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Solving Systems of Polynomial Equations by Bounded and Real Homotopy

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Summary. The homotopy method for solving systems of polynomial equations proposed by Chow, Mallet-Paret and Yorke is modified in two ways. The first modification allows to keep the homotopy solution curves bounded, the second one to work with real polynomials when solving a system of real equations. For the first method numerical results are presented.

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1. Introduction

In [1] a method of computation of all solutions of a system of polynomial equations is proposed. The method of [1] belongs to the class of homotopy methods: it solves the problem by imbedding it into a family of problems depending on a parameter t which for some value t_0 of the parameter can be solved simply. A curve representing the solution as a function of the parameter emanates from the solution for t_0 and eventually leads to the solution of the original problem. Of course there are various reasons for which this procedure may fail. However, in [1] the authors prove that if the homotopy is chosen properly then all isolated zeros of the system of polynomials are obtained in this way – in some sense with probability one.

Let us now recall briefly the results of [1]: The polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ ($\mathbb{C}$ being the complex plane) defining the system of equations

$$P(z)=0 \tag{1}$$

to be solved is imbedded into a family of maps $H\colon \mathbb{C}^n \times [0;1] \times \mathbb{C}^n \times \mathbb{C}^{n^2} \to \mathbb{C}^n$ defined by

$$H_k(z,t,a,b)=tP_k(z)+(1-t)\,Q_k(z_k,b_k)+t(1-t)\,R_k(z,a), \tag{2}$$

where H_k, P_k, z_k, Q_k, R_k, b_k stand for the k-th component of H, P, z, Q, R, b, respectively, and

$$Q_k(z_k, b_k) = z_k^{d_k} - b_k,$$
$$R_k(z, a) = \sum_{j=1}^{n} a_{jk} z_j^{d_k},$$

where d_k stands for the degree of P_k and a_{jk} for the (j,k)-th component of a, respectively. It is proved in [1] that for almost all choices of b, a (either in the topological or measure-theoretical sense) there is a family of smooth disjoint curves emanating from the $D = \prod_{k=1}^{n} d_k$ zeros of the polynomial $H(\cdot, 0, a, b) = Q(z, b)$ which exist for $t \in [0, 1)$ and each isolated zero of P is the limit point of a curve of this family for t approaching 1.

In this paper we modify the homotopy method of [1] in two ways. Using the method of [1] one cannot exclude the possibility that some of the curves escape to infinity for t approaching 1. From the computational point of view it is difficult to distinguish an escaping path from a one ending in a distant root of the original system. This is the main reason why we have to follow such a path "far enough". However, following such a path far enough could be numerically difficult. The first modification of the homotopy method allows to keep the path in a ball centred at the origin with a radius prescribed in advance. This is achieved by working in $\mathbb{C}^{n+1}$ instead of $\mathbb{C}^n$. The degree argument of [1] is used to prove that each isolated zero of the extended zero set in the complex projective n-space $\mathbb{P}^n$ contains a limit point of at least one path generated by the method, i.e. as in [1] we "find" each isolated zero by our homotopy (the extended zero set is the set of zeros of P completed by its improper zeros).

In addition we prove that the method works for all b (with $b_k \neq 0$ for all $k = 1, \dots, n$) and almost all (a, r) (a, b, r-auxiliary parameters of our homotopy) in a stronger than in [1] sense: for a fixed b the set of "bad" (a, r) appears to be a stratified set of codimension greater or equal to 1. For this we employ some results of the theory of semialgebraic sets. The terminology and results needed for the understanding of the proofs are summarized in the Appendix.

Since [1] does not include any numerical examples we cannot make definite statements as to which of the methods is less time consuming or more reliable. Still we believe that the extra computations due to our modification are by far outweighted by the fact that all the paths are uniformly bounded on $[0; 1]$ by a bound we can prescribe in advance. The numerical experiments we have carried out by a program realizing our method (Sect. 4) indicate that it is reliable and fairly effective for small problems. The total computing time depends strongly on the size of the system and on the degree of the polynomials in the system being solved.

It can be easily shown (even in the 1-dimensional case) that the method of [1] does not work if one restricts the choice of the parameters a, b to the reals. Still it would be much more convenient if for P real one could design a method in which Q and R would be also real. For, in this case, one could decrease computational complexity by using real arithmetic in case one follows a real path and following only one half of the imaginary paths (since they occur in conjugate pairs). In Sect. 3, which deals with the second modification, we show that at least theoretically this is possible if one allows the paths to intersect for some $t \in (0;1]$. Again we believe that in most cases there are not too many intersections of paths and hence not too many computations are needed to go through them. So far the implementation of an algorithm based on these ideas has not been finished and hence no numerical results can be presented.

2. Bounded Homotopy

First we introduce some useful notations. For the k-th component of the given polynomial mapping P: $\mathbb{C}^n \to \mathbb{C}^n$ the notation

$$P_k(x) = \sum_{i=1}^{m_k} p_{ik} x^{c_{ik}}$$

will be used, where $p_{ik} \in \mathbb{C}$ ($k=1, \ldots, n$, $i=1, \ldots, m_k \geqq 1$) and $c_{ik} = (c_{ik}^1, \ldots, c_{ik}^n)$ is a multiindex with all c_{ik}^j being nonnegative integers. The symbol $x^{c_{ik}}$, where $x \in \mathbb{C}^n$, stands for the term

$$x_1^{c_{ik}^1} \cdot x_2^{c_{ik}^2} \ldots x_n^{c_{ik}^n}.$$

We denote the degree of the polynomial function P_k by d_k. It is obvious that $d_k = \max_{1 \leq i \leq m_k} |c_{ik}|$, where $|c_{ik}| = c_{ik}^1 + \ldots + c_{ik}^n$. We suppose here $d_k \geqq 1$ for each $k = 1, \ldots, n$. The n-vector $d = (d_1, \ldots, d_n)$ will be referred to as the degree of the mapping P. We introduce further two quantities which are connected with the degree of the mapping P, namely

$$D = \prod_{k=1}^{n} d_k \quad \text{and} \quad \mathscr{D} = \sum_{k=1}^{n} d_k.$$

Let us now recall that a scalar polynomial P_k: $\mathbb{C}^n \to \mathbb{C}^n$ is said to be homogeneous if $d_k = |c_{ik}|$ for each $i = 1, \ldots, m_k$. A polynomial mapping will be called homogeneous if each of its components is a homogeneous polynomial (we do not require that its components are of the same degree).

Given P: $\mathbb{C}^n \to \mathbb{C}^n$ polynomial, we associate with it two homogeneous polynomials $\overset{+}{P}$: $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ and $\bar{P}$: $\mathbb{C}^n \to \mathbb{C}^n$, which we shall call the (+)- and (−)-homogenization of P, respectively, as follows:

$$\overset{+}{P}_k(x, x_0) = \sum_{i=1}^{m_k} p_{ik} x^{c_{ik}} \cdot x_0^{d_k - |c_{ik}|},$$

$$\bar{P}_k(x) = \sum_{|c_{ik}| = d_k} p_{ik} x^{c_{ik}}.$$

It is clear that if P itself is homogenous, $\overset{+}{P}$ does not depend on x_0 and we have

$$\overset{+}{P} = P = \bar{P},$$

where $\overset{+}{P}$ is understood as a mapping $\mathbb{C}^n \to \mathbb{C}^n$ obtained in a natural way. Let us now list some useful obvious facts which relate the homogenizations introduced above to a polynomial mapping F:

(H1) For any homogeneous polynomial mapping F: $\mathbb{C}^m \to \mathbb{C}^n$ the set $F^{-1}(0)$ contains together with each element z also all its complex multiples (i.e. $z \in F^{-1}(0)$ implies $\text{Line}\, z \subset F^{-1}(0)$, where $\text{Line}\, z = \{\lambda z | \lambda \in \mathbb{C}\}$).

(H2) $x \in F^{-1}(0)$ implies $(x, 1) \in \overset{+}{F}{}^{-1}(0)$.

(H3) If $(x, x_0) \in \overset{+}{F}{}^{-1}(0)$ and $x_0 \neq 0$ then $(1/x_0) \cdot x \in F^{-1}(0)$.

(H4) $(x, 0) \in \overset{+}{F}{}^{-1}(0)$ implies $x \in \bar{F}^{-1}(0)$.

Remark 1. With every subset N of $\mathbb{C}^{n+1}$ having the homogenity property (H1) (i.e. $x \in N$ implies $\lambda x \in N$ for each $\lambda \in \mathbb{C}$) we can associate a subset ${}^pN$ of the complex projective n-space $\mathbb{P}^n$ by

$$\mathbf{x} \in {}^pN \text{ if and only if } \rho^{-1}(\mathbf{x}) \in N,$$

where ρ is the natural projection of $\mathbb{C}^{n+1} \setminus \{0\}$ onto $\mathbb{P}^n$ associating with each $(x_0, \ldots, x_n) \neq 0$ the equivalence class $\mathbf{x}$ it defines under the equivalence $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if $y_i = \lambda x_i$ for $i = 0, \ldots, n, 0 \neq \lambda \in \mathbb{C}$. For F polynomial, the set ${}^p(\overset{+}{F}{}^{-1}(0))$ is considered as the extended zero set of F. The (proper) zeros of F appear as points $\mathbf{x}$ of ${}^p(\overset{+}{F}{}^{-1}(0))$ with $x_0 \neq 0$, if $\mathbf{x} \in {}^p(\overset{+}{F}{}^{-1}(0))$ and $x_0 = 0$, $\mathbf{x}$ is considered to be an improper zero of F.

