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Connecting orbits in scalar reaction diffusion equations

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CONTENTS

1 INTRODUCTION

We consider the flow of a one-dimensional reaction diffusion equation

(1.1) $$u_t = u_{xx} + f(u), \; x \in (0, 1)$$

with Dirichlet boundary conditions

(1.2) $$u(t, 0) = u(t, 1) = 0$$

Let v, w denote stationary, i.e. t-independent solutions. We say that v connects to w, if there exists an orbit $u(t, x)$ of (1.1), (1.2) such that

$$\lim_{t \to -\infty} u(t, \cdot) = v \qquad (1.3)$$

$$\lim_{t \to +\infty} u(t, \cdot) = w$$

i.e. $u(t, \cdot)$ is a heteroclinic orbit connecting v to w. In this report we address the following question:

(*) Given v, which stationary solutions w does it connect to?

For a certain class of ordinary differential equations, the question of orbits connecting stationary solutions arose from a study of shock waves via the viscosity method (Gelfand [13], 1959). Later, the main tool to analyze connecting orbits was Conley's topological index ([9], 1978) which found extensive applications to ODE travelling wave problems arising in the thoery of shocks as well as reaction diffusion systems (see [27, §24]). For PDE-flows, Henry [16, §5.3] studied the connection problem of equation (1.1) in its own right, using elementary geometrical arguments and invariant manifold theory. Some other special cases were treated by Conley and Smoller using Conley's index (see [10], [27, §24.D] and the references there). Their approach relied solely on the variational structure of (1.1), and did not exploit maximum principles. A more detailed discussion is postponed to §6.

Apparently it was Hale [14], 1981, who first recognized the importance of maximum principles, notably Matano's result on lap numbers [20], for revealing the Morse–Smale structure of the flow (1.1). Angenent [1] and Henry [17] then showed that stable and unstable manifolds of stationary solutions of (1.1), (1.2) necessarily intersect transversely, if they intersect at all. Note that v connects to w, iff the unstable manifold of v does intersect the stable manifold of w, provided v and w are hyperbolic stationary solutions. Under the additional assumption

$$f(0) = 0 < f'(0) \text{ and } s \cdot f''(s) < 0 \quad \text{for all } s \neq 0, \qquad (1.4)$$

which was also used in the work of Conley and Smoller, Henry [17] completely solved the connection problem by this transversality approach. His result is contained in our main theorem 1.1 below.

Condition (1.4) appears in the work of Chafee and Infante [8], 1974 on the global bifurcation of stationary solutions of

$$u_t = u_{xx} + \alpha^2 \mathrm{f}(u), \qquad (1.1)_\alpha$$

with Dirichlet boundary conditions. In particular, it guarantees that the nontrivial stationary branches bifurcating from zero at $\alpha_k = k\pi/\surd[f'(0)]$, $k = 1, 2, \ldots$ are globally parametrized over $\alpha \in (\alpha_k, \infty)$ (we assume sublinear

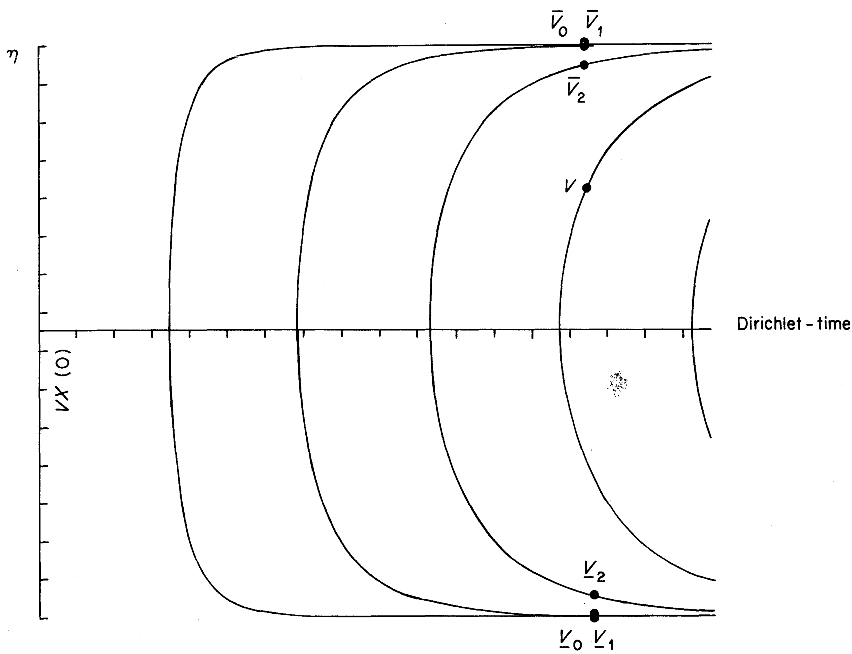

Fig. 1. Time map for $f(u) = -(u+1) \cdot u \cdot (u+1)$, Dirichlet problem.

growth of f, here) Stated loosely: nontrivial branches have no wiggles (cf. Fig. 1). In slightly different language, these results are contained in [3, ch. VI.10], 1959 already.

In this paper we investigate connecting orbits, even if the stationary branches have wiggles (cf. Fig. 2). We allow $f(0) \neq 0$ and drop assumption (1.4). This greatly increases the complexity of the problem, because it introduces many additional solutions. Besides the Dirichlet case, we also consider Neumann boundary conditions (§6).

Before we state our main result, we fix the technical setting of our investigation and pin-point the precise ingredients to our analysis. For the nonlinearity f we assume only

$$f \in C^2, \limsup_{|s| \to \infty} f(s)/s < \pi^2. \tag{1.5}$$

By $\mathscr{F}$ we denote the set of nonlinearities f satisfying (1.5). We endow $\mathscr{F}$ with the weak Whitney topology (cf. [19]). For $f \in \mathscr{F}$, (1.1), (1.2) define a strongly continuous semiflow on the solution space

$$u(t, \cdot) \in X := H^2 \cap H_0^1$$

(cf. [16]). Let $|\cdot|$ denote the H^2-norm on X. The growth condition (1.5) on f just ensures that solutions stay bounded in X for all time [16]. The topology

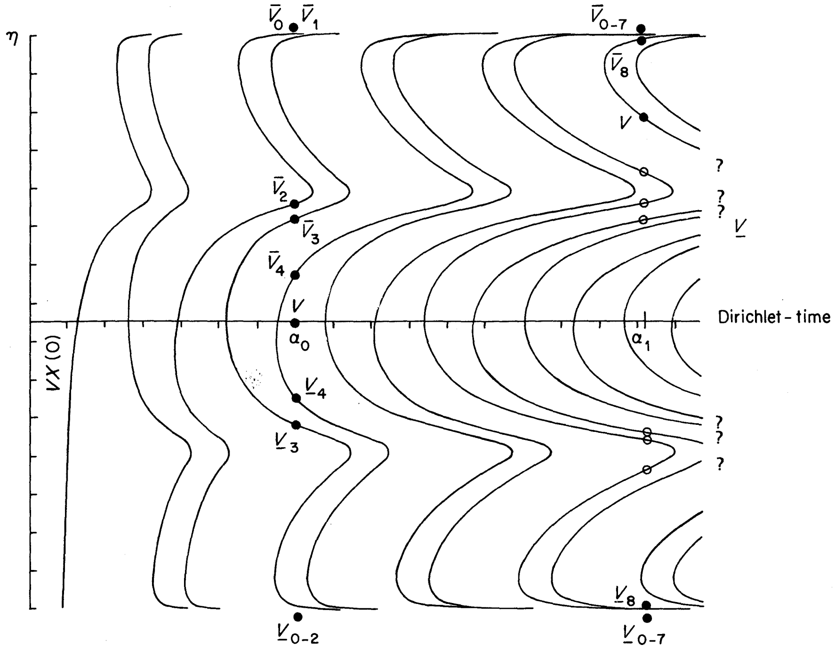

Fig. 2. Time map for $f(u) = -(u+10.2)\cdot u\cdot((u-4)^2+1.75^2)\cdot(u-10)$. Dirichlet problem

on $\mathscr{F}$ is suitable for continuity arguments (see lemmata 2.1 and 3.1 below).

For $f \in \mathscr{F}$ the *gradient structure* of (1.1) guarantees that every orbit tends to some equilibrium via the Ljapunov functional

$$V(u) := \int_0^1 \left(\tfrac{1}{2}u_x^2 - F(u)\right)\mathrm{d}x, \quad F'(s) := f(s) \tag{1.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V(u(t,\cdot)) = -\int_0^1 u_t^2\,\mathrm{d}x$$

[16]. Remember that V was the starting point for the Conley index approach.

Another (discrete) decreasing functional, going back to Nickel [23] essentially, is the *zero number* z. For continuous $\Phi: [0,1] \to \mathbb{R}$, the zero number $z(\Phi)$ for $\Phi \not\equiv 0$ is the maximal integer $n \leqslant \infty$ such that there exist $0 < x_0 < x_1 < \cdots < x_n < 1$ with

$$\Phi(x_i)\cdot\Phi(x_{i+1}) < 0 \quad (0 \leqslant i < n);$$

$z(0) := 0$. By maximum principle arguments (cf. §7 for more details) $t \to z(\tilde{u}(t,\cdot))$ is decreasing along solutions $\tilde{u}(t,\cdot)$ of

$$\tilde{u}_t = \tilde{u}_{xx} + g(x,\tilde{u}) \tag{1.7}$$

with mixed boundary conditions

(1.8) $$\cos\gamma_j \cdot \tilde{u}(t,j) - \sin\gamma_j \cdot \tilde{u}_x(t,j) = 0, \quad j = 0, 1$$

if $g(x,0)=0$. To avoid confusion of solutions $\tilde{u}(t,x)$ of (1.7) with solutions $u(t,x)$ of (1.1) we distinguish solutions as $\tilde{u}$ and u. On g, we assume

(1.9) $$g \in C^2, \limsup_{|s|\to\infty} g(x,s)/s < \infty \quad \text{uniformly in } x \in [0,1],$$

to guarantee global existence of solutions, and denote the set of those g by $\mathscr{G}$. Note that $\mathscr{F}$ may be viewed as a subset of $\mathscr{G}$. For arguments involving $z(u(t,\cdot))$ we also consider

(1.10) $$\mathscr{G}_0 := \{ g \in \mathscr{G} \mid g(x,0) = 0 \quad \text{for all } x \}.$$

Again, $\mathscr{G}$, $\mathscr{G}_0$ are endowed with the weak Whitney topology. Replacing F in (1.6) by $G(x,U)$ with $G_u = g$, equation (1.7) has a gradient structure again.

