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Existence of Regular Synthesis for General Control Problems

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1. INTRODUCTION

The concept of regular synthesis has been introduced by Bolt'anski in his classical paper [1] (cf. also [2]) on the sufficiency of the Pontrjagin maximum principle for time-optimal control problems. It has been used as an assumption on a closed-loop control to generate open-loop optimal controls. Using the theory of subanalytic sets it has been proved in [3] that every normal linear system admits a regular time-optimal synthesis. Subsequently, in [8], Sussmann has been able to dispose of the normality condition and to extend the theorem to a certain class of nonlinear systems.

All of the mentioned papers deal entirely with the time-optimal control problem for systems which are linear in the control, with polyhedral control domains. The present paper constitutes an extension towards optimal control problems with general performance criteria and general control domains. The abstract theorem proved in this paper is modelled after an important class of problems—linear-quadratic optimal control problems with linear control constraints. This problem will be dealt with in a forthcoming paper.

An important requirement in Bolt'anski's definition of regular synthesis which is followed in [3, 4] as well as [8, 9] is that the optimal trajectories enter the switching surfaces (called cells) transversally. This requirement has to be dropped not only in the linear-quadratic optimal control problem but also in the linear time-optimal control problem with a control domain having a piecewise smooth curvilinear boundary, as the following example demonstrates:

Consider the system

$$\dot{x}_1 = x_2 + u_1 ,$$
$$\dot{x}_2 = u_2 \tag{1}$$

with the control domain $U = \{(u_1, u_2) \mid -1 + u_2^2 \leqslant u_1 \leqslant 1 - u_2^2\}$ (Fig. 1) and the time-optimal control problem of steering this system to the target state O.

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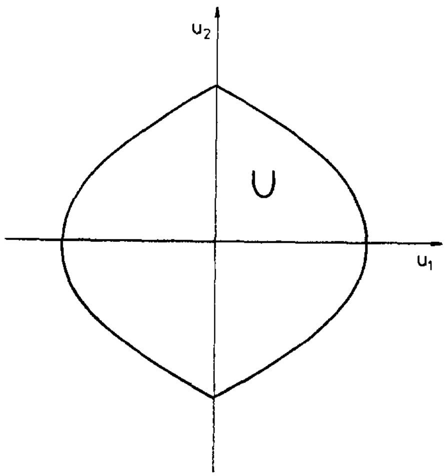

FIGURE 1

Given an initial state x, the maximum principle requires that the optimal controls $u(t)$, $t \in [0, T]$ satisfy the condition

$$\psi_1(t)\, u_1(t) + \psi_2(t)\, u_2(t) = \max_{u \in U}(\psi_1(t)\, u_1 + \psi_2(t)\, u_2) \tag{2}$$

for $t \in [0, T]$, where $\psi(t)$ is a non-zero solution of the adjoint system

$$\dot{\psi}_1 = 0, \qquad \dot{\psi}_2 = -\psi_1, \tag{3}$$

the solution of which has the form $\psi_1(t) = c$, $\psi_2(t) = -ct + d$ with $c \neq 0$ or $d \neq 0$.

The condition (2) can be rewritten in the form

$$u(t) = u \qquad \text{if and only if } \psi(t) \in W(u),$$

where $W(u)$ is the cone of outward normals to U at u.

The cones $W((0, 1))$, $W((0, -1))$ have a non-empty interior while the normal cones at the points of the open arcs joining the points $(0, 1)$ and $(0, -1)$ reduce to halflines. Furthermore, obviously the points of the arcs are continuous functions of their normal vectors. As usual, we construct the synthesis by following backwards the trajectories of (1) through 0, satisfying (2), for various solutions of the adjoint equation. All the solutions of the adjoint equation with $\psi_1(0) = c < 0$, $\psi_2(0) = d > 0$ such that $(c, d) \in \operatorname{int} W((0, 1))$ stay in $W((0, 1))$ for $t < 0$ sufficiently small and then leave it for some finite $\tau < 0$. The corresponding control $u(t)$ will be equal to $(0, 1)$ for $t \in [\tau, 0]$ and for $t < \tau$ it will move continuously along the left boundary arc of U away from $(0, 1)$. Therefore,

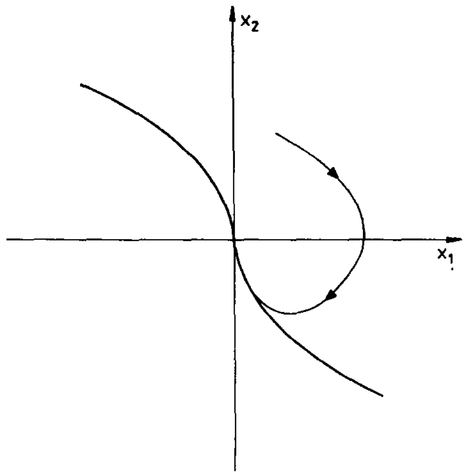

FIGURE 2

the corresponding trajectory will follow the trajectory of the equation $\dot{x}_1 = x_2$, $\dot{x}_2 = 1$ (which is the halfparabola $x_1 = \frac{1}{2}x_2^2$, $x_1 \geqslant 0$) and at $t = \tau$ leave it in a tangent way. In the synthesis terminology the parabola constitutes a cell which is joint by the optimal trajectories in a tangent way (Fig. 2).

In this paper we extend the concept of regular synthesis to problems in which non-transversal meeting of switching surfaces by optimal trajectories cannot be excluded (Section 2) and we formulate (Section 3) and prove (Section 4) an existence theorem for such problems. The Appendix contains a transcription of Bolt'anski's proof of the optimality of a regular synthesis to general problems under the extended concept of synthesis.

2. REGULAR SYNTHESIS FOR GENERAL PERFORMANCE CRITERIA

Consider the control system

$$\dot{x} = f(x, u), \qquad x \in \mathbf{R}^n, \quad u \in U \subset \mathbf{R}^m, \tag{4}$$

$f: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ being C^1, and the cost function

$$J = \int_0^T f^0(x, u)\, dt, \tag{5}$$

where $f^0: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is C^1. Given an initial point x_0 and target point $\hat{x}$, by an admissible control we understand a piecewise continuous function $[0, T] \to U$ such that the solution $x(t)$ of the equation

$$\dot{x} = f(x, u(t)) \tag{6}$$

(called the response of u) satisfying $x(0) = x_0$ exists on $[0, T]$. If $x(T) = \hat{x}$, we say that the control $u(t)$ steers the system from x_0 to $\hat{x}$ (in time T) An admissible control $u: [0, T] \to U$ will be called optimal if it minimizes J among all admissible controls steering the system from x_0 to $\hat{x}$.

It will be convenient at some places (as usual in optimal control theory) to include the performance variable x^0 into the state variables of the system. That is, we add to the system equations (4) the equation

$$\dot{x}^0 = f^0(x, u), \qquad x^0(0) = 0.$$

Also, we denote $\tilde{x} = \mathrm{col}(x^0, x)$, $\tilde{f} = \mathrm{col}(f^0, f)$.

A control $u(t)$ and its response $x(t)$ will be called extremal (with respect to a non-zero solution $\psi(t)$ of the adjoint equation)

$$\dot{\psi} = -\left(\frac{\partial \tilde{f}}{\partial \tilde{x}}(x(t), u(t))\right)^* \psi \tag{7}$$

(the asterisk standing for transpose) if the triple x, u, ψ satisfies Pontrjagin's maximum condition

$$0 = H(x(t), u(t), \psi(t)) = \max_{u \in U} H(x(t), u, \psi(t)) \tag{8}$$

where

$$H(x, u, \psi) = \langle \psi, \tilde{f}(x, u) \rangle.$$

If for a given x there is a unique extremal control u_x steering x to $\hat{x}$ we denote by Ψ_x the set of all non-zero solutions of (7) with respect to which u_x is extremal and by $\Psi_x(t)$ the set of their values at time t.

Let us also note that by a piecewise continuous function we understand a function which is continuous except for a finite number of jump discontinuities. As the value of the function at a point of discontinuity we shall always take its right-hand limit.

Given a C^1 manifold M and two sets $G \subset H \subset M$, by a stratification of G in H we understand a locally finite (in H) partition $\mathscr{P}$ of G into C^1 submanifolds of M (called strata) such that if $P, Q \in \mathscr{P}$, $P \cap \bar{Q} \neq \varnothing$ and $P \neq Q$ then $P \subset \bar{Q}$ and $\dim P < \dim Q$. By the dimension of a stratification we understand the maximum of the dimensions of its strata. We shall call G a stratified set of dimension m if it admits a stratification of dimension m.

Let $G \subset \mathbf{R}^n$ be an open domain such that $\hat{x} \in \bar{G}$. By a regular synthesis in G of the control problem (5), (6) with target point $\hat{x}$ we shall understand a 6-tuple $(\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \Pi, \Sigma, v)$, where

φ is a union of the one-point set $\{\hat{x}\}$ and a locally finite (in G) partition of $G \backslash \{\hat{x}\}$ into C^1 connected submanifolds of $\mathbf{R}^n$ (called cells),

$\mathscr{S}$ is a disjoint union of $\mathscr{S}_1$ (cells of type I) and $\mathscr{S}_2$ (cells of type II),
Π: $\mathscr{S}_1 \to \mathscr{S}$, Σ: $\mathscr{S}_2 \to \mathscr{S}_1$ and v: $G \to U$ (the closed loop control) are maps,

such that the following properties are satisfied:

A. The set $G' = \bigcup \{S \in \mathscr{S} \mid \dim S < n\} \backslash \{\hat{x}\}$ is a stratified subset of G of dimension $<n$ (if $\mathscr{S}$ is a family of sets, we use the notation $\bigcup \mathscr{S} = \bigcup \{S \mid S \in \mathscr{S}\} = \{x \mid x \in S \in \mathscr{S}\}$).

B. The function v is C^1 on each cell. If $S_1 \in \varphi_1$ and $x \in S_1$, then $f(x, v(x)) \in T_x S_1$ (the tangent space to S_1 at x) and there exists a $\tau(x)$ such that the trajectory ξ_x of the equation

$$\dot{x} = f(x, v(x)) \tag{9}$$

in S_1 with $\xi_x(0) = x$ satisfies $\xi_x(t) \in S_1$ for $t \in [0\ \tau(x))$ and $\lim_{t \to \tau(x)} \xi_x(t) \in \Pi(S_1)$. If $S_2 \in \mathscr{S}_2$ then v is continuous on $S_2 \cup \Sigma(S_2)$ and for each $x \in S_2$ there is a unique trajectory ξ_x of (9) such that $\xi_x(0) = x$ and $\xi_x(t) \in \Sigma(S_2)$ for $t > 0$ small. The time $\tau(x)$ for which ξ_x reaches $\Pi(S_1)$ for $x \in S_1$ and $\Pi(\Sigma(S_2))$ for $x \in S_2$ is a C^1 function on S_1, S_2 respectively and the trajectory $\xi_x(t)$ and the control $u_x(t)$ generated by

$$u_x(t) = v(\xi_x(t)) \tag{10}$$

are C^1 functions of t, x for $x \in S_1$ ($x \in S_2$ respectively) and $t \in [0, \tau(x))$ and can be extended to C^1 functions for $t \geqslant \tau(x)$ close to $\tau(x)$.

C. The trajectory $\xi_x(t)$ of (9) which is uniquely defined by B until it stays in G, is extremal and reaches $\hat{x}$ in finite time $T(x)$ passing only finitely many times from one cell to another.