Definition (of the bounded homotopy mapping). For a given polynomial mapping P: $\mathbb{C}^n \to \mathbb{C}^n$ with degree of its k-th component equal to d_k we define an associate mapping

$$H: \mathbb{C}^n \times \mathbb{C} \times [0; 1] \times \mathbb{C}^{n(n+1)} \times \mathbb{C}^n \times \mathbb{R}^+ \to \mathbb{C}^n \times \mathbb{R}$$

($\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+ = \{\lambda \in \mathbb{R} | \lambda \geq 0\}$, $\mathbb{R}_0 = \{\lambda \in \mathbb{R} | \lambda \neq 0\}$ and, analogically, $\mathbb{C}_0 = \{\lambda \in \mathbb{C} | \lambda \neq 0\}$) as follows: For $k = 1, \ldots, n$ we define

$$H_k(x, t, a, b, r) = t \overset{+}{P}_k(x) + (1-t) Q_k(x, b) + t(1-t) R_k(x, a), \tag{3}$$

where

$$Q_k(x, b) = x_k^{d_k} - b_k x_0^{d_k} \qquad (b_k \neq 0), \tag{4}$$

$$R_k(x, a) = \sum_{i=0}^{n} a_{ik} x_i^{d_k}, \tag{5}$$

$x = (x^T, x_0)^T \in \mathbb{C}^{n+1}$, $t \in [0; 1]$, $a \in \mathbb{C}^{n(n+1)}$, $b \in \mathbb{C}^n$. The last component H_{n+1} of H is real-valued and defined by

$$H_{n+1}(x, t, a, b, r) = N(x, r) = \sum_{i=0}^{n} x_i \bar{x}_i - (1 + r)^2 \tag{6}$$

with $r \in \mathbb{R}^+$ and $\bar{x}_i$ being the complex conjugate to x_i.

We shall refer to H as to the bounded homotopy mapping and to the parameters t, (a,b,r) as to the homotopy and auxiliary paramaters, respectively. Further we shall call the condition $H_{n+1}=0$ the norm condition. The column vector of the complex components $H_1,\ldots,H_n$ of H we denote by $\tilde{H}$, so $H=(\tilde{H}^T,N)^T$, where T stands for the transposition.

The restriction of a mapping obtained by fixing some variable ξ will be denoted by the corresponding subscript, e.g. $H(\xi,.)=H_\xi(.)$. Further, Df will denote the differential of a function or a mapping f and $D_x f, D_t f$ etc. will denote the partial differential of f with respect to x, t etc., respectively. Sometimes complex mappings with complex values and complex variables will be regarded as real mappings of real variables (each complex variable $z=x+i y$ will be replaced by $(x,y)^T$, each complex component F_k will be replaced by $(\operatorname{Re} F_k(x,y), \operatorname{Im} F_k(x,y))^T$). In this case complex mappings will be denoted by capital letters and the corresponding real mappings by the corresponding lower case letters. Correspondingly, to distinguish complex and real kernel, dimension and rank we shall use the notations Ker, Dim, Rank, ker, dim, rank for complex kernel, complex dimension, complex rank, real kernel, real dimension and real rank, respectively. Further we shall use the same notations for a complex vector $(v_1,\ldots,v_m)$ and the corresponding real vector $(\operatorname{Re} v_1, \operatorname{Im} v_1,\ldots,\operatorname{Re} v_m, \operatorname{Im} v_m)$. It will be allways clear from the context in which sense the vector should be understood.

Remark 2. Unlike $\tilde{H}_{t,a,b,r}$ the norm component N_r is not a polynomial function in the complex variables $x_0,x_1,\ldots,x_n$.

Lemma 1. *Let M be an $r\times s$ complex matrix. Then the associated real $2r\times 2s$ matrix m which represents the mapping $\mathbb{R}^{2r}\to\mathbb{R}^{2s}$ generated by M satisfies*

$$\dim\ker m = 2\operatorname{Dim}\operatorname{Ker} M \tag{7}$$

and, if $r=s$, also

$$\det m = |\det M|^2 \geqq 0. \tag{8}$$

Proof. The equality (8) is proven in [1, Lemma 3.1]. From (8) it follows easily that $\operatorname{rank} m = 2\operatorname{Rank} M$. Q.E.D.

It may be useful to recall here the relation between m and M from [1]. If the (k,j)-th entry of M is the complex number $w_{kj}=u_{kj}+i v_{kj}$ then m can be written in a block form as an r by s array of two by two blocks, where the (k,j)-th block of m is the real matrix

$$\begin{pmatrix} u_{kj} & -v_{kj} \\ v_{kj} & u_{kj} \end{pmatrix}.$$

Proposition 1. *Let $b\in\mathbb{R}_0^{2n}$ be a fixed vector and let $U\subset\mathbb{R}^{2n(2n+2)}\times\mathbb{R}^+$, $V\subset\mathbb{R}^{2n+2}\setminus\{0\}$ be open semialgebraic sets. The set $S\subset U$ of those (a,r), for which the matrices*

$$D_x\tilde{h}_{a,b,r}(x,t) \quad \textit{and} \quad D_x h_{a,b,r}(x,t)$$

are of full rank for all (x,t) *from* $M=h_{a,b,r}^{-1}(0)\cap\{(x,t)|t\in(0;1), x\in V\}$ *contains an open dense subset of* U.

Remark 3. The formulation of the Proposition 1 may look strange for it follows trivially from the case $V=\mathbb{R}^{2n+2}\setminus\{0\}$, $U=\mathbb{R}^{2n(2n+2)}\times\mathbb{R}^+$. This formulation we have chosen in order to make further reference easier. Following carefully the proof of Proposition 1 it can be easily seen that all that is needed to be known about H there is that $D_a r(x,a)$ has full rank on $V\times U$.

Remark 4. Note that $S=S(b)$ may depend on b. Also note that the set of (a,b,r) for which the rank property of Proposition 1 holds true contains an open dense subset of $\mathbb{R}^{2n(2n+3)+1}$.

Proof. Denote $M=\{(x,t,a,r)|h_b(x,t,a,r)=0,\ t\in(0;1),\ x\in V,\ (a,r)\in U,$ $\operatorname{rank} D_x h_{a,b,r}(x,t)\leqq 2n\}$. The condition $\operatorname{rank} D_x h_{a,b,r}(x,t)\leqq 2n$ can be equivalently expressed by "all subdeterminants of $D_x h_{a,b,r}(x,t)$ of order $2n+1$ are zero". This, as well as the condition $h_b(x,t,a,r)=0$, is an algebraic equality so $\{(x,t,a,r)|h_b(x,t,a,r)=0, \operatorname{rank} D_x h_b(x,t,a,r)\leqq 2n\}$ is a semialgebraic set. Since V, $(0;1)$, U are semialgebraic, so is $V\times(0;1)\times U$ and we have M a semialgebraic set. Denote by π the natural projection $(x,t,a,r)\mapsto(a,r)$ and by C the set

$$C=\left\{(x,t,a,r)|0<t<1,\left|\sum_{i=0}^{n} x_i\bar{x}_i-(1+r)^2\right|<\varepsilon,\ x\in V,(a,r)\in U\right\}$$

for some $\varepsilon>0$. Obviously C is open and $\pi|_{\mathrm{cl}C}$ is proper, where cl denotes the closure. By A3 there is a one-one stratification $(\mathscr{A},\mathscr{B})$ of $\pi|_C$ such that $\mathscr{A}$ is compatible with the family $\{M, C\setminus M\}$. This implies that $\mathscr{A}_M=\{A\in\mathscr{A}|A\cap M\neq\emptyset\}$ and $\mathscr{B}_M=\{B\in\mathscr{B}|B\cap\pi(M)\neq\emptyset\}$ are locally finite partitions into differentiable submanifolds of M, $\pi(M)$, respectively and for each $A\in\mathscr{A}_M$, $\pi(A)\in\mathscr{B}_M$, $\pi|_A$ is differentiable, $\operatorname{rank}\pi|_A=\dim\pi(A)$ and $\pi|_A$ is one-one as soon as $\dim\pi(A)=\dim A$. We have $S=U\setminus\pi(M)$ and we prove that

$$\operatorname{codim} B>0 \quad \text{for all } B\in\mathscr{B}_M.$$

Suppose the contrary. Then some $B\in\mathscr{B}_M$ would be open in U. Let $A\in\mathscr{A}$ be such that $\pi(A)=B$.

