $\overline{W}_I$ for I admissible is a finite covering of R_0^{n+1} by closed, convex cones with non-intersecting interiors.*

4. BOUNDEDNESS OF THE NUMBER OF SWITCHINGS

In this section we shall not refer to the optimal control problem at all. We shall prove a de la Vallé–Poussin type theorem [2] bounding the number of zeros of a component of a solution of a differential equation, for a piecewise linear differential equation. This result will be used for the proof of the uniform local boundedness of the number of the switching points of the extremals which is assumed in the synthesis theorem of [1].

Let K_0 be a cone in R^n and let $\mathscr{K} = \{K_i\}_{i=1}^r$ be a covering of K_0 by convex closed polyhedral cones with non-intersecting interiors. Let $F_i : R^n \to R^n$, $i = 1,..., r$ be linear operators such that if $x \in K_i \cap K_j$ then $F_i x = F_j x$ for $i, j = 1,..., r$. Then, the function $F: K_0 \to R^n$ given by $F(x) = F_i x$ for $x \in K_i$, $i = 1,..., r$ is well defined and continuous. We shall say that F is normal, if for any i, j and any normal c of any $(n-1)$-dimensional face of K_i the pair (F_j, c) is observable, i.e., $\det(c, F_j^* c,..., F_j^{*n-1} c) \neq 0$.

Consider the differential equation

$$\dot{x} = F(x) \tag{18}$$

on K_0. Let $x(t)$ be a solution of (18). We shall call t^* a switching point of $x(t)$, if $x(t^*)$ lies in some face of dimension $<n$ of some of the cones K_i, $i = 1,..., r$. We prove

THEOREM 2. *Let F be normal. Then, there are constants N, $\delta > 0$ such that the number of switching points of any non-trivial solution $x(t)$ of (18) on any interval of length $\leqslant \delta$ does not exceed N.*

For the proof of this theorem we shall need the following

LEMMA 8. *Let $\varphi_k : I_k \to R^n$, $k = 1, 2, 3,...$ be a sequence of functions defined on intervals I_k. Assume that the number of points at which the value of at least one of the components of φ_k is zero tends to infinity as $k \to \infty$. Then, for every N there exists a k and an interval $J_k \subset I_k$ such that the number of zeros of any component of φ_k on J_k is either zero or $>N$.*

Proof. We prove the lemma by induction in n. For $n = 1$ the statement of the lemma is trivial. Assume that for a given $n > 1$ it is valid for all positive integers up to $n - 1$. By passing to a subsequence if necessary we may achieve that the number of zeros of each of the components of φ_k either tends to infinity or is bounded. This follows by induction from the obvious fact that if the sequence $\{\varphi_k\}$ has been reduced in such a way that the number of zeros of the components $1, ..., m$ is either bounded or tends to infinity then there is a subsequence of $\{\varphi_k\}$ whose $m + 1$st component shares this property.

Thus, without loss of generality we may assume that the components are ordered in such a way that the number of zeros of the components $1, ..., m$ ($m \geqslant 0$) is bounded while the numbers of zeros of the components $m + 1, ..., n$ tend to infinity as $k \to \infty$. If $m = 0$, the statement of the lemma is trivial. If $m > 0$, we remove the zeros of the components $1, ..., m$ from the intervals I_k to obtain for each k a finite collection of subintervals of I_k which are free of zeros of the components $1, ..., m$ and the number of which is bounded for $k \to \infty$. Therefore, we can choose for every k one of those subintervals (we denote it by I_k') in such a way that the number of points, in which at least one of the components $m + 1, ..., n$ is zero, tends to infinity as $k \to \infty$. By the induction hypothesis there exist a k and $J_k \subset I_k'$ such that the number of zeros of each of the components $m + 1, ..., n$ is either $> N$ or zero on J_k. Since none of the components $1, ..., m$ has a zero on J_k, this completes the proof.

Proof of Theorem 2. Assume that the statement of the theorem does not hold. Then, for any fixed $\delta > 0$ there exists a sequence of intervals I_k of length $\leqslant \delta$ and trajectories $x_k(t)$ of (18) on I_k such that the numbers of switching points of x_k tend to infinity as $k \to \infty$.

