Systems & Control Letters 5 (1984) 75–80
North-Holland

November 1984

On the assignment of invariant factors by time-varying feedback strategies

J.L. WILLEMS *

University of Gent, Grotesteenweg 2, 9710 Gent (Zwijnaarde), Belgium

V. KUČERA

Institute of Information Theory and Automation, Czechoslovak Academy of Sciences, 18208 Prague, Czechoslovakia

P. BRUNOVSKÝ

Institute of Applied Mathematics, Comenius University, 84215 Bratislava, Czechoslovakia

Received 2 July 1984
Revised 20 August 1984

In this paper the realization of arbitrary invariant factors or cyclic components of the closed-loop system matrix is discussed. It is shown that the use of time-varying feedback strategies enables one to realize arbitrary invariant factors in contrast with time-invariant feedback control.

Keywords: Invariant factors, Time-varying feedback.

1. Introduction

One of the most celebrated and famous results in the theory of linear time-invariant feedback control systems is the pole assignment property: if a linear time-invariant control system of dimension n is controllable, then for any given symmetric set S of n complex numbers there is a time-invariant linear state feedback control strategy such that S is the set of eigenvalues of the closed-loop system matrix [1]. A related interesting result has been reported by Rosenbrock [2, p. 190] and Kučera [3]: if the set S requires coinciding eigenvalues, then the multiplicity of these eigenvalues as zeros of the minimal polynomial cannot arbitrarily be assigned. In other words, there are limitations on the sizes of the cyclic components (the blocks in the Jordan canonical form) of the closed-loop system matrix or on its invariant factors.

In this note it is investigated whether the limitations cannot be relaxed by means of time-varying feedback strategies. In particular periodic strategies are considered; then the concept of system eigenvalues still makes sense, namely the eigenvalues of the transition matrix computed over one period [4]. The main result is that for discrete-time systems periodic feedback control leads to a completely unrestricted assignment of the eigenvalues and the invariant factors; some partial but incomplete results are obtained for continuous-time systems.

2. Problem statement and preliminary results

Consider the linear time-invariant discrete-time system ($t \in \mathbb{Z}$)

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state of the system, $u_t \in \mathbb{R}^m$ the input, and A and B constant matrices of appropriate dimension. The system (1) is assumed to be controllable; the (trivial) case that B has linearly dependent columns is excluded. Controllability ensures that pole assignment is possible; for any prescribed symmetric set of n complex numbers $\{\lambda_i\}$ (such that $\{\lambda_i\} = \{\lambda_i^*\}$), there exists a real feedback matrix K such that the spectrum $\sigma(A+BK)$ of the closed-loop system matrix is equal to the given set $\{\lambda_i\}$. However [2,3] there are restrictions on the sizes of the cyclic components or the invariant factors of $A+BK$ that can be achieved; these limitations can explicitly be expressed in terms of the *controllability indices* of (1). These limitations can obviously not be overcome by means of linear time-invariant dynamic state feedback.

* J.L. Willems acknowledges research support from the Belgian Fund for Joint Basic Research (FKFO Grant).

The use of time-varying periodic feedback strategies makes it possible to overcome the restrictions. This is illustrated by the use of the periodic dynamic feedback strategy

$$u_{kn+m} = K_m x_{kn} \tag{2}$$

with $k \in \mathbb{Z}$, $m \in \mathbb{Z}$, $0 \leqslant m \leqslant n-1$. The system equation yields

$$x_{(k+1)n} = \phi x_{kn} \tag{3}$$

where

$$\phi := A^n + A^{n-1}BK_0 + A^{n-2}BK_1 + \cdots + ABK_{n-2} + BK_{n-1}.$$

Equation (3) corresponds to a linear time-invariant closed-loop system with a sampling interval equal to n times the sampling interval of the original system and with system and input matrices

$$A_1 := A^n, \tag{4}$$

$$B_1 := \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}. \tag{5}$$

It is clear from the classical pole assignment result that the spectrum of ϕ can arbitrarily be assigned by suitable choice of the feedback matrices $K_0, K_1, \ldots, K_{n-1}$. Moreover, since the matrix B_1 has rank n, all controllability indices of the pair (A_1, B_1) are equal to 1. Hence not only the cyclic components or the invariant factors can arbitrarily be assigned, but even *any transition matrix* ϕ can be obtained.