As a first application of the zero number z we consider a hyperbolic stationary solution v of (1.1), (1.2). By hyperbolic we mean that zero is not an eigenvalue of the linearization L at v

(1.11) $$Lu := u_{xx} + f'(v(x))u$$

(1.12) $$u(0) = u(1) = 0.$$

In our setting, L is a closed, densely defined linear operator on $X = H^2 \cap H_0^1$ with domain $\mathscr{D}(L) = H^4 \cap H_0^1$, cf. [16]. More specifically, L is self-adjoint on the Hilbert space X and Sturm–Liouville theory applies to L (see e.g. [2, §8; 3, ch. II; 15. ch. XI] for anything on Sturm–Liouville theory). Thus L has discrete spectrum consisting of simple real eigenvalues $\lambda_0 > \lambda_1 > \cdots$ accumulating at $-\infty$. The corresponding eigenfunctions are denotes by φ_k. By Sturm–Liouville theory, φ_k has exactly k sign changes, i.e. $z(\varphi_k) = k$. To investigate the zero number z in a slightly more nonlinear situation, let $W^{\mathrm{u}}(v)$ resp. $W^{\mathrm{s}}(v)$ denote the unstable resp. stable manifold of v (cf. [16]). These manifolds consist of those solutions $u(t,x)$ which tend to v as t tends to $-\infty$ (resp. $+\infty$). Let $i(v) := \dim W^{\mathrm{u}}(v)$ denote the instability index (Morse index) of v. Because the tangent space to $W^{\mathrm{u}}(v)$ resp. $W^{\mathrm{s}}(v)$ at v is given by the span of those eigenfunctions φ_k for which λ_k is positive resp. negative, $i(v)$ is just the number of positive eigenvalues of the linearization L in short:

$$\lambda_k > 0 \text{ implies } k < i(v).$$

If u is a solution of (1.1), (1.2) then $\tilde{u} := u - v$ is a solution of (1.7), (1.2) putting $g(x,\tilde{u}) := f(\tilde{u} + v(x)) - f(v(x))$, and $z(\tilde{u}(t,\cdot))$ is decreasing. Using this fact, it was proved in [5] that

(1.13) $$z(u_0 - v) < i(v) \text{ for any } u_0 \in W^{\mathrm{u}}(v)$$

and

$$z(u_0 - v) \geqslant i(v) \text{ for any } u_0 \in W^s(v)\backslash\{v\}.$$

Indeed,

$$\lim \frac{\tilde{u}(t,\cdot)}{|\tilde{u}(t,\cdot)|} = \pm\varphi_k$$

as $t \to -\infty$ exists for initial data $u_0 \in W^u(v)\backslash\{v\}$ and equals an eigenfunction φ_k of L with positive eigenvalue λ_k. By Sturm–Liouville theory, $z(\varphi_k) = k$ and we conclude that for $0 > t \to -\infty$

$$\begin{aligned} z(u_0 - v) &= z(\tilde{u}(0,\cdot)) \leqslant z(\tilde{u}(t,\cdot)) \\ &= z\left(\frac{\tilde{u}(t,\cdot)}{|\tilde{u}(t,\cdot)|}\right) \to z(\pm\varphi_k) = k < i(v) \end{aligned}$$

for $u_0 \in W^u(v)\backslash\{v\}$. A similar argument can be given for the stable manifold.
As another relation between i and z we mention

(1.14) $$i(v) \in \{z(v), z(v)+1\}$$

for any stationary solution $v \not\equiv 0$ of (1.1), (1.2). This is proved in §5, lemma 5.1 and serves to distinguish the possible cases in our main theorem below. For example, $i(v) = z(v)$ for $v \not\equiv 0$ and any 'Chafee–Infante–f' satisfying assumption (1.4) above.
For hyperbolic stationary v we define

(1.15) $$\Omega(v) := \{w \mid v \text{ connects to } w \not\equiv v\}$$

and for $0 \leqslant k < i(v)$
$\bar{v}_k$ is the stationary solution $\hat{v}$ with $z(\hat{v}) = k$ such that

$$\hat{v}_x(0) > |v_x(0)| \text{ is minimal,}$$

$\underline{v}_k$ is the stationary solution $\hat{v}$ with $z(\hat{v}) = k$ such that

$$\hat{v}_x(0) < -|v_x(0)| \text{ is maximal.}$$

Note that v enters into the definition of $\bar{v}_k$ and $\underline{v}_k$. Among other things, the proof of our main theorem 1.1 below will also guarantee existence of those $\bar{v}_k$, $\underline{v}_k$ which come up on its various statements. Basically, our growth condition (1.5) on the nonlinearity f is responsible for that. In case (1.5) is violated, some of the $\bar{v}_k$, $\underline{v}_k$ may not exist — see §6 for further discussion At any rate, the $\bar{v}_k$, $\underline{v}_k$ are uniquely defined. With this notation we can state our main result.

1.1 Main theorem

Let $f \in \mathscr{F}$ satisfy assumption (1.5) and let v be a hyperbolic stationary solution of (1.1), (1.2). Then v connects to other stationary solutions as follows.

(i) If $v=0$, or if $v \not\equiv 0$ and $i(v)=z(v)$, then

$$\Omega(v)=\{\underline{v}_k, \bar{v}_k \mid 0 \leqslant k < i(v)\}.$$

(ii) If $v_x(0)>0$ and $i(v)=z(v)+1$, then

$$\Omega(v)=\Omega_1 \cup \Omega_2 \cup \Omega_3$$

where

$\Omega_1=\{\bar{v}_k \mid 0 \leqslant k < i(v)\}$

$\Omega_2=\{\underline{v}_k \mid 0 \leqslant k < i(v)-1\}$ and either

$\Omega_3=\{\underline{v}_k \mid k=i(v)-1\}$ or

Ω_3 consists of one or several stationary solutions w with $-v_x(0) \leqslant w_x(0) \leqslant v_x(0)$ and $i(w)<i(v)$

(iii) If $v_x(0)<0$ and $i(v)=z(v)+1$, then similarly

$$\Omega(v)=\Omega_1 \cup \Omega_2 \cup \Omega_3$$

where

$\Omega_1=\{\underline{v}_k \mid 0 \leqslant k < i(v)\}$

$\Omega_2=\{\bar{v}_k \mid 0 \leqslant k < i(v)-1\}$ and either

$\Omega_3=\{\bar{v}_k \mid 0 \quad k=i(v)-1\}$ or

Ω_3 consists of one or several stationary solutions w with $v_x(0)<w_x(0) \leqslant -v_x(0)$ and $i(w)<i(v)$.

For w not necessarily hyperbolic, $i(w)$ denotes the number of strictly positive eigenvalues here.

We illustrate our results, first in a Chafee–Infante situation where f satisfies (1.4) and then for a more general f, $f(0) \neq 0$, which exhibits wiggles. For a more in-depth discussion see §6.

Any required information on v as well as the solution $\bar{v}_k$, $\underline{v}_k$ can be read off from the stationary global bifurcation diagram of

$$u_t=u_{xx}+\alpha^2 f(u) \qquad (1.1)_\alpha$$

with Dirichlet conditions. For a discussion of these difurcation diagrams with nonlinearity f in a generic class see [4, 24, 30]. We represent any stationary solution by a pair (α, η) where $\eta=\alpha^{-1} \cdot v_x(0)$ determines $v(\cdot)$. Scaling $\xi := \alpha x$, $v^*(\xi) := v(x)$, any solution v^* of

$$0=v^*_{\xi\xi}+f(v^*) \qquad (1.16)$$

with initial data $v^*(0)=0$, $v^*_x(0)=\eta$ which satisfies $v^*(\xi)=0$ at 'time' $\xi=T>0$ yields a solution ; of $(1.1)_\alpha$ for $\alpha := T$ with $v_x(0)=T\eta=\alpha\eta$. The

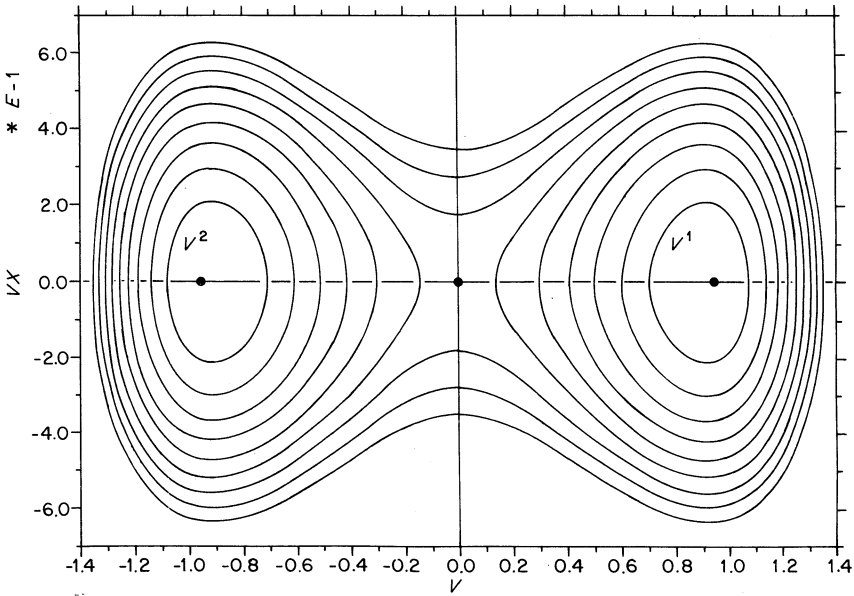

Fig. 3. Phase plane for $f(u) = (u+1)u(u-1)$

dependence $T = T(\eta)$ of the first zero of v^* on the initial data $\eta = v_x^*(0)$ is usually called the 'time-map' [4, 26, 27, 29–31].

In Figs 3, 4, 5 we present a selection of typical phase portraits of (1.16). Note that the Hamiltonian

$$H(v^*, v_\xi^*) := \tfrac{1}{2}(v_\xi^*)^2 + F(v^*) \tag{1.17}$$

is a first integral of (1.16), with $F' = f$ as before. To obtain the phase portrait, we just draw level curves of (1.17). To determine the time-map $T(\eta)$, however, we have to integrate (1.16).