Let c_i, $i = 1, ..., s$, be the normal vectors of all the $(n - 1)$-dimensional faces of the cones K_j, $j = 1, ..., r$. We apply Lemma 8 to the functions $\varphi_k = (\varphi_{k1}, ..., \varphi_{ks})$: $I_k \to R^s$ defined by $\varphi_{ki}(t) = \langle c_i, x_k(t) \rangle$, $i = 1, ..., s$. Obviously, every switching point of x_k is a zero of φ_{ki} for some i. By Lemma 8, for every m we may find a k and an interval $J_k \subset I_k$ in such a way that the number of zeros of each φ_{ki}, $i = 1, ..., s$, is either zero or $> N$. Henceforth we shall fix this k, drop the subscripts k at $x_k(t)$ and J_k and assume that $1, ..., \sigma$ are the components having zeros on J.

We associate with the covering $\mathscr{K}$ a labeled edge graph G with vertices $1, ..., r$ corresponding to the sets K_i, $i = 1, ..., r$, (μ, ν) being an edge labeled by j if and only if K_μ, K_ν have an $(n - 1)$-dimensional intersection the normal vector of which is c_j.

Let (μ, ν) be an edge of G labeled by j. Then, $K_\mu \cap K_\nu$ contains an open subset of the linear subspace $\langle c_j, x \rangle = 0$. Since F is continuous, we have $F_\mu x = F_\nu x$ for $x \in K_\mu \cap K_\nu$ and, consequently, also for each x such that $\langle c_j, x \rangle = 0$. It follows that there exists a $v \in R^n$ such that

$$F_\mu x = F_\nu x + v \langle c_j, x \rangle. \tag{19}$$

Let us choose a cone K_i which is passed by $x(t)$; let it be K_1. Denote by G'

the subgraph of G generated by the edges labeled by $1, \ldots, \sigma$. It is obvious that if $x(t)$ passes some K_m then there is a path in G' joining the vertices 1 and m. It follows from (19) that there are vectors v_{mj}, $j = 1, \ldots, \sigma$ such that

$$F_m x = F_1 x + \sum_{j=1}^{\sigma} v_{mj} \langle c_j , x \rangle.$$

Denote $z_{ij} = z_{ij}(x) = \langle c_i , F_1^{j-1} x \rangle$, $i = 1, \ldots, \sigma, j = 1, \ldots, n$. Instead of $z_{ij}(x(t))$ we shall briefly write $z_{ij}(t)$. Let $t \in J$ be such that $x(t) \in K_m$. Then, we have

$$\frac{dz_{i1}}{dt}(t) = \langle c_i , F_1 x(t) \rangle + \sum_{j=1}^{\sigma} \langle c_i , v_{mj} \rangle \langle c_j , x(t) \rangle$$

$$= z_{i2}(t) + \sum_{j=1}^{\sigma} \langle c_i , v_{mj} \rangle \, z_{j1}(t),$$

$$\frac{dz_{i2}}{dt}(t) = \langle c_i , F_1^2 x(t) \rangle + \sum_{j=1}^{\sigma} \langle c_i , F_1 v_{mj} \rangle \langle c_j , x(t) \rangle \tag{20}$$

$$= z_{i3}(t) + \sum_{j=1}^{\sigma} \langle c_i , F_1 v_{mj} \rangle \, z_{j1}(t), \text{ etc.,}$$

$$\frac{dz_{in}}{dt}(t) = \langle c_i , F_1^n x(t) \rangle + \sum_{j=1}^{\sigma} \langle c_i , F_1^{n-1} v_{mj} \rangle \, z_{j1}(t). \tag{21}$$

By the Cayley–Hamilton theorem we have $F_1^n = \alpha_{n-1} F_1^{n-1} + \cdots + \alpha_0 E$ for suitable $\alpha_0, \ldots, \alpha_{n-1}$. Substituting for F_1^n into the last equation of (20) we obtain

$$\frac{dz_{in}}{dt}(t) = \sum_{j=1}^{n} \alpha_{j-1} z_{ij}(t) + \sum_{j=1}^{\sigma} \langle c_i , F_1^{n-1} v_{mj} \rangle \, z_{j1}(t).$$

Denote $z_i = \mathrm{col}(z_{i1}, \ldots, z_{in})$, $z = \mathrm{col}(z_1, \ldots, z_n)$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} \end{pmatrix},$$

$$B_{mij} = \begin{pmatrix} \langle c_i , v_{mj} \rangle & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \langle c_i , F_1^{n-1} v_{mj} \rangle & 0 & \cdots & 0 \end{pmatrix}$$

$$D_m = \begin{pmatrix} A + B_{m11} & B_{m12} & \cdots & B_{m1\sigma} \\ \cdot & \cdot & \cdot & \cdot \\ B_{m\sigma 1} & & \cdots & A + B_{m\sigma\sigma} \end{pmatrix}.$$

