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Connecting Orbits in Scalar Reaction Diffusion Equations II. The Complete Solution

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1. INTRODUCTION

In [4], we have studied the one-dimensional reaction–diffusion equation

$$u_t = u_{xx} + f(u), \qquad x \in (0, 1) \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t, 0) = u(t, 1) = 0. \tag{1.2}$$

We have obtained a partial answer to the following question:

(*Q*) *Given a stationary* (*i.e. time-independent*) *solution of* (1.1), (1.2), *which other stationary solutions does it connect to*?

To recall, we say that a stationary solution v connects to a stationary solution $w \neq v$ if there is a solution $u(t, x)$, $t \in (-\infty, \infty)$ of (1.1), (1.2) such that

$$\lim_{t \to -\infty} u(t) = v, \qquad \lim_{t \to \infty} u(t) = w \tag{1.3}$$

(when there is no danger of misunderstanding we shall drop the argument x in $u(t, x)$).

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For the history and motivation of the problem we refer the reader to [4]. For additional motivation, not mentioned in [4], note that under mild growth conditions on f at infinity (cf. (1.5) below), (1.1), (1.2) has a unique maximal compact invariant set $\mathscr{A}$ which consists of stationary solutions and their connections. Knowing the flow on $\mathscr{A}$ is instrumental for the understanding of the dynamics of (1.1), (1.2) [7].

Following [8] we can consider (1.1), (1.2) as an abstract differential equation

$$du/dt + Au = F(u) \tag{1.4}$$

with $(Au)(x) = -u_{xx}(x)$, $F(u)(x) = f(u(x))$.

As in [4] we assume that f is C^2 and

$$\limsup_{|s| \to \infty} s^{-1} f(s) < \pi^2. \tag{1.5}$$

By $\mathscr{F}$ we denote the set of $f \in C^2$ satisfying this condition endowed by the C^2 (weak or strong) topology. For $f \in \mathscr{F}$, (1.1), (1.2) define a strongly continuous semiflow S_t, $t \geqslant 0$ on $X = H^2 \cap H_0^1$. All trajectories of S_t stay bounded in the H^2-norm $|\cdot|$ for $t \geqslant 0$ and are relatively compact [8].

In addition we shall assume that all stationary solutions v of (1.1), (1.2) are hyperbolic; i.e., 0 is not in the spectrum of $A - F'(v)$ or, equivalently, $y \equiv 0$ is the only solution of the linearized problem

$$y_{xx} + f'(v(x))\, y = 0, \qquad y(0) = y(1) = 0. \tag{1.6}$$

The set of those f for which (1.1), (1.2) has the above hyperbolicity property we shall denote by $\mathscr{G}$. Let us note that this additional condition on f is not very restrictive—$\mathscr{G}$ is open dense in $\mathscr{F}$ [2, 9, 11].

Recall that by the zero number $z(v) \leqslant \infty$ of a continuous function v on $[0, 1]$ we understand the number of its strict sign changes in $(0, 1)$. By the instability (Morse) index $i(v)$ of a stationary solution v we understand the dimension of the eigenspace of $A - F'(v)$ corresponding to the part of its spectrum lying in the left open complex halfplane. If v is hyperbolic, $i(v)$ is the dimension of the unstable manifold $W^u(v)$ of v and the codimension of its stable manifold $W^s(v)$ [8].

In [4] we have developed several principles for establishing and excluding connections. We have used them to identify stationary solutions w which v connects to. It turned out, however, that those principles have not been powerful enough to carry out this identification completely in all cases.

It is the purpose of the present paper to answer question (Q) completely by closing this gap (see Theorems 1.3, 1.5 below). To this aim we had to develop new tools to establish additional connections. These tools rely on the fact that, by [1, 9], S_t is a Morse–Smale semiflow for $f \in \mathscr{G}$ and on detailed information on the relative ordering of initial slopes of stationary solutions with particular zero numbers and instability indices.

We will not need any additional principles for excluding connections beyond those used in [4]. In fact, we have been able to reduce the number of exclusion principles to two, which we formulate as Propositions 1.1 and 1.2 below. By E we denote the set of stationary solutions of (1.1), (1.2).

1.1. PROPOSITION. *If v, $w \in E$ satisfy $i(w) \geqslant i(v)$, $v \neq w$, then v does not connect to w.*

This proposition is an immediate corollary of the fact that the semiflow S_t is Morse–Smale: since $W^u(v)$ intersects $W^s(w)$ transversely and since $\dim W^u(v) = i(v) \geqslant i(w) = \operatorname{codim} W^s(w)$, we have $\dim W^u(v) \cap W^s(w) \leqslant 0$. Since $v \neq w$, it follows that v cannot connect to w (cf. Lemma 3.4 (i)).

1.2. PROPOSITION [4, Lemma 4.1.] (Blocking Lemma). *Suppose $v, w, \bar{w} \in E$ are such that $\bar{w}'(0)$ is strictly between $w'(0)$ and $v'(0)$ and $z(w - \bar{w}) \geqslant z(v - \bar{w})$. Then, v does not connect to w.*

In the situation of Proposition 1.2 we shall say that $\bar{w}$ blocks the connection from v to w.

We are now able to formulate our main result.

1.3. THEOREM. *A given $v \in E$ connects to all those $w \in E$ which are not excluded by Propositions* 1.1 *and* 1.2. *Specifically, these are the stationary solutions w with $i(w) < i(v)$ for which there is no $\bar{w}$ with $\bar{w}'(0)$ between $v'(0)$ and $w'(0)$ satisfying $z(v - \bar{w}) \leqslant z(w - \bar{w})$.*

As in [4], we denote

$$\Omega(v) := \{w \neq v : v \text{ connects to } w\}.$$

Theorem 1.3 has a simple formulation but it does not give an explicit description of $\Omega(v)$. Rather, it gives conditions for a particular stationary solution to belong to $\Omega(v)$. The following two theorems (the first of which is in fact the main theorem of [4]) describe $\Omega(v)$ explicitly in terms of zero numbers of solutions and their initial slopes.

For $0 \leqslant k < i(v)$ we denote by

$\bar{v}_k$ the stationary solution w with $z(w) = k$ such that $w'(0) > |v'(0)|$ is minimal,

$\underline{v}_k$ the stationary solution w with $z(w) = k$ such that $w'(0) < -|v'(0)|$ is maximal.

We rephrase the main theorem of [4] as

1.4. THEOREM. *Let $f \in \mathcal{G}$ and let v be a stationary solution of* (1.1), (1.2). *Then v connects to other stationary solutions as follows*:

(i) *If $v \equiv 0$ or if $v \not\equiv 0$ and $i(v) = z(v)$ then*

$$\Omega(v) = \{\underline{v}_k, \bar{v}_k : 0 \leqslant k < i(v)\}.$$

(ii) *If $v'(0) > 0$ and $i(v) = z(v) + i$ then*

$$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_1 = \{\bar{v}_k : 0 \leqslant k < i(v)\},$$
$$\Omega_2 = \{\underline{v}_k : 0 \leqslant k < i(v) - 1\},$$

and either $\Omega_3 = \{\underline{v}_{i(v)-f})$ or Ω_3 consists of one or several stationary solutions with $-v'(0) \leqslant w'(0) < v'(0)$ and $i(w) < i(v)$.

We skip the explicit formulation of the case $v'(0) < 0$, $i(v) = z(v) + 1$ which is symmetric to Case (ii) and formally can be obtained by passing from $f(u)$ to $-f(-u)$.

From [4, Lemma 5.1] we know that

$$i(v) \in \{z(v), z(v) + 1\}. \tag{1.7}$$

Therefore, the only cases not settled completely by Theorem 1.4 are $v'(0) > 0$, $i(v) = z(v) + 1$, $\Omega_3 \neq \{\underline{v}_{i(v)-1}\}$, and the symmetric case $v'(0) < 0$, $i(v) = z(v) + 1$, $\Omega_3 \neq \{\bar{v}_{i(v)-1}\}$. Note that $\Omega_3(v)$ consists of those $w \in \Omega(v)$ which satisfy $w'(0) < v'(0)$ and $z(v - w) = z(v)$.

To identify Ω_3 we need even more notation. By I_n we denote the set of those $v \in E$ with $i(v) = n$. Further we denote

$$Z_n := \{v \in E \colon z(v) = n \text{ or } v \equiv 0\}$$

if $f(0) = 0$, n is even, and $0 \in I_n \cup I_{n+1}$. In all other cases we put

$$Z_n := \{v \in E \colon z(v) = n\}.$$

This definition will be motivated in Section 2. In fact, the presence of the zero solution in the nongeneric case $f(0) = 0$ complicates the arguments and requires special attention. Therefore, we suggest the reader ignores this case at first.

The stationary solutions v are naturally ordered by their slope $v'(0)$ at 0. Referring to this ordering we shall freely use the expressions "below", "above", "between" for stationary solutions in their natural sense. We call v_1, v_2 adjacent (or neighbors) in $M \subseteq E$ if there is no $w \in M$ between v_1 and v_2.

Given $J \subset \mathbb{R}$, by EJ we denote the set of those $w \in E$ for which $w'(0) \in J$. Given $v \in E[0, \infty)$, by $\underline{v}, \underline{\underline{v}}$ we denote the maximal element of $E(-v'(0), v'(0)) \cap Z_{z(v)}$, $E(-\infty, -v'(0)] \cap Z_{z(v)}$ respectively (provided it exists).

1.5. THEOREM. *Let $v'(0) > 0$ and $n = z(v) = i(v) - 1$.*

(i) *If $E(-v'(0), v'(0)) = \varnothing$, then $\Omega_3 := \{\underline{\underline{v}}\}$.*

(ii) *If $E(-v'(0), v'(0)) \neq \varnothing$, then*

$$\Omega_3 := \{\underline{v}\} \cup \bigcup_{k<n} W_k$$

where

$$W_k := \{w \in Z_k : \underline{v}'(0) < |w'(0)| < v'(0)\}.$$

Note that $\Omega_3 \neq \varnothing$ by Theorem 1.4. Therefore, $\underline{\underline{v}}$ always exists in Case (i). On the other hand, $\underline{v}$ may not exist in Case (ii). Then, we understand $\{\underline{v}\} = \varnothing$, i.e., $\Omega_3 = \bigcup_{k<n} W_k$ and $W_k = \{w \in Z_k : |w'(0)| < v'(0)\}$.

Comparing Theorem 1.5 to Conjecture 6.2 of [4] we see that the latter has not been correct. In particular the definition of $\underline{v}$ (denoted by $\underline{v}$ in [4]) had to be altered.

For the convenience of the reader we reproduce Fig. 1.2 of [4] in which Conjecture 6.2 is illustrated (Fig. 1.1). For v in the right part of the diagram the candidates for the elements of Ω_3 have been marked by question marks. By Theorem 1.5 all those solutions except the one marked by the cross indeed belong to Ω_3.

The rest of the paper contains the proof of Theorems 1.3 and 1.5 and a discussion of the results. Sections 2 and 3 prepare the proofs of the theorems, which are then given in Section 4. Section 2 deals with the time map associated with (1.1), (1.2) [12, 13]. Via the time map we clarify the relations between zero numbers, instability indices, and the ordering by initial slope. In Section 3 we use a Conley index argument and the Morse–Smale property of S_t as building blocks establishing additional connections (Lemmas 3.4, 3.6, 3.8). In Section 5 we discuss extensions to the nongeneric case and to the Neumann problem.