How to obtain the stationary global bifurcation diagram for $(1.1)_\alpha$ once we know $T(\eta)$ for all η? Given the first positive zero $T(\eta)$ of v^* with $v^*(0) = 0$, $v_\xi^*(0) = \eta$ we may obtain the second zero at

$$\xi = T(\eta) + T(-\eta),$$

because $v^*(T(\eta)) = 0$ by definition, and $v_\xi^*(T(\eta)) = -v_\xi^*(0) = -\eta$ by the first integral (1.17). Here we assume $T(\eta)$ and $T(-\eta)$ are both finite, of course. Proceeding in this manner, we may in fact determine all zeros of v^* from $T(\eta)$ and $T(-\eta)$. Translating this information back to $(1.1)_\alpha$, we can now determine all values α such that the stationary boundary value problem $(1.1)_\alpha$ has a solution v with $v_x(0) = \alpha\eta$. Thus the time-map $T(\eta)$ generates the complete stationary bifurcation diagram of $(1.1)_\alpha$. Note that the ordering of solutions $v(\cdot)$ by $v_x(0)$ which determines $\bar{v}_k$, $\underline{v}_k$ is just the ordering of the corresponding

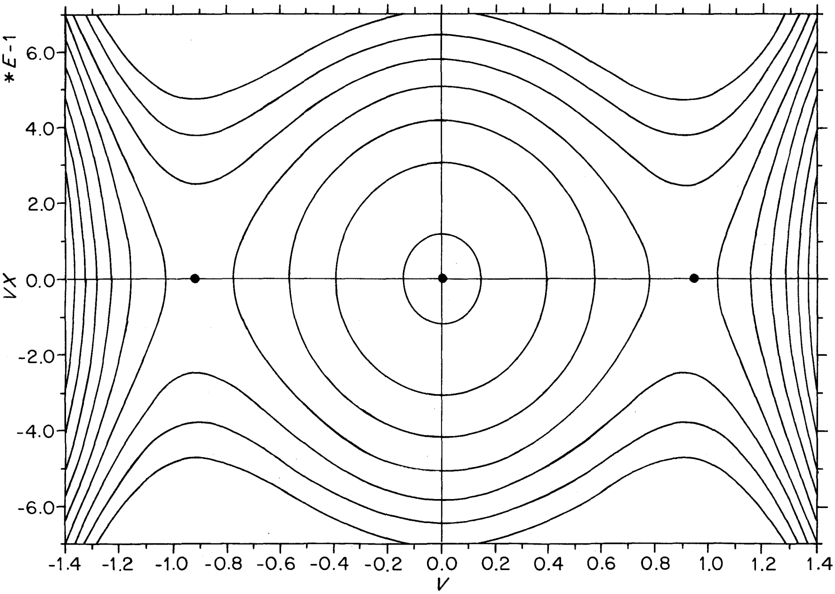

Fig. 4. Phase plane for $f(u) = -(u+1)u(u-1)$

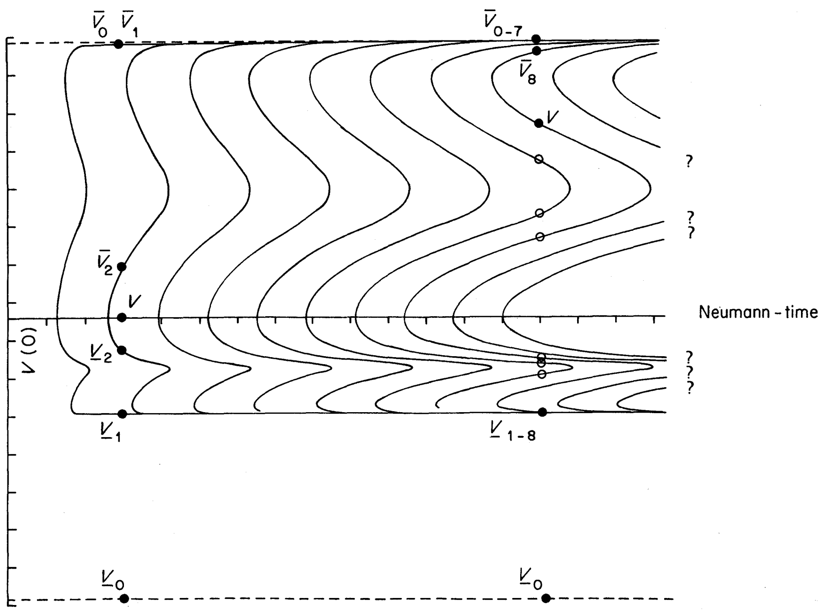

Fig. 5. Phase plane for $f(u) = -(u + 10.2) \cdot u \cdot ((u-4)^2 + 1.75^2) \cdot (u - 10)$

numbers η. This justifies it to represent any stationary solutions of $(1.1)_\alpha$ by the corresponding pair (α, η). For a few typical numerical examples see Figs 1, 2 and 6.

The numerical computations of the time-maps in Figs 1, 2, and 6 used the 27 January 1982 version of the package LSODAR due to L. R. Petzold (Sandia Nat. Lab.) and A. C. Hindmarsh (Lawrence Livermore Nat. Lab.) [18]. It includes automatic method switching between stiff and non-stiff problems, and root-finding. The runs were in double precision on the IBM 3081 D at Universitätsrechenzentrum Heidelberg with a required local relative accuracy of 10^{-4}. Implementation was done jointly by R. Schaaf.

Let us continue to extract from our bifurcation diagrams the information required by our main theorem 1.1. By uniqueness of solutions of the initial value problem (1.16), branches (α, η) with $\eta \neq 0$ are globally parametrized over η as $(\alpha(\eta), \eta)$ and do not intersect (the only possible intersections occur at $\eta = 0$) Moreover any stationary solution v with $\eta \neq 0$ has only simple zeros. Thus the zero number $z(v)$ is invariant along each branch. Put differently, the first, second, kth intersection point with the stationary diagram (α, η) on a line $\eta \equiv$ const. $\neq 0$ starting at $\alpha = 0$ corresponds to a solution v with $z(v) = 0, 1, k-1$ respectively. With this in mind, for given v, the $\bar{v}_k$, $\underline{v}_k$ are easily determined from diagrams like Figs 1, 2 as indicated.

Next we determine $i(v)$ from the stationary bifurcation diagram. It should

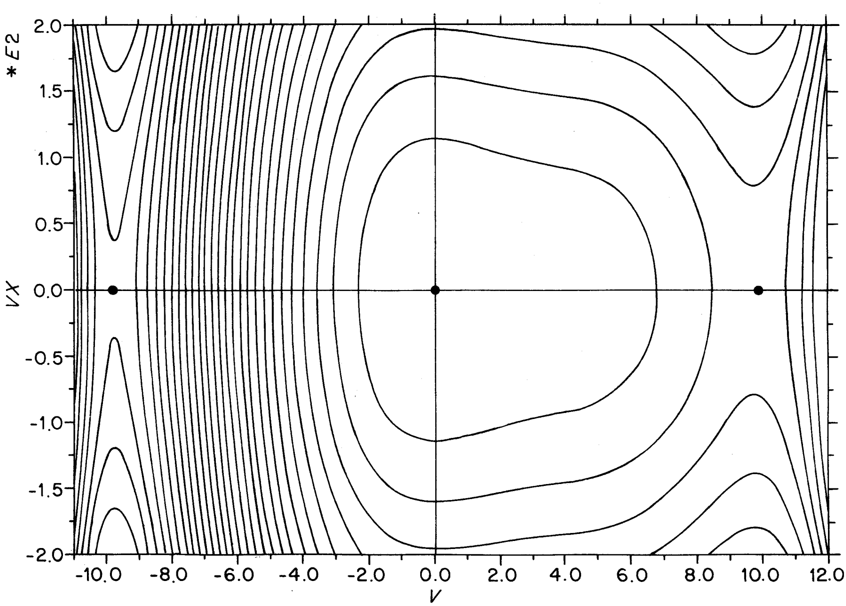

Fig. 6. Time map for $f(u) = -(u + 10.2) \cdot u \cdot ((u-4)^2 + 1.75^2) \cdot (u - 10)$. Neumann problem

be a little bit surprising that this is at all possible because $i(v)$, being the dimension of the unstable manifold at v, relates to the dynamics of $(1.1)_\alpha$ rather than to the plain stationary case. In lemma 5.1 below we prove that

$$i(v) \in \{z(v), z(v)+1\}$$

for hyperbolic stationary $v \equiv 0$, using Sturm–Liouville theory. Moreover it can be shown for (α_0, v) represented by (α_0, η_0) on the branch $(\alpha(\eta), \eta)$, $\eta_0 \neq 0$ that v is hyperbolic iff $\alpha'(\eta_0) \neq 0$ (cf. [27, 29]), and

$$\eta_0 \cdot \alpha'(\eta_0) > 0 \Rightarrow i(v) = z(v)$$
$$\eta_0 \cdot \alpha'(\eta_0) < 0 \Rightarrow i(v) = z(v)+1.$$

This determines which of the alternatives of the main theorem 1.1 applies to $v \not\equiv 0$. If on the other hand $v \equiv 0$, then the eigenvalues of the linearization L from (1.11), (1.12) of $(1.1)_\alpha$ are determined explicitly as

$$\lambda_k = -k^2 + \alpha^2 f'(0).$$

Denoting the bifurcation points from the trivial solution as $(\alpha_k, 0)$ with $\alpha_k^2 = k^2/f'(0)$ we see that $i(v) = \mathrm{k}$ for $v \equiv 0$ and α between α_k and α_{k+1}, provided $f'(0) > 0$. If $f'(0) < 0$ then trivially $i(v) = 0$ for all α.

As a first example suppose f satisfies (1.4), i.e. $f(0) = 0 < f'(0)$, $sf''(s) < 0$ for $s \neq 0$ in addition to growth condition (1.5). Then Birkhoff and Rota [3] and later Chafee and Infante [8] have proved that $\eta \cdot \alpha'(\eta) > 0$ for each nontrivial branch, and that for $\alpha > \alpha_k = k\pi^2/f'(0)$ there exist exactly two nontrivial solutions v with $z(v) = k-1$, one with $\eta > 0$ and one with $\eta < 0$. The typical bifurcation diagram is given in Fig. 1. We illustrate a case where $z(v) = 3$. We know $\eta_0 \cdot \alpha'(\eta_0) > 0$, hence $i(v) = z(v) = 3$ and case (i) of the main theorem implies that

$$\Omega(v) = \{\bar{v}_0, \bar{v}_1, \bar{v}_2, \underline{v}_0, \underline{v}_1, \underline{v}_2\}$$

as depicted in Fig. 1. In general, only case (i) occurs — this is the case analyzed by Henry in [17] and we recover his result.

Now we consider an example which exhibits wiggles (Fig. 2) A somewhat simplistic reason for this phenomenon is that f does not satisfy (1.4), this time. First we pick $\alpha = \alpha_0$ and $v = 0$ with $i(v) = 5$. Again case (i) of the theorem applies. This time, however, there is more than one stationary solution of a given zero number $z = 2, 3$ with $\eta > 0$ resp. $\eta < 0$. Thus minimality (maximality) in the definition of $\bar{v}_k(\underline{v}_k)$ comes into effect and we are left with

$$\Omega(v) = \{\bar{v}_k, \underline{v}_k \mid 0 \leqslant k \leqslant 4\}.$$

Next we pick $\alpha = \alpha_1$ and v with $z(v) = 8$ but $i(v) = 9$, $v_x(0) > 0$. Then

$$\Omega_1 = \{\bar{v}_0, \ldots, \bar{v}_8\}$$
$$\Omega_2 = \{\underline{v}_0, \ldots, \underline{v}_7\}$$

and either $\Omega_3 = \{\underline{v}_8\}$ or Ω_3 consists of one or several solutions w with

$$-v_x(0) \leqslant w_x(0) < v_x(0), \qquad i(w) < i(v) = 9.$$

The possible candidates for w are denoted by '?' in Fig. 2. Obviously, theorem 1.1 does not determine completely which other stationary solutions v connects to. For a conjecture how to resolve this problem we refer the reasonably impatient reader to §6. On the other hand, R. Schaaf [26] provides detailed information on the global bifurcation picture, if $f(0) < 0 < f''(0)$ and f is a polynomial with only real zeros. We believe that out theorem can solve the connection problem completely in that case. Note that f in Fig. 2 violates the above condition.