D. The value

$$J(x, u_x) = \int_0^{T(x)} f^0(\xi_x(t), u_x(t))\, dt$$

is a continuous function of x in G.

The point where the definition of regular synthesis had to be substantially altered with respect to [3] is B. The fact that the transversal entering of $\Pi(S)$ by the trajectories of (9) cannot be required in general has been demonstrated by the example in Section 1. Consequently, it cannot be required that v can be extended to a C^1 function in a neighbourhood of S since the trajectories of (9) could then never enter $\Pi(S)$ tangentially (this would violate the uniqueness theorem for ordinary differential equations). Because of this lack of uniqueness the maps Π, Σ are not automatically defined by $\mathscr{S}$ and v and we had to include them into the definition of the synthesis (note that we require only the uniqueness of the distinguished trajectory ξ_x satisfying the requirements of B for a given x).

Let us note that the transversality assumptions in Bolt'anski's definition are needed to establish the C^1 dependence of $J(x, u_x)$ and some estimates on a cell of dimension n in the sufficiency proof. Nevertheless, as shown in the Appendix, the assumptions of B are still sufficient for carrying out the proof.

The formulation of B is rather cumbersome and one may wonder how its C^1 dependence requirements can be verified. However, there is a standard way to do this (via an auxiliary partition in the product space of the state space and the space of adjoint variables) which will become clear in the proof of the existence theorem.

3. The Theorem

We consider the optimal control problem (4), (5) with the target point $\hat{x}$.

THEOREM. *Let $G \subset \mathbf{R}^n$ be open and let $\hat{x} \in \bar{G}$. Assume that*

1. *The functions f, f^0 are analytic in x, u.*

2. *There exists a covering of $\bar{G} \times (\mathbf{R}^{n+1}\backslash\{0\})$ by closed in $\mathbf{R}^n \times (\mathbf{R}^{n+1}\backslash\{0\})$ sets $N_1, \ldots, N_r$ with conical x-sections such that for every $i = 1,\ldots, r$ there exists an analytic function $w_i(x, \psi)$ in some neighbourhood of N_i satisfying Pontrjagin's maximum condition*

$$H(x, w_i(x, \psi), \psi) = \max_{u \in U} H(x, u, \psi) \qquad \textit{for} \quad (x, \psi) \in N_i \tag{11}$$

(by saying that N_i has a conical x-section we understand that for any fixed x the set of points $(x, \psi) \in N_i$ completed by the point $(x, 0)$ is a cone).

3. *For every $x \in G \cup \{\hat{x}\}$ there is a unique extremal control $u_x(t)$, $t \in [0, T(x)]$ steering x to $\hat{x}$ and such that its response ξ_x with $\xi_x(0) = x$ satisfies $\xi_x(t) \in G$ for $t \in [0, T(x))$. Further, for each $x \in G$ there exists a unique $\mu(x) \in \{1,\ldots, r\}$ such that if $\psi \in \Psi_x$, then $(\xi_x(t), \psi(t))$ is a solution of the system*

$$\begin{aligned} \dot{x} &= f(x, u), \\ \dot{\psi} &= -\left(\frac{\partial f}{\partial \tilde{x}}(x, u)\right)^* \psi, \\ u &= w_i(x, \psi), \end{aligned} \tag{12}$$

with $i = \mu(x)$ satisfying $(\xi_x(t), \psi(t)) \in N_i$ for $t \geqslant 0$ from some neighbourhood of 0.

4. *The number of points (which we shall call switching points of u_x) t such that $\mu(\xi_x(s)) \neq \mu(\xi_x(t))$ for $s < t$ near t is uniformly locally bounded in the following sense: for every compact $K \subset G \subset \{\hat{x}\}$ there exists a $\nu = \nu(K) > 0$ such that u_x has not more than ν switching points for $x \in K$.*

5. *The value* $J(x, u_x)$ *of the performance index* J*, computed along the extremals, is a continuous function of* x *in* G.

6. *For every compact* $K \subset G \cup \{\hat{x}\}$ *there exists a* $\gamma(K) > 0$ *such that* $T(x) < \gamma(K)$ *for* $x \in K$.

7. *There exists an* $\eta > 0$ *such that the solution* $(x(t), \psi(t))$ *of the system* (12) *with* $x(0) \in G$ *and* $(x(0), \psi(0)) \in N_i$ *exists on the interval* $[-\eta, 0]$ *and satisfies* $x(t) \in G$ *for* $t \in [-\eta, 0]$ *for any* $i = 1,..., r$.

Then, the problem (4), (5) *with the target point* $\hat{x}$ *admits a regular synthesis in* G.

This theorem differs from that of [8] in several ways. While in [8] it is assumed that the optimal control can be found among finite concatenations of controls generated by a collection of closed-loop controls $u = v_i(x)$ (which is natural for the class of problems linear in the control, which are studied there) we do not assume a priori that the extremal controls u_x are optimal—their optimality follows from the regularity of the synthesis by Bolt'anski's theorem. Further, in the class of problems we have in mind the solution of the maximum condition (11) as a rule depdns on the adjoint variable too.

On the other hand we assume unique covering of G by the extremals which is a rather restrictive condition. Although formally we do not exclude singular extremals (we do not assume that $w_i(x, \psi)$ is the unique solution of (11)), they are often excluded by Assumption 3.

Due to the linearity of H in ψ we may without loss of generality assume that the functions w_i are constant along the rays of N_i, i.e., $w_i(x, c\psi) = w_i(x, \psi)$ for $c > 0$. Finally, let us note that Assumption 7 can be slightly relaxed: it suffices to have an $\eta > 0$ satisfying the requirements of this assumption for each compact subset of $G \cup \{\hat{x}\}$.

The assumptions of the theorem (2, 3 in particular) are rather complicated. However, they are based on an abstraction of those features of the linear-quadratic optimal control problem with a polyhedral control domain which have been found to be essential for the existence of the regular synthesis. This abstraction has been slightly modified in order that the theorem could be applied to other problems, including some particular ones with singular extremals. We complete this section by two examples on which we shall illustrate the hypotheses of the theorem, the first example being the linear-quadratic problem.

It is not claimed that the theorem contributes to the proof of the sufficiency of the Pontrjagin maximum principle in these particular problems—the sufficiency follows immediately from the existence of optimal controls and the unicity of extremals. Indeed, it cannot be expected that the theorem, in which the unicity of the extremals is assumed, would contribute significantly to the sufficiency problem. Rather, it justifies the idea about the structure of the optimal feedback (piecewise smoothness with regular switching surfaces) one gets from the simple examples in which the latter can be constructed explicitely.

Unlike in the formulation of the general optimal control problem, the time T is fixed in both the examples. However, the case $T = \hat{T}$ fixed can be reduced to the case of T free by including the time into the state variables, the equation $\dot{t} = 1$ into the system equation and the equality $t(T) = \hat{T}$ into the definition of the target point. Of course, this means that the synthesis will be constructed in $\mathbf{R}^{n+1}$, v becoming time-dependent in general.

EXAMPLE 1. Consider the problem (4), (5) with $f(x, u) = Ax + Bu$, A, B, constant and $f^0(x, u) = x^*Qx + u^*Ru$, Q, R, symmetric constant, $Q \geqslant 0$, $R > 0$, $\hat{x} = 0$, $\hat{T} = 0$. We assume that U is a convex compact polytope, $U = \{u \mid \langle c_i, u\rangle \leqslant d_i,\ i = 1,..., p\}$ containing the origin in its interior and that the system (4) is controllable, i.e., $\operatorname{rank}(B, AB,..., A^{n-1}B) = n$.

For G we take the interior of the domain of controllability to $(0, 0)$, which is the set of those points (t, x) from which the system can be steered to $(0, 0)$. It follows from [10] that G is non-empty, for each $(t, x) \in G$ there is a unique extremal joining (t, x) with $(0, 0)$ and this extremal lies entirely in G. Also, $\psi_0 < 0$ holds for such an extremal.

The expression which is to be minimized in the maximum condition (11) reduces to $L(u, \psi) = u^*Ru + \langle\psi', Bu\rangle$, where $\psi' = (\psi_1, ..., \psi_n)$. Let I be a subset of the set $\{1,..., p\}$ of cardinality $\leqslant m$. Then, $P_I = \{u \mid \langle c_i, u\rangle = d_i,\ i \in I\}$ is the affine hull of the face $W_I = \{u \mid \langle c_i, u\rangle = d_i,\ i \in I, \langle c_i, u\rangle < d_i,\ i \notin I\}$ of U (provided there are no redundant constraints). We define w_I by

$$L(w_I(\psi), \psi) = \min_{u \in P_I} L(u, \psi).$$

Using the Lagrange multiplier rule it can be proved that the functions w_I are affine. We define

$$N_I = \mathbf{R}^{n+1} \times \operatorname{cl}\{\psi \mid \psi_0 < 0, w_I(\psi) = w(\psi)\}$$

(the closure to be taken in $R^{n+1}\backslash\{0\}$) where $w(\psi)$ is uniquely defined by

$$L(w(\psi), \psi) = \min_{u \in U} L(u, \psi).$$

The sets N_I and the functions w_I are the sets and the functions the existence of which is asserted in Assumption 2. Under certain "normality" conditions it can be proved that Assumptions 3 and 4 hold (the normality condition consists in certain polynomial inequalities which are satisfied for almost all problems). The proofs (expecially that of the boundedness of the number of switchings) are rather complicated. They will be carried out in detail in a separate paper dealing with this problem. The remaining hypotheses either follow immediately from the formulation of the problem or can be obtained by simple arguments based on the standard theory.

EXAMPLE 2. Consider the optimal control problem

$$\begin{aligned} \dot{x}_1 &= x_2 , \\ \dot{x}_2 &= u_1 - u_2 - kx_2 , \\ f^0(x, u) &= x_2 u_1 , \\ U &= \{u = (u_1 , u_2) \mid u_1 \in [0, \beta], u_2 \in \epsilon[0, \alpha]\}, \end{aligned}$$

$k, \alpha, \beta > 0$ with target point $\hat{T} = 0$, $\hat{x} = 0$ and G being the intersection of the set $F = \{(t, x_1 , x_2) \mid t < 0, x_1 < 0, x_2 > 0\}$ with the set of points that can be steered to the target point along a trajectory lying entirely in F. This problem is a simplified model of the minimum energy control problem of a train on an ideally straight line where x_1 stands for distance, x_2 for velocity, u_1 and u_2 for the tracking and braking force respectively. The term $-kx_2$ represents resistance and $x_2 u_1$ (the integral of which is the work of the tracking force) is assumed to be proportional to the energy consumption (cf. [11]).

From the standard existence theory it follows that the optimal control exists for each $(t, x) \in G$. The solution of the maximum condition (11) is unique except for $\psi_2 = 0$ or $\psi_2 = -\psi_0 x_2$. It is given by the formula

$$\begin{aligned} u &= (\beta, 0) && \text{if } \psi_2 > -\psi_0 x_2 \\ &= (0, 0) && \text{if } 0 < \psi_2 < -\psi_0 x_2 \\ &= (0, \alpha) && \text{if } \psi_2 < 0. \end{aligned}$$

As shown in [11] no optimal control can contain a singular interval with $\psi_2 = 0$ but for $\psi_2 = -\psi_0 x_2$ a singular control can occur, namely $u = (kx_2 , 0)$. Also, it can be proved that for every $(t, x) \in G$ there is precisely one extremal control steering the system from (t, x) to $(0, 0)$ and that this control acquires the values $(\beta, 0)$, $(kx_2 , 0)$, $(0, 0)$, $(0, \alpha)$ in this order (some of them possibly missing).