We can write (20) in the form

$$\frac{dz}{dt}(t) = D_m z(t).$$

Denote $L_{i1}(z) = z_{i1}$, $i = 1,\ldots, \sigma$ and define inductively the family of linear forms $L_{ij}^{m_1,\ldots,m_{j-1}}$ for $i = 1,\ldots, \sigma$, $j = 2,\ldots, n$, $1 \leqslant m_\nu \leqslant r$ for $\nu = 1,\ldots, j$ by

$$L_{ij}^{m_1,\ldots,m_{j-1}}(z) = L_{i,j-1}^{m_1,\ldots,m_{j-2}}(D_{m_{j-1}} z).$$

The forms are defined in such a way that if $x(t) \in K_m$ for some m then

$$\frac{d}{dt} L_{ij}^{m_1,\ldots,m_{j-1}} z(t) = L_{ij}^{m_1,\ldots,m_{j-1},m} z(t).$$

Also, it follows immediately from the structure of D_m that for $j < n$ and any $m_1,\ldots, m_{j-1}$

$$L_{ij}^{m_1,\ldots,m_{j-1}}(z) = z_{ij} + \lambda_{ij}^{m_1,\ldots,m_{j-1}}(z_{11},\ldots, z_{\sigma 1},\ldots, z_{1j-1},\ldots, z_{\sigma,j-1}), \qquad (22)$$

i.e., the coefficient at z_{ij} is one and the form does not contain any other variable $z_{\mu\nu}$ with $\nu \geqslant j$.

Let M be the maximum of the absolute values of the coefficients of all the forms L and let Q be the maximum of the absolute values of the entries of all the matrices D_m. We complete the proof of the theorem by specifying a $\delta > 0$ such that if J has length $\leqslant \delta$ then it is impossible that all $z_{i1}(t)$, $i = 1,\ldots, \sigma$, have more than $N = \sigma^n + (n-1)\sigma^{n-1}$ zeros on J.

Assume the contrary. Then, $(dz_{i1}/dt)(t) = (d/dt) L_{i1}(z(t))$ has more than $\sigma^n + (n-1)\,\sigma^{n-1} - 1$ zero. Since for each t, $(d/dt) L_{i1}(z(t))$ is equal to some $L_{i2}^m(z(t))$, there exists an m such that $L_{i2}^{m_{i1}}$ has more than $\sigma^{-1}(\sigma^n + (n-1) \times \sigma^{n-1} - 1) \geqslant \sigma^{n-1} + (n-2)\,\sigma^{n-2}$ zeros. Again, this means that $(d/dt) L_{i2}^{m_{i1}}(z(t))$ has more than $\sigma^{n-1} + (n-2)\,\sigma^{n-2} - 1$ zero from which we conclude that there exists an m_2 such that $L_{i3}^{m_{i1},m_{i2}}(z(t))$ has more than $\sigma^{n-2} + (n-3)\,\sigma^{n-3}$ zeros on J. By a straightforward induction argument we obtain that there exists a sequence $m_{i1},\ldots, m_{i,n-1}$ such that $L_{ij}^{m_{i1},\ldots,m_{i,j-1}}(z(t))$ has more than $\sigma^{n+1-j} + \sigma^{n-j}(n-j)$ zeros for $j = 1,\ldots, n$. In particular, $L_{in}^{m_{i1},\ldots,m_{i,j-1}}(z(t))$ has at least one zero on J.

Denote $\mu_{ij} = \sup_{t\in J} z_{ij}(t)$, $\mu_j = \sum_{i=1}^{\sigma} \mu_{ij}$, $\bar{\mu}_j = \sum_{\nu=1}^{j} \mu_\nu$, $j = 1,\ldots, n$. From (20) it follows that

$$|\dot{z}_{ij}(t)| \leqslant \mu_{i,j+1} + Q\mu_1 \qquad (23)$$

for all $t \in J$, $i = 1,\ldots, \sigma$, $j = 1,\ldots, n-1$ and from (21) it follows that

$$|\dot{z}_{in}(t)| \leqslant Q(\mu_{i1} + \cdots + \mu_{in} + \mu_1). \qquad (24)$$