Remark 1. The period of the feedback strategy can be reduced to r sampling intervals, with r the largest controllability index of the original system (1). The limitations on the cyclic components become monotonically weaker if the period grows from 1 sampling interval (time-invariant feedback) to r sampling intervals.

3. Main result

In the previous section it was shown that the eigenvalues as well as the invariant factors can arbitrarily be assigned by a periodic feedback of the state at the beginning of the period. This corresponds to a dynamic control strategy. The question then arises whether or not the same result can be achieved using only periodic feedback of the present state, that is static feedback. This would lead to a straightforward generalization of the classical pole-assignment result.

Therefore the periodic feedback

$$u_{kn+m} = K_m x_{kn+m} \tag{6}$$

with $k \in \mathbb{Z}$, $m \in \mathbb{Z}$, $0 \leqslant m \leqslant n-1$, is considered. Then

$$x_{(k+1)n} = \phi(n) x_{kn} \tag{7}$$

with

$$\phi(n) := (A + BK_{n-1})(A + BK_{n-2}) \cdots (A + BK_1)(A + BK_0).$$

Equation (7) governs the state at times kn, $k \in \mathbb{Z}$. It can be viewed as having resulted from the system

$$x_{(k+1)n} = A_1 x_{kn} + B_1 V_{kn}$$

where A_1 and B_1 are defined by (4) and (5), by applying the control

$$v_{kn} := \begin{bmatrix} u_{kn}^{\mathrm{T}} & u_{kn+1}^{\mathrm{T}} & \cdots & u_{(k+1)n-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Hence

$$v_{kn} = \bar{K} x_{kn} = \begin{bmatrix} \bar{K}_0 \\ \bar{K}_1 \\ \vdots \\ \bar{K}_{n-1} \end{bmatrix} x_{kn}$$

with

$$\bar{K}_i = K_i \phi(i), \tag{8}$$

$$\phi(i) := (A + BK_{i-1})(A + BK_{i-2}) \cdots (A + BK_0)$$

for $i = 1, 2, \ldots, n-1$, and $\phi(0)$ the identity matrix.

As pointed out in the previous section it is possible to select the $\bar{K}_i$ such that the matrix $A_1 + B_1\bar{K}$ has any symmetric set of eigenvalues and any compatible set of invariant factors. However it is not obvious that the same can be achieved by means of the matrices K_i. Indeed (8) may not be solvable with respect to the matrices K_i. It can occur that for some j the solutions $K_0, K_1, \ldots, K_j$ of (8) yield a singular $\phi(j+1)$; then a solution K_{j+1} may not exist. This is even true when the desired $\phi(n)$ has no zero eigenvalues. The issue is settled by means of the following theorem.

Theorem 1. *If the pair (A, B) is controllable, then there exists a feedback strategy (6) such that the closed-loop system matrix $\phi(n)$ has a given symmetric set of eigenvalues and a prescribed compatible set of invariant factors.*

Proof. Since system (1) is controllable, a preliminary time-invariant feedback exists such that all eigenvalues of the closed-loop system matrix are zero and the matrix is cyclic.

Since the closed-loop system is cyclic and controllable, it is controllable from a single input channel of (1). Assume that the closed-loop system is transformed to the standard controllable form [4] with respect to such single input u'_t, where no further input is applied to the other input channels. If the transformed state is denoted by x'_t, the system equation becomes

$$x'_{t+1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} x'_t + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u'_t. \tag{9}$$

For this system the above defined matrices A_1 and B_1 are respectively the null matrix and the identity matrix of order n.

Let M be a real matrix having the desired eigenvalues and invariant factors. Denote its rank by r. Suppose that by means of a similarity transformation, which does not alter the eigenvalues and the invariant factors, the matrix M is transformed to M' whose r leading principal minors are non-zero. This can be achieved by the algorithm discussed in the appendix. It is shown below that there exists a periodic feedback control u'_t such that $\phi(n)$ is equal to M'.