Therefore, if we denote $N_1 = \{(x, \psi) \mid \psi_2 \geqslant -\psi_0 x_2\}$, $N_2 = \{(x, \psi) \mid \psi_2 = -\psi_0 x_2\}$, $N_3 = \{(x, \psi) \mid 0 \leqslant \psi_2 \leqslant -\psi_0 x_2\}$, $N_4 = \{(x, \psi) \mid \psi_2 \leqslant 0\}$ (note that $N_2 \subset N_1$, $N_2 \subset N_3$), $w_1(x, \psi) = (\beta, 0)$, $w_2(x, \psi) = (kx_2 , 0)$, $w_3(x, \psi) = (0, 0)$, $w_4(x, \psi) = (0, \alpha)$, then Assumptions 2, 3, 4 are satisfied. It is not difficult to verify that this is the case also for the remaining assumptions.

4. THE PROOF

As in the proof of similar theorems in [3, 8] some results of the theory of subanalytic sets [5, 6] will be used. For the necessary material the reader is referred to [7–9]. where it is summarized in a form which is most convenient for our purposes.

The proof follows a similar pattern as the proof of the similar theorem in [3]. Following the extremal trajectories backwards we construct inductively the cells of the partition $\mathscr{S}$. In each induction step we show that they are CASA sets (i.e., connected analytic submanifolds of R^n which are subanalytic) satisfying the conditions B–D. After that we prove that they cover G locally finitely from which it immediately follows that $\mathscr{S}$ has property A (a locally finnite union of subanalytic sets is subanalytic and thus a stratified set). However, in order to prove B we have to construct an auxiliary partition of the subanalytic set $D(S)$ in $\mathbf{R}^{2n+1}$ (to be defined below) associated with a cell $S \in \mathscr{S}$ into CASA sets which are met transversaly by the entering and exiting trajectories of the system (12).

The terms ancestor and descendant will be used for the cells as in [3, 4]. That is, a cell S will be said to be an ancestor of S if the extremals starting in S pass S' before reaching $\hat{x}$; S will be called a descendant of S' if S' is an ancestor of S. The cells constructed at the kth induction step will be called cells of order k, the unique cell of order 0 being $\{\hat{x}\}$. By $\mathscr{S}_k = \mathscr{S}_k^1 \cup \mathscr{S}_k^2$ we denote the set of cells of order $\leqslant k$ and by G_k we denote their union. Among the cells of $\mathscr{S}_k$ we distinguish a certain class of cells, called border cells, the set of which we denote by $\mathscr{S}'_k$.

Let $x \in G$. We denote $E(x) = \Psi_x(0)$, i.e., $E(x)$ is the set of those $\psi \in \mathbf{R}^{n+1}$ which are initial values for the non-zero solutions of the adjoint equation with respect to which u_x is extremal; further, we denote $E(\hat{x}) = \mathbf{R}^{n+1}\setminus\{0\}$, $E^0(x) = \{\psi \in E(x) \mid |\psi| = 1\}$, $|\cdot|$ standing for the Euclidean norm. Under the assumptions of the theorem the set $E(x)$ has the following properties:

E1. $E(x) \cup \{0\{$ is a convex cone for any $x \in G$.

E2. If $\psi \in E(x)$ and $(x(t), \psi(t))$ is a solution of (12) on $[\tau, 0]$, $\tau < 0$ (or, $[0, \tau]$, $\tau > 0$) such that $x(0) = x$, $\psi(0) = \psi$, $(x(t), \psi(t)) \in N_i$, $x(t) \in G \cup \{\hat{x}\}$ for $t \in [\tau, 0]$ (or, $t \in [0, \tau]$, respectively) then $\psi(\tau) \in E(x(\tau))$.

Assume now that $\mathscr{S}_k = \mathscr{S}_k^1 \cup \mathscr{S}_k^2$ (and the restrictions of Π, Σ and v to $\mathscr{S}_k$) has been constructed in such a way that the following induction hypotheses are satisfied:

I1. $\mathscr{S}_k$ is a finite partition of G_k into relatively compact CASA sets.

I2. For every $S \in \mathscr{S}_k$ the set $D(S) = \{(x, \psi) \mid x \in S,\ \psi \in E^0(x)\}$ is subanalytic.

I3. For every cell $S \in \mathscr{S}_k$, $\mu(x)$ is constant over S and the formula

$$v(x) = w_{\mu(x)}(x, \psi) \qquad \text{for} \quad \psi \in E(x) \tag{13}$$

defines uniquely an analytic function v: $S \to U$ (we shall occasionally use the notation $\mu(S)$ instead of $\mu(x)$ for $x \in S$).

I4. $\mathscr{S}_k$ (together with the restrictions of Π, Σ, v on G_k) has the properties B–D on G_k.

I5. The set $\{t \geqslant 0 \mid \xi_x(t) \in G_k\}$ is closed for every $x \in G$.

I6. If $\xi_x(t_1) \in G_{k-1}$ for some $t_1 \in [0, \eta]$, $\mu(\xi_x(t)) = i$ for some $1 \leqslant i \leqslant r$ and all $0 \leqslant t < t_1$, then $x \in G_k$.

I7. If $S \in \mathscr{S}_k \backslash \mathscr{S}'_k$, $\xi_x(t) \in S$ for some $x \in G$ and $t > 0$, $\mu(\xi_x(s)) = \mu(\xi_x(t))$ for all $s \leqslant t$ from some neighbourhood of t then $\xi_x(s) \in G_k$ for $s \leqslant t$ from some neighbourhood of t then $\xi_x(s) \in G_k$ for $s \leqslant t$ sufficiently near t.

I8. If $x' \in \mathscr{S}'_k \backslash \mathscr{S}'_{k-1}$, $x' = \xi_x(t)$ for some $x \in G$, $t > 0$ then $x \notin G_k$.

I9. If S is of order $i \leqslant k$, its descendants are of order $\geqslant i$, its immediate descendant being of order $\leqslant i + 1$; if S' is a descendant of S of order i, then $\mu(S') = \mu(S)$.

Let S be a cell of order k and $1 \leqslant i \leqslant r$. We shall call (S, i) an admissible pair if either S is a border cell or $i \neq \mu(S)$. The cells of order $k + 1$ will be obtained by the construction described below as descendants of the cells S of order k associated with $i \in \{1,..., r\}$ for all admissible pairs (S, i).

Let (S, i) be an admissible pair. Denote by F^i the flow of the system of equations (12). Since $w_i(x, c\psi) = w_i(x, \psi)$ for $c > 0$ and $E(x) \cup \{0\}$ is a cone for $x \in G$, if $(x(t), \psi(t))$ is a solution of (12) and $\psi(0) \in E(x)$, then the same is true for $(x(t), c\psi(t))$ for all $c > 0$. It follows that if we consider (12) as a differential equation on $G \times (\mathbf{R}^{n+1}\backslash\{0\})$ (which we can—because we deal only with non-zero adjoint vectors) there exists an analytic flow Φ^i on $G \times \mathbf{S}^n$ (the n-sphere) which is the radial projection of F^i. By this we mean that if χ: $G \times (\mathbf{R}^{n+1}\backslash\{0\}) \to G \times \mathbf{S}^n$ is the projection $\chi(x, \psi) = (x, |\psi|^{-1}\psi)$, then $\Phi^i{}_t \circ \chi(x, \psi) = \chi \circ F^i{}_t(x, \psi)$ for each t for which $F^i{}_t$ is defined (cf. [3, Lemma 7]).

We denote by $N_i^0 = \chi(N_i)$, $\Psi_x^0 = \{|\psi(\cdot)|^{-1}\psi(\cdot) \mid \psi \in \Psi_x\}$. Because of the homogeneity of the condition (8) in ψ the trajectories of the system of equations (12) can be replaced in the formulation of the theorem by the trajectories of Φ^i in an obvious way, N_i, Ψ_x replaced by N_i^0, Ψ_x^0 respectively (cf. [3, Remark 1]).

Denote

$$H' = \{\Phi^i_{-\eta}(x, \psi) \mid (x, \psi) \in D(S) \text{ and } \Phi^i{}_s(x, \psi) \in N_i^0 \text{ for } -\eta \leqslant s < 0\},$$

$$H'' = \{\Phi^i{}_t(x, \psi) \mid (x, \psi) \in D(S), -\eta < t < 0 \text{ and } \Phi^i{}_s(x, \psi) \in N_i^0 \text{ for } t \leqslant s < 0\}.$$

Since t ranges over a bounded interval in the expression for H', it follows from the characterization of subanalytic sets in [9] that H', H'' as well as $H = H' \cup H''$ are subanalytic. We define further $K'' = \pi_x(H'')$, $K = \pi_x(H')$, $K = K' \cup K'' = \pi_x(H)$, where π_x is the natural projection on the x-space. Since $\mathbf{S}^n$ is compact, π_x is proper on $\mathbf{R}^n \times \mathbf{S}^n$, which implies that K', K'' and K are subanalytic. Also, since S is relatively compact and t in the expression for H' ranges over a bounded interval, K is relatively compact.

The descendants of S associated with i will be obtained by a sequence of partitions of the sets K, K' into CASA sets, the subsets of K' becoming the border cells.

Let X be a vector field on a manifold M. A submanifold A of M will be called parallel (transversal) to X (or, to its its flow) if $X(x) \in T_xA$ ($X(x) \notin T_xA$ respectively) for all $x \in A$. A collection $\mathscr{A}$ of submanifolds of M will be called compatible with X if each member of $\mathscr{A}$ is either parallel or transversal to X. A partition $\mathscr{A}$ of a manifold M will be said to be compatible with $B \subset M$, if, for every $A \in \mathscr{A}$, either $A \subset B$ or $A \cap B = \varnothing$; $\mathscr{A}$ will be said to be compatible with a family of subsets $\mathscr{B}$ of M if it is compatible with each $B \in \mathscr{B}$.

Let A, B be subsets of analytic manifolds M, N respectively. By a CASA stratification of an analytic map $f: A \to B$ we understand a pair $(\mathscr{P}, \mathscr{R})$, where $\mathscr{P}$ is a CASA stratification of A, $\mathscr{R}$ is a CASA stratification of B such that $f(P) \in \mathscr{R}$ and $\operatorname{rank} f|_P = \dim f(P)$ for each $P \in \mathscr{P}$.

Note that it is important to indicate the set B in the last definition as the set which is stratified by $\mathscr{R}$. Further, note that it follows immediately from the definition that if $R \in \mathscr{R}$ and $f^{-1}(R) \neq \varnothing$, then there exists a $P \in \mathscr{R}$ such that $f(P) = R$. We shall call $(\mathscr{P}, \mathscr{R})$ a CASA quasistratification of $f: A \to B$ if it satisfies all the conditions for a stratification but for the rank condition which is replaced by the following weaker condition: for each $R \in \mathscr{R}$ such that $f^{-1}(R) \neq \varnothing$ there exists a $P \in \mathscr{P}$ such that $f(P) = R$ and $\operatorname{rank} f|_P = \dim R$.

We shall need several lemmas for the proof of the theorem.