For every $i = 1, \ldots, \sigma$ and $j = 1, \ldots, n$ there exists a $t_{ij} \in J$ such that

$$L_{ij}^{m_{i1}, \ldots, m_{i,j-1}}(z(t_{ij})) = 0.$$

From this and (22) it follows

$$| z_{ij}(t_{ij})| \leqslant M\bar{\mu}_{j-1}$$

and from (23), (24) it follows

$$\mu_{ij} \leqslant M\bar{\mu}_{j-1} + \delta(\mu_{i,j+1} + Q\mu_1)$$

and, consequently,

$$\mu_1 \leqslant \sigma M\bar{\mu}_{j-1} + \delta(\mu_{j+1} + \sigma Q\mu_1) \tag{25}$$

for $j < n$,

$$\mu_{in} \leqslant M\bar{\mu}_{n-1} + \delta Q(\mu_{i1} + \cdots + \mu_{in} + \mu_1).$$

From the last inequality we obtain for $\delta < Q^{-1}$

$$\mu_{in} \leqslant \frac{M + \delta Q}{1 - \delta Q}\,\bar{\mu}_{n-1}$$

and, since i is arbitrary,

$$\mu_n \leqslant R_n(\delta)\,\bar{\mu}_{n-1}\,,$$

where $R_n(\delta) = \sigma(1 - \delta Q)^{-1}(M + \delta Q)$ is bounded for $\delta \to 0$. Assume now that for some $1 < j < n$ we have

$$\mu_j \leqslant R_j(\delta)\,\bar{\mu}_{j-1}\,,$$

where $R_j(\delta)$ is bounded for $\delta \to 0$. Then, from (25) it follows

$$\mu_{j-1} \leqslant \sigma M\mu_{j-2} + \delta(R_j(\delta)\,\bar{\mu}_{j-1} + \sigma Q\mu_1)$$

and, since $\bar{\mu}_{j-1} = \bar{\mu}_{j-2} + \mu_{j-1}$,

$$\mu_{j-1} - \delta R_j(\delta)\,\mu_{j-1} \leqslant [\sigma M + \delta R_j(\delta)]\,\bar{\mu}_{j-1} + \sigma Q\mu_1\,.$$

For $\delta > 0$ so small that $\delta R_j(\delta) < 1$ we obtain

$$\mu_{j-1} \leqslant R_{j-1}(\delta)\,\bar{\mu}_{j-2}\,,$$

where

$$R_{j-1}(\delta) = \frac{\sigma(M + \delta R_j(\delta) + Q)}{1 - \delta R_j(\delta)}$$

is bounded for $\delta \to 0$.

Since $\mu_1 = \bar{\mu}_1$, by induction we obtain

$$\mu_2 \leqslant R_2(\delta)\, \mu_1 , \tag{26}$$

where $R_2(\delta)$ is bounded for $\delta \to 0$; since $\bar{\mu}_0 = 0$, we obtain from (25), (26)

$$\mu_1 \leqslant \delta(R_2(\delta) + \sigma Q)\, \mu_1 . \tag{27}$$

For $\delta > 0$ sufficiently small $\delta(R_2(\delta) + \sigma Q) < 1$, so (27) is possible only if $\mu_1 = 0$ which means $\langle c_i , x(t)\rangle = 0$ for all $t \in J$ and all $i = 1,..., \sigma$.

Let $t_0 \in J$ be such a point that $x(t_0)$ lies in a face S of some set K_i. The linear hull of this face is defined by a part of the equalities $\langle c_j , x\rangle = 0$, $j = 1,..., \sigma$, say $\langle c_1 , x\rangle = \cdots = \langle c_\rho , x\rangle = 0$. Since the faces are relatively open, $x(t) \in S \subset K_i$ for t near t_0. Thus, $x(t)$ satisfies $\dot{x}(t) = F_i x(t)$ for t near t_0 from which it follows that

$$\frac{d^\nu}{dt^\nu} \langle c_j , x(t_0)\rangle = \langle F_i^{*\nu} c_j , x(t_0)\rangle = 0$$

for all $j = 1,..., \sigma$ and $\nu = 0,..., n - 1$. From the observability of the pair (F_i , c_j) it follows that $x(t_0) = 0$, contrary to our assumption. Since there is only a finite number of possible choices of the subgraphs G' of G, this contradiction completes the proof of the theorem.

5. Existence of Regular Synthesis

As we mentioned in Section 1, the regular synthesis theorem which we formulate and prove in this concluding section requires from the optimal control problem a certain normality condition. We formulate this condition first.