Let the periodic feedback be

$$u'_{kn+m} = K'_m x'_{kn+m}$$

and let the rows of M' be denoted by $\bar{K}'_i$:

$$M' = \begin{bmatrix} \bar{K}'_0 \\ \bar{K}'_1 \\ \vdots \\ \bar{K}'_{n-1} \end{bmatrix}.$$

Then the equations (8) are

$$K'_0 = \bar{K}'_0, \quad K'_1 Q_1 = \bar{K}'_1, \quad \dots, \quad K'_{n-1} Q_{n-1} = \bar{K}'_{n-1}, \tag{10}$$

where

$$Q_i = \begin{bmatrix} q_{i+1} \\ \vdots \\ q_n \\ \bar{K}'_0 \\ \vdots \\ \bar{K}'_{i-1} \end{bmatrix}$$

for $i = 1, \dots, n-1$, with q_k a row vector with zero entries except for the k-th entry which is equal to one.

Since the r leading principal minors of M' do not vanish, the matrices $Q_1, \dots, Q_{r-1}$ are nonsingular; hence the equations (10) can be solved for $K'_0, K'_1, \dots, K'_{r-1}$. This concludes the analysis if none of the desired eigenvalues is zero, since then $r = n$. If there are zero eigenvalues, and hence $r < n$, the equations for $K'_r, K'_{r+1}, \dots, K'_{n-1}$ can also be solved, since the rows $\bar{K}'_r, \bar{K}'_{r+1}, \dots, \bar{K}'_{n-1}$ of M' are linearly dependent on the preceding rows $\bar{K}'_0, \bar{K}'_1, \dots, \bar{K}'_{r-1}$; the latter being rows of $Q_r, Q_{r+1}, \dots, Q_{n-1}$ as well.

This concludes the proof of Theorem 1 since the sum of the preliminary linear time-invariant feedback and the designed linear periodic feedback yields obviously a linear periodic feedback. □

Remark 2. Theorem 1 proves that any invariant factors can be assigned to the closed-loop system matrix $\phi(n)$. However this does not mean that an arbitrary $\phi(n)$ can be realized; the reason is that a similarity transformation on the desired closed-loop system matrix had to be introduced in the proof of the theorem. This remark is illustrated by the following example. Let the system data be

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This system is controllable. Let the desired closed-loop system matrix be

$$\phi(n) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For this example the set of equations (8) is

$$K_0 = [0 \quad 1], \qquad K_1 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = [1 \quad 0].$$

Obviously this set of equations has no solution.

Theorem 1 only states that in the orbit of any matrix with respect to the similarity transformation, the closed-loop system matrix can be assigned to at least one element. The eigenvalues and the invariant factors are clearly invariant on such orbit.

Remark 3. The preceding remark points out an essential difference between what can be achieved by means of periodic feedback of the present state and by means of periodic feedback of the state at the beginning of each period. The limitations for the former case are discussed in Remark 2. The analysis of Section 2 shows that in the latter case any closed-loop system matrix can be obtained by the feedback strategy (2).

Remark 4. The feedback strategy used in the proof of Theorem 1 consists of a time-invariant feedback in all input channels and an additional periodic feedback in a single input channel. Any input channel can be used for this time-varying feedback. The period is n. It is readily checked that the period can be reduced if time-varying feedback is used in more than one input channel.

4. Continuous-time systems

The next question considered is whether the same results can be derived for the controllable continuous-time system described by the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{11}$$

The feedback strategy using the state at the beginning of each period can also be applied here; consider the impulsive periodic control, whose period is denoted by T, which in $(kT, kT+T)$, $k \in \mathbb{Z}$, is given by

$$\begin{aligned} u(t) = {} & K_0 x(kT)\delta_0(t-kT-\tau) \\ & + K_1 x(kT)\delta_1(t-kT-\tau) \\ & + \cdots + K_{n-1}x(kT)\delta_{n-1}(t-kT-\tau) \end{aligned} \tag{12}$$

with $0 < \tau < T$, where δ_0 denotes the Dirac impulse and δ_i its i-th derivative. From (11) one obtains

$$x(kT+T) = \phi(T)x(kT)$$

with

$$\begin{aligned} \phi(T) := {} & \exp(AT) \\ & + \exp[A(T-\tau)]\big(BK_0 + ABK_1 \\ & + \cdots + A^{n-1}BK_{n-1}\big). \end{aligned}$$

Since the matrix exponential is nonsingular, the eigenvalues and the invariant factors of $\phi(T)$ can arbitrarily be assigned, as discussed in Section 2. The impulsive controls can then be replaced by an equivalent smooth periodic feedback control strategy, as in [5].