LEMMA 1. *Let A be a subanalytic subset of an analytic manifold M and let Φ be the flow of a vectorfield X such that $\Phi_t(x)$ is defined for all $x \in A$ and $t \in [-\eta, 0]$. Then, there exists a CASA stratification of $\Phi_{[-\eta,0]}(A)$ such that the function $\sigma: \Phi_{[-\eta,0]}(A) \to \mathbf{R}^1$ defined by $\sigma(x) = \inf_{t \geqslant 0}\{t \mid \Phi_t(x) \in A\}$ is analytic on each stratum of $\mathscr{P}$.*

Proof. By [9, Corollary 2] it suffices to prove that the graph of σ is subanalytic. This follows from the fact that the point (x, t) belongs to the graph of σ if and only if $x \in \Phi_{[-\eta,0]}(A) \wedge (\forall s)(0 < s < t \Rightarrow \Phi_s(x) \in M \backslash A) \wedge (\forall \epsilon)\ [0 < \epsilon < 1 \Rightarrow (\exists \tau)(0 < \tau < \epsilon \wedge \Phi_{t+\tau}(x) \in A$. The subanalyticity of this set follows from the characterization of subanalytic sets given in [9].

LEMMA 2. *Let, M, N be analytic manifolds, $A \subset M$, $B \subset N$ and let $(\mathscr{S}, \mathscr{T})$ be a CASA stratification of the analytic map $f: A \to B$.* If *$\mathscr{P}$, $\mathscr{R}$ are CASA stratifications of A, B compatible with $\mathscr{S}$, $\mathscr{T}$ respectively such that $f(P) \in \mathscr{R}$ for every $P \in \mathscr{P}$ then $(\mathscr{P}, \mathscr{R})$ is a quasistratification of $f: A \to B$.*

Proof. Let $R \in \mathscr{R}$, $f^{-1}(R) \neq \varnothing$. Then, there exist $T \in \mathscr{T}$, $S \in \mathscr{S}$ such that $R \subset T = f(S)$. Denote $\dim S = s$, $\dim T = t$, $\dim R = r$. Because of the rank condition, the set $f^{-1}(R) \cap S$ is a submanifold of S of dimension $s - t + r$. On the other hand, $f^{-1}(R) \cap S$ is a locally finite union of members of $\mathscr{P}$. Let P be the one of them with the highest dimension which has to be $s - t + r$. If $\operatorname{rank} f|_P < r$ then $\operatorname{rank} f|_S \leqslant \operatorname{rank} f|_P + \dim S - \dim P < r + s - s +$

$t - r = t$ which contradicts the assumption that $(\mathscr{S}, \mathscr{T})$ is a stratification of $f: A \to B$.

LEMMA 3. *Let M, N be analytic manifolds, N compact, $H \subset M \times N$ subanalytic and let X be an analytic vector field on $M \times N$. Let $\mathscr{C}$, $\mathscr{D}$ be locally finite collections of subanalytic subsets of $M \times N$, M respectively. Then, there exists a pair of CASA stratifications $\mathscr{P}, \mathscr{R}$ of H, $K = \pi_M(H)$ (where π_M is the natural projection of $M \times N$ on M) respectively such that*

1. *$\pi_M(P) \in \mathscr{R}$ for every $P \in \mathscr{P}$,*
2. *$\mathscr{P}, \mathscr{R}$ are compatible with $\mathscr{C}$, $\mathscr{D}$ respectively,*
3. *$\mathscr{P}$ is compatible with X.*

This lemma we obtain by induction from the following

LEMMA 4. *Let M, N, H, K, X, $\mathscr{C}$, $\mathscr{D}$ be as in Lemma* 3. *Assume that H, K are CASA and* $\dim K = k$. *Then, there exist CASA stratifications $\mathscr{P}, \mathscr{R}$ of H, K respectively, satisfying* 1, 2 *of Lemma* 3 *and*

3′. $\mathscr{P}' = \{P \in \mathscr{P} \mid \dim \pi_M(P) = k\}$ *is compatible with X.*

Proof. By [7, Theorem 8] there exists a CASA stratification $\mathscr{E}$ of H compatible with X and $\mathscr{C}$. Further, there is a CASA stratification $\mathscr{F}$ of K compatible with $\mathscr{D} \cup \{\pi_M(E) \mid E \in \mathscr{E}\}$. Denote $\mathscr{R}' = \{F \in \mathscr{F} \mid \dim F = k\}$, $\mathscr{P}' = \{E \cap \pi_M^{-1}(F) \mid F \in \mathscr{R}', E \in \mathscr{E}\}$. For $F \in \mathscr{R}'$, the set $\pi_M^{-1}(F) \cap H$ is open in H, so $E \cap \pi_M^{-1}(F)$ is an open submanifold of E for $E \in \mathscr{E}$, and therefore remains compatible with X. Thus, $\mathscr{P}'$ is compatible with X.

There exists a CASA stratification $(\mathscr{P}'', \mathscr{R}'')$ of $\pi_x : H\backslash\bigcup \mathscr{P}' \to K\backslash\bigcup \mathscr{R}'$ such that $\mathscr{P}''$ is compatible with $\mathscr{E}$ and $\{\bar{D} \cap (H\backslash\bigcup \mathscr{P}') \mid P \in \mathscr{P}'\}$ and $\mathscr{R}''$ is compatible with $\mathscr{F}$. Since $\operatorname{cl}(H\backslash\bigcup \mathscr{P}') \cap \bigcup \mathscr{P}' = \varnothing$, $\operatorname{cl}(K\backslash\bigcup \mathscr{R}') \cap \bigcup \mathscr{R}' = \varnothing$, the pair $(\mathscr{P}, \mathscr{R})$, $\mathscr{P} = \mathscr{P}'' \cup \mathscr{P}'$, $\mathscr{R} = \mathscr{R}'' \cup \mathscr{R}'$ will satisfy the requirements of the lemma.

Proof of Lemma 3. As the induction statement we employ the following one: There exist CASA stratifications $(\mathscr{P}_k, \mathscr{R}_k)$ of H, K respectively satisfying conditions 1, 2 and

3_k. $\{P \in \mathscr{P}_k \mid \dim \pi_M(P) \geqslant k\}$ is compatible with X.

By [7, Theorem 4] there exists a CASA stratification $(\mathscr{S}, \mathscr{T})$ of $\pi_x : H \to K$ such that $\mathscr{S}$ is compatible with $\mathscr{C}$, and $\mathscr{T}$ is compatible with $\mathscr{D}$. Since condition 3_k is trivially satisfied for $k > \dim K$, we can take $\mathscr{P}_k = \mathscr{S}$, $\mathscr{R}_k = \mathscr{T}$ for $k = \dim K + 1$. We prove that from the existence of $(\mathscr{P}_k, \mathscr{R}_k)$ the existence of $(\mathscr{P}_{k-1}, \mathscr{R}_{k-1})$ follows by Lemma 4.

By Lemma 4, for each $B \in \mathscr{R}_k$ such that $\dim B = k - 1$, there exist CASA stratifications $\mathscr{P}_B$, $\mathscr{R}_B$ of $H \cap \pi_x^{-1}(B)$ and B respectively such that $\pi_M(P) \in \mathscr{R}_B$

for each $P \in \mathscr{P}_B$ and $\mathscr{P}'_B \equiv \{P \in \mathscr{P}_B \mid \dim \pi_x(P) = k - 1\}$ is compatible with X. By [7, Theorem 4] there exists a CASA stratification $(\mathscr{P}', \mathscr{R}')$ of $\pi_x : \bigcup \{P \in \mathscr{P}_k \mid \dim \pi_x(P) < k - 1\} \to \bigcup \{R \in \mathscr{R}_k \mid \dim R < k - 1\}$ such that $\mathscr{P}'$ is compatible with $\mathscr{P}_k$ and all $\bar{P}$ for $P \in \mathscr{P}_B$ and $\mathscr{R}'$ is compatible with $\mathscr{P}_k$ and all $\bar{R}$ for $R \in \mathscr{R}_B$, for all $B \in \mathscr{R}_k$, $\dim B = k - 1$. Since $\bar{A}_1 \cap A_2 = \varnothing$ and $\bar{B}_1 \cap B_2 = \varnothing$ for $B_1, B_2 \in \mathscr{R}_k$, $\dim B_1 = k - 1$, $\dim B_2 \geqslant k - 1$, $A_1, A_2 \in \mathscr{P}_k$, $\dim \pi_M(A_1) = k - 1$ and $\dim \pi_M(A_2) \geqslant k - 1$, if we denote $\mathscr{P}_{k-1} = \{P \in \mathscr{P}_k \mid \dim \pi_M(P) \geqslant k\} \cup \bigcup \{\mathscr{P}_B \mid B \in \mathscr{R}_k, \dim B = k - 1\} \cup \mathscr{P}'$, $\mathscr{R}_{k-1} = \{R \in \mathscr{R}_k \mid \dim R \geqslant k\} \cup \bigcup \{\mathscr{R}_B \mid B \in \mathscr{R}_k, \dim B = k - 1\} \cup \mathscr{R}'$, then $(\mathscr{P}_{k-1}, \mathscr{R}_{k-1})$ will satisfy conditions 1, 2 as well as 3_{k-1}.

We will now return to the proof of the theorem.

From Lemma 1 and [7, Theorem 4] it follows that there exists a CASA stratification $\mathscr{P}_1$ of H, compatible with the partition $\{H', H''\}$ and such that the function $\sigma: H \to D(S)$ defined by $\sigma(x, \psi) = \inf\{t > 0, \Phi_t^i(x, \psi) \in D(S)\}$ is analytic on each stratum of $\mathscr{P}_1$. Further, there exists a CASA stratification $(\mathscr{P}_2, \mathscr{R}_2)$ of $\pi_x : H \to K$ such that $\mathscr{P}_2$ is compatible with $\mathscr{P}_1$.

It follows directly from the definitions of H, K and from E2 that for every $x \in K$ we have $\{\psi \mid (\psi, x) \in H\} \subset E^0(x)$ and that for $(\psi, x) \in H$ the control $u(t)$ defined by

$$\begin{aligned} u(t) &= w_i(\Phi^i{}_t(x, \psi)) \qquad \text{for} \quad 0 \leqslant t < \sigma(x, \psi) \\ &= u_y(t - \sigma(x, \psi)) \qquad \text{for} \quad \sigma(x, \psi) \leqslant t \leqslant \sigma(x, \psi) + T(y), \end{aligned} \tag{14}$$

where $y = \pi_x(\Phi^i_{\sigma(x,\psi)}(x, \psi))$, is an extremal control steering x to $\hat{x}$. From Assumption 3 it follows that $u(t) = u_x(t)$ for $0 \leqslant t \leqslant \sigma(x, \psi) + T(y)$, $\pi_x(\Phi^i{}_t(x, \psi)) = \xi_x(t)$ for $0 \leqslant t \leqslant \sigma(x, \psi)$, independently of ψ such that $(\psi, x) \in H$; also, it follows that $\mu(x) = i$, $\{\psi \mid (x, \psi) \in H\} = E^0(x)$ for $x \in K$ and that σ is independent of ψ on H. This means that the formula

$$v(x) = w_i(x, \psi) = w_{\mu(x)}(x, \psi) \qquad \text{for} \quad (x, \psi) \in H \tag{15}$$

defines uniquely a function v: $K \to U$.

If $R \in \mathscr{R}_2$, there exists a $P \in \mathscr{P}_2$ such that $\pi_x(P) = R$. Since rank $\pi_x |_P =$ rank R, for any point $(x, \psi) \in P$ there exists an analytic submanifold Q of P containing (x, ψ) such that $\pi_x |_Q : Q \to \pi_x(Q)$ is an isomorphism and $\pi_x(Q)$ contains a neighbourhood of x in R. We have $v(x) = (\pi_x |_Q)^{-1} w_i(x, \psi)$ in some neighbourhood of x in R from which it follows that v is analytic in this neighbourhood. Thus, v is analytic on each stratum of $\mathscr{R}_2$.