The rest of the paper is organized as follows. In §2 we construct the basic tool to establishing connections: the y-map. For given initial datum $\tilde{u}_0$, $y(\tilde{u}_0)$ completely describes the behaviour of $z(\tilde{u}(t, \cdot))$ and of sign $\tilde{u}_x(t, 0)$ along the semi-orbit $\tilde{u}(t, \cdot)$ of $\tilde{u}_0$ under equations (1.7) and (1.8). In §3 we show that y restricted to the unstable manifold $W^{\mathrm{u}}(v)$ induces an essential mapping of spheres. As a corollary, we obtain orbit connections to equilibria w with prescribed $z(w - v)$ and $\operatorname{sign}(w_x(0) - v_x(0))$ (cf (3.4 a,b)). The problem remains to identify w. This boils down to two basic lemmata given in §4. They allow us to pass from $z(w - v)$ to $z(w)$ and account for the minimality (maximality) occurring in the definition of $\tilde{v}_k$ ($\underline{v}_k$). Fitting everything together we prove our main theorem in §5. We devote §6 to a detailed discussion, including a comparison of our approach to those of Henry, and Conley, Smoller, an extension to Neumann boundary conditions (theorem 6.1), a conjecture on the complete answer to the connecting-orbit-problem in the Dirichlet case, and some open questions. In §7 we collect some background material on the behavior of the zero number along solutions $u(t, \cdot)$ of (1.1), (1.2).

2 THE y-MAP

In this section we construct a continuous mapping

$$y: \{\tilde{u}_0 \in X \mid z(\tilde{u}_0) \leqslant n, \quad \tilde{u}_0 \not\equiv 0\} \to S^n$$

where S^n denotes the standard n-sphere in $\mathbb{R}^{n+1}$. Knowing $y(\tilde{u}_0)$ we will know $z(\tilde{u}(t, \cdot))$, $t \geqslant 0$, all along the orbit $\tilde{u}(t, \cdot)$ of (1.7) starting at $\tilde{u}_0$. Moreover, y depends continuously on the nonlinearity $g \in \mathscr{G}_0$ defined in (1.10). Restricting y to an n- dimensional sphere Σ^n in the unstable manifold of $v \equiv 0$ will provide us with an essential mapping of spheres. With these properties in mind, we will immediately establish existence of connections in §3.

Throughout §2, we consider the equation

$$\text{(2.1)} \qquad \begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + g(x, \tilde{u}), \qquad 0 < x < 1 \\ \cos\gamma_j \cdot \tilde{u}(t, j) &- \sin\gamma_j \cdot \tilde{u}_x(t, j) = 0, \qquad j = 0, 1 \end{aligned}$$

under the restrictive assumption

$$\cos \gamma_0 \neq 0$$

i.e. we exclude the pure Neumann condition here. For ease of notation, we prefer to write u instead of $\tilde{u}$ in this section. Recall from the introduction that $z(u(t, \cdot))$ is non-increasing with t..

We construct the y-map. For $u_0 \in X, u_0 \not\equiv 0, z(u_0) \leqslant n$ with orbit $u(t, \cdot)$, define $t_k \in [0, \infty]$ to be the first time that the zero number $z(u(t, \cdot))$ drops below the k-level:

$$t_k := \inf\{t \geqslant 0 \mid z(u(t, \cdot)) \leqslant k\}, \ \tau_k := \tanh t_k \in [0, 1] \tag{2.2a}$$

Note that $0 = \tau_n \leqslant \tau_{n-1} \leqslant \cdots \leqslant \tau_0$. Further we define

$$\sigma_k := \begin{cases} \operatorname{sign} u_x(t, 0) \text{ for some } t \in (t_k, t_{k-1}), & \text{if } t_k < t_{k-1} \\ 0 & \text{otherwise} \end{cases} \tag{2.2b}$$

The sign σ_k is well defined because $u_x(t, 0) \neq 0$ for $t_k < t < t_{k-1}$ by lemma 7.4. The components of the map $y = (y_0, \ldots, y_n)$ are defined as

$$\begin{aligned} y_0 &:= \sigma_0 (1 - \tau_0)^{1/2} \\ y_k &:= \sigma_k (\tau_{k-1} - \tau_k)^{1/2}, \qquad 1 \leqslant k \leqslant n. \end{aligned} \tag{2.3}$$

By construction, y maps into S^n.

Suppose we know $y(u_0)$. Then we can reconstruct the dropping times t_k above uniquely. Moreover we obtain the signs σ_k in case $t_k \neq t_{k-1}$. As an important special case, suppose $y(u_0) = \sigma e_k$ where e_k denotes the kth unit vector, $\sigma \in \{-1, 1\}$. This implies $t_0 = \cdots = t_{k-1} = \infty$, $t_k = 0$ and therefore for all $t > 0$:

$$\begin{aligned} z(u(t, \cdot)) &= k \\ \sigma \cdot u_x(t, 0) &> 0. \end{aligned} \tag{2.4}$$

2.1 LEMMA The y-map (2.3) depends continuously on $g \in \mathscr{G}_0$ and on $u_0 \in X \backslash \{0\}$ with $z(u_0) \leqslant n$.

Proof Throughout the proof we use that the solution $u(t, \cdot)$ of (2.1), viewed as a C^1-function of x, depends continuously on g, u_0 and t. To be more specific, let this solution be denoted by $u(t, x) = u(t, x; g, u_0)$ emphasizing its actual dependence on g and u_0. Then the map

$$\begin{aligned} \mathscr{G} \times X \times [0, \infty) &\to C^1([0, 1], \mathbb{R}) \\ (g, u_0, t) &\mapsto (x \mapsto u(t, x; g, u_0)) \end{aligned}$$

is continuous, because it is the composition of the analogous map

$$\mathscr{G} \times X \times [0, \infty) \to H^2([0,1], \mathbb{R}) \cap H_0^1([0,1], \mathbb{R})$$

which is continuous by Henry [11], and the continuous Sobolev embedding

$$H^2([0,1], \mathbb{R}) \cap H_0^1([0,1], \mathbb{R}) \to C^1([0,1], \mathbb{R}).$$

Note that we use the weak Whitney topology on $\mathscr{G}$ here. This is sufficient because continuous dependence on initial data is a local property.

We show that $\tau_k \geqslant 0$ depends lower semicontinuously on $(g, u_0) \in \mathscr{G}_0 \times X$. First note that

$$\begin{aligned} z: C^0([0,1], \mathbb{R}) &\to \mathbb{Z} \\ \phi &\to z(\phi) \end{aligned}$$

is lower semicontinuous by definition of z. Together with continuity of $u(t, \cdot; g, u_0)$ and the definition of τ_k this implies: for any $\varepsilon > 0$ such that $\tau_k - \varepsilon > 0$, and for t defined by $\tanh t = \tau_k - \varepsilon$ there exists a neighborhood U of (g, u_0) in $\mathscr{G} \times X$ such that for any $(\hat{g}, \hat{u}_0) \in U$ we have

$$\begin{gathered} z(u(t, \cdot, \hat{g}, \hat{u}_0)) \geqslant z(u(t, \cdot; g, u_0)) > k, \text{ and hence} \\ \tau_k(\hat{g}, \hat{u}_0) \geqslant \tanh t = \tau_k(g, u_0) - \varepsilon. \end{gathered}$$

Thus τ_k is lower semicontinuous.

We show that $\tau_k \geqslant 0$ is upper semicontinuous if t_k is finite. In that case, lemma 7.3 implies that for any $\varepsilon > 0$ there exists some t such that $\tau_k < \tanh t < \tau_k + \varepsilon$ and all zeros of $x \to u(t, x; g, u_0)$ are simple. Using continuity of $u(t, \cdot; g, u_0) \in C^1$ and the definition of τ_k again, this implies: there exists a neighborhood U of $(g, u_0) \in \mathscr{G} \times X$ such that for any $(\hat{g}, \hat{u}_0) \in U$ we have

$$\begin{gathered} z(u(t, \cdot; \hat{g}, \hat{u}_0)) = z(u(t, \cdot, g, u_0)) \leqslant k, \text{ and hence} \\ \tau_k(\hat{g}, \hat{u}_0) \leqslant \tanh t < \tau_k(g, u_0) + \varepsilon. \end{gathered}$$

Thus τ_k is upper semicontinuous and, consequently, continuous.

Finally, we claim that each component y_k of the y-map depends continuously on $(g, u_0) \in \mathscr{G}_0 \times X$. We already know that $\tau := (\tau_0, \ldots, \tau_{n-1})$ depends continuously on (g, u_0). If $\tau_k < \tau_{k-1}$ at (g, u_0) then lemma 7.4 implies

$$u_x(t, 0; g, u_0) \neq 0$$

for any $t \in (t_k, t_{k-1})$. Fixing any such t, there exists a neighborhood U of (g, u_0) in $\mathscr{G} \times X$ such that for any $(\hat{g}, \hat{u}_0) \in U$ we have

$$u_x(t, 0; \hat{g}, \hat{u}_0) \neq 0,$$

by continuous dependence of $u(t, \cdot; g, u_0) \in C^1$. Hence σ_k is constant on U, and y_k is continuous by continuity of τ_k, τ_{k-1}. If on the other hand $\tau_k = \tau_{k-1}$, then $y_k = 0$ at (g, u_0) and continuity of τ implies that

$$|y_k| < \varepsilon$$

for all $(\hat{g}, \hat{u}_0)$ in some neighborhood U of (g, u_0) in $\mathscr{G} \times X$, no matter which sign σ_k takes. Therefore, y is again continuous and the proof is complete. □

In order to actually find u_0 with $y(u_0) = \sigma e_k$, we are interested in surjectivity of the y-map (2.3) for nonlinear g. As usual, it is much easier to discuss y for linear g. But deforming g by a homotopy from the linear to the nonlinear, surjectivity might be destroyed. Fortunately, topology helps us to bridge this gap. In lemma 2.2 below, we prove that for linear g

$$y: \Sigma^n \to S^n$$

is an *essential* mapping between spheres Σ^n and S^n. Essential means that there is no homotopy from y to the constant map. Charmingly, this property is invariant under homotopies to nonlinear g — by definition. Moreover it implies that y remains surjective; or else the image of y would miss some point in S^n and could therefore be contracted to a single point in contradiction to y being essential. Conversely, if $n = 0$ and y is surjective then y is also essential. However, this does not hold for $n > 0$, in general. With this in mind, we turn to the case of linear g.

Specialize $g(x, u) = a(x) \cdot u$, $a \in C^2$, and denote the (Sturm–Liouville) eigenvalues and eigenfunctions of

$$\lambda u = u_{xx} + a(x)u \tag{2.5}$$

with boundary condition (2.1) by $\lambda_0 > \lambda_1 \ldots$ and $\varphi_0, \varphi_1 \ldots$ as in the introduction. We take $\varphi_k(x)$ normalized to unit length in X with the additional sign convention $\varphi_k{}'(0) > 0$. Assume that

$$\lambda_n > 0 \tag{2.6}$$

i.e. $u \equiv 0$ has Morse-index $i(u \equiv 0) \geqslant n + 1$. Denoting

$$W_n = \operatorname{span}\{\varphi_0, \ldots, \varphi_n\},$$

it is known from Sturm–Liouville theory that $z \leqslant n$ on W_n, see also [5]. Let Σ^n denote a sphere centered at 0 in W_n.