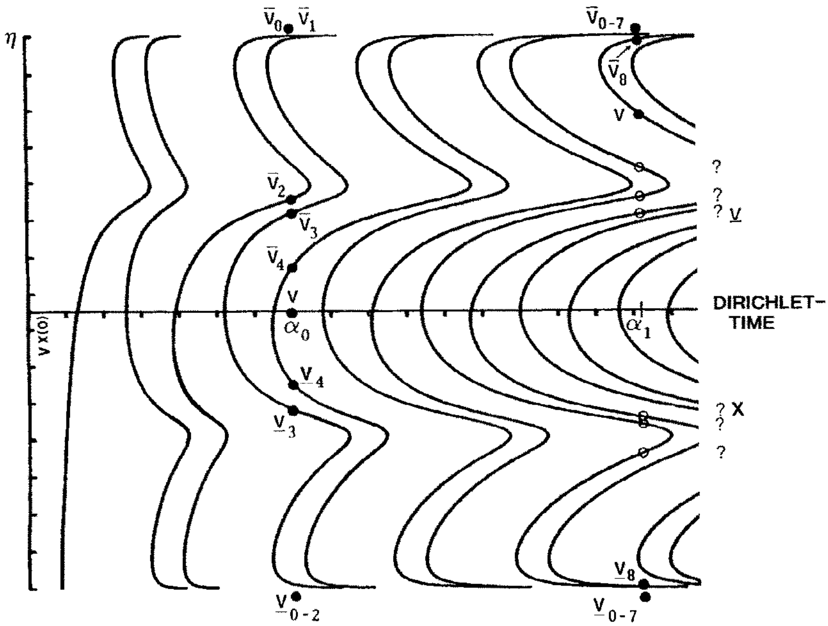

FIG. 1.1. Time map for $f(u) = -(u+10.2) \cdot u \cdot ((u-4)^2 + 1.75^2) \cdot (u-10)$, Dirichlet problem.

2. The Time-Map and the Location of Stationary Solutions

Throughout this section we assume $f \in \mathscr{F}$, see (1.5).

In [12, 13] a useful tool has been introduced for the study of stationary solutions—the time-map. In this section we use the time-map to obtain some information about the possible orderings of initial slopes of stationary solutions with particular zero numbers and instability indices. We suggest the reader makes himself familiar with the discussion of the time-map in [4, Section 1] before reading this section.

The time-map is associated with the family of boundary value problems

$$v'' + f(v) = 0 \tag{2.1}$$

$$v(0) = v(L) = 0 \tag{2.2}$$

with $L > 0$. Stationary solutions of (1.1), (1.2) are the solutions of this problem for $L = 1$. We extend the definitions of zero number, instability index, and hyperbolicity to solutions of (2.1), (2.2) for $L \neq 1$ in the natural way.

The *nth time map* T_n associated with (2.1), (2.2) is defined as follows: $T_n(\eta)$ is the nth positive zero of the solution $v(x)$ of (2.1) satisfying

$$v(0)=0, \qquad v'(0)=\eta, \tag{2.3}$$

whenever this zero exists.

The domain of definition dom T_n is an open interval (α_n, β_n) with possibly some isolated points removed [2, Lemma 4.1]. These points can be determined from the associated planar system

$$v'=w, \qquad w'=-f(v) \tag{2.4}$$

(cf. [4, Figs. 3, 4, 6] for typical phase plane diagrams). Then $\eta \in (\alpha_n, \beta_n)\backslash\text{dom } T_n$ iff the trajectory of (2.4) through $(0, \eta)$ is

— a separatrix of a saddle point in one of the halfplanes $\pm v > 0$ for $n=1$ and, respectively, $\pm\eta \geqslant 0$,

— a separatrix of a saddle point with $v \neq 0$ for $n>1$ [2, Lemma 4.1],

— the origin.

It follows that $\text{dom } T_1 \supset \text{dom } T_m = \text{dom } T_n$ for $m, n > 1$. Note that $T_m(\eta) < T_n(\eta)$ for $m < n$ and $0 \neq \eta \in \text{dom } T_n$. Moreover, $T_n(\eta) \to \infty$ for $\eta \to \eta^* \in [\alpha_n, \beta_n]\backslash\text{dom } T_n$ (see [2, Lemma 4.2]). For $v'(0) \neq 0$ all the zeros of the solutions of (2.1) are simple. The implicit function theorem then implies that T_n are C^2 in dom $T_n\backslash\{0\}$.

The collection of the graphs of T_n, $n \geqslant 1$ is called the *time-map diagram* of the problem (1.1), (1.2) (Fig. 1.1). In this diagram $v \in Z_n$, $v \not\equiv 0$ is represented by the intersection point $\eta = v'(0)$, $L=1$ of the graph of T_{n+1} with the line $L=1$. In order to emphasize this representation we shall write $\eta(v) = v'(0)$.

By [2, Theorem 2.5], v is hyperbolic precisely if the intersection of the graph of T_{n+1} with the line $L=1$ at the point $(\eta(v), 1)$ is transverse, i.e., $T_{n+1}(\eta(v)) \neq 0$. Moreover, the instability index of v can be determined from the sign of $T'_{n+1}(\eta(v))$ as follows:

2.1. Lemma. *Let $f \in \mathscr{G}$ and let v_0 be a stationary hyperbolic solution of* (2.1), (2.2) *at $L=L_0$ with $\eta_0 = v_0'(0) \neq 0$. Put $n = z(v_0)$. If $\eta_0 T'_{n+1}(\eta_0) > 0$ then $i(v_0) = n$, otherwise $i(v_0) = n+1$.*

Proof. By [2, Theorems 2.5–2.7], $T'_{n+1}(\eta_0) \neq 0$ because v_0 is hyperbolic. Moreover, $i(v_0) \in \{n, n+1\}$ by [4, Lemma 5.1]. Let $v(\cdot, \eta)$ denote the solution of (2.1) with $v(0, \eta) = 0$, $v_x(0, \eta) = \eta$. Differentiating $v(T_{n+1}(\eta), \eta) = 0$ with respect to η we obtain

$$v_x(T_{n+1}(\eta), \eta)\, T'_{n+1}(\eta) + v_\eta(T_{n+1}(\eta), \eta) = 0.$$

Because sign $v_x(T_{n+1}(\eta), \eta) = (-1)^{n+1}$ sign η, this implies

$$\operatorname{sign}(\eta T'_{n+1}(\eta)) = (-1)^n \operatorname{sign} v_\eta(T_{n+1}(\eta), \eta). \tag{2.5}$$

Since

$$v_\eta(0, \eta) = 0, \qquad v_{x\eta}(0, \eta) = 1,$$

we have $v_\eta(x, \eta) > 0$ for $x > 0$ small. Moreover, $v_\eta(\cdot, \eta)$ has only simple zeros. Therefore, by (2.5),

$$\operatorname{sign}(\eta T'_{n+1}(\eta)) = (-1)^{n + z(v_\eta)} = (-1)^{z(v_0) + z(v_\eta)}. \tag{2.6}$$

Let $r = i(v_0)$. Let $\lambda_0 < \cdots < \lambda_{r-1} < 0 < \lambda_r$ be the first $r+1$ eigenvalues of the problem

$$y_{xx} + [f'(v_0(x)) + \lambda]\, y = 0 \tag{2.7}$$

$$y(0) = y(L_0) = 0 \tag{2.8}$$

and let $\phi_0, ..., \phi_r$ be respectively their eigenfunctions. From the Sturm–Liouville theory we know that $z(\phi_i) = i$ for all i. Differentiating (2.1) with respect to η we see that $\psi := v_\eta|_{\eta=\eta_0}$ solves (2.7) with $\lambda = 0$. The Sturm–Liouville comparison theorem implies that between any two zeros of ψ there is a zero of ϕ_r and between any two zeros of ϕ_{r-1} there is a zero of ψ.

Therefore $r - 1 < z(\psi) \leqslant r$, i.e., $z(\psi) = r = i(v_0)$. Together with (2.6), this proves the lemma. ∎

For the remainder of this section we assume that $f \in \mathscr{G}$. As an immediate consequence of the properties of the time map and of Lemma 2.1 we have

2.2. LEMMA. *Let $v \in Z_n$ and let w be a neighbor of v in E, $\eta(v)\,\eta(w) > 0$. Then v and w can be related in the following alternative ways:*

(i) $w \in Z_n$, $i(w) \neq i(v)$

(ii) $w \in Z_{n-1}$, $i(w) = i(v) - 1$

(iii) $w \in Z_{n+1}$, $i(w) = i(v) + 1$.

To find out the possible relations between adjacent stationary solutions with opposite signs of η we need some information about bifurcation of stationary solutions at $\eta = 0$ which we shall briefly call the *0-bifurcation.*

First note that whenever v is a solution of (2.1), (2.2) then so is

$$\hat{v}(x) := v(L - x).$$

It follows that T_n is even for n odd. Now, suppose $f(0) < 0$, $0 \in \operatorname{dom} T_1$. Let

v_0 be a solution of (2.1), (2.2) at $L=L_0$ with $v_0'(0)=z(v_0)=0$. Since $v_0''(0)=-f(0)>0$, $v_0(x)>0$ for $x\in(0, L_0)$. The trajectory of (2.4) representing v_0 passes through the origin and is periodic with period L_0. For small $\eta>0$ the closed trajectory passing through $(0, \eta)$ represents four solutions of (6.1), (6.2) with L near L_0:

— one solution v_1 with $z(v_1)=0$, $L=T_1(\eta)$, $v_1'(0)=\eta>0$,

— two solutions $v_2, \hat{v}_2$ with $z(v_2)=z(\hat{v}_2)=1$, $v_2'(0)=-\hat{v}_2'(0)=\eta$, $L=T_2(\eta)=T_2(-\eta)=T_1(\eta)+T_1(-\eta)$,

— one solution v_3 with $z(v_3)=2$, $L=T_3(-\eta)=T_1(\eta)+2T_1(-\eta)$ and $v_2'(0)=-\eta<0$ (Fig. 2.1).

Furthermore, by [2, Lemma 4.3], $T_1(\eta)<L_0$ and $T_3(-\eta)>L_0$ for $\eta>0$ small. Also, note that by [2, Lemma 4.3], $T_1(\eta)$ is discontinuous at $\eta=0$ and $T_1(\eta)\to 0$ for $\eta\to 0-$ (Fig. 2.2).

We leave the description of the zero bifurcation for the analogous cases $f(0)>0$ and $z(v_0)>0$ to the reader.

We finish our discussion of the 0-bifurcation by considering the case $f(0)=0$. If $f(0)=0$ and $f'(0)>0$, then by [8, 5.3] the solution $v\equiv 0$ of (2.1), (2.2) satisfies $i(v)=n$ precisely if there are n 0-bifurcation points $0<L_1<\cdots<L_n$ left to L. In addition we have $\lim_{\eta\to 0} T_n(\eta)=L_n$ for each $n>0$ (cf. also [4, Section 1]). If $f(0)=0$ and $f'(0)<0$, the origin is a saddle point of (2.4). Then, all T_n, if defined near $\eta=0$, satisfy $\lim_{\eta\to 0} T_n(\eta)=\infty$.