By Lemma 3 there exists a pair of CASA stratifications $(\mathscr{P}_3, \mathscr{R}_3)$ of H, K respectively such that $\mathscr{P}_3$ is compatible with $\mathscr{P}_2$ and Φ^i, $\mathscr{R}_3$ is compatible with $\mathscr{R}_2$ and (consequently), v is analytic on each stratum of $\mathscr{R}_3$. By Lemma 2, $(\mathscr{P}_3, \mathscr{R}_3)$ is a CASA quasistratification of $\pi_x : H \to K$.

To obtain the descendents of S associated with i we have to split $\mathscr{R}_3$ further in such a way that each member of the resulting partition of K has uniquely defined ancestors.

In order to do so we project the members of $\mathscr{P}_3$ on $D(S)$ along the trajectories of Φ^i, i.e., we consider the sets $\rho(P)$, $P \in \mathscr{P}_3$, where $\rho(x) = \Phi^i_{\sigma(x)}(x, \psi)$ (we have $\rho(x) \in D(S)$ because of I5 and $\xi_x(t) = \pi_x(\Phi^i{}_t(x, \psi))$ for $0 \leqslant t \leqslant \sigma(x)$. From the expression

$$\rho(P) = \pi_{x,\psi}\{(t, \Phi^i{}_t(t, x, \psi) \mid (x, \psi) \in P, 0 < t \leqslant \eta, \Phi^i{}_t(x, \psi) \\ \in D(S), \Phi^i{}_s(x, \psi) \notin D(S) \text{ for } 0 < s < t\},$$

where $\pi_{x,\psi}$ is the natural projection of the (t, x, ψ)-space on the (x, ψ)-space, it follows that the sets $\rho(P)$ are subanalytic. Therefore, there exists a CASA stratification $\mathscr{C}$ of $\rho(H)$ compatible with Φ^i and the family of the sets $\{\rho(P) \mid P \in \mathscr{P}_3\} \cup \{\bar{D} \cap D(S) \mid P \in \mathscr{P}\}$. Since no point of a parallel member of $\mathscr{C}$ can be the first point at which a trajectory of Φ^i starting in H meets $D(S)$, all the members of $\mathscr{C}$ have to be transversal and this will remain true for each stratification of $\rho(H)$ compatible with $\mathscr{C}$.

Denote ρ_1 the projection of K on S along the trajectories ξ, i.e. $\rho_1(x) = \pi_x(\rho(x, \psi)) = \xi_x(\sigma(x))$ for $(x, \psi) \in H$. By [7, Theorem 4] there exists a CASA stratification $(\mathscr{M}, \mathscr{N})$ of $\pi_x : \rho(H) \to \rho_1(K)$ such that $\mathscr{M}$ is compatible with $\mathscr{C}$. We now denote $\mathscr{P}, \mathscr{R}$ the partitions of H, K respectively, the members of which are connected components of the sets $P \cap \rho^{-1}(M)$, $P \in \mathscr{P}_3$, $M \in \mathscr{M}$ and $R \cap \rho_1^{-1}(N)$, $R \in \mathscr{R}_3$, $N \in \mathscr{N}$ respectively. Obviously, $\mathscr{P}$ is compatible with Φ^i. Further, we have $\rho(P_1) = \rho(P_2)$ if $\rho(P_1) \cap \rho(P_2) \neq \varnothing$, $P_1, P_2 \in \mathscr{P}$ and, similarly, $\rho_1(R_1) = \rho_1(R_2)$ if $\rho_1(R_1) \cap \rho_1(R_2) \neq \varnothing$, $R_1, R_2 \in \mathscr{R}$. Also, if $P \in \mathscr{P}$, $P \in P_3 \cap \rho^{-1}(M)$, then $\pi_x(P)$ is obviously contained in some connected component of $\pi_x(P_3) \cap \rho_1^{-1}(\pi_x(M))$.

It is obvious that the members of $\mathscr{P}, \mathscr{R}$ are subanalytic. We prove that they are analytic submanifolds and that for each $x \in R \in \mathscr{R}$ there exist $p \in P \in \mathscr{P}$ such that $x = \pi_x(p)$ and rank $\pi_x |_P (p) =$ rank R (consequently, $\pi_x(P)$ covers a neighbourhood of x in R.

First, take a $P \in \mathscr{P}$, $P \in P_3 \cap \rho^{-1}(M)$, $P_3 \in \mathscr{P}_3$, $M \in \mathscr{M}$ with P_3 transversal. Since $\mathscr{M}$ is compatible with $\mathscr{C}$ and $\{\rho(P) \mid P \in \mathscr{P}_3\}$, Φ_i is transversal also to M and we have $M \subset \rho(P_3)$. This means that for any $p \in P$, $q \in M$, $\rho \mid P_3$ has a local right inverse at p that maps q onto p and is analytic. Consequently, $\rho |_P$ is a local analytic isomorphism of P and M at p, which means that P has to be an analytic submanifold of P_3 and, thus, also of $\mathbf{R}^{2n+1}$.

If P_3 is parallel, for a given $p \in P$ we take an analytic transversal submanifold T of P_3 of codimension 1 in P_3 through p. The same argument as for P_3 trans versal proves that $T \cap P$ (and, consequently, also P) is an analytic submanifold.

Let now $x \in R \in \mathscr{R}$, $R = R_3 \cap \rho_1^{-1}(N)$, $R_3 \in \mathscr{R}_3$, $N \in \mathscr{N}$. Since $(\mathscr{P}_3, \mathscr{R}_3)$ is a quasistratification of $\pi_x : H \to K$, there exists a $P_3 \in \mathscr{P}_3$ such that $\pi_x |_{P_3} = R_3$

and rank $\pi_x|_{P_3} = \text{rank}\, R_3$. Therefore, there exists an analytic submanifold P' of P_3 such that $(\pi_x|_{P'})^{-1}$ is a local isomorphism of R_3 and P' at x. Furthermore, since the diagram

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \rho(H) \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi_x} \\ K & \xrightarrow[\rho_1]{} & \rho_1(K) \end{array} \tag{16}$$

commutes we see that $\varphi = \pi_x \circ \rho \circ (\pi_x|_{P'})^{-1}$ is an analytic map of a neighbourhood of x in R_3 on a neighbourhood of $\rho_1(x)$ in $\rho_1(R_3)$. On the other hand, since $(\mathcal{M}, \mathcal{N})$ is a stratification of $\pi_x : \rho(H) \to \rho_1(K)$, there exists a submanifold M' of some $M \in \mathcal{M}$, $M \subset \rho_3(P_3)$ such that $\pi_x|_{M'}$ is a local isomorphism of M' and $\rho_1(R) = N$ at $\varphi(x)$.

Assume first that P_3 is transversal. Since both P_3 and $\rho(P_3)$ are transversal, $\rho^{-1}(M') \cap P_3$ is isomorphic to M' and from the commutativity of (16) it follows that $\pi_x \circ \rho^{-1} \circ (\pi_x|_{M'})^{-1}$ is locally at $\varphi(x)$ analytic and equal to φ^{-1}. This means that φ is a local isomorphism of R and N at x which proves that R is an analytic submanifold of R_3 and, thus, also of $\mathbf{R}^n$.

If P_3 is parallel then $f(x, v(x))$ is obviously parallel to R_3 for $x \in R_3$. We consider an analytic submanifold R_3' of R_3 of codimension 1 in R_3 through x which is transversal to $f(x, v(x))$ for $x \in R_3'$. As in the case of P_3 transversal we can prove that $R \cap R_3'$ is a submanifold of R_3' from which it immediately follows that R is a submanifold of R_3 and, thus, also of $\mathbf{R}^n$.

Keeping the meaning of x, R, P_3, R_3 let now $P \in \mathcal{P}$ be such that $\dim(P \cap \pi_x^{-1}(x)) = \dim P_3 - \dim R_3$. Such a member exists since $\mathcal{P}$ is locally finite and $\pi_x^{-1}(x) \cap P_3$ is subanalytic. Then, there has to exist a $p \in P$ such that $\dim(T_pP \cap T_p(\pi_x^{-1}(x) \cap P_3)) = \dim P_3 - \dim R_3$. By the dimension argument used in the proof of Lemma 2 we obtain rank $\pi_x|_P(p) = \dim R$.

The partition $\mathcal{R}$ is the final partition of K, the members of which we take as the descendants of S associated with i. Those of the members which are subsets of K' we take as border cells. We now show that by this construction the induction hypotheses remain valid for $k + 1$, with Π, Σ properly extended.

In order to distinguish the induction hypotheses from the statements to be proved we shall label them according to the order in which they are considered ($\mathrm{I1}_k$, $\mathrm{I1}_{k+1}$, etc.). Further, henceforth we shall denote K, H, $\mathcal{P}$, $\mathcal{R}$, etc., the unions over all admissible pairs (S, i) of the sets K, H, $\mathcal{P}$, $\mathcal{R}$, etc., respectively which we have clnsidered for a particular admissible pair (S, i). By $K(S, i)$, etc., we shall denote that part of them which consists of descendants of $S \in \mathcal{S}_k$ associated with i. By ρ, ρ_1 we denote the maps $H \to \bigcup \{D(S) \mid S \in \mathcal{S}_k\}$, $K \to G_k$ respectively, the restriction of which to $H(S, i)$, $K(S, i)$ we have formerly denoted by the same symbols.

To establish $\mathrm{I1}_{k+1}$ we have to prove that two cells of $\mathcal{S}_{k+1}$ cannot intersect.

Let $S_1, S_2 \in \mathscr{S}_{k+1}$, $x \in S_1 \cap S_2$. If $S_1 \in \mathscr{S}_k$, $S_2 \in \mathscr{S}_{k+1}\backslash\mathscr{S}_k$ then from Assumption 3 it follows $\mu(S_1) = \mu(S_2)$. Let $S_2 \in \mathscr{R}(S, i)$. Then, since $x \in S_1$, S_1 is a descendant of S of order k and by $I9_k$, $\mu(S_1) = \mu(S) = \mu(S_2) = i$. Since (S, i) is admissible, S is a border cell, which violates $I8_k$.

So, $S_1, S_2 \in \mathscr{S}_{k+1}\backslash\mathscr{S}_k$. Since the members of $\mathscr{R}(S, i)$ do not intersect for any admissible pair (S, i), if $S_1 \cap S_2 \neq \varnothing$, either $\mu(S_1) \neq \mu(S_2)$ or S_1, S_2 are descendants of different cells of $\mathscr{S}_k$. Both possibilities obviously contradict Assumption 3. This proves $I1_{k+1}$.

By E2 we have for each $S \in \mathscr{S}_{k+1}\backslash\mathscr{S}_k$

$$D(S) = H \cap \pi_x^{-1}(S)$$

from which $I2_{k+1}$ immediately follows.

For the proof of I_{3k+1} assume $S \in \mathscr{R}(S', i)$. Then, it follows directly from the construction that $\mu(S) = i$ and $D(S) \subset N_i$. The analyticity of v on S follows from the compatibility of $\mathscr{R}$ with $\mathscr{R}_2$.

Since D is trivially satisfied by Assumuption 5, in order to establish $I4_{k+1}$ we have to prove that for Π, Σ properly extended to $\mathscr{S}_{k+1}$, B, C are satisfied over $\mathscr{S}_{k+1}$ as well.