2.2 LEMMA Under assumption (2.6), the restriction of the y-map

$$y: \Sigma^n \to S^n$$

is essential, i.e. y is not homotopic to a constant. In particular, y is surjective.

Proof The proof is by induction on n. For W_k, $0 \leqslant k \leqslant n$ observe that

$\Sigma^k := \Sigma^n \cap W_k$ is a k-dimensional sphere centered at 0. For $n = 0$, $\Sigma^0 = \{\pm\varphi_0\}$ for example. In that case $y = y_0 = \sigma_0 = \pm \text{sign}\ \varphi_0'(0)$ and y is surjective, hence essential. Below, we work our way up from Σ^0 to Σ^n where each Σ^{k-1} occurs as an equator in Σ^k. By a Mayer–Vietoris argument, y is essential from Σ^k to S^k if it was essential from Σ^{k-1} to S^{k-1}. For $k = n$, the lemma will be proved.

Suppose now the lemma is proved for $k-1$ already. Identify W_k with $\mathbb{R}^{k+1}$ by $(\eta_0, \ldots, \eta_k) \to \Sigma\eta_j\varphi_j$ and write $\Sigma^k = \{\Sigma\eta_j^2 = 1\}$. Note that W_{k-1} with sphere Σ^{k-1} is a subspace of W_k and Σ^{k-1} becomes an equator of Σ^k. We denote the closed hemisphere as

$$
\begin{aligned}
S_\pm^k &:= \{(y_0, \ldots, y_k) \in S^k \mid \pm y_k \geqslant 0\} \\
\Sigma_\pm^k &:= \{(\eta_0, \ldots, \eta_k) \in \Sigma^k \mid \pm \eta_k \geqslant 0\},
\end{aligned}
$$

and the equators as

$$
\begin{aligned}
S^{k-1} &= \{(y_0, \ldots, y_k) \in S^k \mid y_k = 0\} \\
\Sigma^{k-1} &= \{(\eta_0, \ldots, \eta_k) \in \Sigma^k \mid \eta_k = 0\}.
\end{aligned}
$$

For the restriction we have

$$
y^{k-1} := y\,|_{\Sigma^{k-1}} : \Sigma^{k-1} \to S^{k-1},
$$

because $\tau_{k-1} = 0$ on $\Sigma^{k-1} \subset W_{k-1}$ implies $y_k = 0$. Moreover

$$
\begin{aligned}
y : \Sigma_+^k &\to S_+^k \\
\Sigma_-^k &\to S_-^k.
\end{aligned}
$$

Indeed, $u_0 \in \Sigma_+^k$ with $\eta_k > 0$ and $u(t, \cdot) = \Sigma \exp(\lambda_j t)\eta_j\varphi_j$, $\lambda_0 > \cdots > \lambda_k$ imply

$$
\lim_{t \to -\infty} \frac{u(t, \cdot)}{|u(t, \cdot)|} = \text{sign}\ \eta_k \cdot \phi_k = \varphi_k
$$

and, using that $z(u(t, \cdot)$ is decreasing for all real t, consequently

either $\qquad \tau_{k-1} = 0,$

or $\qquad \tau_{k-1} > 0, \qquad \sigma_k = \text{sign}\ \varphi_k'(0) = +1.$

Note that for $\tau_{k-1} > 0$ the sign σ_k may be evaluated for t near $-\infty$. In the first case $y_k = 0$, whereas in the second case $y_k > 0$. Hence y maps Σ_+^k into S^k_+; the case of Σ_-^k is analogous.

Now y^{k-1} is essential by induction hypothesis, hence its Brouwer degree $\deg y^{k-1}$ is nonzero (cf. [12] for a definition of degree and the topological background used below). Let us consider the Mayer–Vietoris sequence for

y, $k \geqslant 1$ [12]

$$\begin{array}{ccccccc} 0 \to H_k(\Sigma^k) & \to & H_{k-1}(\Sigma^{k-1}) & \to & H_{k-1}(\Sigma^k_+) & \oplus & H_{k-1}(\Sigma^k_-) \\ \downarrow \deg y & & \downarrow \deg y^{k-1} & & & \downarrow & \\ 0 \to H_k(S^k) & \to & H_{k-1}(S^{k-1}) & \to & H_{k-1}(S^k_+) & \oplus & H_{k-1}(S^k_-). \end{array}$$

The homologies of hemispheres are trivial, the other homologies are just $\mathbb{Z}$, hence

$$\deg y = \deg y^{k-1} \neq 0$$

and y is essential [12]. This completes the induction step and the proof of the lemma. □

A quicker, less explicit proof may be sketched as follows.. Because we consider a linear flow $u(t, \cdot)$, the y-map in lemma 2.2 is odd. Thus y is essential by the Borsuk–Ulam Theorem (which can be proved by the Mayer–Vietoris sequence given above).

3 ESTABLISHING CONNECTIONS

We use the y-map constructed in §2, to establish connections from a stationary hyperbolic solution v to at least $2i(v)$ distinct other stationary solutions; $i(v)$ denotes the Morse index of v, as before. Below, we employ homotopy-invariance of the y-map to see that the y-map induces an essential mapping from a sphere Σ^n of dimension $n = i(v) - 1$ around v in the unstable manifold of v, and mapping into the standard n-sphere S^n. Indeed, we investigated the linear case in lemma 2.2 and our result is obtained by standard homotopy to the nonlinear case.

Throughout §3, we again consider the equation

$$(3.1) \qquad u_t = u_{xx} + g(x, u), \; 0 < x < 1$$

$$\cos \gamma_j \cdot u(t, j) - \sin \gamma_j \cdot u_x(t, j) = 0, \quad j = 0, 1$$

for $\cos \gamma_0 \neq 0$ and $g \in \mathscr{G}_0$ (in particular $g(x, 0) = 0$, cf. (1.10)). replacing $\tilde{u}$ by u as in §2.

3.1 LEMMA Suppose $v \equiv 0$ is a hyperbolic stationary solution of (3.1) with unstable manifold W^u of dimension $i(v) > 0$. Let $\Sigma \subset W^u \backslash \{v\}$ be homotopic in $W^u \backslash \{v\}$ to a small sphere centered at v in W^u of dimension $n = i(v) - 1$.

Then for any finite sequence

$$0=\delta_n\leqslant\delta_{n-1}\leqslant\delta_{n-2}\leqslant\cdots\leqslant\delta_0\leqslant\infty$$
$$s_k\in\{1,-1\},\quad 1\leqslant k\leqslant n$$

there exists an initial datum $u_0\in\Sigma$ such that the graph $t\to z(u(t,\cdot))$ is characterized by (δ_k). More precisely for any $0\leqslant t<\infty$:

$$t\geqslant\delta_k\Leftrightarrow z(u(t,\cdot))\leqslant k \tag{3.2a}$$

$$\delta_k<t<\delta_{k-1}\Rightarrow \operatorname{sign} u_x(t,0)=s_k \tag{3.2b}$$

Proof First suppose that the restricted y-map

$$y:\Sigma\to S^n$$

is essential (cf. [12] for the topological facts used). Then y is surjective. Now define η just as the y-map in (2.2–2.3), but replacing t_k by δ_k, and σ_k by s_k. By surjectivity of y, there exists an initial datum $u_0\in\Sigma$, such that $y(u_0)=\eta$. But knowing y, the dropping times t_k and signs σ_k associated to the orbit $u(t,\cdot)$ of u_0 are uniquely determined as

$$t_k=\delta_k$$

and, in case $\delta_k<\delta_{k-l}$,

$$\sigma_k=s_k.$$

Therefore, it only remains to prove that y is essential. How to achieve this? By a homotopy, of course! We deform g into its linearization, defining

$$g_\beta(x,u):=\beta g(x,u)+(1-\beta)g_u(x,0)\cdot u$$

with homotopy parameters $0\leqslant\beta\leqslant1$. Simultaneously this deforms the unstable manifold $W^{\mathrm{u}}(g_\beta)$ associated to the stationary solution $v\equiv0$ of g_β. Note that our homotopy leaves the linearization at $v\equiv0$ unchanged. Moreover, $g_\beta\in\mathscr{G}_0$ depends continuously on β, because $\mathscr{G}_0$ carries the weak Whitney topology Let

$$W^{\mathrm{u}}_{\mathrm{loc}}(g_0):=\operatorname{span}\{\varphi_0,\dots,\varphi_n\}\cap\{u_0\in X\mid |u_0|<2\varepsilon\}$$

denote the (cut-off) tangent space of $W^{\mathrm{u}}(g_\beta)$ at $v\equiv0$. Then the local unstable manifolds of g_β are parametrized by diffeomorphisms

$$p_\beta: W^{\mathrm{u}}_{\mathrm{loc}}(g_0)\to W^{\mathrm{u}}_{\mathrm{loc}}(g_\beta)$$

where P_β^{-1} is induced by the orthogonal projection onto $\operatorname{span}\{\varphi_0,\dots,\varphi_n\}$. Note that P_β depends continuously on β in the uniform C^0-topology. Fix a sphere

$$\Sigma^n:=\{u\in W^{\mathrm{u}}_{\mathrm{loc}}(g_0)\mid |u|=\varepsilon\}$$

and let y^β denote the restriction to $P_\beta(\Sigma^n)$ of the y-map associated to g_β. After a homotopy, we may assume $\Sigma = P_1(\Sigma^n)$. Finally, define

$$\tilde{y}^\beta := y^\beta \cdot P^\beta : \Sigma^n \to S^n.$$

This mapping is well-defined (recalling from (1.13) that $z \leqslant n$ on $W_u(g_\beta)$), continuous and depends continuously on β by lemma 2.1. Lemma 2.2 implies that

$$\tilde{y}^0 = y_0 \circ P_0 = y_0 : \Sigma^n \to S^n$$

is essential. By homotopy-invariance, this implies that $\tilde{y}^1 = y^1 \circ P_1 = y \circ P_1$, and hence y, is essential — completing the proof. □

As a corollary to lemma 3.1, we obtain connections from v to at least $2i(v)$ different other stationary solutions under a growth restriction on g.

3.2 COROLLARY Suppose v is a hyperbolic stationary solution of (3.1) with Morse index $i(v) > 0$. In addition, let $g \in \mathscr{G}$ satisfy the growth condition

$$\overline{\lim_{|u| \to \infty}} \; g(x, u)/u \leqslant 0 \tag{3.3}$$

uniformly in x (we do not require $g(x, 0) = 0$, here).

Then for any $0 \leqslant k < i(v)$, $\sigma \in \{1, -1\}$, there exists a stationary solution $w \neq v$ such that v connects to w and

$$z(w - v) = k \tag{3.4a}$$

$$\operatorname{sign}(w_x(0) - v_x(0)) = \sigma. \tag{3.4b}$$

Proof Without loss of generality, we may assume $v \equiv 0$. Indeed, let u be a solution of (3.1) with $g \in \mathscr{G}$. Then $u := u - v$ satisfies (3.1), replacing the nonlinearity g there by

$$\tilde{g}(x, \tilde{u}) := g(x, \tilde{u} + v(x)) - g(x, v(x));$$

note that $\tilde{g} \in \mathscr{G}_0$.