We are now able to motivate the definition of Z_n for n even in case $f(0)=0$. Since $f\in\mathscr{G}$, $v\equiv 0$ is a hyperbolic solution of (2.1), (2.2) with $L=1$. Therefore, it has a unique local continuation to a family of solutions $\{v^g\}$ of the problem

$$v^g_{xx}+g(v^g)=0,\ v^g(0)=v^g(1)=0$$

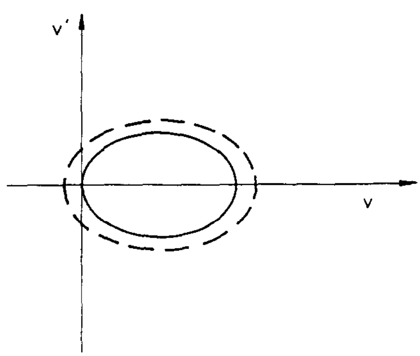

FIG. 2.1. Trajectory of a solution with zero initial slope (—) and with nonzero initial slope (- - -)

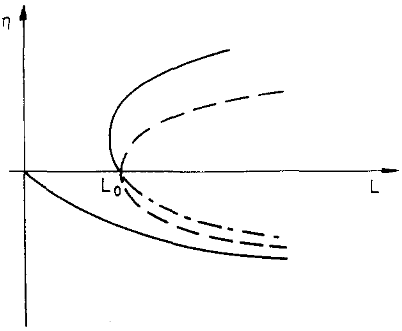

FIG. 2.2. The 0-bifurcation.

for $g \in \mathscr{F}$ near f. If $g(0) \neq 0$, from the time map diagram we immediately conclude

$$v^g \in Z_n \text{ iff } n \text{ is even and } v \in I_n \cup I_{n+1} \tag{2.9}$$

(cf. Fig. 2.3). Therefore, it is natural to extend (2.9) to $g = f$.

We also note that for $g \in \mathscr{G}$ there is always an n even and a solution $v \in Z_n$ separating w and $\hat{w}$ for any $w \in Z_k$, k odd. If $f(0) = 0$, this is of course the zero solution.

The following conclusions are now immediate (recall that we assume $f \in \mathscr{G}$ just above Lemma 2.2.)

2.3. LEMMA. *If $f(0) \neq 0$ then for each $v \in E$ we have $\eta(v) \neq 0$.*

2.4. LEMMA. *If $v \in Z_n$, $w \in Z_m$ are neighbors in E such that $\eta(v)\,\eta(w) \leqslant 0$ then $|n-m| \leqslant 1$, $|i(v) - i(w)| = 1$ and at least one of the integers m, n is even.*

2.5. LEMMA. *Let v, w be neighbors in Z_n. Then $i(w) \neq i(v)$ unless n is odd and $\eta(v)\,\eta(w) < 0$; in the latter case $w = \hat{v}$.*

2.6. LEMMA. *Let $v \in Z_n$, $\eta(v) \geqslant 0$. Denote*

$$\alpha_+ := \max\{0, \max\{\eta(w) \colon w \in E(-\infty, \eta(v)) \cap Z_n\}\},$$

$$\beta_+ := \min\{\eta(w) \colon w \in E(\eta(v), \infty) \cap Z_n\}.$$

(i$_+$) *If $i(v) = n$ then $Z_k \cap E(\alpha_+, \eta(v)) = \varnothing$ for $k \leqslant n$ and $Z_k \cap E(\eta(v), \beta_+) = \varnothing$ for $k \geqslant n$.*

(ii$_+$) *If $i(v) = n+1$ then $Z_k \cap E(\alpha_+, \eta(v)) = \varnothing$ for $k \geqslant n$ and $Z_k \cap E(\eta(v), \beta_+) = \varnothing$ for $k \leqslant n$.*

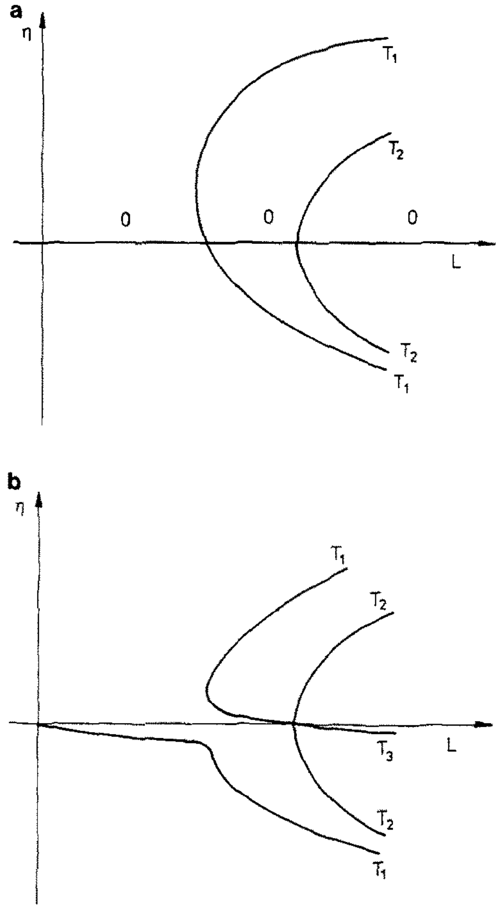

FIG. 2.3. Time map diagram: (a) $g=f$; (b) $g(0)\neq 0$.

Similarly, let $v\in Z_n$, $\eta(v)\leqslant 0$,

$$\alpha_- := \min\{0, \min\{\eta(w)\colon w\in E(\eta(v), \infty)\cap Z_n\}\},$$
$$\beta_- := \max\{\eta(w)\colon w\in E(-\infty, -\eta(v))\cap Z_n\}.$$

(i$_-$) *If* $i(v)=n$ *then* $Z_k\cap E(\eta(v), \alpha_-)=\varnothing$ *for* $k\leqslant n$ *and* $Z_k\cap E(\beta_-, \eta(v))=\varnothing$ *for* $k\geqslant n$.

(ii)$_-$) *If* $i(v)=n+1$ *then* $Z_k\cap E(\eta(v), \alpha_-)=\varnothing$ *for* $k\geqslant n$ *and* $Z_k\cap E(\beta_-, \eta(v))=\varnothing$ *for* $k\leqslant n$.

3. Components of the Proof

Throughout this section we assume $f \in \mathscr{G}$. If v connects to $w \neq v$ we shall briefly write $v \searrow w$. We recall that for $v \in E$ we write $\eta(v) = v'(0)$.

First, we recall three facts from [4].

3.1. Lemma. *If $v \neq w \in E$ and $z(v-w) \geqslant i(v)$ then v does not connect to w.*

This lemma is not formulated explicitly in [4] but is in fact proved in the initial paragraph of Section 5.

3.2. Lemma [4, Lemma 4.2]. *If $v_1 \neq v_2 \in E$ and $|\eta(v_1)| \geqslant |\eta(v_2)|$ then $z(v_1 - v_2) = z(v_1)$ and all zeros of $v_1 - v_2$ are simple.*

3.3. Lemma [4, Corollary 3.2]. *Let $v \in E$. Then, for each $k < i(v)$ and $\sigma \in \{-1, 1\}$, v connects to a solution $w \in E$ with $z(v-w) = k$ and $\sigma \cdot (\eta(v) - \eta(w)) > 0$.*

Since S_t is a gradient semiflow [8, 13] the stable and unstable manifolds of the stationary solutions are imbedded submanifolds of X. By [1, 9] they intersect transversely, so S_t is a Morse–Smale semiflow [7, Section 10]. Therefore, we have

3.4. Lemma [7, Section 10; 9].

(i) *If $v_- \searrow v_+ \in E$ then $i(v_-) > i(v_+)$ and $W^u(v_-) \cap W^s(v_+)$ is an imbedded submanifold of X of dimension $i(v_-) - i(v_+)$.*

(ii) *If $v_1 \searrow v_2 \searrow v_3 \in E$ then $v_1 \searrow v_3$.*

(iii) *Let $v_- \searrow v_+ \in E$. Then $\mathrm{cl}(W^u(v_-) \cap W^s(v_+))$consists of v_-, v_+, all $w \in E$ such that $v_- \searrow w \searrow v_+$, and their connections.*

Now we collect additional auxiliary results which will be used in the proofs of Theorems 1.3 and 1.5.

3.5. Lemma. *Let $v_-, v_+ \in E$, $i(v_-) = i(v_+) + 1$. Then, there exists at most one trajectory connecting v_- to v_+.*

Proof. Let $W^u(v_-) \cap W^s(v_+) \neq \varnothing$. Then, $\dim(W^u(v_-) \cap W^s(v_+)) = 1$ by Lemma 3.4 (i). Since trajectories connecting v_- to v_+ lie in $W^u(v_-) \cap W^s(v_+)$ they are isolated (i.e., any point u of any of the connecting trajectories admits a neighborhood U such that no other trajectory connecting v_-, v_+ passes through U except of the trajectory of u).

Let $u_i(t)$ be solutions such that

$$\lim_{t \to \pm\infty} u_i(t) = v_\pm,$$

$i=1, 2$. Since $\eta(u_i(t))$ depends continuously on t, there are t_1, t_2 such that $\eta(u_1(t_1))$ is between $\eta(v_-)$ and $\eta(v_+)$ and $\eta(u_2(t_2))=\eta(u_1(t_1))$.

Let $w(t)=u_2(t_2+t)-u_1(t_1+t)$. Then $\eta(w(0))=0$ and $w(t)$ is a solution of the linear equation

$$w_t = w_{xx} + a(t, x)w, \qquad w(t, 0) = w(t, 1) = 0, \tag{3.1}$$

where

$$a(t, x) = \int_0^1 f'((1-\theta)\, u_1(t_1+t, x) + \theta u_2(t_2+t, x))\, d\theta.$$

Write (3.1) in the abstract form

$$w_t + Aw = B(t)w \tag{3.2}$$

with $Aw = -w_{xx}$, $(B(t)w)(x) = a(t, x)\, w(x)$. Note that

$$\lim_{t\to\pm\infty} B(t) = F'(v_\pm),$$

F defined by (1.4).

Now, we indirectly prove $w(t)\equiv 0$. Suppose $w(t)\not\equiv 0$. We have $w(t)\to 0$ for $t\to\pm\infty$, and, therefore,

$$\lim_{t\to\pm\infty} w(t)/|w(t)| = \phi_\pm$$

where ϕ_-, ϕ_+ are eigenvectors of some negative eigenvalue of $A-F'(v_-)$ and some positive eigenvalue of $A-F'(v_+)$, respectively (cf. [9, Theorem 3] or [1, Lemmas 6, 7]). Thus, $z(w(t)) < i(v_-)$ for t near $-\infty$ while $z(w(t)) \geqslant i(v_+) = i(v_-)-1$ for t near $+\infty$. Since $w(t)$ solves (3.1), $z(w(t))$ does not increase with t [4, Section 1]. Hence $z(w(t)) = i(v_+)$ for all t. This contradicts $\eta(w(0))=0$ since by [4, Lemma 7.4] $z(w(t))$ drops at $t=0$ in such case.

Therefore, $w(t)\equiv 0$, or, $u_1(t)$ and $u_2(t)$ have the same trajectories. ∎

3.6. LEMMA. *Let $v \searrow w_1 \searrow w \in E$, $i(w_1) = i(v)-1 = i(w)+1$. Then, there exists a $w_2 \in E$ such that $w_2 \neq w_1$, $i(w_2) = i(w_1)$, and $v \searrow w_2 \searrow w$.*

Proof. Our original proof used the λ-lemma [10, 7] to show that the intersection of $W^u(v) \cap W^s(w)$ with a small sphere Γ in $W^u(v)$ around v is a continuous curve with two distinct limit points u_1 and u_2, with u_1 lying on the orbit connecting v to w_1. The existence of w_2 then followed from $u_2 \in \mathrm{cl}(W^u(v) \cap W^s(w)) \backslash (W^u(v) \cap W^s(w))$ and Lemma 3.4 (iii).