Since $\operatorname{rank}\pi|_A=\operatorname{rank} B$ for each $(x^*,t^*,a^*,r^*)\in A$ from the implicit function theorem we easily obtain that there exists a submanifold U^* of A of real dimension $2n(2n+2)+1$ such that $V^*=\pi(U^*)$ is a neighbourhood of (a^*,r^*) and $\pi|_{U^*}$ is a diffeomorphism. This means that there are smooth maps $x=\xi(a,r)$, $t=\tau(a,r)$ on V^* such that $x^*=\xi(a^*,r^*)$, $t^*=\tau(a^*,r^*)$ and $(\xi(a,r),\tau(a,r),a,r)\in A$ for all $(a,r)\in V^*$. Thus we have

$$h_b(\xi(a,r),\tau(a,r),a,r)=0 \tag{9}$$

and

$$\operatorname{rank} D_x h_b(\xi(a,r),\tau(a,r),a,r)<2n+1 \tag{10}$$

for $(a,r)\in V^*$. Differentiating (9) with respect to a and r we obtain

$$\begin{aligned} D_x h_b\cdot D_a\xi+D_t h_b\cdot D_a\tau+D_a h_b=0\\ D_x h_b\cdot D_r\xi+D_t h_b\cdot D_r\tau+D_r h_b=0 \end{aligned} \tag{11}$$

(in (11) and in the rest of the proof we skip the arguments (a,r) and understand $(a,r)\in V^*$). Since $D_t N=0$, (11) can be written as

$$D_x \tilde{h}_b \cdot D_a \xi + D_t \tilde{h}_b \cdot D_a \tau = -D_a \tilde{h}_b, \tag{12}$$

$$D_x N \cdot D_a \xi = 0, \tag{13}$$

$$D_x \tilde{h}_b \cdot D_r \xi + D_t \tilde{h}_b \cdot D_r \tau = 0, \tag{14}$$

$$D_x N \cdot D_r \xi = -2(1+r). \tag{15}$$

Since $\operatorname{rank} D_t \tilde{h}_b \cdot D_a \tau \leqq 1$, $\operatorname{rank} D_a \tilde{h}_b = 2n$ on $V \times (0;1) \times U$ and, by Lemma 1, $\operatorname{rank} D_x \tilde{h}_b < 2n$ implies $\operatorname{rank} D_x \tilde{h}_b \leqq 2n-2$, (12) can be satisfied only if $\operatorname{rank} D_x \tilde{h}_b = 2n$. Thus $D_t \tilde{h}_b$ is a linear combination of the columns of $D_x \tilde{h}_b$ and we have from (12–15)

$$\begin{aligned}
2n+1 &= \operatorname{rank} D_{a,r} h_b = \operatorname{rank} \begin{pmatrix} D_a \tilde{h}_b & 0 \\ 0 & -2(r+1) \end{pmatrix} \\
&= \operatorname{rank} \begin{pmatrix} D_x \tilde{h}_b \cdot D_a \xi + D_t \tilde{h}_b \cdot D_a \tau & D_x \tilde{h}_b \cdot D_r \xi + D_t \tilde{h}_b \cdot D_r \tau \\ D_x N \cdot D_a \xi & D_x N \cdot D_r \xi \end{pmatrix} \\
&= \operatorname{rank} \left[\begin{pmatrix} D_x \tilde{h}_b & D_t \tilde{h}_b \\ D_x N & 0 \end{pmatrix} \cdot \begin{pmatrix} D_a \xi & D_r \xi \\ D_a \tau & D_r \tau \end{pmatrix} \right] \\
&\leqq \operatorname{rank} \begin{pmatrix} D_x \tilde{h}_b & D_t \tilde{h}_b \\ D_x N & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} D_x \tilde{h}_b \\ D_x N \end{pmatrix} = \operatorname{rank} D_x h_b
\end{aligned}$$

which contradicts (10). Q.E.D.

Remark 5. For the bounded homotopy we will always assume the choice of the sets U and V from Proposition 1 as follows: U is the unit ball in $\mathbb{R}^{2n(2n+2)}$ and $V=\mathbb{R}^{2n+2}\setminus\{0\}$. Then the conclusion of the Proposition 1 holds with $M = h_{a,b,r}^{-1}(0)$.

Corollary 1.1. *For each* $b \in \mathbb{C}_0^n$, $(a,r) \in S$ *the set* $\tilde{h}_{a,b,r}^{-1}(0)$ *is a* 3-*dimensional submanifold of* $\mathbb{R}^{2n+3}$, $h_{a,b,r}^{-1}(0)$ *is a* 2-*dimensional submanifold of* $\mathbb{R}^{2n+3}$, *both manifolds intersecting each plane* $t=\text{const}<1$ *transversally;* $h_{t,a,b,r}^{-1}(0)$ *is a* 1-*dimensional submanifold of* $\mathbb{R}^{2n+2}$ *for* $t<1$.

A connected component of $h_{a,b,r}^{-1}(0)$ is obviously closed. Since $h_{a,b,r}^{-1}(0) \cap \{(x,t) | t \in [0;1]\}$ is bounded, the set of those $t^* \in [0;1]$ for which a given component of $h_{a,b,r}^{-1}(0)$ intersects the plane $t=t^*$ is closed as well. By Corollary 1.1 for $(a,r)\in S$ this set is open in $[0;1)$ as well. Thus we have

Corollary 1.2. *Let* $b \in \mathbb{C}_0^n$, $(a,r) \in S$. *Then, any connected component* K *of* $h_{a,b,r}^{-1}(0)$ *has a non-empty intersection with each plane* $t=\text{const} \in [0;1]$.

This corollary means that for almost all (a,r) each connected component of $h_{t,a,b,r}^{-1}(0)$ for $t \in (0;1)$ can be continued both sides (for t increasing and decreasing) within the set $h_{a,b,r}^{-1}(0)$ until the limits of the interval $[0;1]$.

Proposition 2. *Let* $b^* \in \mathbb{C}_0^n$, $(a^*,r^*) \in S$, $t^* \in (0;1)$ *be fixed. Then the set* $h_{t^*,a^*,b^*,r^*}^{-1}(0)$ *consists of exactly* D *disjoint circles all centred at* 0 *with radius* $(1+r^*)$.

Proof. For each $t \in (0;1)$ fixed, $\tilde{H}_{t,a,b,r}$ is a homogeneous polynomial mapping $\mathbb{C}^{n+1} \to \mathbb{C}^n$ which by virtue of Propositition 1 and Remark 3 has the matrix $D_x \tilde{H}_{t,a,b,r}$ of full rank on $\mathbb{C}^{n+1} \setminus \{0\}$. This implies that $\tilde{H}^{-1}_{t,a,b,r}(0)$ consists of isolated complex lines of roots Line z, where $z \in \tilde{H}^{-1}_{t,a,b,r}(0)$.

From Bézout's theorem for homogeneous systems of polynomial equations in a complete algebraic field it follows that there are exactly D 1-dimensional complex subspaces of roots for such a system (including multiplicity) having only the zero-vector in common. It is easy to see that restricting the roots of

$$\tilde{H}_{t^*,a^*,b^*,r^*}(x) = 0$$

by the norm equation to

$$h^{-1}_{t^*,a^*,b^*,r^*}(0) = \tilde{h}^{-1}_{t^*,a^*,b^*,r^*}(0) \cap N^{-1}_{r^*}(0)$$

one obtains exactly D (including multiplicity) circles of radius $(1+r^*)$ centred at 0. Each two of these circles are either identical or disjoint. For $b \in \mathbb{C}^n_0$, $h^{-1}_{0,a^*,b^*,r^*}(0)$ consists of exactly D disjoint circles and $D_x h_{a^*,b^*,r^*}(x,t)$ has full rank on $h^{-1}_{a^*,b^*,r^*}(0) \cap \{(x,t) \mid t \in [0;1)\}$. We can continue these circles according to Corollary 1.2; by Corollary 1.1 no two circles can merge for any $t \in [0;1)$. Q.E.D.

Proposition 3. *Let* $(a,b,r) \in \mathbb{C}^{n(n+1)} \times \mathbb{C}^n_0 \times \mathbb{R}^+$, $0 \leqq \tau_0 < \tau_1 \leqq 1$ *and let* K *be a connected component of* $h^{-1}_{a,b,r}(0) \cap \{(x,t) \mid t \in [\tau_0;\tau_1)\}$ *such that for all* $t^* \in [\tau_0;\tau_1)$ *one has* $K \cap \{(x,t) \mid t = t^*\} \neq \emptyset$. *Assume that* K *is a manifold of dimension* 2. *Then* $\mathrm{cl}\, K \setminus K = K_0 \times \{\tau_1\}$, *where* K_0 *is a circle of radius* $(1+r)$ *centred at* 0.

Remark 6. Due to Corollary 1.2, in the context of bounded homotopy, Proposition 3 will be used merely for $\tau_1 = 1$. Its more general form will be needed in Sect. 3.

Proof. The set $h^{-1}_{a,b,r}(0)$ and, consequently, also K and $\mathrm{cl}\, K$ are algebraic sets. By A 3, $\mathrm{cl}\, K$ has a finite stratification $\mathscr{A}$ compatible with the family $\{K, \mathrm{cl}\, K \setminus K\}$. Since K has dimension 2, so has $\mathscr{A}$. Since each $A \in \mathscr{A}$ such that $A \subset \mathrm{cl}\, K \setminus K$ satisfies $A \subset \mathrm{cl}\, B$ for some $B \in \mathscr{A}$, $B \subset K$, we have by Corollary 1.1

$$\dim A < \dim B \leqq 2. \tag{16}$$

Obviously K_0 is compact. We prove now that it is connected.