A different approach consists of converting the continuous-time system into a discrete-time system by sampling and considering constant inputs between sampling times. If the sampling interval is suitably chosen, the obtained discrete-time system is controllable. Both algorithms discussed in Sections 2 and 3 can then be applied to assign the invariant factors. This yields a state feedback which is not impulsive, but a finite gain feedback of either the state at the beginning of the period or the state at the sampling times; the control inputs then remain constant between sampling times.

The question that remains unanswered is what can be achieved by time-varying feedback of the instantaneous state, i.e. by inputs of the form

$$u(t) = K(t)x(t). \tag{13}$$

Compared to the discrete-time case, the pole assignment problem for the transition matrix $\phi_K(T)$ of the system

$$\dot{x}(t) = [A + BK(t)]x(t)$$

for some fixed positive T has an obvious intrinsic restriction: Liouville's theorem shows that the product of the eigenvalues of $\phi_K(T)$ equals

$$\det \phi_K(T) = \exp\left\{\int_0^T \mathrm{tr}[A + BK(t)]\,\mathrm{d}t\right\}$$

and is positive; the question remains whether for any $(n \times n)$-matrix M with positive determinant there exists a feedback matrix $K(t)$ such that $\phi_K(T)$ is similar to M.

One way to construct a periodic feedback matrix was suggested by Brunovský [5] for the problem of pole assignment for controllable linear periodic systems. In that paper the feedback control consists of impulsive feedbacks of the instantaneous state at n suitably chosen times within a period, where n is the dimension of the state vector. The limitations of that approach for the present problem are discussed by means of an illustrative example. The second-order system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{14}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

is considered. Let the period T be equal to 2π. We try to find a periodic feedback strategy such that $\phi_K(T)$ is on the similarity orbit of $-I$. However, this orbit consists only of $-I$. For $t_1 < t_2$ in $(0, T)$ and the feedback control in that period

$$u(t) = [k_1\delta_0(t - t_1) + k_2\delta_0(t - t_2)]\, x(t), \tag{15}$$

we obtain [5]

$$\phi_K(T) = \exp(AT)\exp(-At_2)\exp(bk_2) \cdot \exp(At_2 - At_1)\exp(bk_1)\exp(At_1).$$

The question is whether t_1, t_2, k_1 and k_2 can be chosen such that $\phi_K(T)$ is equal to $-I$. Using

$$\exp(bk_i) = I + \psi(k_i b) bk_i$$

where

$$\psi(y) := [\exp(y) - 1]/y \quad \text{for } y \neq 0$$

and

$$\psi(0) := 1,$$

one readily obtains

$$\phi_K(T) = [I + \psi(\tilde{k}_2\tilde{b}_2)\tilde{b}_2\tilde{k}_2][I + \psi(\tilde{k}_1\tilde{b}_1)\tilde{b}_1\tilde{k}_1]$$

with, for $i = 1, 2$,

$$\tilde{b}_i := \exp(-At_i)b, \qquad \tilde{k}_i := k_i \exp(At_i).$$

The condition

$$\phi_K(T) = -I$$

is hence equivalent to

$$\tilde{b}_1\tilde{k}_1\psi(\tilde{k}_1\tilde{b}_1) + \alpha\tilde{b}_2\tilde{k}_2\psi(\tilde{k}_2\tilde{b}_2) = -2I \tag{16}$$

with

$$\alpha = [1 + \tilde{k}_2\tilde{b}_2\psi(\tilde{k}_2\tilde{b}_2)]^{-1}$$

where t_2 and k_2 should be such that

$$1 + \tilde{k}_2\tilde{b}_2\psi(\tilde{k}_2\tilde{b}_2) \neq 0.$$

Equation (16) yields

$$\tilde{k}_1\psi(\tilde{k}_1\tilde{b}_1) = \left[\frac{2\cos(t_2)}{\sin(t_1 - t_2)} \quad \frac{2\sin(t_2)}{\sin(t_1 - t_2)}\right], \tag{17}$$

$$\alpha\tilde{k}_2\psi(\tilde{k}_2\tilde{b}_2) = \left[\frac{2\cos(t_1)}{\sin(t_2 - t_1)} \quad \frac{2\sin(t_1)}{\sin(t_2 - t_1)}\right]. \tag{18}$$