The following more elegant topological proof was suggested to us by K. Mischaikow. For a background see [6].

It follows from Lemma 3.4 (iii) that $S=\mathrm{cl}(W^u(v)\cap W^s(w))$ is an isolated invariant set admitting a Morse decomposition $S=M_1\cup M_2\cup M_3$ where $M_1=\{v\}$, $M_2=\{\bar{w}: v\searrow\bar{w}\searrow w\}$, $M_3=\{w\}$. By Lemma 3.4 (i) all the elements of M_2 have the same instability index $i(w_1)$. Hence, they do not connect to each other.

The proof now proceeds indirectly. Assume $M_2=\{w_1\}$. From the uniqueness and transversality of the orbits connecting v to w_1 and w_1 to w it follows that the connection matrix [6] corresponding to this Morse decomposition has the form

$$\begin{array}{cccc} & M_1 & M_2 & M_3 \\ M_1 & 0 & 1 & \alpha \\ M_2 & 0 & 0 & 1 \\ M_3 & 0 & 0 & 0 \end{array}$$

with coefficients in $\mathbf{Z}_2$. This matrix, however, cannot be a connection matrix since its square does not vanish. This contradiction proves the lemma. ∎

3.7. Lemma. *Let v, w_1, w_2, w be connected as in Lemma 3.6. Assume $n=i(w_1)=i(w_2)=z(v)$, $w\in Z_{n-1}\cup Z_n$, and $\eta(v)>|\eta(w)|$. Then w_1, w_2 are below v and w is between w_1 and w_2.*

Proof. The proof is indirect. Assume $\eta(w_j)>\eta(v)$ for $j=1$ or 2. By Lemma 2.1 we have $w_j\in Z_{n-1}\cup Z_n$. By Lemma 3.2, $z(w_j-v)\leqslant\max\{z(w_j), z(v)\}=n$ and, because $n(v)>|\eta(w)|$, $z(v-w)=n$. Hence v blocks $w_j\searrow w$, a contradiction.

So both w_1 and w_2 are below v. Next we suppose $\eta(v)>\eta(w_1)>\eta(w_2)>\eta(w)$. By Lemma 3.2, $z(v-w_1)=z(v)=n$, $z(w_2-w)\geqslant\min\{z(w_2), z(w)\}=n-1$. Therefore, by Proposition 1.2, $v\searrow w_2$, $w_1\searrow w$ imply $z(w_1-w_2)<n$, $z(w_1-w_2)\geqslant n$, respectively, a contradiction.

Finally, suppose $\eta(v)>\eta(w)>\eta(w_1)>\eta(w_2)$. Since $z(v-w_1)\leqslant n$, $z(w_1-w)\geqslant n-1$, and $v\searrow w_2$, $w_2\searrow w$, Proposition 1.2 implies $z(w_1-w_2)<n$, $z(w_1-w_2)\geqslant n$ respectively, a contradiction.

This completes the proof since, up to interchanging w_1 and w_2, the only possibility left is the one mentioned in the lemma. ∎

3.8. Lemma. *Let $v_-\searrow v_+\in E$, $i(v_+)<z(v_+-v_-)=i(v_-)-1$. Then, there exists a $w\in E\cap\mathrm{cl}(W^u(v_-)\cap W^s(v_+))$ such that $v_-\searrow w\searrow v_+$.*

Proof. The existence of w follows from Lemma 3.4 (ii) provided we prove that the set $\{v_-\}\cup(W^u(v_-)\cap W^s(v_+))\cup\{v_+\}$ is not closed. We introduce the proof of this latter fact by a short outline.

Denote $n=z(v_+-v_-)$, $m=i(v_+)$. Further, denote by $\lambda_0^\pm<$

$\lambda_1^{\pm} < \ldots, \phi_0^{\pm}, \phi_1^{\pm}, \ldots,$ respectively, the eigenvalues and the corresponding normalized eigenvectors of $A - F'(v_{\pm})$. We consider a small sphere Γ around v_- in $W^u(v_-)$ which is crossed transversely by the trajectories and show that the restriction to $W^u(v_-) \cap W^s(v_+) \cap \Gamma$ of the orthogonal projection $X \to \operatorname{span}\{\phi_m^-, \ldots, \phi_{n-1}^-\}$ is a local diffeomorphism. This will prove that $W^u(v_-) \cap W^s(v_+) \cap \Gamma$, and, therefore, also $\{v_-\} \cup (W_u(v_-) \cap W^s(v_+)) \cup \{v_+\}$ is not closed.

Next, we give the details. Let $u(t)$ be a solution connecting v_- to v_+. Since $W^u(v_-)$, $W^s(v_+)$ are C^1, we have

$$T_{u(t)} W^u(v_-) \to \operatorname{span}\{\phi_j^- : j = 0, \ldots, n\} \text{ for } t \to -\infty, \tag{3.3}$$

and

$$\begin{aligned} T_{u(t)}^{\perp} W^s(v_+) &\to T_{v_+}^{\perp} W^s(v_+) \\ &= \operatorname{span}\{\phi_j^+ : j = 0, \ldots, m-1\} \qquad \text{for} \quad t \to \infty \end{aligned}$$

(orthogonality understood in $L_2(0, 1)$).

Now, fix t_0 and consider the sphere $\Sigma^{m-1} = T_{u(t_0)}^{\perp} W^s(v_+) \cap \{\psi_0 : |\psi_0| = 1\}$. By [9] we have $z(\psi_0) < m$ for each $\psi_0 \in \Sigma^{m-1}$. It follows from [9] that, for all t, $T_{u(t)}^{\perp} W^s(v_+)$ is spanned by the vectors $\psi(t)$, which are, by definition, the solutions of the adjoint equation

$$\dot{\psi} - A\psi = -F'(u(t))\psi \tag{3.4}$$

with $\psi(t_0) \in \Sigma^{m-1}$.

Note that the zero number of the solutions of (3.4) does not increase with decreasing t. Therefore, $z(\psi(t)) < m$ for all $t < t_0$ and we can associate with (3.4) a continuous essential map $y: \Sigma^{m-1} \to S^{m-1}$ as in [4, Section 2] but with t replaced by $-t$. Specifically, the map y is defined as follows. For $0 < k < m$, let t^k denote the first time $t < t_0$ such that $z(\psi(t))$ drops to k or below. Let $\tau_k := \tanh(t_0 - t^k)$. The sign $\sigma_k := \operatorname{sign} \psi_x(t, 0)$ is constant for t between t^{k-1} and t^k; put $\sigma_k = 0$ in case $t^{k-1} = t^k$. The components $y = (y_0, \ldots, y_{m-1})$ of the map y are defined as

$$y_k \sigma := \tau_k(\tau_{k-1} - \tau_k)^{1/2}, \qquad 0 \leqslant k < m,$$

with the convention $\tau_{-1} := 1$. By construction, y maps into $S^{m-1} \subset R^m$. Observe that y is indeed essential since [4, Lemma 2.2] extends to the time-dependent equation (3.4).

By [9, Theorem 4], for each solution $\psi(t)$ of (3.4) there exists an eigenvector ϕ_j^-, $j \geqslant 0$ such that

$$\lim_{t \to -\infty} \psi(t)/|\psi(t)| = \phi_j^- . \tag{3.5}$$

Since $z(\psi(t)) < m$ for all t near $-\infty$ and $z(\phi_j^-) = j$, we have $j < m$. Since y is essential, it is surjective. In particular, for each $0 \leqslant j < m$ there exists a $\zeta_j \in \Sigma^{m-1}$ such that $y_j = 1$, $y_\nu = 0$ for $\nu \neq j$, where $y = (y_0, ..., y_{m-1})$. This implies in particular that the solution $\psi_j(t)$ of (3.4) with $\psi_j(t_0) = \zeta_j$ satisfies (3.5). Using (3.3) and the fact that ϕ_j^-, $j \geqslant 0$ are orthogonal we obtain for $t \to -\infty$

$$\begin{aligned} &T_{u(t)}(W^u(v_-) \cap W^s(v_+)) \\ &\quad = T_{u(t)} W^u(v_-) \cap T_{u(t)} W^s(v_+) \\ &\quad = T_{u(t)} W^u(v_-) \cap (\operatorname{span}\{\psi_j(t): j = 0, ..., m-1\})^\perp \\ &\quad \to \operatorname{span}\{\phi_j^- : j = 0, ..., n\} \cap (\operatorname{span}\{\phi_j^- : j = 0, ..., m-1\})^\perp \\ &\quad = \operatorname{span}\{\phi_j^- : m \leqslant j \leqslant n\}. \end{aligned} \tag{3.6}$$

Since $u(t) \to v_+$ for $t \to \infty$ we have $z(u(t) - v_-) = z(v_+ - v_-) = n$ for large t and, consequently, $z(u(t) - v_-) \geqslant n$ for all t. By [3, Theorem 2.1] this implies

$$\lim_{t \to -\infty} (u(t) - v_-)/|u(t) - v_-| = \pm\phi_n^-. \tag{3.7}$$

A sufficiently small neighborhood of v_- on $W^u(v_-)$ can be considered as an open subset of $\mathbb{R}^{n+1}$, the coordinates $\chi = (\chi_0, ..., \chi_n)$ chosen in such a way that $\chi_k(u) = \langle u - v_-, \phi_k^- \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(0, 1)$. Then, locally at v_-, the restriction of S_t to $W^u(v_-)$ is generated by the ordinary differential equation

$$d\chi/dt = C\chi + g(\chi), \tag{3.8}$$

where $C = \operatorname{diag}\{-\lambda_0^-, ..., -\lambda_n^-\}$, g is C^1, and

$$g'(0) = 0. \tag{3.9}$$

Denote $\Gamma(\rho) = \{u \in W^u(v_-): \|\chi(u)\| \leqslant \rho\}$, where $\|\chi\| = \langle \chi, \chi \rangle^{1/2}$ for $\chi \in \mathbb{R}^{n+1}$. By (3.9), $\|g(\chi)\| = o(\|\chi\|)$ for $y \to 0$. There exists a $\rho_0 > 0$ such that if $\chi(t)$ is a solution of (3.8) and $0 \neq \|\chi(t)\| \leqslant \rho_0$, we have

$$\begin{aligned} 1/2\, d/dt\, \|\chi(t)\|^2 &= -\sum_{j=0}^{n} \lambda_j^- \chi_j^2 + \langle \chi, g(\chi) \rangle \\ &\geqslant [-\lambda_n^- \|\chi(t)\| - \|g(\chi(t))\|]\, \|\chi(t)\| > 0. \end{aligned}$$

This means that $\{\Gamma(\rho): 0 \leqslant \rho \leqslant \rho_0\}$ is a nested family of C^1 cylinders containing v_- which is crossed transversely by the trajectories $W^u(v_-)$. Note that $T_u \Gamma(\rho) \to \operatorname{span}\{\phi_0^-, ..., \phi_{n-1}^-\}$ for $u = \rho\phi_n^-$, $\rho \to 0$.

The set $M(\rho) := \Gamma(\rho) \cap W^s(v_+)$ is a manifold of dimension $n-m$. We finish the proof by showing that for sufficiently small ρ the assumption of $M(\rho)$ being closed leads to a contradiction. Note that if $M(\rho)$ is closed for some $\rho \in (0, \rho_0]$ then it is closed and compact for all such ρ.