Assume that it is not. Then $K_0 = K_1 \cup K_2$, where K_1, K_2 are compact and there is an $\eta > 0$ such that their distance is not less than η. Denote by G the set of those x for which $\|x\|_2 = (1+r)$ ($\|\cdot\|_2$ being the standard Euclidean norm in real or complex vector space) and the distance of which to both K_1 and K_2 is not less than $1/3\,\eta$. The set G is non-empty and compact. Let $x_1 \in K_1$, $x_2 \in K_2$. Then there are sequences of points $\{(y^k_i, t^k_i)\}^\infty_{k=1}$ from K such that

$$(y^k_i, t^k_i) \xrightarrow[k \to \infty]{} (x_i, \tau_1)$$

for $i = 1,2$. There is a path in $K \cap \{(x,t) \mid t$ between $t^k_1, t^k_2\}$ connecting (y^k_1, t^k_1) with (y^k_2, t^k_2). Since for large k we have $\|y^k_i - x_i\|_2 \leqq 1/3\,\eta$, this path has to contain a point (z_k, t_k) such that t_k is between t^k_1, t^k_2 and $z_k \in G$. Passing to a subsequence we have $z_k \to z$, $t_k \to \tau_1$ for some $z \in G$, so $G \cap K_0 \neq \emptyset$ which is impossible.

Because of (H2) $h_{t,a,b,r}^{-1}(0)$ contains with each x also the entire circle $\{x \mid \|x\|_2 = 1+r\} \cap \text{Line } x$. Hence K_0 contains at least one entire circle with radius $(1+r)$ centred at 0. Since K_0 is connected and, due to (16), 1-dimensional, it consists of exactly one such circle. Q.E.D.

Remark 7. By continuity, the limit circle K_0 for $\tau_1 = 1$ consists of roots of $\overset{+}{P}(x)$ which by (H2), (H3) correspond to a unique root of P (proper, if $x_0 \neq 0$, and improper, if $x_0 = 0$). Proposition 3 improves the correspondent result of [1] in that it asserts that K_0 is uniquely defined even if the root is not isolated.

Summarizing Propositions 1–3 we have

Theorem 1. *Given $b \in \mathbb{C}_0^n$, there exists an open dense subset S of $\mathbb{R}^{2n(2n+2)} \times \mathbb{R}^+$ such that for all $(a,r) \in S$ and each $y \in q_b^{-1}(0)$ there exists a uniquely defined connected component $K(y)$ of $h_{a,b,r}^{-1}(0) \cap \{(x,t) \mid t \in [0;1)\}$ which has the following properties:*

1. *$K(y) \cap \{(x,t) \mid t \in (0;1)\}$ is a (real) analytic submanifold of $\mathbb{R}^{2n+2} \times \mathbb{R}^+$ with (real) dimension two.*
2. *$K(y)$ intersects each plane $t = \text{const} \in [0;1)$ transversally in a circle of radius $(1+r)$ centred at 0 which lies in Line x for some $x \in \mathbb{C}^{n+1}$.*
3. *The sets K_0, K_1 defined by $K_0 \times \{0\} = K(y) \cap \{(x,t) \mid t = 0\}$ and $K_1 \times \{1\} = \text{cl}\, K(y) \cap \{(x,t) \mid t = 1\}$ are circles such that $K_0 \subset \text{Line } y$ and $K_1 \subset \text{Line } x$ for some $x \in \overset{+}{P}{}^{-1}(0)$.*
4. *The manifolds $K(y)$ do not intersect in $\{(x,t) \mid t < 1\}$ for different y's.*

Essentially, a path following algorithm consists in numerical integration of an ordinary differential equation the trajectory of which is a curve joining a root of the auxiliary equation $Q(x) = 0$ with a root of the equation to be solved $P(x) = 0$.

Theorem 1 asserts the existence of a tube with constant diameter joining the circle K_0 of the norm restricted roots of the (+)-homogenisation of Q with a similar circle for $\overset{+}{P}$. To separate a path from this "cylinder" that could be numerically followed we have to define a vector field on $K(y)$ the integral curves of which would lead from K_0 to K_1. Of course, there are various ways to define such a vector field. A natural way to do so is to define a vector field v: $K(y) \to TK(y)$ ($TK(y)$ being the tangent bundle to $K(y)$) which has the property that for each $(x,t) \in K(y)$, $v(x,t)$ has the maximal t-component among all vectors of $T_{(x,t)}K(y)$ with the same length. This is obviously the vector orthogonal to the tangent vector to the circle $K(y) \cap \{\mathbb{C}^{n+1} \times \{t\}\}$ which is $(i\, x, 0)$. Thus, v can be defined by setting its t-component equal to 1 and choosing its x-component w to satisfy the conditions

$$(w(x,t), 1) \in T_{(x,t)} K(y) = \ker D h_{a,b,r}(x,t), \tag{17}$$

$$(w(x,t), 1) \perp (i\, x, 0). \tag{18}$$

Now (17), (18) mean

$$D_x h_{a,b,r}(x,t) \cdot w(x,t) + D_t h_{a,b,r}(x,t) = 0, \tag{19}$$

$$\langle (i\, x, 0), (w(x,t), 1) \rangle = 0, \tag{20}$$

respectively (here $\imath x$ is understood as the real vector $(-\operatorname{Im} x_0, \operatorname{Re} x_0, \ldots, -\operatorname{Im} x_n, \operatorname{Re} x_n)$ and $\langle\cdot,\cdot\rangle$ stands for the real inner product. Since $(\imath x, 0) \in T_{(x,t)} K(y)$ (19), (20) are independent and one has

$$w(x,t) = \begin{pmatrix} D_x h_{a,b,r}(x,t) \\ \imath x^T \end{pmatrix}^{-1} \cdot \begin{pmatrix} D_t h_{a,b,r}(x,t) \\ 0 \end{pmatrix}. \tag{21}$$

Note that the definition (21) of w can be extended from $K(y)$ to its neighbourhood. Moreover, since (17) holds for $(x,t) \in K(y)$, $K(y)$ is an invariant set for the ordinary differential equation

$$\frac{dx}{dt} = w(x,t). \tag{22}$$

Thus we have

Theorem 2. *Let $b \in \mathbb{C}_0^n$, $(a,r) \in S$. For each $y \in Q_b^{-1}(0)$ any solution $\varphi(t,\tilde{y})$ of (22) satisfying $\varphi(0,\tilde{y}) = \tilde{y}$ for some $\tilde{y} \in \operatorname{Line} y$, $\|\tilde{y}\|_2 = 1+r$, satisfies $\|\varphi(t,\tilde{y})\|_2 = 1+r$ for $t \in [0;1)$. Further, the limit set $\Lambda(y)$ of $\varphi(t,\tilde{y})$ for $t \to 1$, $\Lambda(y) = \bigcap_{\varepsilon > 0} \operatorname{cl}\{(t, \varphi(t,\tilde{y})) \mid t - \varepsilon < t < 1\}$, is contained in a single circle $K_1 \subset \operatorname{Line} z$ with radius $1+r$ centred at 0, where $z \in \overset{+}{P}{}^{-1}(0)$.*

For a given point $x \in \mathbb{C}^{n+1}$, denote $C(x) = \{\lambda x \mid \lambda \in \mathbb{C}, |\lambda| = 1\}$. The following theorem asserts that for each isolated zero $\mathbf{x} \in {}^{p}(\overset{+}{P}{}^{-1}(0))$ (proper or improper – cf. Remark 1.) there exists a curve of zeros of H starting at some zero of Q_b for $t=0$ that ends up in $C(\rho^{-1}(\mathbf{x}))$ for $t=1$.

Theorem 3. *Assume that* $\operatorname{Rank} D_x \overset{+}{P}(x)$ *is not everywhere less than n. Let $b \in \mathbb{C}_0^n$, $(a,r) \in S$ and let $\hat{\mathbf{z}}_0$ be an isolated zero of the extended zero set ${}^{p}(\overset{+}{P}{}^{-1}(0))$. Let $z_0 \in \rho^{-1}(\hat{\mathbf{z}}_0)$, $\|z_0\|_2 = r+1$. Then, there exists an $y \in Q_b^{-1}(0)$ such that $\Lambda(y) \subset \{1\} \times C(z_0)$, where Λ is as in Theorem 2 and ρ is as in Remark 1.*

Proof. Let $z_0 \in \rho^{-1}(\hat{\mathbf{z}}_0)$ be such that $\|z_0\|_2 = r+1$. The set of those $(x,z) \in \mathbb{C}^{2n+2}$ for which

$$\det \begin{pmatrix} D_x \overset{+}{P}(x) \\ \bar{z}^T \end{pmatrix} = 0$$

is an algebraic variety V in $\mathbb{C}^{2n+2}$. Since $\operatorname{Rank} D_x \overset{+}{P}(x)$ is not everywhere less than n, $\mathbb{C}^{2n+2} \setminus V \neq \emptyset$, so $\mathbb{C}^{2n+2} \setminus V$ is open dense in $\mathbb{C}^{2n+2}$. It follows that in every neighbourhood of z_0 there exists a z_1 such that

$$\det \begin{pmatrix} D_x \overset{+}{P}(x) \\ \bar{z}_1^T \end{pmatrix} \not\equiv 0. \tag{23}$$

In particular z_1 can be chosen so close to z_0 that it is not orthogonal to $\operatorname{Line} z_0$ (or, equivalently, each hyperplane $\bar{z}_1^T x = c$ intersects $\operatorname{Line} z_0$ transversally in a single point). As in [1] we obtain from (23)

$$\deg\left(\begin{pmatrix}\overset{+}{p}(x)\\ \bar{z}_1^T x\end{pmatrix}, U, 0\right) > 0$$

where U is a neighbourhood of z_0 in $\mathbb{R}^{2n+2}$ with cl U compact. Since z_1 is not orthogonal to Line z_0, z_0 is the unique zero of the system of equations

$$\begin{aligned}\overset{+}{P}(x)&=0\\ \bar{z}_1^T x&=\bar{z}_1^T z_0.\end{aligned} \tag{24}$$

By the degree argument of [1] (cf. [7]), for each $t<1$ sufficiently close to 1 there exists an $y(t)\in U$ such that

$$\begin{aligned}\tilde{H}_{a,b,r}(y(t),t)&=0,\\ \bar{z}_1^T y(t)&=\bar{z}_1^T z_0\end{aligned}$$

and $y(t)\to z_0$ for $t\to 1$. The point $z^*(t)=(1+r)\dfrac{y(t)}{\|y(t)\|_2}$ satisfies $z^*(t)\in H_{a,b,r,t}^{-1}(0)$ and, $\lim_{t\to 1} z^*(t)=z_0$. Thus, for $t<1$, each point $z^*(t)$ must belong to $\Lambda(y)$ for some $y\in Q_b^{-1}(0)$. The statement of the proposition now follows from the fact that $Q_b^{-1}(0)$ is finite. Q.E.D.