Consider the optimal control problem (1)–(3), the associated cones W_I and the functions w_I that we constructed in Section 3 (Theorem 1). For every $I \subset \{1,..., p\}$ admissible we denote

$$H_I = \tfrac{1}{2} R^{-1} C_I K_I^{-1} C_I^* R^{-1} B^* - B^*,$$

$$h_I = R^{-1} C_I K_I^{-1} d_I .$$

By F_I we denote the $(2n+1) \times (2n+1)$ matrix

$$F_I = \begin{pmatrix} A & Bh_I & BH_I \\ 0 & 0 & 0 \\ -Q & 0 & -A^* \end{pmatrix}.$$

We shall say that the problem (1)–(3) is normal if for every I admissible and any normal vector of any of the faces of the cones W_J for J admissible the pair (F_I, s) is observable.

From the expression (17) of the sets W_I it follows that any such normal s can be expressed in the form

$$s = \text{col}(-d_i + \langle c_i, h_J \rangle, c_i^* H_J) \qquad \text{for some } i = 1,\ldots, p, \gamma \neq 0.$$

This means that the entries of s and F_I are rational functions of the entries of the matrices A, B, Q, R, $C = (c_1, \ldots, c_p)$, $d = (d_1, \ldots, d_p)$. Consequently, the non-observability of some pair(F_I, s) can be expressed in the form $P(A, B, Q, R, C, d) = 0$, where P is a polynomial in the entries of A, B, Q, R, C, d. We can consider the data of the problem (1)–(3), consisting of the matrices A, B, Q, C, d as elements of a finite-dimensional vector space of a suitable dimension. In this space the set of non-normal problems is contained in an algebraic variety that does not coincide with the entire space. Such a variety is nowhere dense and closed which implies that the set of normal problems contains an open dense subset of all problems given by data of the same dimension. In this sense it can be said that almost all problems are normal.

Now we are able to formulate

THEOREM 3. *Let the problem* (1)–(3) *be normal. Then, it admits a regular synthesis in the interior G of the domain of controllability of* 0.

For the concept of regular synthesis the reader is referred to [1]. Let us note that G is a subset of R^{n+1} (the x, t-space), the closed-loop control v will be a function of both x and t and t is considered as a state variable (cf. the paragraph preceding Example 1 in [1]).

Proof. We prove this theorem by verifying the hypotheses of the Theorem of [1].

Hypothesis 1 is satisfied trivially. For the sets N_i and the functions w_i of Hypothesis 2 we take the sets $N_I = R^{n+1} \times (\overline{W}_I \times R) \backslash \{0\}]$ and the functions w_I respectively (recall that the state space is R^{n+1}) for I admissible, where W_I, w_I are defined in Section 3.

By Theorem 1 and Corollary 2 the sets W_I cover R_0^{n+1} and the functions w_I are analytic in R_0^{n+1}. Since $\psi^0 < 0$ and is constant along the solutions of the adjoint equation with respect to which the optimal trajectories in G are extremal it is of no significance for the validity of the theorem that the sets $\overline{W}_I$ do not cover all R^{n+1} and the functions w_I are not defined for $\psi^0 = 0$.

To verify Hypotheses 3, 4 we first note that the system [1, (12)] in our case has the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \dot{\psi}^0 &= 0, \\ \dot{\eta} &= -\psi^0 Qx - A^*\eta, \\ u &= w_I(\psi) \end{aligned} \tag{28}$$

(it is not necessary that we write down the equations for the state variable representing time and the corresponding adjoint variable). By the transformation $y = \psi^0 x$ we can rewrite the system (29) to the form

$$\begin{aligned} y &= Ay + B(h_I\psi^0 + H_I\eta), \\ \dot{\psi}^0 &= 0, \\ \dot{\eta} &= -Qy - A^*\eta, \end{aligned} \tag{29}$$

the matrix of which is F_I. Obviously, the switching points in the sense of [1] are also switching points of the system (29) in the sense of Section 4 provided we take R_0^{n+1} as K_0 and $\mathscr{K} = \{\overline{W}_I \mid I \text{ admissible}\}$. By the normality assumption it follows from Theorem 2 that there exist positive constants δ, N such that no solution of (29) can have more than N switching points on an interval of length $\leqslant \delta$. In particular we obtain that a solution $(t, x(t), \psi(t))$ of (28) cannot stay in an intersection of two different sets N_I, N_J on a non-trivial interval (all points of this interval would be switching points). From this and LQ2 we obtain Hypothesis 3. If we take into account that time is included into the state variables, from the existence of δ, N we obtain also Hypothesis 4.