Now we apply lemma 3.1 to the solutions of (3.1), picking

$$\delta_j := \begin{cases} 0 & \text{for } j \geqslant k \\ \infty & \phantom{\text{for }} j < k \end{cases}$$

$$s_k := \sigma.$$

With initial datum $u_0 \in W^{\mathrm{u}}$ corresponding to this choice, lemma 3.1 asserts for the solution $\tilde{u}(t, \cdot)$ that $z(\tilde{u}(t, \cdot)) = k$, sign $\tilde{u}_x(t, 0) = \sigma$ for all $t > 0$. But $\tilde{u}(t, \cdot)$

converges to a stationary solution w, as $t \to \infty$ by assumption (3.3) and the gradient structure of equation (3.1) Because $w \not\equiv 0$ has only simple zeros (it solves the ordinary differential equation $0 = w_{xx} + \tilde{g}(x, w)$), properties (3.4) are immediate from (3.2). This completes the proof of the corollary.

4 EXCLUDING CONNECTIONS

Suppose we have constructed a connection from a stationary solution v to a stationary solution $w \neq v$ such that

(4.1a) $$z(w - v) = k$$

(4.1b) $$\operatorname{sign}(w_x(0) - v_x(0)) = \sigma,$$

where $0 \leqslant k < i(v)$, $\sigma \in \{1, -1\}$ are given (this we achieved in corollary 3.2). In this section we try to identify the set of all w such that (4.1) holds for some fixed given v, k and σ. In general, this set may contain more than just one element. However, for nonlinearities $g(x, u) = f(u)$ independent of x and for Dirichlet boundary conditions, the w in question is determined uniquely in terms of $z(w)$ and $w_x(0)$ — in most cases. The two lemmata below are the crucial tools to determine w.

4.1 LEMMA Suppose $g \in C^2$, and $v, w, \bar{w}$ are three distinct stationary solutions of (1.7), (1.8), $\cos \gamma_0 \neq 0$, such that $w_x(0)$ lies strictly between $v_x(0)$ and $\bar{w}_x(0)$. Then

(4.2) $$z(v - w) \leqslant z(\bar{w} - w)$$

implies that v does not connect to $\bar{w}$.

Proof We prove the lemma by negation. Let v connect to $\bar{w}$ via an orbit $u(t, \cdot)$, $t \in \mathbb{R}$. Then $\tilde{u} := u - w$ satisfies again an equation of the form (1.7), similarly to the proof of corollary 3.2. Hence we may assume without loss of generality that $w = 0$, $g(x, 0) = 0$, $v_x(0) < 0 < \bar{w}_x(0)$. Non-increase of $z(u(t, \cdot))$ then implies

$$z(v - w) = z(v) \geqslant z(\bar{w}) = z(\bar{w} - w).$$

Finally, $z(v) \neq z(\bar{w})$, because $v_x(0)$ and $\bar{w}_x(0)$ have opposite sign (cf. lemma 7.4). Therefore, $z(v - w) > z(\bar{w} - w)$ if v connects to w — and the lemma is proved. □

The next lemma is based on phase-plane analysis for stationary solutions. This forces us to restrict out attention to autonomous, i.e. x-independent

$g(x, u) = f(u)$ and to Dirichlet boundary conditions (the Neumann case is discussed in §6). Going back to §3 and (4.1) we obtained $z(w - v) = k$ for some w that v connects to It is significant, that the lemma below allows us to replace $z(w - v)$ by $z(w)$ itself, if $|w_x(0)| \geqslant |v_x(0)|$. This enables us to describe connections in terms of $z(w)$, $z(v)$, $i(v)$ alone — rather than $z(w - v)$, which can not be read off from time-map bifurcation diagrams as given in the introduction.

4.2 LEMMA Consider equation (1.1) with Dirichlet boundary conditions (1.2). Let v^1 and v^2 be two distinct stationary solutions. Then $|v_x^1(0)| \geqslant |v_x^2(0)|$ implies

$$z(v^1 - v^2) = z(v^1) \tag{4.3}$$

Proof Recall that any stationary solution v of (1.1), (1.2) has only simple zeros, or else $v \equiv 0$. If $v^1 \equiv 0$, then $v_x^2 = 0$ implies $v^2 \equiv 0$, and v^2 cannot be distinct from v^1. Hence we consider $v^1 \not\equiv 0$, only. We partition the non-empty set

$$\{x \in [0, 1] \mid v_x^1(x) \neq 0\} = I_0 \cup \cdots \cup I_{n+1}$$

into its disjoint connected components (intervals) I_j, $0 \leqslant j \leqslant n + 1$, $n = z(v^1)$. Note that $0 \in I_0$, $1 \in I_{n+1}$. It is sufficient to show that each I_j contains exactly one zero of $v^1 - v^2$, this zero is automatically simple.

Each interval I_j contains at least one zero of $v^1 - v^2$. Indeed, for $j = 0, n + 1$ these are given by $x = 0, 1$, respectively. For $0 < j < n + 1$ this follows because

$$\operatorname{sign}(v^1 - v^2) = \operatorname{sign} v^1$$

at the endpoints of I_j. To see this, just note that v^1 attains both its extreme values at the endpoints of I_j and that the orbit of $(v^1(x), v_x^1(x))$ does not lie inside the orbit of $(v^2(x), v_x^2(x))$ for the Hamiltonian system

$$0 = v_{xx} + f(v) \tag{4.4}$$

in the (v, v_x)-plane, by assumption (cf. e.g. Fig. 4).

Each interval I_j contains at most one zero of $v^1 - v^2$. This is again immediate from the fact that the orbit of $(v^1(x), v_x^1(x))$ does not lie inside the orbit of $(v^2(x), v_x^2(x))$, which implies that

$$\operatorname{sign}(v_x^1(x) - v_x^2(x)) = \operatorname{sign} v_x^1(x)$$

whenever $v^1(x) = v^2(x)$ for some $x \in I_j$ (cf. Fig. 4).

Hence each interval I_j contains exactly one (simple) zero of $v^1 - v^2$. This implies $z(v^1 - v^2) = n = z(v^1)$ and the proof is finished. □

5 PROOF OF THEOREM 1.1

We combine the results of Sections 3 and 4 to prove theorem 1.1. First, we use corollary 3.2 to establish connections from v, stationary and hyperbolic with positive Morse index $i(v)$, to some stationary $w \not\equiv v$ such that

(5.1a) $$z(w-v)=k$$

(5.1b) $$\operatorname{sign}(w_x(0)-v_x(0))=\sigma,$$

where $0 \leqslant k < i(v)$, $\sigma \in \{1, -1\}$ can be prescribed arbitrarily. Vice versa, any stationary solution w that v connects to has to satisfy (5.1) for some appropriately chosen $0 \leqslant k < i(v)$ and $\sigma \in \{1, -1\}$ because, by (1.13), $z(u_0 - v) < i(v)$ for any initial data u_0 in the unstable manifold W^{u} of v. Below we employ lemmata 4.1 and 4.2 to identify w as described in theorem 1.1. Remember that lemma 4.2 required autonomous phase plane analysis; therefore we consider equation (1.1) with Dirichlet boundary condition (1.2) throughout this section.

As a final preparation to the proof of theorem 1.1, we show that its cases (i)–(iii) are the only possible ones.

5.1 LEMMA Let $v \not\equiv 0$ be a hyperbolic stationary solution of (1.1) with Dirichlet boundary condition (1.2). Then Morse index $i(v)$ and zero number $z(v)$ are related by

$$i(v) \in \{z(v), z(v)+1\}.$$

Proof By Rolle's theorem and phase-plane analysis of

$$0 = v_{xx} + f(v)$$

we have $z(v_x) = z(v) + 1$ for Dirichlet boundary conditions on v.

Now consider the linearization (1.11), (1.12) and its eigenvalues λ_n, eigenfunctions φ_k as in (2.5) with $a(x) := f'(v(x))$. For $n := i(v) - 1$ we have

$$\lambda_n > 0 > \lambda_{n+1},$$

$$z(\varphi_n) = n, z(\varphi_{n+1}) = n+1$$

On the other hand, $u := v_x$ also satisfies (2.5) with $\lambda = 0$. By the Sturm–Liouville comparison theorem [3, 15], between any two consecutive zeros of φ_n there is a zero of v_x, and between any two consecutive zeros of v_x there is a zero of φ_{n+1}— all these zeros being simple. This respectively implies

$$n+1 \leqslant z(v_x), \text{ and } z(v_x) - 1 \leqslant n+1, \text{ i.e.}$$

$$i(v) = n+1 \leqslant z(v_x) = z(v) + 1, \text{ and}$$

$$i(v) = n+1 \geqslant z(v_x) - 1 = z(v),$$

completing the proof of the lemma. A similar idea can be found in [27, Lemma 24.16]. □

Proof of theorem 1.1:

Using corollary 3.2 as above, it remains to identify those stationary w satisfying (5.1) which v does connect to. We have to consider three cases. First we address $v \not\equiv 0$ and $0 \leqslant k < z(v)$. Then we analyze $v \equiv 0$ (in case $f(0) = 0$) for $0 \leqslant k < i(v)$ These two cases result in part (i) of theorem 1.1 (note that $z(v) = i(v)$ is assumed there for $v \not\equiv 0$). Finally, we consider $v \not\equiv 0$, $k = z(v) = i(v) - 1$ as the only remaining case. Replacing $f(s)$ by $-f(-s)$ we may in fact assume $v_x(0) > 0$ in that case without loss of generality.

Case 1: $v \not\equiv 0$, $0 \leqslant k < z(v)$

By lemma 4.2, $k < z(v)$ implies

(5.2a) $$k = z(w - v) = z(w),$$

(5.2b) $$|\, w_x(0) \,| > |\, v_x(0) \,|.$$

If $\sigma = +1$ (resp. -1) then $w_x(0)$ is above (resp. below) $v_x(0)$ and, by (5.2b), also above (resp. below) $\pm\, |\, v_x(0) \,|$. On the other hand w is the minimal (resp. maximal) stationary solution with that property by lemma 4.1 (recall that v does not connect to w by assumption). Therefore $w = \bar{v}_k$ (resp. $\underline{v}_k$).

Case 2: $v \equiv 0$, $0 \leqslant k < i(v)$

Rereading case 1, (5.2a–b) are automatic for $v \equiv 0$. Leaving the remaining arguments of case 1 unchanged we conclude again $w = \bar{v}_k$ (resp. $\underline{v}_k$).