Remark. One can easily see that the rank assumption of Theorem 3 is weaker than the condition in [1] which says that Rank $D_x P(x)$ is not everywhere less than n. This allows us to find complex 1-dimensional components of roots of a homogeneous map $P\colon \mathbb{C}^n\to\mathbb{C}^n$ which correspond to isolated improper roots of P.

3. Real Homotopy

In this section we propose a homotopy method (to be called real homotopy) for solving a system of real polynomial equations using the bounded homotopy map H of Sect. 2 with a, b real.

For a given polynomial mapping $P\colon \mathbb{R}^n\to\mathbb{R}^n$ (which of course can be considered as a mapping $\mathbb{C}^n\to\mathbb{C}^n$ satisfying $P(x)=P(\bar{x})$) we consider the mapping H defined by (3), (5), (6) and with Q_k defined by

$$Q_k(x,b)=\prod_{i=1}^{d_k}(x_k-b_{ki}\cdot x_0) \tag{25}$$

instead of (4), where throughout the entire section we assume the parameters $b\in\mathbb{R}^{\mathscr{D}}$ to satisfy

$$b_{ki}\neq b_{kj}\quad \text{for all } k=1,\dots,n \text{ and } i\neq j;\quad i,j\in\{1,\dots,d_k\}. \tag{26}$$

However, unlike in Sect. 2, we consider H as a mapping

$$H: \mathbb{C}^n \times \mathbb{C} \times \mathbb{U} \times \mathbb{R}^{n(n+1)} \times \mathbb{R}^{\mathcal{D}} \times \mathbb{R}^+ \to \mathbb{C}^n \times \mathbb{R},$$

where $\mathbb{U} = \{u \in \mathbb{C} \mid 0 \leqq \operatorname{Re} u \leqq 1, \ |\operatorname{Im} u| < \eta\}$ for some $\eta > 0$ (i.e. we take a, b real and we allow the homotopy parameter to be complex from some neighbourhood of the real interval $[0; 1]$). In order to keep the intuitive meaning of t we shall use the letter u for the complex variable corresponding to t and reserve the letter t for its real part $\operatorname{Re} u$.

Proposition 4. *Let $b \in \mathbb{R}^{\mathcal{D}}$ be fixed, $V \subset \mathbb{R}^{2n+2}$ and $W \subset \mathbb{R}^{n(n+1)} \times \mathbb{R}^+$ be open semialgebraic sets. The set $G \subset W$ of those (a, r) for which $D_{x,u} h_{a,b,r}(x, u)$ and $D_{x,u} \tilde{h}_{a,b,r}(x, u)$ are of full rank for all $(x, u) \in h_{a,b,r}^{-1}(0) \cap \{(x, u) \mid 0 < t < 1, \ x \in V\}$ contains an open dense set in W.*

Remark 8. As in Remark 3 it is sufficient for R to satisfy: $D_a r(x, a)$ has full rank on $V \times W$. As in Remark 5 we assume for the real homotopy H given by (3), (5), (6), (25) the choice of the sets V and W as follows:

$$V = \mathbb{R}^{2n+2} \setminus \{0\}, \qquad W = \text{unit ball in } \mathbb{R}^{n(n+1)} \times \mathbb{R}^+.$$

Proof. Let $b \in \mathbb{R}^{\mathcal{D}}$ satisfying (26) be fixed. Denote

$$Z = \{(x, u, a, r) \mid 0 < t < 1, h_b(x, u, a, r) = 0, \operatorname{rank} D_{x,u} h_{a,b,r}(x, u) \leqq 2n\}.$$

As in the proof of Proposition 1 we conclude that Z is a semialgebraic set and so is $G = W \setminus \pi(Z)$. Following the argument of Proposition 1 we further find that if $\dim \pi(Z) = n(n+1) + 1$ then there exists an open set $W^* \subset W$ and smooth mappings $x = \xi(a, r)$, $u = \sigma(a, r)$ on W^* such that

$$h_b(\xi(a, r), \sigma(a, r), a, r) = 0 \tag{27}$$

and

$$\operatorname{rank} D_{x,u} h(\xi(a, r), \sigma(a, r), a, r) \leqq 2n \tag{28}$$

for all (a, r) from W^*.

Differentiating with respect to a, r we obtain (the arguments (a, r) skipped)

$$D_x \tilde{h}_b \cdot D_a \xi + D_u \tilde{h}_b \cdot D_a \sigma + D_a \tilde{h}_b = 0, \tag{29}$$

$$D_x N \cdot D_a \xi = 0, \tag{30}$$

$$D_x \tilde{h}_b \cdot D_r \xi + D_u \tilde{h}_b \cdot D_r \sigma = 0, \tag{31}$$

$$D_x N \cdot D_r \xi + 2(r+1) = 0. \tag{32}$$

As the complex subspace of $\mathbb{C}^n$ generated by the columns of $D_a \tilde{H}_b$ for $(x, u, a, r) \in V \times (\mathbb{U} \setminus \{0, 1\}) \times W$ is the entire space $\mathbb{C}^n$, for each $y \in \mathbb{C}^n$ there exists a vector $q \in \mathbb{C}^{n(n+1)}$ such that

$$D_a \tilde{H}_b \cdot q = y.$$

Using (29) we obtain

$$y = (-D_x \tilde{H}_b \cdot D_a \xi - D_a \tilde{H}_b \cdot D_a \sigma) q = -D_{x,u} \tilde{H}_b \cdot \begin{pmatrix} D_a \xi \cdot q \\ D_a \sigma \cdot q \end{pmatrix}$$

which implies that $\operatorname{Rank} D_{x,u} \tilde{H}_b = n$.

Now, it follows that (28) is possible only if $D_{x,u}N=(D_x N,0)$ is a linear combination of the rows of $D_{x,u}\tilde{h}_b$. By (29) and (32) this means that there is a real $2n$-vector w such that $w^T\cdot D_x\tilde{h}_b=D_x N$ and $w^T\cdot D_u\tilde{h}_b=0$. Using (29–32) we obtain

$$\begin{aligned} w^T(-D_a\tilde{h}_b,0) &= w^T(D_x\tilde{h}_b\cdot D_a\xi+D_u\tilde{h}_b\cdot D_a\sigma, D_x\tilde{h}_b\cdot D_r\xi+D_u\tilde{h}_b\cdot D_r\sigma)\\ &= w^T(D_x\tilde{h}_b, D_u\tilde{h}_b)\cdot\begin{pmatrix} D_a\xi & D_r\xi\\ D_a\sigma & D_r\sigma\end{pmatrix}=(D_x N,0)\cdot\begin{pmatrix} D_a\xi & D_r\xi\\ D_a\sigma & D_r\sigma\end{pmatrix}\\ &= (D_x N\cdot D_a\xi, D_x N\cdot D_r\xi)=(0,-2(r+1))\end{aligned}$$

which is impossible for $r\geqq 0$. This completes the proof. Q.E.D.

Corollary 4.1. *For each b as in Proposition 4 and $(a,r)\in G$ fixed the intersection of the sets $\tilde{Z}=\tilde{h}_{a,b,r}^{-1}(0)$ and $Z=h_{a,b,r}^{-1}(0)$ with the set $\{(x,u)|0<t<1\}$ considered as the subsets of the (x,u)-space are 4- and 3-dimensional real analytic submanifolds of $\mathbb{R}^{2n+4}$, respectively. Alternatively, $\tilde{Z}$ can be considered as the 2-dimensional complex submanifold $\tilde{H}_{a,b,r}^{-1}(0)$ of $\mathbb{C}^{n+2}$.*

We shall call a point $(x,u)\in\tilde{Z}$ singular if $\operatorname{Rank} D_x\tilde{H}_b(x,u)<n$, and regular otherwise. Obviously if (x,u) is singular (regular) then also $\operatorname{Line} x\times\{u\}$ consists of singular (regular, respectively) points only. Hence passing to H we can speak about singular and regular circles in a natural way.