First we consider property C. Since $\mathscr{S}_k$ has property C and for each $x \in K(S, i)$, ξ_x can pass from one cell of $\mathscr{S}_{k+1}\backslash\mathscr{S}_k$ to another only if $\Phi_t{}^i(x, \psi)$ for $(x, \psi) \in H$ passes from one member of $\mathscr{P}(S, i)$ to another, it suffices to prove that the latter can happen at most finitely many times. Since $\mathscr{P}(S, i)$ is finite, if this were not the case, there would exist a $P \in \mathscr{P}(S, i)$ such that $P \cap \Phi^i_{[0,\sigma(x))}(x, \psi)$ would have an infinite number of connected components. This, however, is impossible since $P \cap \Phi^i_{[0,\sigma(x))}(x, \psi)$ as an intersection of two subanalytic sets is subanalytic and therefore has finitely many connected components.

To verify B we note first that by $I3_{k+1}$, which we have already established, v is analytic on every cell. As the cells of type I we take those members of $\mathscr{R}$ which contain the projection of at least one parallel member of $\mathscr{P}$, the remaining cells to become cells of type II. It is obvious that for S_1 of type I every $x \in S_1$ is covered by a π_x-projection of some parallel member of $\mathscr{P}$ and that $f(x, v(x)) \in T_x S_1$ for each x S_1.

Let $S_1 \in \mathscr{R}(S, i)$. We denote by $\Pi(S_1) = \Pi(S_1, x)$ that member of $\mathscr{S}_{k+1}$ which is met first by $\xi_x(t)$ after leaving S_1, for $x \in S_1$. For a fixed x this member is well defined because of C, which we have already verified. In order to prove that the definition of $\Pi(S_1)$ is independent of the choice of $x \in S_1$ it obviously suffices to prove that for any $x \in S_1$, $\Pi(S_1, y)$ is independent of y from some neighbourhood of x in S_1.

To prove this we take a $p \in P \in \mathscr{P}$ such that $x = \pi_x(p)$ and rank $\pi_x \mid_P(p) =$ rank S_1. Obviously, P has to be parallel. We denote by P' the first member of $\mathscr{P} \cup \mathscr{M}$ met by $\Phi_t{}^i(p)$ for $t > 0$ the π_x-projection of which lies outside S_1. P' is well defined since, as we have shown when verifying C, $\Phi_t{}^i(p)$ can pass from one member of $\mathscr{P}$ to another at most finitely many times before reaching

$D(S)$. We prove that the last member P'' of $\mathscr{P}$, which $\Phi_t^i(p)$ meets before meeting P', is parallel from which it immediately follows that P' is transversal.

Assume the contrary. From the transversality of P'' it follows that there exists a $t_1 > 0$ such that $\Phi_{t_1}^i(p) \in P''$ and $\Phi_t^i(p) \in P'$ for $t > t_1$ near t. Denote $x' = \pi_x(\Phi_{t_1}^i(p))$. There exist a $Q \in \mathscr{P}$ parallel and a point $q \in Q$ such that $\pi_x(q) = x'$. Since Q is parallel, we have $\xi_{x'}(t) = \pi_x(\Phi_t^i(q)) \in S_1$ for $t > 0$ small. On the other hand, we have also $\xi_{x'}(t) = \pi_x(\Phi_{t_1+t}^i(p)) \notin S_1$ for $t > 0$, which is a contradiction.

Now let τ_0 be such that $\Phi_{\tau_0}^i(p) \in P'$, $\pi_x(\Phi_t^i(p)) \in S_1$ for $t < \tau_0$. From the transversality of P' it follows that there exists an analytic function τ defined on some neighbourhood W of p in P such that $\tau(p) = \tau_0$ and $\Phi_{\tau(q)}^i(q) \in P'$ for $q \in W$. Since rank $\pi_x|_P(p) =$ rank S_1, $\pi_x(W)$ is a neighbourhood of x. For every $q \in W$ we have $\xi_{\pi_x(q)}(\tau(q)) \in \pi_x(P')$. Since $\pi_x(P') \in \Pi(S_1, x)$, we have $\xi_y(\tau(y)) \in \Pi(S_1, x)$ for every $y \in \pi_x(W)$, where $\tau(y) = \tau(q)$ for $q \in W \cap \pi_x^{-1}(y)$. This means $\Pi(S_1, y) = \Pi(S_1, x)$ provided we prove that $\xi_y(t)$ does not meet any $R \in \mathscr{R}$, $R \neq S_1$ for $0 < t < \tau(y)$. We prove that there exists a neighbourhood $W_0 \subset \pi_x(W)$ of x in S_1 such that the latter is impossible for $y \in W_0$.

Indeed, were this not the case, there would exist a $Q \in \mathscr{P}$ transversal, sequences $p_\nu \to p, p_\nu \in P, t_\nu \to t^* \in [0, \tau(p)]$ such that $q_\nu = \Phi_{t_\nu}^i(p_\nu) \in Q, q = \Phi_{t^*}^i(p) \in \bar{Q} \backslash Q$. Let $Q \subset P_3 \in \mathscr{P}_3$. From the construction of P from $\mathscr{P}_3$ and from $p_\nu, p \in S_1$ it follows $q \in \bar{P}_3 \backslash P_3$. Since $\mathscr{P}_3$ is a stratification of H, either q belongs to some member of $\mathscr{P}_3$ of dimension $<$ dim P_3 or it belongs to $\bar{P}_3 \cap D(S)$ the dimension of which is also $<$ dim P_3. Since $\mathscr{M}(S, i)$ is a stratification compatible with the sets $\rho(P_3)$ and $\bar{P}_3 \cap S$, $P_3 \in \mathscr{P}_3$, in both cases $\rho(q)$ cannot belong to the same member of $\mathscr{M}$ as $\rho(q_\nu)$, so p_ν cannot belong to the same member of $\mathscr{P}$ as p which contradicts $p_\nu \in P$.

Now, let S_2 be of type II, $x \in S_2$, $x = \pi_x(p)$, $p \in P \in \mathscr{P}_3$, rank $\pi_x \mid_P (p) =$ rank S_2. Then, P is necessarily transversal, so $\Phi_t^i(p) \in P'$ for $t > 0$ sufficiently small for some P' parallel. Since P' is parallel, $\pi_x(P') \cap S_2 = \varnothing$ and, consequently, $\xi_x(t) \in \pi_x(P') \not\subset S_2$ for $t > 0$ small. We denote by $\Sigma(S_2) = \Sigma(S_2, x)$ that member of $\mathscr{R}$ which contains $\pi_x(P')$. Again, in order to prove that $\Sigma(S_2)$ is well defined, it suffices to prove that for a given $x \in S_2$, $\Sigma(S_2, y)$ does not depend on y for y from some neighbourhood of x in S_2. However, were this not the case, it would be easy to conclude that there would exist a $Q \in \mathscr{P}$ transversal such that $p \in \bar{Q} \backslash Q$ and the contradiction could be obtained in a manner similar to that in the case of Π.

Next we show that the regularity requirements of B are satisfied.

Take a cell $S_1 \in \mathscr{R}(S, i)$ of type I (the modifications for a cell of type II are obvious) and any $x \in S_1$. Since there exist $p \in P \in \mathscr{P}$ such that $x = \pi_x(p)$ and rank $\pi_x|_P(p) =$ rank S_1, as we have mentioned above, there exists a neighbourhood W of x in S_1 and an analytic submanifold M of P such that $\pi_x|_M : M \to W$ is an isomorphism. Thus, for $y \in W$ there exists a unique ψ_y depending analytically on y such that $(y, \psi_y) \in M$.

We have

$$\begin{aligned} \xi_y(t) &= \pi_x(\Phi_t^i(y, \psi_y)) \qquad \text{for} \quad t \in [0, \sigma(y)] \\ u_y(t) &= w_i(\Phi_t^i(y, \psi_y)). \end{aligned} \tag{17}$$

Since $\Phi_t^i(y)$ can be extended beyond $\sigma(y)$, the required analytic dependence of ξ_y and u_y on y and t follows immediately from (17). The analyticity of τ we have in fact proved when verifying that Π is well defined.

To verify $I5_{k+1}$ assume $x \notin G_{k+1}$ and denote $t_0 = \inf\{t \mid \xi_x(t) \in G_{k+1}\}$. We have to prove $\xi_x(t_0) \in G_{k+1}$. If $\xi_x(t) \in G_k$ for $t > t_0$ close to t_0 this follows from $I5_k$ so we assume $\xi_x(t) \in G_{k+1} \backslash G_k$ for $t > t_0$ near t_0. Then, there exists a cell $S \in \mathscr{S}_{k+1} \backslash \mathscr{S}_k$ such that $\xi_x(t) \in S$ and $\mu(\xi_x(t)) = \mu(S)$ for $t > t_0$ near t_0. Since $N_{\mu(S)}$ is closed, we have $\mu(\xi_x(t_0)) = \mu(S)$ and from the construction of the cells of order $k + 1$ it follows immediately that $\xi_x(t_0) \in G_{k+1}$.

Because of $I5_k$ we may, without loss of generality, assume for the proof of $I6_{k+1}$ that $x' = \xi_x(t_1) \in S \subset G_k$ but $\xi_x(t) \notin G_k$ for $t < t_1$. From the assumption and from E2 it follows that $\Phi_t^i(x, \psi) \in N_i$, $\xi_x(t) = \pi_x(\Phi_t^i(x, \psi))$ for $0 \leqslant t < t_1$ and $\Phi_{t_1}^i(x, \psi) \in D(S)$, provided $\psi \in E^0(x)$. Since $t_1 \leqslant \eta$ this means $\xi_x(t) \in K(S, i)$ for $t \in [0, t_1)$. Because of $I7_k$ either $i \neq \mu(S)$ or $S \in \mathscr{S}'_k$ which means that (S, i) is an admissible pair and therefore $K(S, i) \subset G_{k+1}$.

For the proof of $I7_{k+1}$ assume that $S \in \mathscr{S}_{k+1} \backslash \mathscr{S}'_{k+1}$, $S \in \mathscr{R}(S', i)$, (S', i) admissible, $S' \in \mathscr{S}_k$. Then, $\xi_x(s) \in K(S', i)$, $\mu(\xi_x(s)) = i$ for $t \leqslant s < t + \sigma$, $\xi_x(t + \sigma) \in S'$. From E2 it follows that $(x', \psi') \in D(S')$ for $\psi \in \Psi_x$ where $x' = \xi_x(t + \sigma)$, $\psi' = |\psi(t + \sigma)|^{-1} \psi(t + \sigma)$. Since $\mu(\xi_x(s)) = i$ for $t - \delta < s \leqslant t + \sigma$ for some $\delta > 0$ we have $\Phi^i_{t+\sigma-s}(x', \psi') = (\xi_x(s), |\psi(s)|^{-1} \psi(s)) \in N_i$ for $t - \delta_1 < s < t + \sigma$ and, consequently, $\xi_x(s) = \pi_x(\Phi^i_{t+\sigma-s}(x', \psi')) \in K(S', i) \subset G_{k+1}$ for $t - \delta_1 < s \leqslant t + \sigma$, where $\delta_1 = \min\{\delta, \eta - \sigma\}$.

The properties $I8_{k+1}$, $I9_{k+1}$ follow immediately from the construction of the set K and the unicity of extremals. This completes the proof of the induction step.