Let $u \in M(\rho)$ for some ρ. Then $\lim_{t \to \infty} S_t(u) = v_+$, hence $z(S_t(u) - v_-) = z(v_+ - v_-) = n$ for t large. Consequently, $z(S_t(u) - v_-) \geqslant n$ for all t. Since $u \in W^u(v_-)$, we also have $z(S_t(u) - v) \leqslant n = i(v_-) - 1$ for all t. Hence, $z(S_t(u) - v) = n$ for all t.

By [3, Theorem 2.1], there exists an n-dimensional submanifold W_{n-1} of $W^u(v_-)$ through v_- such that for all $u \in W^u(v_-) \backslash W_{n-1}$, $u(t) := S_t(u)$, one has (3.7). It follows that if $M(\rho)$ is closed, then $M(\rho)$ is a disjoint union of the compact sets

$$M^{\pm}(\rho) := \{u \in M(\rho) \colon \lim_{t \to -\infty} (S_t(u) - v_-)/|S_t(u) - v_-| = \pm\phi_n^-\},$$

at least one of which is non-empty.

Let $N(\rho)$ be any of the sets $M^{\pm}(\rho)$ which is not empty. It is a submanifold of $\Gamma(\rho)$ of dimension $n-m$. Let $Y := \operatorname{span}\{\phi_m^-, ..., \phi_{n-1}^-\}$ and let P be the spectral projection $X \to Y$. Since $N(\rho_0)$ is compact, (3.7) holds uniformly for $u(t) = S_t(u)$, $u \in N(\rho_0)$. Therefore,

$$T_u \Gamma(\rho) \to \operatorname{span}\{\phi_j^- : 0 \leqslant j < n\} \qquad \text{for} \quad \rho \to 0, u \in N(\rho).$$

Hence, by (3.6), it follows that

$$T_u N(\rho) = T_u W^s(v_+)) \cap T_u \Gamma(\rho) \to \operatorname{span}\{\phi_j^- : m \leqslant j < n\} = Y$$

for $\rho \to 0$, $u \in N(\rho)$. This means that for $\rho > 0$ sufficiently small and $u \in N(\rho)$

$$P|_{T_u N(\rho)} \text{ is an isomorphism.} \tag{3.10}$$

Since $N(\rho)$ is compact, so is $P(N(\rho))$. On the other hand, by (3.10), $P(N(\rho))$ is open in Y, a contradiction. ∎

3.9. Lemma. *Let $v \searrow w \in E$. Suppose*

(i) $i(v) = z(v) + 1$, $i(w) < z(v)$.

(ii) $|\eta(v)| \geqslant |\eta(w)|$.

Then there exists a $\underline{w}$ such that $|\eta(\underline{w})| < |\eta(v)|$, $i(\underline{w}) = n$, and $v \searrow \underline{w} \searrow w$.

Proof. Let $n = z(v)$. Without loss of generality assume $\eta(v) > 0$. By Lemma 3.2, $z(v - w) = z(v) = n$. Hence Lemma 3.8 applies (v_-, v_+ replaced by v, w, respectively), i.e., there is another stationary point

$w_1 \in \mathrm{cl}(W^u(v) \cap W^s(w))$ such that $v \searrow w_1 \searrow w$. Note that by Lemma 3.4 (i), $i(w_1) \leqslant n$.

We claim that $z(w_1 - v) = n$. Indeed, $z(v - w) = n = i(v) - 1$ implies

$$z(u-v) \geqslant n \qquad \text{for all } u \in W^u(v) \cap W^s(w). \tag{3.11}$$

Also, Lemma 3.2 implies $z(v - w) = z(v) = n$. The zeros of $w_1 - v$ are simple by Lemma 3.2 and $w_1 \in \mathrm{cl}(W^u(v) \cap W^s(w))$, hence (3.11) implies $z(w_1 - v) \geqslant n$. On the other hand $z(w_1 - v) \leqslant n = i(v) - 1$ by Lemma 3.1. This proves the claim.

Next we claim that $|\eta(w_1)| < \eta(v)$. Indeed, suppose $\eta(w_1) > \eta(v)$. Then v blocks $w_1 \searrow w$ by Proposition 1.2 because $z(w_1 - v) = n = z(w - v)$. On the other hand, suppose that $\eta(w_1) \leqslant -\eta(v)$. Then $z(v - w) = n = z(w_1 - w)$ since Lemma 3.2 yields $z(w_1 - w) = z(w_1) = z(w_1 - v) = n$. Hence w blocks $v \searrow w_1$. These two contradictions prove our claim.

Since $v \searrow w_1 \searrow w$ we have $i(w_1) \leqslant n$ and, therefore, also $z(w_1) \leqslant n$. If $i(w_1) = n$, we put $\underline{w} := w_1$. Suppose therefore that $i(w_1) < n$. Then we can repeat the argument given above, with w replaced by w_1, to find a solution w_2 such that $v \searrow w_2 \searrow w_1$ and $|\eta(w_2)| < \eta(v)$. By Lemma 3.4 (ii) we also obtain $w_2 \searrow w$. Since $i(w_2) \geqslant i(w_1) + 1$ by Lemma 3.4 (i) it is now obvious that after a finite number of steps we arrive at a solution $\underline{w}$ with the required properties. ∎

Synthesizing Lemmas 3.6, 3.7, and 3.9 we obtain

3.10. Corollary. *Let $v \searrow w \in E$ be such that*

(i) $i(v) = z(v) + 1$, $i(w) = z(v) - 1$,

(ii) $|\eta(w)| < \eta(v)$.

Then there exist w_1, w_2 such that $i(w_j) = z(v)$, $\eta(w_j) < \eta(v)$, $v \searrow w_j \searrow w$, $j = 1, 2$, and w is between w_1 and w_2.

4. Proofs of Theorems 1.3, 1.5

Proof of Theorem 1.5

We first carry out the proof under the additional assumption $f(0) \neq 0$. Note that in this case $Z_n = \{w \in E: z(w) = n\}$. We postpone the discussion of the (non-generic) case $f(0) = 0$ to the end of the proof.

We proceed by induction on $n = z(v) = i(v) - 1$. Recall that as before $\eta(v)$ denotes the initial slope $v'(0)$ of v. We also recall that the stationary solutions $\underline{v}, \underline{\underline{v}}$ are defined immediately before Theorem 1.5.

Consider $n = 0$ first. By Lemma 3.3, v connects to some element

$w \in E(-\infty, \eta(v)) \cap I_0 \subseteq E(-\infty, \eta(v)) \cap Z_0$. From Proposition 1.2 (or the maximum principle) it follows that w is the maximal element of $E(-\infty, \eta(v)) \cap Z_0$. If $E(-\eta(v), \eta(v)) = \emptyset$ then $w = \underline{v}$ by definition of $\underline{v}$. Any solution w_1 from $E(-\eta(v), \eta(v))$ blocks the connections from v to any solution w_2 below. Indeed, $z(v - w_1) = z(v) = 0$ while trivially $z(w_2 - w_1) \geqslant 0$. Thus, if $E(-\eta(v), \eta(v)) \neq \emptyset$ then $w \in E(-\eta(v), \eta(v))$. In that case $w = \underline{v}$ by definition of $\underline{v}$.

Assume now that $n > 0$ and that the statements of Theorem 1.5 (and of its counterpart for $\eta(v) < 0$) hold for all $v \in Z_k \cap I_{k+1}$, $k = 0, ..., n-1$.

So we consider the case $v \in Z_n \cap I_{n+1}$, $\eta(v) > 0$. If $E(-\eta(v), \eta(v)) = \emptyset$, Theorem 1.4 (ii) reduces to the alternative $\Omega_3 = \underline{v}_n$ or $\Omega_3 = \{w\}$, where $w \in E$, $\eta(w) = -\eta(v)$. To prove $\Omega_3 = \{\underline{v}\}$ we have to show that if $\Omega_3 = \underline{v}_n$ then $\underline{v}_n$ is the maximal element of $E(-\infty, \eta(v)) \cap Z_n$, i.e., there is no w such that $\eta(w) = -\eta(v)$.

This claim we prove indirectly. If w exists then by Lemma 3.2 $z(v - w) = z(v) = z(\underline{v}_n) = z(\underline{v}_n - w)$. Hence w blocks $v \searrow \underline{v}_n$, a contradiction to $\Omega_3 = \{\underline{v}_n\}$.

Assume now $E(-\eta(v), \eta(v)) \neq \emptyset$. Denote $L := \{\underline{v}\} \cup \bigcup_{k<n} W_k$ and recall that Ω_3 is the set of those $w \in \Omega(v)$ which satisfy $z(v - w) = n$, $\eta(w) < \eta(v)$. In four steps we prove that $\Omega_3 = L$. First we prove that if $w \notin L$, $\eta(w) < \eta(v)$ and $z(v - w) = n$ then v does not connect to w. In other words, $\Omega_3 \subseteq L$. Using the induction hypothesis, in Step 2 we prove that v connects to all elements of L provided v is known to connect to all elements of

$$K := L \cap (Z_{n-1} \cup Z_n)$$

In Step 3 we prove that v connects to some element of K. Finally, using the induction hypothesis once more, we conclude the proof in Step 4 by showing that if v connects to some solution from K then it connects to its neighbors in K as well.

When using the induction hypothesis we shall sometimes refer to solutions and sets of solutions (like $\underline{v}$, L, etc.) in the definition of which v is replaced by another solution w. In such cases we shall write respectively $\underline{v}(w)$, $L(w)$, etc.

Step 1. If $w \notin L$, $z(v - w) = n$ and $\eta(w) < \eta(v)$ then v does not connect to w. In other words, $\Omega_3 \subseteq L$.

Let w be as above. By Lemma 3.2 and the definition of L we have the following possibilities:

(i) $\eta(w) < -\eta(\underline{v})$, $z(w) = n$,

(ii) $|\eta(w)| \leqslant \eta(v)$, $z(w) = n$, $w \neq \underline{v}$,

(iii) $|\eta(w)| < \eta(\underline{v})$, $z(w) < n$.

In case (i), any element $w_1 \in E(-\eta(v), \eta(v))$ satisfies $z(v-w_1)=n=z(w-w_1)$ by Lemma 3.2 and therefore blocks $v \searrow w$ by Proposition 1.2.

In the cases (ii), (iii), we have $\eta(w)<\eta(\underline{v})<\eta(v)$; in both cases Lemma 3.2 implies $z(v-\underline{v})=n=z(w-\underline{v})$. Therefore, $\underline{v}$ blocks $v \searrow w$ by Proposition 1.2.

Step 2. If v connects to all elements of K then v connects to all elements of L.

Let $w \in L \backslash K$, i.e., $w \in L$, $z(w)<n-1$. We distinguish two cases, recalling that $\eta(w) \neq 0$ by Lemma 2.3.

Case (i). $\eta(w)>0$. From Lemma 2.2 it follows that Z_{n-1} has elements between w and v; let w_1 be the minimal one. Note that $w_1 \in K$ because $z(w_1)=n-1$, $\eta(v)>\eta(w_1)>\eta(w)$, and, by definition of L, $\eta(w)>\eta(\underline{v})$ if $\underline{v}$ exists. Therefore, $v \searrow w_1$ by assumption. By Lemma 2.6 applied to w_1 we have $i(w_1)=n$, hence $w_1 \in Z_{n-1} \cap I_n$. Since there is no element from Z_{n-1} between w_1 and w, we have $w \in L(w_1)$. Thus, $w_1 \searrow w$ by the induction hypothesis and $v \searrow w$ by Lemma 3.4 (ii).