Proposition 5. *Let Z_0 a connected component of Z and let $(x^0,u^0)\in Z_0$ be singular, $0<t^0<1$. Then one has the following alternative:*

(a) *$u=u^0$ for each $(x,u)\in Z_0$,*

(b) *the singular circle $C(x^0,u^0)=\operatorname{Line} x^0\times\{u^0\}\cap\left\{(x,u)\,\middle|\,\sum_{i=0}^{n}x_i\bar{x}_i=1+r\right\}$ is isolated (i.e. there is a neighbourhood of $C(x^0,u^0)$ in the (x,u)-space such that any singular point from this neighbourhood lies on $\operatorname{Line} x^0\times\{u^0\}$).*

Proof. There is a neighbourhood V_1 of (x^0,u^0) such that $\tilde{Z}\cap V_1=\tilde{Z}_0\cap V_1$. Therefore all singular points of $\tilde{Z}$ in V_1 are contained in $\tilde{Z}_0\cap V_1$. Take any 1-dimensional complex analytic submanifold M of Z_0 through (x^0,u^0) that is transversal to $\operatorname{Line} x^0\times\{u^0\}$, i.e. such that $(x^0,u^0)\in M$ and

$$T_{(x^0,u^0)}M+\operatorname{Line} x^0\times\{u^0\}=T_{(x^0,u^0)}Z_0. \tag{33}$$

There is a neighbourhood $V_2\subset V_1$ such that (33) remains valid with (x^0,u^0) replaced by any $(x,u)\in V_2\cap M$. For $\varepsilon>0$ sufficiently small the set $V_3=\{(c\,x,u)|1-\varepsilon<|c|<1+\varepsilon,\ c\in\mathbb{C},\ (x,u)\in M\cap V_2\}$ is a neighbourhood of (x^0,u^0) in $\tilde{Z}_0$. This follows immediately from (33) by the inverse function theorem. Since with (x,u) singular all points $(c\,x,u)$ with $c\in\mathbb{C}$ are singular in order to prove the proposition, it suffices to prove the following alternative:

(a′) u is constant over M

(b′) there is a neighbourhood V_4 of (x^0,u^0) such that the only singular point of $M\cap V_4$ is (x^0,u^0).

Assume (b') does not hold. Then there is a sequence of singular points $(x^k, u^k) \in M$ such that $(x^k, u^k) \neq (x^0, u^0)$ and $(x^k, u^k) \to (x^0, u^0)$ for $k \to \infty$. In other words, all subdeterminants of order n of $D_x \tilde{H}_{a,b,r}$ vanish at (x^k, u^k). Since these subdeterminants are analytic functions on M they have to vanish everywhere in M.

Let (M_0, φ) be a complex analytic coordinate neighbourhood of (x^0, u^0) in M, i.e. φ is an analytic diffeomorphism of M_0 with an open subset of $\mathbb{C}$. This means that M_0 can be analytically parametrized by a complex parameter ξ, i.e. one has $(x, u) \in M_0$ if and only if $x = x(\xi)$, $u = u(\xi)$. Since $\tilde{H}_{a,b,r}(x, u) = 0$ for $(x, u) \in M_0$, differentiating with respect to ξ we obtain

$$D_x \tilde{H}_{a,b,r}(x(\xi), u(\xi)) \cdot \frac{dx}{d\xi}(\xi) + D_u \tilde{H}_{a,b,r}(x(\xi), u(\xi)) \cdot \frac{du}{d\xi}(\xi) = 0. \tag{34}$$

Since $\operatorname{Rank} D_x \tilde{H}_{a,b,r}(x(\xi), u(\xi)) < \operatorname{Rank} D_{x,u} \tilde{H}_{a,b,r}(x(\xi), u(\xi))$ (34) is possible only if $du/d\xi(\xi) = 0$, which implies $u(\xi) = \text{const}$ or $u = u^0$ on M_0. Q.E.D.

We will use the real homotopy in the following way to construct an algorithm. Each connected component $K(y)$ of Z ($y \in Q_b^{-1}(0)$) can be partitioned into two subsets:

$$K^r(y) = \{(x, u) \mid (x, u) \text{ regular point of } K(y)\},$$
$$K^s(y) = \{(x, u) \mid (x, u) \text{ singular point of } K(y)\},$$

where $K^r(y)$ is obviously open in $K(y)$ and $K^s(y)$ closed in $K(y)$. As in Sect. 2 we can conclude that for $(x, u) \in K^r(y)$ there is a unique $w \in \mathbb{C}^{n+1}$ satisfying

$$(w, 1) \in T_{(x,u)} K^r(y) = \ker D h_{a,b,r}(x, u), \tag{35}$$
$$(w, 1) \perp (i\, x, 0), \tag{36}$$

(note that in (35), (36) 1 and 0 are understood as complex constants). The solution of (22), (21) yields a path in $K^r(y)$ with u real that can be parametrized by t. This means that while we are in $K^r(y)$ the homotopy solution curve can be followed as in the case of bounded homotopy. The problem is how to continue a curve once we have reached a singular point. In order to propose such a continuation algorithm we need some more information about the local behaviour of the solution curves near a singular point.

Let $(x^0, t^0) \in K^s(y)$ then by Proposition 5 it has to be a point from an isolated singular circle. Since $(x^0, 0) \in \operatorname{Ker} D_{x,u} \tilde{H}_{a,b,r}(x^0, t^0)$ we have

$$\operatorname{Rank} \begin{pmatrix} D_{x,u} \tilde{H}_{a,b,r}(x^0, t^0) \\ (x^0, 0) \end{pmatrix} = n + 1$$

and the set

$$F = \{(x, u) \mid \tilde{H}_{a,b,r}(x, u) = 0,\ \langle (x^0, 0), (x, u) \rangle_C = 0\}$$

(where $\langle \cdot, \cdot \rangle_C$ is the complex inner product) can be parametrised by a variable x_i, say x_n. So

$$F = \{(\varphi_0(x_n), \ldots, \varphi_{n-1}(x_n), x_n, \psi(x_n)) \mid \varphi_0, \ldots, \varphi_{n-1}, \psi \text{ are analytic functions of } x_n \text{ with } \psi'(x_n^0) = 0\} \tag{37}$$

Because of the fact that on the entire connected component $K(\mathscr{y})$ u is constant only on the unique circle in Line $\mathscr{x} \times \{u\}$ where $(\mathscr{x}, u) \in K(\mathscr{y})$, ψ has in its expansion into Taylor series at least one non-zero coefficient, i.e. one has

$$\psi(x_n) = t^0 + \frac{1}{k!}\psi_k(x_n - x_n^0)^k + o(x_n - x_n^0), \tag{38}$$

where $\psi_k \neq 0$.

In analogy to bifurcation theory one could expect generically $k=2$ but we have not been able to prove it. Therefore, we have to deal with arbitrary k.

The homotopy can be continued after having reached $(\mathscr{x}^0, t^0)$ if we find a point of F near $(\mathscr{x}^0, t^0)$ with $u=t$ real, $t>t^0$. By (37), to find such points one has to solve the equation

$$\operatorname{Im} \psi(x_n) = 0. \tag{39}$$

The following proposition gives the structure of the solution set of (39). It can be obtained by using the classical degenerate implicit function theorems (cf. [8]). Nevertheless, since it is perhaps shorter to prove it directly, we give an outline of its proof.

Proposition 6. *All solutions of* (39) *are locally at* $(\mathscr{x}^0, t^0)$ *given by*

$$x_n = x_n^0 + s(z_j + g_j(s)), \qquad -\delta < s < \delta,\ \delta > 0,\ j = 0, \dots, 2k-1, \tag{40}$$

where g_j are analytic, $g_j(0)=0$ and

$$z_j = \exp\left(i\frac{1}{k}(-\operatorname{Arg}\psi_k + j\pi)\right).$$

Moreover, t is increasing along (40) *for j even and decreasing for j odd.*

Outline of Proof. The numbers z_j are obviously all the solutions of

$$\operatorname{Im}(\psi_k z_j^k) = 0 \tag{41}$$

of modulus 1.

The functions g_j are obtained by the (non-degenerate) implicit function theorem for the system of equations

$$\begin{aligned} &\operatorname{Im}((z_j + g_j)^k + o(s)) = 0 \\ &\operatorname{Re} z_j \cdot \operatorname{Re} g_j + \operatorname{Im} z_j \cdot \operatorname{Im} g_j = 0 \end{aligned} \tag{42}$$

at the point $s=0$, $g_j=0$. The first equation of (42) is obtained from (39) by substituting for x_n from (40), the second equation is added to make the correction term g_j orthogonal to z_j. From the unicity part of the implicit function theorem it follows that locally the solutions given by (40) are the unique solutions of (39) in the sectors $|\operatorname{Arg}(x_n - x_n^0) - \operatorname{Arg} z_j| < \varepsilon$ for some $\varepsilon > 0$. Therefore, should (40) not represent all solutions of (39), there would be a sequence $x_n^\nu \to x_n^0$ for $\nu \to \infty$ satisfying (39) such that $(x_n - x_n^0)/|x_n - x_n^0| \to z \neq z_j$. Substituting into (39) and passing to the limit one immediately sees that z would have to solve (41) which is impossible. We have

$$\psi(x_n^0 + s(z_j + g_j(s))) = t^0 + s^k \cdot |\psi_k| \cdot e^{j\pi i} + o(s^k)$$

which proves that t increases for j even and decreases for j odd.