Since

$$\tilde{b}_1 = \begin{bmatrix} -\sin(t_1) \\ \cos(t_1) \end{bmatrix},$$

equation (17) shows that $\tilde{k}_1\tilde{b}_1$ should satisfy

$$\tilde{k}_1\tilde{b}_1\psi(\tilde{k}_1\tilde{b}_1) = -2.$$

This is impossible for the function ψ defined above. The obtained result is a special case of a general solvability condition derived by Brunovský [5]. Hence the impulsive state feedback strategy cannot realize a matrix $\phi_K(T)$ in the similarity orbit of the matrix $-I$. On the other hand the linear time-invariant feedback strategy

$$u(t) = \tfrac{3}{4}x_1(t)$$

yields the closed-loop system equations

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 0 \end{bmatrix} x(t)$$

and hence

$$\phi_K(2\pi) = -I.$$

This example shows that the type of periodic feedback strategies used by Brunovský [5] does not exhaust all possibilities of the periodic feedback for the assignment of invariant factors and that a new idea for constructing suitable periodic feedback controls still has to be found. The problem whether or not the invariant factors can arbitrarily be assigned (provided $\det \phi_K(T) > 0$) has not been completely solved.

5. Conclusion

It has been shown in this paper that linear time-varying state feedback for linear time-invariant discrete-time systems enables one to realize not only arbitrary eigenvalue assignment, but also assignment of arbitrary invariant factors. Partial results are discussed for the same problem for continuous-time systems.

Appendix

Lemma 1. *For any real square matrix A of order n and rank r, there exists a real nonsingular matrix T of the same order, such that the first r leading principal minors of $T^{-1}AT$ are non-zero.*

The proof of this lemma is a direct consequence of the following two propositions and the induction argument.

Proposition 1. *There exists a similarity transformation such that, for any $k \leqslant r$, the k-th leading principal minor is non-zero.*

Proof. (i) There exists a similarity transformation involving only elementary row and corresponding column operations such that the first k rows of the transformed matrix A' are linearly independent. Let A' be partitioned as follows:

$$A' := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with

$$A_{11} \in \mathbb{R}^{k \times k}, \qquad A_{12} \in \mathbb{R}^{k \times (n-k)},$$

$$A_{21} \in \mathbb{R}^{(n-k) \times k}, \qquad A_{22} \in \mathbb{R}^{(n-k) \times (n-k)}.$$

Thus

$$\operatorname{rank}[A_{11} \quad A_{12}] = k.$$

(ii) There exists a matrix $N \in \mathbb{R}^{(n-k) \times k}$ such that $A_{11} + A_{12}N$ is non-singular.

(iii) Let M be defined by

$$M := \begin{bmatrix} I_k & 0 \\ N & I_{n-k} \end{bmatrix}$$

where I_m denotes the identity matrix of order m. Then

$$M^{-1} = \begin{bmatrix} I_k & 0 \\ -N & I_{n-k} \end{bmatrix}.$$

The leading principal minor of order k of $M^{-1}A'M$ is the determinant of $A_{11} + A_{12}N$, which does not vanish. □

Proposition 2. *If the m-th leading principal minor of a square matrix is non-zero, then there exists a similarity transformation such that the $(m-1)$-th leading principal minor is non-zero, and such that the m-th, $(m+1)$-th, ..., leading principal minors are not affected.*

Proof. Let A_m be the upper left $(m \times m)$ submatrix of A; this submatrix is nonsingular. Apply Proposition 1 to A_m for $k = m - 1$, and denote by T_m the resulting transformation matrix. Then the matrix

$$T = \begin{bmatrix} T_m & 0 \\ 0 & I_{n-m} \end{bmatrix}$$

defines the required similarity transformation for A. □

References

[1] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd. ed. (Springer, New York, 1979).
[2] H.H. Rosenbrock, *State-Space and Multivariable Theory* (Wiley, New York, 1970).
[3] V. Kučera, Assigning the invariant factors by feedback, *Kybernetika* **17** (1981) 118–127.
[4] R.W. Brockett, *Finite Dimensional Linear Systems* (Wiley, New York, 1970).
[5] P. Brunovský, Controllability and linear closed-loop controls in linear periodic systems, *Journal of Differential Equations* **6** (1969) 296–313.