Case 3: $v_x(0) > 0$, $k = z(v) = i(v) - 1$

If $\sigma = +1$, then $|\, w_x(0) \,| = w_x(0) > v_x(0) = |\, v_x(0) \,|$, hence (5.2a–b) hold and case 1 applies identifying w as $\bar{v}_k$. Likewise, if $\sigma = -1$ and $w_x(0) < -v_x(0)$, we conclude that $w = \underline{v}_k$. However, if $\sigma = -1$ and $w_x(0) \geqslant -v_x(0)$ (i.e. $-v_x(0) \leqslant w_x(0) < v_x(0)$) complications arise. In this one remaining case, we first claim that $i(w) < i(v)$. From (1.13) it is immediate that

$$z(v - w) < \dim W^{\mathrm{u}}(v) = i(v).$$

Now follow $\tilde{u}(t, \cdot) := u(t, \cdot) - w$ for $t \to \pm\infty$ along an orbit $u(t, \cdot)$ connecting v to w. Similarly to the argument given in the introduction, $\tilde{u}(t, \cdot)/|\, \tilde{u}(t \cdot)\,|$ converges to an eigenfunction φ_k with eigenvalue $\lambda_k \leqslant 0$ of the linearization at

w as $t \to +\infty$ and

$$z(\tilde{u}(t,\cdot)) = z(\tilde{u}(t,\cdot)/|\tilde{u}(t,\cdot)|) \geqslant z(\varphi_k) = k \geqslant i(w)$$

for all real t. In the limit $t \to -\infty$ this yields

$$z(v - w) \geqslant \dim W^{\mathrm{u}}(w) = i(w).$$

Finally we claim that there is no additional connection to any stationary $\bar{w}$ with

$$z(\bar{w}) = k, \bar{w}_x(0) < -v_x(0).$$

Indeed, $w_x(0)$ is between $\bar{w}_x(0)$ and $v_x(0)$. Further, by lemma 4.2

$$z(v - w) = z(v) = k = z(\bar{w}) = z(\bar{w} - w).$$

Now lemma 4.1 implies that v does not connect to $\bar{w}$. This completes the proof of theorem 1.1. □

6 DISCUSSION

We compare our result with previous results by Henry [17] for Dirichlet boundary conditions. Then we present a theorem for the Neumann case, sketching the necessary modifications Finally we indicate some open problems.

Under the additional assumption (1.4): $f(0) = 0 < f'(0)$, $s \cdot f''(s) \geqslant 0$ for $s \neq 0$, Henry [17, §4] works out all the connecting orbits for the Dirichlet problem (1.1), (1.2). As outlined in the introduction, (1.4) implies $i(v) = z(v)$ for any nonzero stationary solution v — and Henry's result is recovered by case (i) of theorem 1.1

The method of Henry [17] is in some sense complementary to ours. One establishes, for a general class of scalar parabolic equations including nonlinearities $f(x, u, u_x)$ and nonlinear boundary conditions that stable and unstable manifolds of hyperbolic equilibria always intersect transversely (cf. [17, theorem 7], degenerate equilibria are treated as well, and [1]). Consequently, if $\bar{v}_k$, $\underline{v}_k$ are hyperbolic, too, the sets $C(v, \bar{v}_k)$ and $C(v, \underline{v}_k)$ of orbits connecting v to $\bar{v}_k$ resp. $\underline{v}_k$ are seen to be immersed submanifolds of X of dimension $i(v) - k$, provided these sets are guaranteed to be non-empty by theorem 1.1. On the other hand, our approach and in particular lemma 3.1 shows, that the graph of $z(u(t,\cdot) - v)$ can be prescribed arbitrarily on orbits in $C(v, \bar{v}_k)$ resp $C(v, \underline{v}_k)$, along with the signs of $u_x(t, 0)$ where $z(u(t,\cdot) - v)$ is locally constant. Of course, the obvious limitation $i(v) > z \geqslant k$ has to be observed. Even under restrictive assumptions on f, this was previously unknown.

The first results on connecting orbits were obtained by Conley and Smoller (cf [10, 27] and references there) by an entirely different method. Their approach relied solely on the Ljapunov functional V from (1.6), rather than the discrete 'Ljapunov' functional z. Given V, Conley's index theory could be applied to establish connections from v to w, typically in cases where $i(v) = i(w) + 1$. For an exposition on Conley's index see e.g. [9], [27, §§22–24] and in particular [27, lemma 24.12 and theorem 24.14]. A very rudimentary account would run as follows. Consider Fig. 1 and take a parameter α between the first and second bifurcation point. We have three stationary solutions: $v \equiv 0$, $\bar{v}_0$ and $\underline{v}_0$ in our notation. The Conley index may be defined as follows. Let S be an isolated invariant set with isolating neighborhood N_1 (i.e. S is the maximal invariant subset of N_1 and is contained in the interior of N_1). Let N_2 denote the exit set of N_1, i.e. those points of ∂N_1 which leave N_1 in forward time. Then $h(S)$, the Conley-index of S, is the homotopy type of N_1/N_2, i.e. of N_1 with the exit set collapsed to a (distinguished) point. In our example, take S to be successively v, $\bar{v}_0$, $\underline{v}_0$, and finally the maximal compact invariant set $\mathcal{A}$. Then

$$\begin{aligned} h(v) &= \Sigma^1 \\ h(\bar{v}_0) = h(\underline{v}_0) &= \Sigma^0, \\ h(\mathcal{A}) &= \Sigma^0, \end{aligned}$$

where Σ^k denotes the k-sphere with some distinguished point, and $h(\mathcal{A})$ is computed by homotopy to parameters α below the first bifurcation point. It is a result of Conley that

$$h(\mathcal{A}) = h(v) \vee h(\bar{v}_0) \vee h(\underline{v}_0),$$

if $\mathcal{A} = \{v, \bar{v}_0, \underline{v}_0\}$; here v denotes the wedge product. But

$$\Sigma^0 \neq \Sigma^1 \vee \Sigma^0 \vee \Sigma^0,$$

hence $\mathcal{A}$ cannot consist of stationary solutions only — it must also contain connecting orbits. By symmetry and because $\bar{v}_0$, $\underline{v}_0$ are both stable, these orbits have to connect v to both v_0 and $\underline{v}_0$.

Henry's result, as well as ours, contain all information that was originally gained by Conley's index. But curiously enough, these later approaches are both based on the discrete 'Ljapunov' functional given by the zero number z. In other words, maximum principles are emphasized, rather than the variational structure which plays only a marginal role in the results of Henry and ourselves. The Ljapunov functional V was used only to guarantee convergence of $u(t, \cdot)$ to a critical point.

Indeed, it appears that connecting orbits can be worked out even without any variational structure. As a concrete exmaple, we mention the scalar delay equation

$$\dot{x}(t) = -x(t) + f(x(t-1)) \tag{6.1}$$

with negative feedback: $f(0)=0$, $f'(0)<0$, $x\cdot f(x)<0$ for $x \neq 0$. Mallet-Paret [22] uses a discrete 'Ljapunov' functional with properties quite similar to our zero number — we call it z, again — in order to work out connecting orbits for (6.1). But (6.1) is not a variational problems solutions $x(t)$ cannot be expected to converge to equilibrium for $t\to\infty$, in general. Thus, the role of 'stationary solutions' has to be replaced by the 'maximal compact invariant subset of $\{z=k\}$' for the various integers k. Such sets with different k may be connected by solutions of (6.1). In fact, the 'Ljapunov' functional z alone still enabled Mallet-Paret [22] to detect some connecting orbits via Conley's index method (the analogous approach, based on z, was never tried for our reaction diffusion problem). Again, Conley's method seems to be limited to establishing connections between maximal invariant subsets with adjacent k. Recently, this difficulty has been circumvented by topological considerations which use only z, but are not explicitly related to Conley's index. Summarizing, both equation (6.1) and equations (1.1), (12) exhibit a discrete 'Ljapunov' functional z with the enticing property that it takes on many different values near $v\equiv 0$, e.g. on $W^{\mathrm{u}}(v)$, which can be studied by homotopy to the linear case.

Returning to the variational setting once more, we notice that $V(v)>V(w)$ if v connects to w. Knowing which w our v connects to, it should be possible to obtain this relation directly from the phase portrait of the Hamiltonian system (4.4). Except — we do not know, how. In this context it should be noted that R. Schaaf [26] has proved monotonicity of V along those parts of the stationary bifurcation diagram which consist of nondegenerate hyperbolic solutions.

For scalar reaction diffusion equations with Neumann boundary condition

$$u_t = u_{xx} + f(u), \quad 0<x<1 \tag{1.1}$$

$$u_x(t,0) = u_x(t,1) = 0, \tag{6.2}$$

and growth condition

$$\overline{\lim_{|s|\to\infty}}\ f(s)/s < 0, \quad f\in C^2 \tag{6.3}$$

we present an analogue to theorem 1.1. For any C^1-function $v\colon[0,1]\to\mathbb{R}$, we define the lap number $l(v)$ (cf. [20]) by

$$l(v) = \begin{cases} z(v_x)+1, & \text{if } v_x \not\equiv 0 \\ 0 \quad , & \text{if } v_x \equiv 0. \end{cases} \tag{6.4}$$

Given a stationary solution v, let $\tilde{v}_k$ (resp. $\underset{\sim}{v}_k$) denote the stationary solution $\tilde{v}$ with minimal $\tilde{v}(0) >$ range (v) (resp. maximal $\tilde{v}(0) <$ range (v)) As in the Dirichlet case, the proof of theorem 6.1 below in particular proves existence of all $\tilde{v}_k$, $\underset{\sim}{v}_k$ which occur in its statements. Uniqueness of the $\tilde{v}_k$, $\underset{\sim}{v}_k$ is obvious viewing the stationary boundary value problem as an initial value problem.

Given v, the solutions $\bar{v}_k$, $\underset{\sim}{v}_k$ and $l(v)$, $i(v)$ can be identified from global bifurcation pictures as given, e.g. in [24]. As in the Dirichlet-case, numerical bifurcation diagrams can be obtained by rescaling (1.1) to (1.16) and appropriately defining the Neumann-time $\hat{T} = \hat{T}(v^*(0))$ as the first positive zero of $v_x^*(\xi)$. Implementation details were the same as before, and a concrete example is shown in Fig. 6.

6.1 THEOREM Under assumption (6.1) above, let v be a hyperbolic stationary solution with lap-number $l(v)$ and Morse-index $i(v) > 0$ of equation (1.1) with Neumann boundary condition.

Then v connects to other stationary solutions (denoted by $\Omega(v)$ again) as follows.

(i) If $v \equiv$ constant, or if $i(v) = l(v)$, then

$$\Omega(v) = \{\bar{v}_k, \underset{\sim}{v}_k \mid 0 \leqslant k < i(v)\}.$$

(ii) If $v(0) = \max v \neq \min v$ and $i(v) = l(v) + 1$, then

$$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_1 = \{\bar{v}_k \mid 0 \leqslant k < i(v)\}$$

$$\Omega_2 = \{\underset{\sim}{v}_k \mid 0 \leqslant k < i(v) - 1\} \quad \text{and}$$

either $\Omega_3 = \{\underset{\sim}{v}_k \mid k = i(v) - 1\}$

or Ω_3 consists of one or several stationary solutions w with range$(w) \subset$ range(v) and $i(w) < i(v)$.