Remark 9. Note that the vectors $z_0, \ldots, z_{2k-1}$ are located counterlockwise on the unit circle forming angles π/k. According to Proposition 6 they are tangent to the x_n-components of the solution curves of the system of equations

$$\begin{aligned} H_{a,b,r}(x, u) &= 0 \\ \langle (x^0, 0), (x, u) \rangle_C &= 0 \\ \operatorname{Im} u &= 0 \end{aligned} \tag{43}$$

at $x_n = x_n^0$. The tangents to the solution curves of (43) are given by $\Phi(z_j) = (\varphi_0'(0) z_j, \ldots, \varphi_{n-1}'(0) z_j, z_j, 0)$, $j = 0, 1, \ldots, 2k-1$; for j increasing curves with t increasing and decreasing alternate.

Proposition 6 indicates how one could continue the homotopy form a singular point (x^0, t^0).

Let us assume that we have a "starting" procedure $S = S(z)$ that works as follows:

Given a unit vector $z \in \mathbb{C}$ it finds a point different from (x^0, t^0) on the solution curve of (43) tangent to $\Phi(z)$ if there is such a curve. We also assume that S is reliable enough to allow us to conclude that there is no solution curve tangent to $\Phi(z)$ if $S(z)$ is not successful.

If we have reached (x^0, t^0) by following a homotopy path we know one of the vectors z_j (denote it by z^*) but we do not know k (we assume that F admits a parametrisation (37)).

If (x^0, t^0) has not been visited before by following another homotopy curve, for $k = 2, 3, 4, \ldots$ do $S(z)$ consecutively for $z = \exp(i\pi/k) \cdot z^*$ until S becomes successful and its resulting point has $t > t^0$. Then, $S(z)$ must yield such points also for all $z = \exp(i(2j+1)\pi/k) \cdot z^*$ for $j = 1, 2, \ldots, k-1$. Consequently, we can start k solution curves from (x^0, t^0).

If (x^0, t^0) has been visited before, denote k_1, z_1^* the k and z^* from our last visit of (x^0, t^0). We first check whether $z^* = z_1^* \cdot \exp(i 2j\pi/k)$ for some $j = 1, \ldots, k-1$. If yes, we stop and start another homotopy curve from $t = 0$. If no, we proceed as in the case of first visit with k running over all prime number multiples of k_1.

Let us note that we always start more than one homotopy curve from a singular point coming in with one curve. Hence, we may reach all roots of P without having to start all homotopy solution curves from $t = 0$.

The proposed continuation procedure becomes rather complicated with k increasing. However, although we have not been able to prove that generically $k = 2$, one can expect that singular points with k large do not appear too often.

We do not present numerical results of computation by the real homotopy method. The reason is that we have so far not been able to work out a reliable program realization of the starting procedure S. Still, we decided to present the idea of the method, since it seems to be interesting and the problem of realization of the procedure S is a well defined independent problem.

4. Numerical Results

The first step of the algorithm for finding all solutions (more precisely all isolated solutions and at least one point from some other connected components of roots of $\overset{+}{P}(x)=0$) based on the theory of Sect. 2 is the choice (at random) of the auxiliary parameters a, b, r. Then for each $y \in Q_b^{-1}(0)$ we solve the equation (22), (21) with the initial condition

$$x(0)=y. \tag{44}$$

By solving (21), (22), (44) on the interval $[0;1]$ we obtain a point $x(1) \in \overset{+}{P}^{-1}(0)$ and using ($H3$) and ($H4$) we obtain either a proper or an improper root of $P(z)=0$. Proposition 1 states that with probability one we shall not have to change our choice of (a, r) because of singularity of $D_x h_{a,b,r}$ during computing all paths given by (21), (22), (44).

The most important problem to solve in the implementation process is the construction of a reliable and effective procedure which solves (21), (22), (44). The use of simple procedure was not successful because jumps to proximate components of zeros of the bounded homotopy occurred. Therefore we decided to solve the implicit modification of (21), (22)

$$\begin{pmatrix} D_x h_{a,b,r}(x,t) \\ i\, x^T \end{pmatrix} \cdot \frac{dx}{ds} + \begin{pmatrix} D_t h_{a,b,r}(x,t) \\ 0 \end{pmatrix} \cdot \frac{dt}{ds} = 0. \tag{45}$$

At first we used the DERPAR procedure form [5]. This procedure chooses the most appropriate component of (x, t) as the independent variable s. After some experience we completely modified the DERPAR procedure keeping other subprocedures without any extensive changes.

Our experience shows that the method requires a very reliable procedure for solving (44), (45). Because of the great number of paths to be followed the procedure has to be also very effective.

We have run our program on several test problems and six of them are listed here. Problems 1–5 were run on the EC 1010 computer and the problem 6 on the SIEMENS 4004/150 in double real and double complex precision. In the following formulations of the test problems we present also the results of the computer runs. All real roots are presented as real $(n+1)$-tuples $(x_1, \ldots, x_n, x_0)$ and complex roots as complex $(n+1)$-tuples $(\mathrm{Re}\, x_1 + i\, \mathrm{Im}\, x_1, \ldots, \mathrm{Re}\, x_0 + i\, \mathrm{Im}\, x_0)$. The results are denoted by BHM and the known exact solution by ES. ENS is the exact number of solutions and ENPS, ENIS is the exact number of proper or improper solutions (components of solutions), respectively. Proper roots were scaled to $x_0 = 1$.

Problem 1.

$$x_1 + 10x_2 = 20$$
$$x_1 + 10x_2 = -20$$

ENS $=1$, ENIS $=1$, ES $=(10, -1, 0)$, BHM $=(1.68985 - i\,0.08936,\ -0.16898 + i\,0.00893, 0 + i\,0)$.

Problems 2–4 are obtained from three well-known test problems for global optimisation ([2]). The polynomial system represents in these cases the stationarity equation: gradient of the optimised function equal to zero.

Problem 2.

$$\begin{aligned} 2x_1(x_1^2-x_2)+2(x_1-1)&=0\\ x_2-x_1^2\qquad\qquad &=0 \end{aligned}$$

(obtained from the simplified Rosenbrock function in [2]) ENS=2, ENIS=1, ENPS=1, ES={(1,1,1), (0,1,0)},

$$\begin{aligned}\mathrm{BHM}=\{(&\ \ 0.99998-i0.00014,\ \ 0.99997-i0.00040,\ \ 1+i0),\\ (&-0.0688-i0.2956,\ \ 1.10951-i1.259,\ \ -0.0414-i0.0167),\\ (&-0.0324+i0.2153,\ \ 1.6841-i0.0909,\ \ -0.027-i0.0096),\\ (&\ \ 0.1315-i0.2355,\ \ -1.6714-i0.175,\ \ 0.025+i0.031),\\ (&-0.1908+i0.324,\ \ -1.5919+i0.4572,\ \ 0.021+i0.0837),\\ (&\ \ 0.1724+i0.3045,\ \ -1.6569-i0.137,\ \ 0.0344-i0.0649)\}\end{aligned}$$

Problem 3.

$$\begin{aligned} 2ax_1+4bx_1^3+6cx_1^5-x_2&=0\\ 2dx_2+4ex_2^3\qquad\qquad -x_1&=0 \end{aligned}$$

where $a=-2$, $b=1.05$, $c=-1/6$, $d=-1$, $e=0$ (obtained from the "three hump camel-back function" in [2]). ENS=5, ENPS=5 (all 5 solutions are real),

$$\begin{aligned}\mathrm{BHM}=\{(&-1.74743+i0.00011,\ \ 0.87371-i0.00005,\ \ 1+i0),\\ (&\ \ 1.07053+i0.00001,\ \ -0.53526-i0,\ \ 1+i0),\\ (&-1.07052+i0.00002,\ \ 0.53526-i0.00001,\ \ 1+i0),\\ (&\ \ 1.74752+i0,\ \ -0.8737+i0,\ \ 1+i0),\\ (&\ \ 0+i0,\ \ 0+i0,\ \ 1+i0)\}\end{aligned}$$

Problem 4. The equations are as in Problem 3 for the following values of the parameters: $a=-4$, $b=2.1$, $c=-1/3$, $d=4$, $e=-4$. This problem was obtained from the "six hump camel-back function" in [2]. ENS=15, ENPS=15, (all 15 solutions are real),

$$\begin{aligned}\mathrm{BHM}=\{(&-1.70411+i0.00021,\ \ 0.79614-i0.00007,\ \ 1+i0),\\ (&-0.08984+i0,\ \ 0.71265+i0,\ \ 1+i0),\\ (&-1.23022+i0,\ \ -0.16233+i0,\ \ 1+i0),\\ (&\ \ 1.60715+i0,\ \ 0.56858+i0.00004,\ \ 1+i0),\\ (&\ \ 1.23018+i0.00001,\ \ 0.16233+i0,\ \ 1+i0),\\ (&\ \ 1.70360+i0,\ \ -0.79608+i0,\ \ 1+i0),\\ (&\ \ 1.29607+i0,\ \ 0.60508+i0,\ \ 1+i0),\\ (&-1.63806+i0,\ \ -0.22867+i0,\ \ 1+i0),\\ (&\ \ 0+i0,\ \ 0+i0,\ \ 1+i0),\\ (&-1.12253+i0.00713,\ \ 0.75822+i0.00103,\ \ 1+i0),\\ (&\ \ 1.63808+i0,\ \ 0.22867+i0.00002,\ \ 1+i0),\end{aligned}$$

$$\begin{array}{lll}
(\ 1.10920+i0, & -0.76826+i0, & 1+i0), \\
(-1.60710+i0, & -0.56865+i0, & 1+i0), \\
(\ 0.08984+i0, & -0.71265+i0, & 1+i0), \\
(-1.29606-i0.00001, & -0.60508+i0, & 1+i0)\}
\end{array}$$