By induction it follows that $\mathscr{S} = \bigcup_{k \geqslant 0} \mathscr{S}_k$ is a regular synthesis provided we prove that every $C \subset G$ compact is covered by a finite number of cells of $\mathscr{S}$. This follows immediately from the following

LEMMA 5. *Let $x \in G$. Then, x is contained in a cell of order $\leqslant \eta^{-1}T(x) + \vartheta(x)$ where ϑ is the number of switchings of the extremal control u_x.*

Proof. We prove this lemma by induction in the number $\eta^{-1}T(x) + \vartheta(x)$. If $T(x) = 0$, the statement is trivial. Assume that it holds for all $x \in G$ such that $\eta^{-1}T(x) + \vartheta(x) \leqslant k$. Let $x \in G$ be such that $\eta^{-1}T(x) + \vartheta(x) \leqslant k + 1$. Let t_1 be the first switching point of u_x, $\tau = \min\{\eta, t_1\}$. Denote $x' = \xi_x(\tau)$. Then, $\eta^{-1}T(\xi_x(\tau)) + \vartheta(\xi_x(\tau)) \leqslant k$, so $\xi_x(\tau) \in G_k$. By I6, $x \in G_{k+1}$, which proves the lemma.

Now, take a compact $C \subset G$. By Assumptions 4 and 6 we have $T(x) \leqslant \gamma(C)$, $\vartheta(x) \leqslant \nu(C)$, so $\eta^{-1}T(x) + \vartheta(x) \leqslant \eta^{-1}\gamma(C) + \nu(C)$ for any $x \in C$. By Lemma 5, x belongs to a cell of order $\leqslant \eta^{-1}\gamma(C) + \nu(C)$ which means that C is covered by the cells of order $\leqslant \eta^{-1}\gamma(C) + \nu(C)$ the number of which is finite. This completes the proof of the theorem.

Appendix

In this appendix we transcribe Bolt'anski's proof of the optimality of the controls generated by a regular synthesis to the case of a general performance index and the modified definition of synthesis.

We consider the system

$$\dot{x} = f(x, u)$$

$x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ with the performance index

$$J = \int_0^T f^0(x, u)\, dt,$$

the control domain $U \subset \mathbf{R}^m$ and the target point $\hat{x}$. By $J(x_0, u)$ we denote the value of the performance index for the control u and the initial state x_0. We assume that f and f^0 are C^1.

Lemma A1. *Let $G \subset \mathbf{R}^n$ be open, $\hat{x} \in \bar{G}$, and let $V\colon G \cup \{\hat{x}\} \to \mathbf{R}^1$ be continuous in $G \cup \{\hat{x}\}$, C^1* in *G and satisfy for every $x \in G$, $u \in U$ the inequality*

$$f^0(x, u) + \frac{\partial V}{\partial x}(x) f(x, u) \geqslant 0. \tag{A1}$$

Let $u(t)$, $t \in [0, T]$ be a control steering the point $x_0 \in G$ to $\hat{x}$ such that its response $x(t)$ starting at x_0 satisfies $x(t) \in G$ for each $t \in (0, T)$. Then,

$$J(x_0, u) \geqslant V(x_0) - V(\hat{x}) \tag{A2}$$

Proof. We have

$$J(x_0, u) = \int_0^T f^0(x(t), u(t))\, dt \geqslant -\int_0^T \frac{\partial V}{\partial x}(x(t)) f(x(t), u(t))\, dt$$
$$= -\int_0^T \frac{d}{dt}(V(x(t)))\, dt = V(x_0) - V(\hat{x}).$$

Lemma A2. *Let $G \subset \mathbf{R}^n$ be open and let $M \subset G$. Let $V\colon G \cup \{\hat{x}\} \to \mathbf{R}^1$ be continuous on $G \cup \{\hat{x}\}$, C^1 on $G \backslash M$ and satisfy* (1) *for every $u \in U$ and $x \in G \backslash M$. Let $u(t)$, $t \in [0, T]$ be a control steering the system from x_0 to $\hat{x}$ such that its response*

$x(t)$ has a finite number of intersections with M and satisfies $x(t) \in G$ for $t \in [0, T)$. Then, (A2) *holds.*

Proof. Let $0 < \vartheta_1 < \cdots < \vartheta_{s-1} < T$ be all the moments of intersection of $x(t)$ with M, $\vartheta_0 = 0$, $\vartheta_s = T$. Then, we can apply Lemma 1 to the restrictions of $u(t)$ to the intervals $[\vartheta_{i-1}, \vartheta_i]$, $i = 1, \ldots, s$ to obtain

$$-V(x(\vartheta_i)) + V(x(\vartheta_{i-1})) \leqslant \int_{\vartheta_{i-1}}^{\vartheta_i} f^0(x(t), u(t))\, dt,$$

$i = 1, \ldots, s$. Adding these inequalities we obtain (A2).

LEMMA A3. *Let G, V be as in Lemma* A2. *Let $u(t)$, $t \in [0, T]$ steer the system from x to $\hat{x}$, its response $x(t)$ satisfying $x(t) \in G$ for each $t \in [0, T)$. Assume that in any neighbourhood of x_0 there exists an $y_0 \in G$ such that the response of $u(t)$ starting at y_0 meets M at finitely many points on* $[0, T]$. *Then,* (A2) *holds.*

Proof. Let $\epsilon > 0$. There exist neighbourhoods W_0, W_1 of x_0, $\hat{x}$ respectively such that $|V(x) - V(x_0)| < \epsilon$, $|V(x) - V(\hat{x})| < \epsilon$ for $x \in W_0$, $x \in W_1 \cap G$ respectively and a $\delta > 0$ such that $x(T - \delta) \in W_1 \cap G$ and

$$\int_{T-\delta}^{T} f^0(x(t), u(t))\, dt > -\epsilon. \tag{A3}$$

There exists a neighbourhood $W_0' \subset W_0$ of x_0 such that the response $y(t)$ of $u(t)$ starting at an arbitrary $y_0 \in W_0'$ satisfies $y(T - \delta) \in W_1 \cap G$, $y(t) \in G$ for $t \in [0, T - \delta]$ and

$$-\int_0^{T-\delta} f^0(y(t), u(t))\, dt + \int_0^{T-\delta} f^0(x(t), u(t))\, dt > -\epsilon. \tag{A4}$$

By the assumption of the lemma there exists a $y_0 \in W_0'$ such that $y(t)$ meets M at at most finitely many points. According to Lemma A2 we have

$$-V(y(T - \delta)) + V(y_0) \leqslant \int_0^{T-\delta} f^0(y(t), u(t))\, dt. \tag{A5}$$

Further, since $y_0 \in W_0'$, we have $y(T - \delta) \in W_1 \cap G$, so

$$-V(y_0) + V(x_0) < \epsilon \tag{A6}$$

$$-V(\hat{x}) + V(y(T - \delta)) < \epsilon. \tag{A7}$$

Adding (A3)–(A6) we obtain

$$-V(x_1) + V(x_0) - 2\epsilon \leqslant \int_0^T f^0(x(t), u(t))\, dt + 2\epsilon.$$

Since $\epsilon > 0$ may be taken arbitrarily small, we have (A2).

LEMMA A4. *Let $G \subset \mathbf{R}^n$ be open and let M be a stratified subset of G of dimension $<n$. Let $u(t)$, $t \in [0, T]$ be a control such that its response $x(t, x_0, u)$ starting at x_0 lies entirely in G. Then, in every neighbourhood of x_0, there exists a y_0 such that $x(t, y_0, u)$ has finitely many intersections with M.*

This lemma is proved in [3] (cf. also Erratum to [3]). As a corollary of Lemmas A3, A4 we obtain

THEOREM A1. *Let $G \subset \mathbf{R}^n$ be open and let M be a closed stratified subset of G of dimension $<n$. Let $\hat{x} \in \bar{G}$, $V: G \cup \{\hat{x}\} \to \mathbf{R}^1$ be continuous on $G \cup \{\hat{x}\}$ and C^1 on $G \backslash M$, $V(\hat{x}) = 0$. For each $x \in G$ let there exist a control $u_x(t)$, $t \in [0, T(x)]$ steering the system from x to $\hat{x}$ such that $x(t, x, u_x) \in G$ for $t \in [0, T(x))$ and $J(x, u_x) = V(x)$. Then, in order that all u_x be optimal for the initial state x, it is necessary and sufficient that V satisfy* (A1) *in $G \backslash M$.*

Proof. The sufficiency follows immediately from Lemmas A3, A4. To prove necessity, assume that there exists a point $x_0 \in G \backslash M$ and $u^* \in U$ such that

$$f^0(x_0, u^*) + \frac{\partial V}{\partial x}(x_0) f(x_0, u^*) < 0. \tag{A8}$$

Since $G \backslash M$ is open, there exists an $\epsilon > 0$ such that $x(t, x_0, u^*) \in G$ for $t \in [0, \epsilon)$ (here by u^* we understand the constant control with value u^*) and that (A8) is satisfied with x_0 replaced by $x(t, x_0, u^*)$, $t \in [0, \epsilon)$. We concatenate the constant control u^* on $[0, \epsilon)$ with the control $u_{x(\epsilon, x_0, u^*)}$ and keep the notation u^* for this control. Its response $x(t, x_0, u^*)$ we denote by $x^*(t)$. Then we have from (A8) for $t \in [0, \epsilon)$

$$f^0(x^*(t), u^*) < -\frac{\partial V}{\partial x}(x^*(t)) f(x(t), u^*) = -\frac{\partial V}{dt}(x^*(t))$$

from which we obtain

$$-V(x^*(\epsilon)) + V(x_0) > \int_0^\epsilon f^0(x^*(s), u^*(s))\, ds. \tag{A9}$$

If we denote by $T = T(x^*(\epsilon))$, we have from (A9)

$$\begin{aligned} J(x_0, u^*) &= \int_0^{T+\epsilon} f^0(x^*(t), u^*)\, dt = \int_0^\epsilon f^0(x^*(t), u^*)\, dt \\ &\quad + \int_0^T f^0(x(t, x^*(\epsilon), u_{x^*(\epsilon)})\, dt < V(x_0) - V(x^*(\epsilon)) \\ &\quad + V(x^*(\epsilon)) - V(\hat{x}) = V(x_0) = J(x_0, u_{x_0}), \end{aligned}$$

which means that u_{x_0} is not optimal.

Now we use Theorem A1 to prove

THEOREM A2. *Let $(\mathscr{S}, v)$ be a regular synthesis in an open domain G, $\hat{x} \in \bar{G}$. Then, for every $x \in G$ the control $u_x(t)$, generated by the closed-loop control v via the equation*

$$u_x(t) = v(\xi_x(t)) \tag{A10}$$

is the optimal control for the initial state x.

Proof. Denote

$$V(x) = \int_0^{T(x)} f(\xi_x(t), u_x(t))\, dt = J(x, u_x).$$

By D, V is continuous in G. By Theorem A1 and Assumption A it remains to prove that V is C^1 in $G \backslash G'$ and satisfies (A1) there.

Obviously, v is C^1 on $G \backslash G'$ if it is C^1 on each of the cells of dimension n. By induction in the number of the passings of ξ_x from one cell to another we prove that this is true for all cells. First we note that it follows from Assumption B that given a cell $S \in \mathscr{S}$ there is a unique sequence of cell $S_1, \ldots, S_q$, $q = q(S)$ such that every trajectory ξ_x for $x \in S$ passes precisely the cells $S_1, \ldots, S_q$ (in this order) until it reaches $\hat{x}$. The statement is satisfied trivially for $q = 0$, since the only cell with $q = 0$ is $\{\hat{x}\}$. Assume that V is C^1 on each cell with $q \leqslant k$. Then, the C^1 dependence of V on x on each cell with $q = k + 1$ follows immediately from the expression

$$V(x) = \int_0^{\tau(x)} f(\xi_x(t), u_x(t))\, dt + V(\xi_x(\tau(x)))$$

(τ defined as in Assumption B) by B and the induction assumption.