(iii) An analogous statement holds for $v(0) = \min(v) \neq \max(v)$

Sketch of proof First we adapt the y-map to Neumann boundary conditions. In fact we only replace $u_x(t, 0)$ by $u(t, 0)$ in the definition (2.2b) of σ_k. As before, y is continuous (cf. also [20]) and essential with this definition and §3 remains valid. In particular, for any $0 \leqslant k < i(v)$, $\sigma \in \{1, -1\}$ there exists an initial datum $u_0 \in W^u(v)$ such that $u(t, \cdot) \xrightarrow[t \to \infty]{} w$ and

$$z(w - v) \equiv k,$$
$$\operatorname{sign}(w(0) - v(0)) = \sigma.$$

Analogously to lemma 4.1, v does not connect to $\bar{w}$ if there exists a stationary w with $w(0)$ between $v(0)$ and $\bar{w}(0)$ satisfying

$$z(v - w) \leqslant z(\bar{w} - w).$$

This accounts for the minimality (maximality) in the definitions of $\tilde{v}_k$ ($\underset{\sim}{v}_k$). The appropriate modification of lemma 4.2 to Neumann conditions tells that

$$z(v^1 - v^2) = \begin{cases} l(v^1) \geqslant 1 \\ 0 \end{cases} \text{if} \quad \begin{array}{l} \text{range}(v^2) \subset \text{range}(v^1) \\ \text{range}(v^2) \cap \text{range}(v^1) = \varnothing \end{array} \tag{6.5}$$

for $v^1 \not\equiv v^2$ and is proved as before. Note that up to an interchange of v^1 and v^2 one of these alternatives has to occur. Relation (6.5) translates $z(v^1 - v^2)$ to the lap numbers $l(v^1)$ or $l(v^2)$, except when the ranges of v^1 and v^2 do not intersect. Taking $v^2 = v$, $v^1 = \tilde{v}$, with $z(\tilde{v} - v) = 0$ and *minimal* $\tilde{v}(0) >$ range v, however, it is immediate that $\tilde{v} \equiv$ const., i.e $l(\tilde{v}) = 0$, from (v, v_x) phase-plane analysis. Just observe that any two closed trajectories either are nested or their intersections with the v-axis are separated by a stationary (saddle type) solution (cf. Fig. 3).

By the final Sturm–Liouville ingredient that

$$i(v) \in \{l(v), l(v) + 1\}$$

for non-constant v, we observe that theorem 6.2 (i)–(iii) cover all possible cases. Replacing $z(v)$, $z(w)$ by $l(v)$, $l(w)$ the proof of theorem 6.1 is completed analogously to §5. □

For mixed boundary conditions we are lacking a bridge to cross the gap separating $z(v - w)$ from $z(w)$ itself. Of course, corollary 3.2 still establishes connections. But we are unable to identify the target w intrinsically, e.g. by looking at the global bifurcation diagram.

Note that our growth condition on f can be weakened for both the Dirichlet and the Neumann case to include arbitrary linear growth at infinity. If we just assume

$$\overline{\lim_{|s| \to \infty}} \; f(s)/s < \infty,$$

some of our stationary solutions v_k (bar or tilde) may not exist any more. In the bifurcation diagram of

$$0 = v_{xx} + \alpha^2 f(v),$$

$v_x(0)$ or $v(0)$ may escape to infinity at some finite value α. This is best seen in case $f(v)$ is linear, $f(v) = f'_\infty \cdot v$, $f'_\infty > 0$, where vertical stationary bifurcation occurs for any α_j such that $-\alpha_j^2 \cdot f'_\infty$ is an eigenvalue of v_{xx} For nonlinear f, this picture is somewhat perturbed but essentially correct. Still, by our y-map, there remains a trajectory $u(t, \cdot)$ in $W^u(v)$ with

$$z(u(t, \cdot) - v) \equiv k$$
$$\text{sign}(u_x(t, 0) - v_x(0)) = \sigma$$

for all $t \geqslant 0$ (here we consider the Dirichlet case, for simplicity) But $u(t, \cdot)$ cannot remain bounded, unless it converges to some equilibrium v_k. If α is such that v_k does not exist, then the y-map leads to an unbounded solution $u(t,\cdot)$ — we have established a connection from v to infinity. For more information on stationary solutions in such problems with $f(x,u)=f(u)+\sin x$ and 'jumping nonlinearities' $0<f'(-\infty)<f'(+\infty)$ we recommend [21].

Penultimately, we propose a conjecture giving, for the Dirichlet case, a complete description of the set $\Omega(v)$ of all stationary solutions w that v connects to. Recall from theorem 1.1 that $\Omega(v)$ consisted of three disjoint subset

$$\Omega(v)=\Omega_1\cup\Omega_2\cup\Omega_3,$$

in case $v_x(0)\neq 0$, $i(v)=z(v)+1$, and there was some unsettled alternative for Ω_3. In order to describe Ω_3 precisely we define:

$\underline{v}$ is the stationary solution $\hat{v}$ with $\hat{v}\equiv 0$ or $z(\hat{v})=z(v)$, such that

$$\hat{v}_x(0)<|v_x(0)| \text{ is maximal}$$

$\bar{v}$ is the stationary solution $\hat{v}$ with $\hat{v}\equiv 0$ or $z(\hat{v})=z(v)$, such that

$$\hat{v}_x(0)>-|v_x(0)| \text{ is minimal.}$$

These stationary solutions are uniquely defined. They are claimed to exist wherever they figure in the conjecture below.

6.2 CONJECTURE Let the assumptions of theorem 1.1 be satisfied..

(i) If $v_x(0)>0$ and $i(v)=z(v)+1$, then
Ω_3 consists of $\underline{v}$ and all stationary solutions w such that

$$\underline{v}_x(0)<|w_x(0)|<v_x(0) \text{ and } z(w)<z(v).$$

(ii) If $v_x(0)<0$ and $i(v)=z(v)+1$, then similarly
Ω_3 consists of $\bar{v}$ and all stationary solutions w such that

$$v_x(0)<-|w_x(0)|<\bar{v}_x(0) \text{ and } z(w)<z(v).$$

The proof of this conjecture is work in progress. Restricting attention to $v_x(0)>0$, the alternative for Ω_3 in theorem 1.1 is resolved by the conjecture as follows. If $\underline{v}_k=\underline{v}$ for $k=i(v)-1=z(v)$, then there are no stationary solutions $\hat{v}$ with

$$z(\hat{v})=z(v), \quad |\hat{v}_x(0)|<v_x(0)$$

by definition of $\underline{v}_k$. Similarly, $\hat{v}\equiv 0$ is not a stationary solution. Using a more detailed analysis of the stationary bifurcation diagram than given in this chapter, this implies that the set of stationary w considered in the conjecture is empty. If, on the other hand, $\underline{v}_k \neq \underline{v}$, then

$$\underline{v}_{k,x}(0) < \underline{v}_x(0) < v_x(0)$$

and lemma 4.1 implies that v does not connect to $\underline{v}_k$. Then the second alternative of theorem 1.1 applies, and the set of w considered there contains the set of w from conjecture 6.2, because any w from conjecture 6.2 satisfies

$$| w_x(0) | < v_x(0), \text{ and}$$
$$i(w) \leqslant z(w) + 1 < z(v) + 1 = i(v)$$

These remarks reconcile conjecture 6.2 and theorem 1.1.

Finally, nothing global is known for higher dimensions of the space variable x. We are lacking an analogue of the zero number $z(\phi)$. Within the class of rotationally symmetric solutions in a ball, the problem seems tractable. But introducing polar coordinates, this is essentially the one-dimentional case again. For Conley's index, using only the variational structure, these problems do not matter — at least in principle. To our knowledge, no attempt has been made so far to push this advantage to its limits. For special systems, i.e. higher dimensional u, but one space dimension for x, Smoller and Shi [28] and Conley and Smoller [11] obtained information on the flow, again by Conley's index. But in general, we suffer from the lack of a zero number.

7 APPENDIX ON z

We collect a few useful facts on the behaviour of the zero number z along solutions of the equation

$$u_t = u_{xx} + g(x, u), \quad x \in (0, 1) \tag{7.1}$$

$$u(t, 0) = u(t, 1) = 0. \tag{7.2}$$

Throughout this section we assume $g \in \mathscr{G}_0$, i.e. $g(x, 0) = 0$ and $g \in C^2$ with the linear growth condition (1.9). We try to convince the reader of most of these facts by an example — rigorous proofs are given in the references as indicated.

7.1 LEMMA [5] For $g \in \mathscr{G}_0$, the map

$$[0, \infty) \to N_0$$
$$t \to z(u(t, \cdot))$$

is non-increasing with t along solutions $u(t, x)$ of (7.1), (7.2).

7.2 EXAMPLE Consider a solution u such that $x \to u(t_0, x)$ has only simple zeros except for a double zero at x_0, i.e.

$$u(t,x)=\tfrac{1}{2}(x-x_0)^2$$

for x near x_0. Then locally near (t_0, x_0)

$$u_t = u_{xx} + g(x,u) = 1 + g(x,u) > 0,$$

because $g(x_0, u(t_0, x_0)) = 0$. Therefore

$$z(u(t_0+\varepsilon, \cdot)) = z(u(t_0, \cdot)) = z(u(t_0-\varepsilon, \cdot)) - 2$$

for small $\varepsilon > 0$, and indeed z decreases by 2 at $t = t_0$. A rigorous proof uses maximum principles like [5, 20, 23, 27, 32], instead. Historically, Nickel [23] first used such arguments and the essential idea of proof is due to him.

7.3 LEMMA [5, 20] Assume $g \in \mathscr{G}_0$ and $z(u(0, \cdot)) < \infty$. Then the set of times $t > 0$, such that $x \to u(t, x)$ has only simple zeros, is open and dense in $\mathbb{R}^+$.

Indeed, openness is obvious by continuity of the flow $t \to u(t, \cdot) \in C^1$. Density again uses maximum principles The benevolent reader may reconsider example 7.2 to get convinced that multiple isolated zeros of $u(t_0, \cdot)$ become simple or disappear immediately for $t \neq t_0$.

7.4 LEMMA Assume $g \in \mathscr{G}_0$ and $z(u(0, \cdot)) < \infty$. Define the dropping times t_k of $z(u(t, \cdot))$ as in (2.2a), and assume $t_k < t_{k-1}$. Then

$$u_x(t,0) \neq 0 \qquad \text{for all } t \in (t_k, t_{k-1})$$

and, in particular, sign $u_x(t,0)$ does not depend on $t \in (t_k, t_{k-1})$.

For lack of reference, a rigorous proof of lemma 7.4 is given below. The idea, however, is again basic. As in example 7.2, the occurrence of a multiple isolated zero of $u(t_0, \cdot)$ at $x = 0$, $t_0 \in (t_k, t_{k-1})$ would force $z(u(t, \cdot))$ to decrease at $t = t_0$ in contradiction to $z(u(t, \cdot))$ being locally constant near t by definition of t_k and t_{k-1}.

Proof of lemma 7.4: We already noted that

$$z(u(t, \cdot)) = k \qquad \textit{for all } t \in (t_k, t_{k-1}).$$

Now [6, lemma 2] implies that for any $t \in (t_k, t_{k-1})$ there exists an $\varepsilon > 0$ such that $u(x', t')$ has one definite sign independent of x', t', provided that

$$0 < x' < \varepsilon$$
$$|t - t'| < \varepsilon.$$

This fact just uses the strong maximum principle [6, 27,32]. But then

$$u_x(t', 0) \neq 0,$$

provided that $|t - t'| < \varepsilon$, again by the strong maximum principle [6, 27,32]. This completes the proof. □

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