Problem 5.

$$\begin{aligned}
4x_1^3-3x_1-x_2&=0\\
x_1^2-\ x_2\qquad &=0
\end{aligned}$$

This is the Problem 2 from [6]. ENS $=4$, ENPS $=3$, ENIS $=1$,
ES $=\{(1,1,1),\ (0,0,1),\ (-0.75,0.5625,1),\ (0,1,0)\}$,
BHM $=\{(0,0,1),\ (-0.75,0.5625,1),$

$$\begin{array}{lll}
(\ 0.99995+i0.00001, & 0.99969+i0.0003, & 1+i0), \\
(-0.04171+i0.01696, & 1.672-i0.30847, & 0.00071-i0.00045), \\
(\ 0.00259-i0.03256, & 1.6598-i0.36898, & -0.00033-i0.00017), \\
(\ 0.03984+i0.01108, & 1.45203-i0.88389, & 0.0004+i0.00077)\}
\end{array}$$

Problem 6.

$$\begin{aligned}
&1000x_1(\lambda_{11}-\lambda)/\lambda_{11}+a_1x_1^3+a_2x_3^3+a_3x_1^2x_3+a_4x_1x_2^2+a_5x_1x_3^2\\
&\qquad\qquad +a_6x_1x_4^2+a_7x_2^2x_3+a_8x_3x_4^2=0\\
&1000x_2(\lambda_{21}-\lambda)/\lambda_{21}+a_4x_1^2x_2+b_1x_2^3+b_2x_2x_3^2+b_3x_2x_4^2+b_4x_1x_2x_3=0\\
&1000x_3(\lambda_{31}-\lambda)/\lambda_{31}+c_3x_1^3+a_5x_1^2x_3+a_7x_1x_2^2+a_8x_1x_4^2+c_1x_3^3\\
&\qquad\qquad +b_2x_2^2x_3+c_2x_3x_4^2+c_4x_1x_3^2=0\\
&1000x_4(\lambda_{22}-\lambda)/\lambda_{22}+a_6x_1^2x_4+d_1x_4^3+b_3x_2^2x_4+c_2x_3^2x_4+d_2x_1x_3x_4=0
\end{aligned}$$

where the values of the parameters and the obtained roots are given in Table 1. This problem is obtained by substituting truncated power series into the von Kármán equation describing the buckling of a square plate simply supported at the boundary, taking into account the geometric nonlinearities. The Bézout number D for this problem is 81 but it was not necessary to run all the 81 paths. From a simple sign analysis of the original polynomial system it follows that with each root (x_1, x_2, x_3, x_4) the vectors

$$(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4),\ (-x_1,x_2,-x_3,x_4),\ (x_1,-x_2,x_3,x_4),\ (x_1,x_2,x_3,-x_4) \tag{46}$$

are roots of this system as well. Hence having a root we may write up to 15 other roots. The computer runs were stopped after all $D=81$ different roots have been obtained by (46) from the roots corresponding to the computed endpoints of the paths. In this way we had to follow only 62 from the 81 paths. Table 1 shows only one root from each group of roots obtained from it by (46). All roots were proper so in the table we omit the homogeneous variable $x_0=1$. NDR denotes the number of different roots obtained from the corresponding root by (46).

Table 1. Constants and results to Problem 6
BHM

x_1	x_2	x_3	x_4	NDR
$0-i28.192$	$94.338+i0$	$0+i77.358$	$0+i0$	4
$-19.094+i14.452$	$-4.81+i14.164$	$24.201+i30.207$	$43.634+i10.935$	16
$14.412-i15.332$	$0+i0$	$72.724+i8.22$	$0+i0$	4
$-15.197+i0$	$0+i0$	$33.9+i0$	$0+i29.085$	4
$11.616-i6.136$	$24.027-i3.932$	$47.901+i13.056$	$0+i0$	8
$0-i5.575$	$0-i19.046$	$0-i28.7$	$48.792+i0$	8
$0+i0$	$0-i24.434$	$0+i0$	$45.35+i0$	4
$-18.939-i15.879$	$0+i0$	$22.815-i33.44$	$42.035-i7.958$	8
$0+i8.504$	$0+i0$	$0+i37.666$	$46.068+i0$	4
$7.669+i0$	$0+i30.21$	$-48.088+i0$	$-35.156+i0$	8
0	0	0	0	1
11.803	0	-34.845	0	2
-20.213	-26.937	-14.453	0	4
0	0	0	13.811	2
0	-51.57	0	0	2
11.803	0	34.845	0	2

$a_1=3.9314884$	$b_1=0.64674841$	$d_1=0.3276746$
$a_2=-0.10265241$	$b_2=0.43085621$	$d_2=0.4312049$
$a_3=-2.0073028$	$b_3=1.0240673$	$\lambda=17\pi^2$
$a_4=2.6326882$	$b_4=0.2911508$	$\lambda_{11}=4\pi^2$
$a_5=1.2427438$	$c_1=0.16362693$	$\lambda_{21}=25\pi^2/4$
$a_6=0.93419692$	$c_2=0.30115442$	$\lambda_{31}=100\pi^2/9$
$a_7=0.1455754$	$c_3=-0.66910093$	$\lambda_{22}=16\pi^2$
$a_8=0.21560245$	$c_4=-0.30795742$	

Our experience shows that the method based on the bounded homotopy is useful for small problems (in size - number of equations - and in the degree of the polynomials). On the other hand it is reliable in finding all complex solutions to any system of polynomials.

The only computer program of a path following algorithm for the solution of polynomial equations we know of is reported in [6]. This program is based on the algorithm presented in [3]. To compare our algorithm to that of [6, 3] note that:
- we did not try anything like its "heuristic mode of algorithm" (which gives no theoretical guarantee to find all solutions),
- we follow only D paths compared to $(d_1+1)\cdot(d_2+1)\dots(d_n+1)$ paths in [6, 3],
- paths to be followed by our method are all bounded unlike in [3], where at least $(d_1+1)\dots(d_n+1)-D$ paths espace to infinity at $t=1$.

Appendix

A 1. Stratifications

Let M be a (C^r, $1<r<\infty$; real analytic) manifold, $A\subset M$. By a (C^r; real analytic) stratification $\mathscr{S}$ of A we understand a locally finite partition of A into connected (C^r; real analytic respectively) manifolds (called strata) with the following property: If P, $Q \in \mathscr{S}$, $P \neq Q$, $P \cap \operatorname{cl} Q \neq \emptyset$, then $P \subset \operatorname{cl} Q$ and $\dim P < \dim Q$.

By the dimension of a stratification we understand the maximum of the dimensions of its strata.

A collection $\mathscr{S}$ of subsets of M will be said to be compatible with $A \subset M$, if for each $S \subset \mathscr{S}$ either $S \subset A$ or $S \cap A = \emptyset$; $\mathscr{S}$ will be said to be compatible with a collection $\mathscr{A}$ of subsets of M if it is compatible with each $A \in \mathscr{A}$.

Let M, N be differentiable manifolds, $A \subset M$, f: $A \to N$ be differentiable. By a stratification of f we understand a pair of stratifications $(\mathscr{S}, \mathscr{T})$ of A, N respectively such that for each $S \in \mathscr{S}$ we have $f(S) \in \mathscr{T}$ and $\operatorname{rank} f|_S = \dim f(S)$. Also, this stratification is said to be one-one if f is one-one on each S such that $\dim S = \dim f(S)$.

A2. Semialgebraic Sets

A subset $A \in \mathbb{R}^n$ that can be defined by a finite number of algebraic equalities and inequalities is called semialgebraic. That is, A can be expressed as

$$A = \{x \in \mathbb{R}^n \mid P_i(x) = 0,\ Q_j(x) < 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q\},$$

where P_i, Q_j are polynomials.

The class of semialgebraic sets has the following important properties:

1. The closure, interior and boundary of a semialgebraic set is semialgebraic.

2. Difference, finite union and finite intersection of semialgebraic sets is semialgebraic.

3. Let $A \subset \mathbb{R}^n$ be semialgebraic, P: $\mathbb{R}^m \to \mathbb{R}^n$, Q: $\mathbb{R}^n \to \mathbb{R}^p$ polynomial. Then, $P^{-1}(A)$, $Q(A)$ are semialgebraic (Tarski-Seidenberg theorem).

The important (well known) property of semialgebraic sets we frequently use in the paper is the following one: A semialgebraic subset of $\mathbb{R}^n$ admits a finite analytic stratification.

A3. A Theorem of Hardt

A less known theorem strengthening considerably the stratification result of $A2$ has been proved by Hardt [4]:

Let P: $\mathbb{R}^m \to \mathbb{R}^n$ be polynomial and let $\mathscr{C}$, $\mathscr{D}$ be finite collection of semialgebraic subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively. Then, there exists a finite analytic one-one stratification $(\mathscr{S}, \mathscr{T})$ of P such that $\mathscr{S}$, $\mathscr{T}$ are compatible with $\mathscr{C}$, $\mathscr{D}$ respectively.

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