Case (ii). $\eta(w)<0$. From Theorem 1.4 applied to v it follows that Z_{n-1} has elements below w; let w_1 be the maximal one. Since $z(w)<n-1$ we have $\bigcup_{k<n-1} Z_k \cap E(\eta(w_1), 0) \neq \varnothing$. Lemma 2.6 ($\mathrm{i}_-$) applied to w_1 yields $i(w_1)>z(w_1)$. Since there is no element of Z_{n-1} between w_1 and w, we have $w_1 \searrow w$ by the induction hypothesis applied to w_1. If $\eta(w_1)>-\eta(v)$ then $\eta(w_1)<\eta(w)<-\eta(\underline{v})$, hence $w_1 \in K$. Thus, $v \searrow w_1$ by assumption. If, on the other hand, $\eta(w_1) \leqslant -\eta(v)$, then $w_1=\underline{v}_{n-1}$ and $v \searrow w_1$ by Theorem 1.4. Since $v \searrow w_1 \searrow w$, $v \searrow w$ by Lemma 3.4 (ii).

Step 3. The solution v connects to some element of K.

By Lemma 3.3, v connects to some solution w below v with $z(v-w)=n$. Since $E(-\eta(v), \eta(v)) \neq \varnothing$ by assumption, $\eta(w)>-\eta(v)$ by Step 1. If $w \in I_n$ then $w \in Z_{n-1} \cap Z_n$ by Lemma 2.1. Hence $w \in K, v \searrow w$, and we are done. Next suppose $w \notin I_n$, i.e., $i(w)<n=z(v)$. Then, by Lemma 3.9, v connects to some solution $\underline{w} \in I_n$ with $|\eta(\underline{w})|<\eta(v)$. As before, $\underline{w} \in Z_{n-1} \cup Z_n$. By Step 1 it follows $\underline{w} \in L \cap (Z_{n-1} \cup Z_n)=K$. Since $v \searrow \underline{w}$, this completes step 3.

Step 4. If v connects to some $w \in K$ then it connects to its neighbors in K (provided they exist).

We begin with preparatory lemmas.

4.1. Lemma. *Let* $z(w)=n-1$ *and let* $v \searrow w$. *Then* $v \searrow \hat{w}$, *where* $\hat{w}(x):=w(1-x)$.

Proof. If $n-1$ is even, the lemma is trivial since $\hat{w}=w$. If $n-1$ is odd, then n is even, hence $v=\hat{v}$. If $u(t, x)$ is a solution which connects v to w then $\hat{u}(t, x)=u(t, 1-x)$ obviously connects $v=\hat{v}$ to $\hat{w}$. ▮

4.2. LEMMA. *Assume that $\underline{v}$ exists. Then*

(i) $i(\underline{v})=n$ *and*

(ii) *the upper (lower) neighbor of $\underline{v}$ in K is given by $\bar{v}_{n-1}(\underline{v})$, ($\underline{v}_{n-1}(\underline{v})$, respectively) whenever this neighbor exists.*

Proof. (i) By definitions of K and $\underline{v}$, v and $\underline{v}$ are neighbors in Z_n and $|\eta(\underline{v})|<\eta(v)$. In particular $\underline{v} \neq \hat{v}$ and, therefore, $i(\underline{v}) \neq i(v)$ by Lemma 2.5. More precisely, $i(\underline{v})=n$ since $i(v)=n+1$ and $i(\underline{v}) \in \{n, n+1\}$.

(ii) We note that $\underline{v}$ is the only element in $Z_n \cap K$. Therefore, any neighbor of $\underline{v}$ in K is from Z_{n-1}. By definition of K, the lower neighbor of $\underline{v}$ is the maximal element from Z_{n-1} with $\eta<|\eta(\underline{v})|$ which is, by definition, $\underline{v}_{n-1}(\underline{v})$. Similarly, the upper neighbor of $\underline{v}$ in K is $\bar{v}_{n-1}(\underline{v})$, provided $\eta(\underline{v})>0$. To verify that the upper neighbor of $\underline{v}$ is still $\bar{v}_{n-1}(\underline{v})$ in the case $\eta(\underline{v})<0$ it suffices to prove that

$$Z_{n-1} \cap E(\eta(\underline{v}), -\eta(\underline{v}))=\varnothing. \tag{4.1}$$

To prove (4.1) note first that $\eta(\underline{v})<0$ forces n to be even. This follows from Lemma 2.5 since $|\eta(\underline{v})|<\eta(v)$ and, therefore, $\underline{v} \neq \hat{v}$.

Suppose now that (4.1) does not hold. Since $i(\underline{v})=z(\underline{v})=n$ and since v, $\underline{v}$ are neighbors in Z_n with $\eta(v)\, \eta(\underline{v})<0$ we conclude from Lemma 2.6 (i) that

$$Z_{n-1} \cap E(\eta(\underline{v}), 0)=\varnothing. \tag{4.2}$$

Suppose now that there exists a $w \in Z_{n-1} \cap E(0, -\eta(\underline{v}))$. Then $\eta(\hat{w}) \in E(\eta(\underline{v}), 0)$ since $n-1$ is odd. Hence $\hat{w} \in Z_{n-1} \cap E(\eta(v), 0)$, which contradicts (4.2). This, together with (4.2), proves (4.1) and concludes the proof of the lemma. ▮

4.3. LEMMA. *If $w_1, w_2 \in Z_{n-1}$ are neighbors in K then they are neighbors in Z_{n-1}.*

Proof. The statement follows immediately from the definition of K if $\eta(w_1)\, \eta(w_2)>0$.

Consider now $\eta(w_1)>0>\eta(w_2)$ and suppose there exists a $\underline{w} \in Z_{n-1} \backslash K$ between w_1 and w_2. Then $\eta(w_1)>\eta(\underline{v})>\eta(\underline{w})>-\eta(\underline{v})>\eta(w_2)$ by definition of K. Thus, $\underline{v} \in K$ is between w_1 and w_2, hence w_1 and w_2 are not neighbors in K, contrary to our assumption. ▮

4.4. LEMMA. *We have*

$$K \subseteq I_{n-1} \cup I_n. \tag{4.3}$$

If w_1, w_2 are neighbors in K then

$$i(w_1) \neq i(w_2) \tag{4.4}$$

unless $w_1 = \hat{w}_2$. If $w_1 = \hat{w}_2$ then n is even, $z(w_1) = z(w_2) = n-1$, and $i(w_1) = i(w_2)$.

Proof. First we note that (4.3) follows immediately from $K \cap Z_n = \{\underline{v}\}$ together with Lemma 4.2 (i) and Lemma 2.1.

Consider the case $w_1 = \hat{w}_2$. Then, $i(w_1) = i(w_2)$ by symmetry. Since $\underline{v}$ is the only element in $Z_n \cap K$, $z(w_1) = z(w_2) = n-1$. Since $w_1 \neq w_2$, $\eta(w_1) = -\eta(w_2)$, and $n-1$ is odd. Therefore, n is even.

Next consider the case $w_1 \neq \hat{w}_2$; we prove (4.4). Since $K \subseteq Z_{n-1} \cup Z_n$ and since $\underline{v}$ is the only element in $K \cap Z_n$, we are left with the following possibilities (up to interchanging w_1 and w_2):

(i) $z(w_1) = n$ (i.e., $w_1 = \underline{v}$) and $z(w_2) = n-1$;

(ii) $z(w_1) = z(w_2) = n-1$.

In Case (i) Lemma 4.2 (ii) implies that w_2 is either $\underline{v}_{n-1}(w_1)$ or $\bar{v}_{n-1}(w_1)$. In both cases $w_1 \searrow w_2$ by Lemma 4.2 (i) and Theorem 1.4. Hence, (4.4) holds by Lemma 3.4 (i).

In Case (ii) Lemma 4.3 implies that w_1, w_2 are neighbors in Z_{n-1}. Since we consider the case $w_1 \neq \hat{w}_2$, (4.4) follows from Lemma 2.5. ∎

For any given $w \in K$, we will denote by $\underline{w}, \bar{w}$ its lower and upper neighbors in K, respectively, provided it exists. When discussing any of these neighbors we will tacitly assume, without further notice, that it actually exists.

4.5. LEMMA (Snowball Principle I). *Assume that v connects to $w \in K$, with $i(w) = n$. Then v connects to $\underline{w}$ and $\bar{w}$.*

Proof. First we consider the case $z(w) = n$. Then $w = \underline{v}$ by definition of K. By Lemma 4.2 we have $\underline{w} = \underline{v}_{n-1}(w)$, $\bar{w} = \bar{v}_{n-1}(w)$. Since $i(w) = z(w) = n$, $w \searrow \underline{v}_{n-1}(w) = \underline{w}$ and $w \searrow \bar{v}_{n-1}(w) = \bar{w}$ by Theorem 1.4. Because $v \searrow w$ by assumption, Lemma 3.4 (ii) implies $v \searrow \underline{w}$, $v \searrow \bar{w}$.

It remains to consider the case $z(w) = n-1$. By Lemma 4.1, v connects to a neighbor of w if that neighbor is $\hat{w}$. Below, we may therefore assume that $\underline{w} \neq \hat{w} \neq \bar{w}$. It is sufficient to prove $w \searrow \underline{w}, \bar{w}$, because Lemma 3.4 (ii) then implies $v \searrow \underline{w}, \bar{w}$.

We may without loss of generality assume $\eta(w)>0$ (since neighbors in $K\cap Z_{n-1}$ are neighbors in Z_{n-1} by Lemma 4.3 the other case is analogous). Then $\bar{w}=\bar{v}_{n-1}(w)$ and $w\searrow\bar{w}$ by Theorem 1.4.

For $\underline{w}$ we distinguish two possibilities:

$$\text{(a)}\ |\eta(\underline{w})|<\eta(w), \qquad \text{(b)}\ \eta(\underline{w})\leqslant -\eta(w).$$

In Case (a) we have $\underline{w}=\underline{v}(w)$ by definition of $\underline{v}$. Since $i(w)=n=z(w)+1$, $w\searrow\underline{w}$ by the induction hypothesis.

Now, assume (b) holds. In Lemma 4.6 below we prove that in this case, under our standing assumption $\underline{w}\neq\hat{w}$, we have

$$E(-\eta(w),\eta(w))=\varnothing \tag{4.5}$$

Using (4.5) we obtain $\underline{w}=\underline{v}(w)$. Thus our induction hypothesis together with (4.5) yields $w\searrow\underline{w}$. Then, $v\searrow\underline{w}$ follows by Lemma 3.4 (ii). ∎

To complete the proof of Lemma 4.5 it remains to prove

4.6. Lemma. *Assume $\eta(w)>0, i(w)=n=z(w)+1$, $\underline{w}$ is the lower neighbor of w in Z_{n-1}, $\eta(\underline{w})\leqslant-\eta(w)$, and $\underline{w}\neq\hat{w}$. Then $E(-\eta(w),\eta(w))=\varnothing$.*

Proof. Since $\underline{w}\neq\hat{w}$ and $\eta(\underline{w})\,\eta(w)<0, z(w)=n-1$ is even by Lemma 2.5. Moreover, $i(\underline{w})=n-1=z(\underline{w})$.