Assume now that $x \in S$, $\dim S = n$ and that $S_1, \ldots, S_r$ are the cells of type I passed by ξ_x in this order before reaching $\hat{x}$. Denote by $\tau_i(x)$ the times for which ξ_x enters S_i, $i = 1, \ldots, r$. It follows from B that τ_i are well defined and C^1 on S and so are $\xi_x(\tau_i(x))$.

Since the controls u_x satisfy Pontrjagin's maximum principle, we have

$$\langle \psi, \tilde{f}(x, u) \rangle \leqslant \langle \psi, \tilde{f}(x, v(x)) \rangle = 0 \tag{A11}$$

for each $\psi \in E(x)$ and $u \in U$. The proof will be finished if we prove

$$\operatorname{col}\left(-1, -\frac{\partial V}{\partial x}(x)\right) \in E(x) \tag{A12}$$

since then (A11) means (A1).

We prove that if $\psi \in E(x)$ then $\psi_0 \neq 0$ and for every $h \in R^n$

$$\psi_0 \frac{d}{d\epsilon} V(x + \epsilon h)|_{\epsilon=0} = \langle \psi', h \rangle,$$

where $\psi = (\psi_0, \psi_1, ..., \psi_n)$, $\psi' = (\psi_1, ..., \psi_n)$. We have then

$$\frac{d}{d\epsilon} V(x + \epsilon h)|_{\epsilon=0} = \langle \psi_0^{-1}\psi', h \rangle$$

for every $h \in R^n$, which implies $(\partial V/\partial x(x))^* = \psi_0^{-1}\psi'$. Since by the maximum principle $\psi_0 < 0$, it follows from E1

$$\mathrm{col}\left(-1, -\frac{\partial V}{\partial x}(x)\right) = \mathrm{col}(-1, -\psi_0^{-1}\psi') = -\psi_0^{-1}\psi \in E(x),$$

which proves (A12).

Denote $\tau_i^\epsilon = \tau_i(x + \epsilon h)$ for $i = 1,..., r$. For $\epsilon > 0$ sufficiently small $x + \epsilon h \in S$ and $\max\{\tau_i^\epsilon, \tau_i^0\} < \min\{\tau_{i+1}^\epsilon, \tau_{i+1}^0\}$ for $i = 0,..., r$, where $\tau_0^\epsilon = 0$, $\tau_{r+1}^\epsilon = T(x + \epsilon h)$. For such ϵ we define $\Delta_i = [\max\{\tau_i^\epsilon, \tau_i^0\}, \min\{\tau_{i+1}^\epsilon, \tau_{i+1}^0\}]$, $i = 0,..., r$, $\Delta_i' = [\min\{\tau_i^\epsilon, \tau_i^0\}, \max\{\tau_i^\epsilon, \tau_i^0\}]$, $i = 0,..., r + 1$. By induction we obtain from the C^1 dependence assumptions in B that $\xi_{x+\epsilon h}(t)$, $u_{x+\epsilon h}(t)$, τ_i^ϵ, $i = 0,..., r + 1$, are C^1 functions of ϵ, uniformly in t. Consequently, if we denote $x^\epsilon = \xi_{x+\epsilon h}$, $u^\epsilon = u_{x+\epsilon h}$, $T^\epsilon = T(x + \epsilon h)$, we have $| \tilde{x}^\epsilon(t) - \tilde{x}^0(t)| = O(\epsilon)$, $| u^\epsilon(t) - u^0(t)| = O(\epsilon)$, $| T^\epsilon - T^0 | = O(\epsilon)$, $| \tau_i^\epsilon - \tau_i^0 | = O(\epsilon)$, $i = 0,..., r + 1$, uniformly in t.

Let first $T^\epsilon \geqslant T^0$. Since $x^\epsilon(T^\epsilon) = x^0(T^0) = \hat{x}$ we have

$$\begin{aligned}
&\psi_0(V(x + \epsilon h) - V(x)) \\
&= \psi_0 \left[\int_0^{T^\epsilon} f^0(x^\epsilon(t), u^\epsilon(t))\, dt - \int_0^{T^0} f^0(x_0(t), u^0(t))\, dt\right] \\
&= \psi_0 \left[\int_0^{T} f^0(x^\epsilon(t), u^\epsilon(t))\, dt - \int_0^{T^0} f^0(x^0(t), u^0(t))\, dt\right] \\
&\quad + \langle \psi'(T^0), x^\epsilon(T^0) \rangle - \langle \psi', x + \epsilon h \rangle - (\langle \psi'(T^0), \hat{x} \rangle \\
&\quad - \langle \psi', x \rangle) + \langle \psi', \epsilon h \rangle + \langle \psi'(T^0), x^\epsilon(T^\epsilon) - x^\epsilon(T^0) \rangle \\
&= \int_0^{T^0} \left[\frac{d}{dt} \langle \psi(t), \tilde{x}^\epsilon(t) \rangle - \frac{d}{dt} \langle \psi(t), x^0(t) \rangle\right] dt \\
&\quad + \int_{T^0}^{T^\epsilon} \left\langle \psi(T^0), \frac{d\tilde{x}^\epsilon(t)}{dt} \right\rangle dt + \langle \psi', \epsilon h \rangle
\end{aligned}$$

$$= \int_0^{T^0} \left[\left\langle \psi(t), \frac{d\tilde{x}^\epsilon(t)}{dt} \right\rangle - \left\langle \psi(t), \frac{d\tilde{x}^0(t)}{dt} \right\rangle + \left\langle \frac{d\psi(t)}{dt}, \tilde{x}^\epsilon(t) - \tilde{x}^0(t) \right\rangle\right] dt$$

$$+ \int_{T^0}^{T^\epsilon} \langle \psi(T^0), \tilde{f}(x^\epsilon(t), u^\epsilon(t))\rangle \, dt + \langle \psi, \epsilon h\rangle$$

$$= \int_0^{T^0} [H(x^\epsilon(t), u^\epsilon(t), \psi(t)) - H(x_0(t), u^0(t), \psi(t))] \, dt$$

$$- \int_0^{T^0} \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t))(\tilde{x}^\epsilon(t) - \tilde{x}^0(t)) \, dt$$

$$+ \int_{T^0}^{T^\epsilon} \langle \psi(T^0), \tilde{f}(x^\epsilon(t), u^\epsilon(t))\rangle \, dt + \langle \psi', \epsilon h\rangle = F(\epsilon) + \langle \psi', \epsilon h\rangle.$$

From the maximum principle it follows that

$$H(x^0(t), u^\epsilon(t), \psi(t)) - H(x^0(t), u^0(t), \psi(t)) \leqslant 0 \qquad \text{for} \quad t \in [0, T^0],$$

$$H(x^0(T^0), u^\epsilon(T^0), \psi(T^0)) = \langle \psi(T^0), \tilde{f}(x^0(T^0), u^\epsilon(T_0))\rangle \leqslant 0,$$

from which we obtain

$$\begin{aligned} g(t, \epsilon) &= H(x^\epsilon(t), u^\epsilon(t), \psi(t)) - H(x^0(t), u^0(t), \psi(t)) \\ &\quad - \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t))(\tilde{x}^\epsilon(t) - \tilde{x}^0(t)) \\ &= H(x^0(t), u^\epsilon(t), \psi(t)) - H(x^0(t), u^0(t), \psi(t)) \\ &\quad + H(x^\epsilon(t), u^\epsilon(t), \psi(t)) - H(x^0(t), u^\epsilon(t), \psi(t)) \\ &\quad - \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t))(\tilde{x}^\epsilon(t) - \tilde{x}^0(t)) \\ &\leqslant \left(\frac{\partial H}{\partial \tilde{x}} (x^0(t), u^\epsilon(t), \psi(t)) - \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t))\right) \\ &\quad \times (\tilde{x}^\epsilon(t) - \tilde{x}^0(t)) + o(\epsilon), \end{aligned} \tag{A13}$$

$$\begin{aligned} \langle \psi(T^0), \tilde{f}(x^\epsilon(t), u^\epsilon(t))\rangle &= \langle \psi(T^0), \tilde{f}(x^0(T^0), u^\epsilon(T^0))\rangle \\ &\quad + \langle \psi(T^0), \tilde{f}(x^\epsilon(T^0, u^\epsilon(T^0)) - \tilde{f}(x^0(T^0), u^\epsilon(T^0))\rangle \\ &\quad + \langle \psi^\epsilon(T^0), \tilde{f}(x^\epsilon(t), u^\epsilon(t)) - \tilde{f}(x^\epsilon(T^0), u^\epsilon(T^0))\rangle \leqslant O(\epsilon) \end{aligned} \tag{A14}$$

provided ϵ is so small that $x^\epsilon(T^0) \in S_r$.

From (A14) we have

$$\int_{T^0}^{T^\epsilon} \langle \psi(T^0), \tilde{f}(x^\epsilon(t), u^\epsilon(t)) \rangle \, dt \leqslant o(\epsilon). \tag{A15}$$

For $t \in \Delta_i$, $i = 0,..., r$ we have

$$\left| \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^\epsilon(t), \psi(t)) - \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t)) \right| = O(\epsilon)$$

and, consequently, from (A13)

$$g(t, \epsilon) \leqslant o(\epsilon); \tag{A16}$$

for $t \in \Delta_i$ we have obviously

$$g(t, \epsilon) \leqslant O(\epsilon) \tag{A17}$$

From (A15)–(A17) we obtain

$$F(\epsilon) \leqslant \sum_{i=0}^{r} o(\epsilon) + \sum_{i=1}^{r+1} \int_{\Delta_i'} O(\epsilon) + o(\epsilon) = o(\epsilon)$$

from which we obtain

$$\psi_0(V(x + \epsilon h) - V(x) \leqslant \epsilon \langle \psi', h \rangle + o(\epsilon). \tag{A18}$$

Now, let $T^\epsilon \leqslant T^0$. Then, we obtain, as in the case $T^0 \leqslant T^\epsilon$,

$$\begin{aligned} &\psi_0(V(x + h) - V(x)) \\ &\quad = \int_0^{T^\epsilon} \Big[H(x^\epsilon(t), u^\epsilon(t), \psi(t)) \\ &\qquad - H(x^0(t), u^0(t), \psi(t)) - \frac{\partial H}{\partial \tilde{x}} (x^0(t), u^0(t), \psi(t))(x^\epsilon(t) - x^0(t)) \Big] \, dt \\ &\qquad \times \int_{T^\epsilon}^{T^0} \langle \psi(T^\epsilon), \tilde{f}(x^0(t), u^0(t)) \rangle \, dt \end{aligned}$$

and

$$\langle \psi(T^\epsilon), \tilde{f}(x^0(t), u^0(t)) \rangle \leqslant O(\epsilon);$$

so (A18) holds also in this case.

Therefore we have

$$\psi_0 \frac{d}{d\epsilon} V(x + \epsilon h)|_{\epsilon=0} \leqslant \langle \psi', h \rangle$$

for all $h \in \mathbf{R}^n$ and, consequently,

$$\psi_0 \frac{d}{d\epsilon} V(x + \epsilon h)|_{\epsilon=0} = \langle \psi', h \rangle$$

for all $h \in \mathbf{R}^n$.

If $\psi_0 = 0$, we have $\langle \psi', h \rangle = 0$ for all $h \in \mathbf{R}^n$, so $\psi = 0$ which violates $\psi \in E(x)$. Therefore, $\psi_0 \neq 0$, which completes the proof.

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