To verify Hypothesis 5 assume $(\tau_0, y_0) \in G$, $(\tau_k, y_k) \in G$ for $k = 1, 2,\ldots$, $(\tau_k, y_k) \to (\tau_0, y_0)$ and denote $x_k(t)$ the optimal trajectory of the initial point (τ_k, y_k), $k = 1, 2,\ldots$ By LQ2, LQ3 and Lemma 1, there are non-zero functions $\psi_k(t) = (\psi_k{}^0, \eta_k(t))$ on $[\tau_k, 0]$ with $\psi_k{}^0 \neq 0$ such that $x_k(t), \eta_k(t)$ satisfy the system of equations

$$\begin{aligned} \dot{x}_k &= Ax_k + Bw(\psi_k), \\ \dot{\eta}_k &= -A^*\eta_k - \psi^0 Qx_k . \end{aligned} \tag{30}$$

Without loss of generality we may assume that the vectors ψ_k are normalized in such a way that $\psi_k{}^0 = -1$. We prove that the sequence $\{\eta_k\}$ is bounded.

Denote $V(\tau, y) = J(\tau, y, u_{\tau,y})$ for $(\tau, y) \in G$. From the boundedness of U it follows that V is bounded on $\bar{G}$. By the transformation $t = -t'$ we can transform the problem (1)–(3) with target state 0 and variable initial point to the endpoint problem on the interval $[0, -\tau]$ with initial state 0 to which we can apply the results of [4, Sect. 3.5]. In the notation of [4, Sect. 3.5] the set of points $\{(y, V(\tau, y) \mid y \in \mathscr{R}(\tau)\}$ coincides with the lower boundary of the convex set $\hat{K}_v$ and we have $\psi \in E(\tau, y)$ for some $y \in \mathscr{R}(\tau)$ $(E(\tau, y)$ defined as in LQ3) if and only if ψ is an outer normal to $\hat{K}_v$ at the point $(y, V(\tau, y))$. Consequently, we have for every $y, y' \in \mathscr{R}(\tau)$ and $\psi = (\psi^0, \eta) \in E(\tau, y)$

$$\psi^0 V(\tau, y') + \langle \psi, y' \rangle \leqslant \psi^0 V(\tau, y) + \langle \psi, y \rangle. \tag{31}$$

Since G is open, there exists a $\delta > 0$ such that $(\tau, y) \in G$ as soon as $|\tau - \tau_0| < 2\delta$, $|y - y_0| < 2\delta$. Assume $|\eta_k| \to \infty$. For k sufficiently large we have $|\tau_k - \tau_0| < \delta$, $|y_k - y_0| < \delta$, so $(\tau_k, y_k + \delta |\eta_k|^{-1} \eta_k) \in G$, By (31) we have

$$-V(\tau_k, y_k + \delta |\eta_k|^{-1}\eta_k) + \langle \eta_k, y_k + \delta |\eta_k| \eta_k \rangle \geqslant -V(\tau_k, y_k) + \langle \eta_k, y_k \rangle$$

from which it follows that

$$V(\tau_k, y_k + \delta |\eta_k|^{-1} \eta_k) \geqslant \delta \langle \eta_k, |\eta_k|^{-1} \eta_k \rangle = \delta |\eta_k|$$

which contradicts the boundedness of V on G.

Since $\{\eta_k\}$ is bounded we may without loss of generality assume $\psi_k(0) \to \chi = (-1, \zeta)$ for $k \to \infty$ for some $\zeta \in R^n$. Since w is Lipschitz continuous (Corollary 1) we have $(x_k(t), \eta_k(t)) \to (x_0(t), \eta_0(t))$, where $(x_0(t), \eta_0(t))$ is the solution of (30) satisfying $(x_0(\tau_0), \eta_0(\tau_0)) = (y_0, \zeta)$. In particular, we have $x_0(0) = 0$. This means that the control $w(\psi_0(t))$, $\psi_0(t) = (-1, \eta_0(t))$ is an extremal control steering y_0 to 0 on $[\tau_0, 0]$. By LQ2, $w(\psi_0(t)) = u_{\tau_0, x_0}(t)$ for almost all $t \in [\tau_0, 0]$ and, consequently,

$$V(\tau_0, y_0) = \lim_{k \to \infty} V(\tau_k, y_k).$$

This completes the verification of Hypothesis 5.

Since time appears as state variable in our case, Hypothesis 6 is satisfied trivially. Hypothesis 7 follows immediately from the fact that the system [1, (12)] which has the form (28) in our case is linear in x, η for every I admissible. This completes the proof of the theorem.

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