Suppose now $E(-\eta(w),\eta(w))=E(-\eta(w),0)\cup E(0,\eta(w))\neq\varnothing$. By Lemma 2.6, all solutions from $E(-\eta(w),0)$ have $z>n-1$ while all solutions from $E(0,\eta(w))$ have $z<n-1$. By Lemmas 2.2, 2.4 the zero numbers of neighbors in E canot differ by more than one. Therefore, if $E(0,\eta(w))\neq\varnothing$ then there exists a $w_1\in Z_{n-2}\cap E(0,\eta(w))$. Similarly, if $E(-\eta(w),0)\neq\varnothing$, then there exists a $w_2\in Z_n\cap E(-\eta(w),0)$. Since both $n-2$ and n are odd, we conclude $\hat{w}_1\in E(-\eta(w),0)$ in the first case, and $\hat{w}_2\in E(0,\eta(w))$ in the second case. Both cases lead to contradictions since $z(\hat{w}_1)=n-2<n-1$ and $z(\hat{w}_2)=n>n-1$. ∎

4.7. Lemma (Snowball Principle II). *Assume that v connects to $w\in K$ with $i(w)=n-1$. Then v, w connect to the w-neighbors $\bar{w}$, $\underline{w}$ in K.*

Proof. Since $i(w)\in\{z(w),z(w)+1\}$ by Lemma 2.1 and since $K\subseteq Z_{n-1}\cup Z_n$ we have $z(w)=n-1$. By Corollary 3.10 there exist $w_1,w_2\in E$ such that $i(w_1)=i(w_2)=n$, $\eta(v)>\eta(w_1)>\eta(w)>\eta(w_2)$, and $v\searrow w_j\searrow w, j=1,2$. We have $z(w_j)\geqslant n-1$, hence $w_j\in K$ by Step 1. This implies that $\underline{w}$, $\bar{w}$ exist. We prove indirectly

$$w_1=\bar{w}, \qquad w_2=\underline{w}. \tag{4.6}$$

Suppose $w_2 \neq \underline{w}$. Then $\underline{w}$ is between w and w_2 and Lemma 3.2 implies

$$z(v-\underline{w}) \leqslant n, \qquad z(\underline{w}-w_2) \geqslant n-1, \qquad z(\underline{w}-w) \geqslant n-1. \tag{4.7}$$

In order that $\underline{w}$ not block $v \searrow w_2$ we need $z(\underline{w}-w_2) < n$. Hence $z(\underline{w}-w_2) = n-1$. Then, however, $z(w-\underline{w}) \geqslant z(\underline{w}-w_2)$ by (4.7). Thus $\underline{w}$ blocks $w_2 \searrow w$, a contradiction.

Now, suppose $w_1 \neq \bar{w}$. Then $\bar{w}$ is between w and w_1. We distinguish two cases:

$$\text{(a) } z(w_1) = n, \qquad \text{(b) } z(w_1) = n-1.$$

In Case (a) we have $z(w_1) = i(w_1)$. Hence Theorem 1.4 applies to w_1. Since $w_1 \searrow w$, this implies $w = \underline{v}_{n-1}(w_1)$. Also, $v \searrow w_1$ implies $w_1 = \underline{v}$ by Step 1. Hence

$$\eta(\bar{w}) < -|\eta(w_1)| \tag{4.8}$$

holds (trivially for $\eta(w_1) < 0$ and by definition of K for $\eta(w_1) > 0$). By (4.8) and Lemma 3.2 this implies

$$z(w_1 - \bar{w}) = z(\bar{w}). \tag{4.9}$$

Since $w_1 = \underline{v}$ is the only element in $K \cap Z_n$ we have $z(\bar{w}) = n-1$. Since $z(w) = n-1$, also, Lemma 3.2 yields

$$z(\bar{w} - w) = n-1. \tag{4.10}$$

By Proposition 1.2, (4.9) and (4.10) imply that $\bar{w}$ blocks $w_1 \searrow w$, a contradiction.

In Case (b) we have $z(w_1) = z(w) = n-1$, $i(w_1) = n > i(w) = n-1$. Hence, by Theorem 1.4 and the induction hypothesis, the following alternatives are possible:

$w = \underline{v}_{n-1}(w_1)$ (in case $\eta(w_1) < 0$)
$w = \underline{\underline{v}}(w_1)$ (in case $\eta(w_1) > 0$ and $E(-\eta(w_1), \eta(w_1)) = \varnothing$)
$w = \underline{v}(w_1)$ (in case $\eta(w_1) > 0$ and $E(-\eta(w_1), \eta(w_1)) \neq \varnothing$).

In any case w and w_1 are neighbors in Z_{n-1}. To prove that they are also neighbors in K it remains to be shown that $\underline{v} \notin E(\eta(w), \eta(w_1))$.

To this aim we first note that if $\underline{v} \in E(\eta(w), \eta(w_1))$ then $\eta(w_1) > 0$. Indeed, $\eta(\underline{v}) < \eta(w_1) < 0$ contradicts Lemma 2.6 (i_-) since $z(\underline{v}) = \mathrm{i}(\underline{v})$ by Lemma 4.2.

Now, since $i(w_1) = z(w_1) + 1$, it follows from Lemma 2.6 (ii_+) that $\underline{v} \notin E(\max\{0, \eta(w)\}, \eta(w_1))$. This proves the claim if $\eta(w) > 0$. Next suppose that $\eta(w) < 0$ and $\underline{v} \in E(\eta(w), 0)$. We will obtain a contradiction, separately, for the cases that n is odd or even.

Indeed, suppose n is odd. Then $\underline{v} \in E(\eta(w), 0)$ implies $\hat{\underline{v}} \in E(0, -\eta(w)) \subset E(0, \eta(v))$. This contradicts the definition of $\underline{v}$ as being the maximal element of $E(-\eta(v), \eta(v)) \cap Z_n$.

If n is even, on the other hand, then $n-1$ is odd. Hence $i(w_1) \neq i(w)$ and $\eta(w_1)\,\eta(w) < 0$ contradicts Lemma 2.5. ∎

Having proved Lemmas 4.2, 4.5, and 4.7 we can easily complete Step 4 and, hence, also the proof of Theorem 1.5 for the case $f(0) \neq 0$.

Indeed, if $w \in K$ then $i(w) \in \{n-1, n\}$ (by Lemma 2.1 for $w \in Z_{n-1}$ and by Lemma 4.2 (i) for $w = \underline{v}$). Now, Step 4 follows from Lemma 4.5 for $w \in I_n \cap K$ and from Lemma 4.7 for $w \in I_{n-1} \cap K$.

To complete the proof of Theorem 1.5 it remains to settle the case $f(0) = 0$.

Let $f \in \mathscr{G}$, $f(0) = 0$. Let $\{f_\varepsilon\}$ (ε near 0) be a family of functions from $\mathscr{F}$ such that $f_0 = f$, $f_\varepsilon \to f$ in $\mathscr{F}$ for $\varepsilon \to 0$, and $f_\varepsilon(0) \neq 0$ for $\varepsilon \neq 0$ (one can take, e.g., $f_\varepsilon(u) = f(u) + \varepsilon$). Since $\mathscr{G}$ is open dense in $\mathscr{F}$, $f_\varepsilon \in \mathscr{G}$ for ε near 0. By S^ε, E^ε we denote the semiflow and the set of stationary solutions, respectively, of the perturbed problem

$$u_t = u_{xx} + f_\varepsilon(u), \qquad u(t, 0) = u(t, 1) = 0. \tag{4.11}$$

Since $S^0 (= S)$ is Morse–Smale and since $E^0 (= E)$ is finite, S^ε is still Morse–Smale for ε from some neighborhood U of 0. Moreover, for each $v \in E$ there exists a local continuation v^ε of v such that $v^0 = v$, $v^\varepsilon \in E^\varepsilon$; for ε small enough all elements of E^ε occur as continuations of some element of E and we have $i(v^\varepsilon) = i(v)$, $v^\varepsilon \in Z_{z(v)}$. The last inclusion simply means $z(v^\varepsilon) = z(v)$ for $v \not\equiv 0$; for $v \equiv 0$ it is a consequence of the definition of Z_n (cf. Fig. 2.3). The ordering of the elements of E by their initial slopes extends to $\varepsilon \neq 0$; i.e., for every $v_1, v_2 \in E$ we have

$$\eta(v_1^\varepsilon) < \eta(v_2^\varepsilon) \qquad \text{iff} \quad \eta(v_1) < \eta(v_2). \tag{4.12}$$

Now let v satisfy the assumptions of the theorem; i.e., $i(v) = z(v) + 1$, $\eta(v) > 0$. From (4.12) and the definition of the set L it follows immediately that

$$L(v^\varepsilon) = \{w^\varepsilon \colon w \in L(v)\} \tag{4.13}$$

for ε sufficiently near 0. Since S is Morse–Smale, for every $v, w \in E$ and ε sufficiently near 0 we have

$$v^\varepsilon \searrow w^\varepsilon \qquad \text{iff} \quad v \searrow w. \tag{4.14}$$

For $|\varepsilon| \neq 0$ small we have $f(0) \neq 0$ and $E(-\eta(v^\varepsilon), \eta(v^\varepsilon)) \neq \emptyset$, hence

$$\Omega_3(v^\varepsilon) = L(v^\varepsilon). \tag{4.15}$$

From (4.13), (4.14), (4.15) it follows that $\Omega_3(v) = L(v)$, and Theorem 1.5 is proved. ∎

Proof of Theorem 1.3

We prove that if $w \notin \Omega(v)$ and $i(w) < i(v)$ (i.e., Proposition 1.1 does not apply) then there exists a $\underline{w} \in E$ that blocks $v \searrow w$ (i.e., Proposition 1.2 applies). We carry out the proof under the generic assumption $f(0) \neq 0$. The extension to the case $f(0) = 0$ follows along the lines of the perturbation argument used in the proof of Theorem 1.5.

Without loss of generality assume $\eta(v) > 0$. Note that $i(w) < i(v)$ implies $z(w) < i(v)$ by Lemma 2.1. We distinguish three cases:

(a) $|\eta(w)| \geqslant \eta(v)$,

(b) $|\eta(w)| < \eta(v)$, $i(v) = z(v) + 1$,

(c) $|\eta(w)| < \eta(v)$, $i(v) = z(v)$.

Case (a). $|\eta(w)| \geqslant \eta(v)$. Let $k := z(w)$. By Theorem 1.4 and by definition of $\bar{v}_k$, $\eta(w) \geqslant \eta(v)$ and $v \neq w \notin \Omega(v)$ means $\eta(w) > \eta(\bar{v}_k) > \eta(v)$. By Lemma 3.2, $z(w - \bar{v}_k) = k = z(\bar{v}_k - v)$, hence $\bar{v}_k$ blocks $v \searrow w$.

For analogous reasons $\underline{v}_k$ blocks $v \searrow w$ if $\eta(w) \leqslant -\eta(v)$ and $k < z(v)$.

If $\eta(w) \leqslant -\eta(v)$ and $k = z(v)$ (which is possible only if $i(v) = z(v) + 1$) then $w \notin \Omega(v)$ means that w is below any element $\underline{w}$ of $\Omega_3(v)$. The altenative of Theorem 1.4 (ii) yields either $\eta(\underline{w}) < -\eta(v)$ and $z(\underline{w}) = k$ (if $\Omega_3(v) = \underline{v}_k$) or $|\eta(\underline{w})| \leqslant \eta(v)$. In both cases $\underline{w}$ is between v and w and, in addition, Lemma 3.2 implies $z(v - \underline{w}) = k = z(\underline{w} - w)$. Thus, $\underline{w}$ blocks $v \searrow w$.

Case (b). $|\eta(w)| < \eta(v)$, $i(v) = z(v) + 1$. In this case Theorem 1.5 applies: from $w \notin \Omega(v)$ it follows that $w \notin L$, and Lemma 3.2 implies $z(v - w) = z(v)$. In this situation the existence of the blocking solution $\underline{w}$ is proved in Step 1 of the proof of Theorem 1.5.

Case (c). $|\eta(w)| < \eta(v)$, $i(v) = z(v)$. We denote $n := z(v)$ and distinguish two cases:

(ca) n is odd (cb) n is even

Case (ca). n is odd. Let $\underline{w}$ denote the maximal element of $Z_n \cap E(-\infty, \eta(v))$. Since n is odd we have $Z_n \ni \hat{\underline{w}} \neq \underline{w}$, $\eta(\hat{\underline{w}}) = -\eta(\underline{w})$, $i(\hat{\underline{w}}) = i(\underline{w})$, where, as before, $\hat{w}(x) = w(1-x)$. Thus Lemma 2.5 implies $\eta(\underline{w}) > 0$ unless $\underline{w} = \hat{v}$.

If $\underline{w} = \hat{v}$ then Lemma 2.6 (i_+) applied to v and Lemma 2.6 (i_-) applied to $\hat{\underline{w}}$ yield

$$i(w_1) \geqslant z(w_1) > n = i(v) \qquad \text{for} \quad w_1 \in E(-\eta(v), \eta(v)). \tag{4.16}$$

In other words, there is no w satisfying both (c) and $i(w) < i(v)$.

If $\eta(\underline{w}) > 0$ we have $i(\underline{w}) = n+1$ by Lemma 2.2 and $i(w_1) \geqslant z(w_1) > n$ for all w_1 with $\eta(\underline{w}) < \eta(w_1) < \eta(v)$ by Lemma 2.6 (i_+).

Thus,

$$\eta(w) < \eta(\underline{w}). \tag{4.17}$$

Also, $\hat{\underline{w}}$ and $\hat{v}$ are neighbors in Z_n, $\eta(\hat{\underline{w}}) = -\eta(\underline{w}) > -\eta(v) = \eta(\hat{v})$, and $i(\hat{v}) = i(v) = n$. Hence $i(w_1) \geqslant z(w_1) > n$ for all w_1 with $\eta(\hat{\underline{w}}) > \eta(w_1) > \eta(\hat{v})$ by Lemma 2.6 (i_-) applied to $\hat{v}$. Thus, $\eta(w) > -\eta(\underline{w})$ and, because of (4.17), $|\eta(w)| < \eta(\underline{w})$. Since $z(v - \underline{w}) = n = z(\underline{w} - w)$ by Lemma 3.2, $\underline{w}$ blocks $v \searrow w$.

Case (cb). n is even. Again let $\underline{w}$ be the maximal element of $Z_n \cap E(-\infty, \eta(v))$. By Lemma 2.5 we have $i(\underline{w}) = n+1$, hence $w \neq \underline{w}$. We distinguish two possibilities:

$$\text{(cba)}\ \eta(\underline{w}) > 0 \qquad \text{and} \qquad \text{(cbb)}\ \eta(\underline{w}) < 0.$$

Case (cba). $\eta(\underline{w}) > 0$. As in case (ca) we prove below that $i(w_1) \geqslant z(v) = i(v)$ if $w_1 \in E(\eta(\underline{w}), \eta(v)) \cup E(-\eta(v), -\eta(\underline{w}))$, i.e., $|\eta(w)| \leqslant \eta(\underline{w})$. The connection $v \searrow w$ is then blocked by $\underline{w}$.

By Lemmas 2.6 (i_+) and 2.1 we have

$$\bigcup_{k \leqslant n} Z_k \cap E(\eta(\underline{w}), \eta(v)) = \varnothing. \tag{4.18}$$

Consider now $w_1 \in E(-\eta(v), -\eta(\underline{w}))$. We prove that $i(w_1) < z(v)$ leads to a contradiction.

Suppose $z(w_1) = n-1$. Since $n-1$ is odd, $\hat{w}_1 \in E(\eta(\underline{w}), \eta(v))$. This contradicts Lemma 2.6 (i_+).

Thus, we can suppose $z(w_1) < n-1$. Indirectly we prove that $Z_{n-1} \cap E(\eta(w_1), \eta(\underline{w})) \neq \varnothing$. From $Z_{n-1} \cap E(\eta(w_1), \eta(\underline{w})) = \varnothing$ it follows that the upper neighbor of the maximal element of $\bigcup_{k<n-1} Z_k \cap E[\eta(w_1), \eta(\underline{w}))$ in E is from Z_m, $m \geqslant n > z(w_1) + 1$—a contradiction to Lemmas 2.2 and 2.4.

Since $n-1$ is odd, we have

$$Z_{n-1} \cap E(0, -\eta(w_1)) = \{\hat{w}_3 : w_3 \in Z_{n-1} \cap E(\eta(w_1), 0)\}.$$

In particular, both $Z_{n-1} \cap E(0, -\eta(w_1))$ and $Z_{n-1} \cap E(\eta(w_1), 0)$ are non-empty. On the other hand, using (4.18) we obtain

$$\begin{aligned} Z_{n-1} \cap E(-\eta(v), \eta(v)) &= Z_{n-1} \cap E(-\eta(\underline{w}), \eta(\underline{w})) \\ &= [Z_{n-1} \cap E(0, \eta(\underline{w}))] \\ &\quad \cup \{\hat{w}_3 : w_3 \in Z_{n-1} \cap E(0, \eta(\underline{w}))\}. \end{aligned} \tag{4.19}$$

Lemma 2.6 (i_+) applied to the maximal element w_4 of $E(0, \eta(\underline{w})) \cap Z_{n-1}$ implies $i(w_4) > z(w_4)$. By (4.19), the minimal element of $Z_{n-1} \cap E(\eta(w_1), 0)$ is $\hat{w}_4$. Since $i(\hat{w}_4) = i(w_4) > z(w_4) = z(\hat{w}_4)$, Lemma 2.6 ($\mathrm{ii}_-$) applied to $\hat{w}_4$ contradicts $z(w_1) < n-1 = z(\hat{w}_4)$.

Case (cbb). $\eta(\underline{w}) < 0$. We prove that no $w \in E$ satisfying $|\eta(w)| < \eta(v)$ and $i(w) < i(v)$ exists in this case.

Since $z(v) = i(v)$, Lemma 2.6 (i_+) implies

$$\bigcup_{k<n} Z_k \cap E(0, \eta(v)) = \varnothing. \tag{4.20}$$

Consider now $E(-\eta(v), 0)$. If $w \in \bigcup_{k<n} Z_k \cap E(-\eta(v), 0)$ then, by Lemmas 2.4 and 2.2, either $w \in Z_{n-1}$ or $Z_{n-1} \cap E(\eta(w), \eta(v)) \neq \varnothing$. It follows that $Z_{n-1} \cap E(-\eta(v), 0) \neq \varnothing$ in any case. However, since $n-1$ is odd, $w_1 \in Z_{n-1} \cap E(-\eta(v), 0)$ implies $\hat{w}_1 \in E(0, \eta(v))$, contrary to (4.20). This completes the proof of Theorem 1.3. ∎

5. Concluding Remarks

Although we did not carry out the details we believe that, employing the same tools as in the generic case, a complete answer to the connection problem can be obtained without the genericity assumption on f. However, we have restricted our attention to the generic case since we do not expect the non-generic case to bring anything new except technical complications.

Propositions 1.1 and 1.2 hold analogously for equation (1.1) with Neumann boundary conditions

$$u_x(t, 0) = u_x(t, 1) = 0. \tag{5.1}$$

In [4, Theorem 6.1] and [5] a counterpart of Theorem 1.4 for the Neumann case is presented. Employing the techniques developed in Section 3 and the properties of the time map diagram for the Neumann problem one can also complete Theorem 6.1 of [4]. We omit the proofs which are analogous to the Dirichlet case.

Theorem 1.3 extends to the Neumann case without change. To formulate the counterpart of Theorem 1.5 for the Neumann case we have to introduce some notation.

Let $\mathscr{G}_N$ be the set of those $f \in \mathscr{F}$ for which all stationary solutions of (1.1), (5.1) are hyperbolic (by [9, 11] this set is also generic). Given a non-constant C^1 function v on $[0, 1]$ let $l(v)$, the *lap number* of v, denote the number of sign changes of the derivative of v in $(0, 1)$ increased by one. In other words, $l(v) := z(v_x) + 1$. Put $l(v) := 0$ for v constant. As in the Dirichlet case, E is the set of stationary solutions of (1.1), (5.1). Let N_n denote the set of stationary solutions with $l(w) = n$. Finally, for a given $v \in N_n$, let M_k denote the set of those solutions $w \in N_k$ satisfying range $w \subset$ range v for which there is no solution $\bar{w} \in N_n$ such that range $w \subsetneqq$ range $\bar{w} \subsetneqq$ range v.

THEOREM 5.1. *Let* $f \in \mathscr{G}_N$ *and let* $v \in N_n, v \not\equiv \text{constant}, i(v) = n + 1, v(0) = \max v$. *Then*

$$\Omega_3 = \{w \in M_n : w(0) = \max w\} \cup \bigcup_{0 \leqslant k < n} M_k$$

(*for the definition of* Ω_3 *cf.* [4, Section 6]).

Note that this theorem excludes the alternative $\Omega_3 = \{\underset{\sim}{v}_n\}$ of [4, Theorem 6.1]. Indeed, there is always a constant solution in range v which blocks $v \searrow \underset{\sim}{v}_n$.

In Fig. 5.1 (which is Fig. 5.1 of [4]), for v in the right part of the

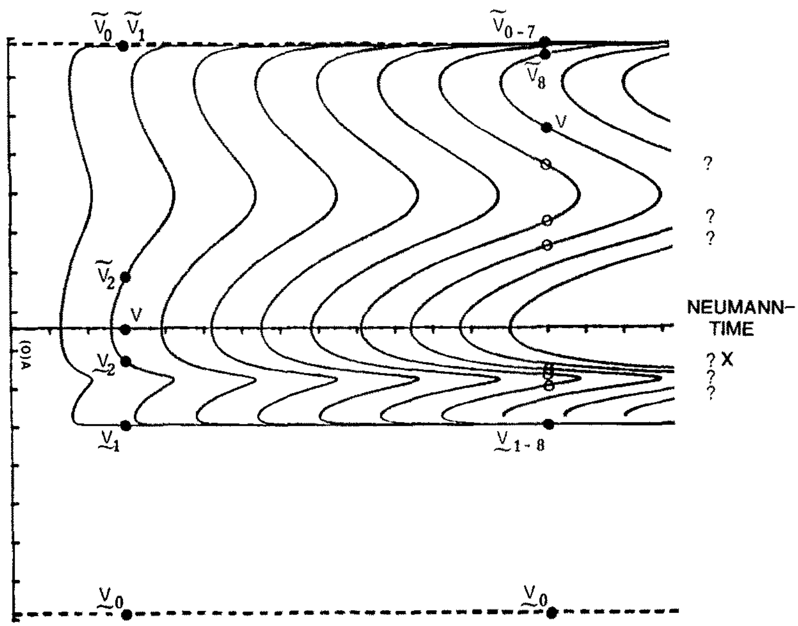

FIG. 5.1. Time map for $f(u) = -(u + 10.2) \cdot u \cdot ((u-4)^2 + 1.75^2) \cdot (u - 10)$, Neumann problem.

diagram the candidates for Ω_3 not excluded by [4, Theorem 6.1] are marked by ? By Theorem 5.1, v connects to all of them except the one marked by X.

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