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The Morse–Smale Structure of a Generic Reaction–Diffusion Equation in Higher Space Dimension

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Transversality of stable and unstable manifolds of equilibria plays an important role in the qualitative study of dynamical systems. In particular, it is closely related to structural stability properties. For example, a gradient flow on a finite dimensional space is known to be structurally stable (qualitatively intact under small perturbations of the vector field), provided all its equilibria are hyperbolic and their stable an unstable manifolds intersect transversally (see [Pali, Pali-S, Pali-M]; see also [Sh] for more recent results and references). By theorems of Kupka and Smale, it is known that a generic finite-dimensional vector field enjoys both hyperbolicity of all equilibria and transversality of their stable and unstable manifolds (see [Pali-M, Rob, Ta, Ku-O]).

Structural stability has also been studied for several classes of dissipative infinite-dimensional dynamical systems. As a rule, one investigates structural stability of the flow restricted to a compact set, usually the attractor, rather than the flow in the original state space (an exception can be found in [Lu2]; see also the related papers [Bat-L, Lu1, Ko]). Here, too, hyperbolicity of equilibria and transversality of their stable and unstable manifolds guarantee the structural stability of gradient flows (see [Hal-M-O]). It is natural to ask whether these properties are generic in the considered class of dynamical systems. To be more specific, consider the Dirichlet problem for the reaction–diffusion equation,

$$u_t = \Delta u + f(x, u), \qquad t > 0,\ x \in \Omega, \tag{1.1}$$

$$u = 0, \qquad t > 0,\ x \in \partial\Omega. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and f is a sufficiently regular function on $\bar{\Omega} \times \mathbb{R}$. Problem (1.1), (1.2) defines a local

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semiflow on an appropriate Banach space, for example, the Sobolev space $W_0^{1,p}(\Omega)$ with $p>N$ (see [He1, Am, Lun2, Da-K]). This semiflow is gradient-like: the energy functional

$$\varphi \mapsto \int_\Omega \left(\tfrac{1}{2}\,|\nabla\varphi(x)|^2 - F(x, \varphi(x))\right) dx,$$

where $F(x, u)$ is the antiderivative of $f(x, u)$ with respect to u, decreases along nonconstant trajectories.

It is relatively easy to prove that generically in a C^r topology ($r \geqslant 1$) all equilibria of (1.1), (1.2) are hyperbolic (see Section 3). As the transversality of stable and unstable manifolds is concerned, a remarkable property is found if $N=1$. In this case, transversality always occurs—for any equation and any two hyperbolic equilibria (see [He2, An]). This fact is closely related to the nodal properties of solutions and the Jordan curve theorem in the x, t plane.

In higher space dimensions, the situation is quite different. Stable and unstable manifolds of hyperbolic equilibria can intersect nontransversally (see [Pol2, Pol3]). One of the main objectives of the present paper is to prove that generically this cannot happen.

To formulate the result precisely, let k be a positive integer and let $\mathfrak{G}$ denote the space of all C^k functions $f\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ endowed with the C^k Whitney topology. This is the topology in which the collection of all the sets

$$\{ g \in \mathfrak{G} : |D^i f(x, u) - D^i g(x, u)| < \delta(u),\ i = 0, ..., k,\ x \in \bar{\Omega},\ u \in \mathbb{R} \},$$

where δ is a positive continuous function on $\mathbb{R}$, forms a neighborhood basis of an element f. Recall that $\mathfrak{G}$ is a Baire space: any residual set (a countable intersection of open and dense sets) is dense in $\mathfrak{G}$ (see [Go-G]).

Our main result reads as follows.

THEOREM 1.1. *There is a residual set $\mathfrak{G}^{MS}$ in $\mathfrak{G}$ such that for any $f \in \mathfrak{G}^{MS}$ all equilibria of* (1.1), (1.2) *are hyperbolic and if φ^-, φ^+ are any two such equilibria then their stable and unstable manifolds intersect transversally*:

$$W^u(\varphi^-) \pitchfork W^s(\varphi^+).$$

We remark that the assertion remains valid if instead of the Whitney (strong) topology one considers the weak C^k topology on $\mathfrak{G}$. The same proof works in this case (the arguments are always independent of the behaviour of functions f outside a compact set in $\bar{\Omega} \times \mathbb{R}$). The proof can also be easily adapted to other boundary conditions (Neumann, Robin).

The method of our paper is based on a study of parabolic differential operators, such as the operator

$$u \mapsto u_t - \Delta u - f(x, u), \tag{1.3}$$

acting on an appropriate space of functions defined for all $t \in \mathbb{R}$. The basic idea is to formulate the transversality of stable and unstable manifolds in terms of regular values of this operator and then apply a parametrized transversality theorem. Much of the underlying technical work will be done in the context of an abstract parabolic equation with parameter. We derive sufficient conditions for transversality of the stable and unstable manifolds of equilibria to be generic with respect to the parameter. We have taken some care to formulate these results in a way suitable for applications in other classes of parabolic (or ordinary) differential equations.

With abstract sufficient conditions for transversality at hand, our main concern becomes their verification for problem (1.1), (1.2), where f is viewed as a parameter. This is perhaps the most interesting part—the core of our paper. Nodal properties of solutions of parabolic and elliptic equations play an important role in these considerations.

The method using differential operators, such as (1.3), has already been used in various contexts. For example, it has been employed in the study of global (homoclinic and heteroclinic) bifurcations. In particular, let us mention the work of Hale and Lin [Hal-L] on delay equations which contains many ideas and refined results in this direction. For other applications of this method see also [Palm, Pe, Bl, Zh, Sa, Sc] and references therein. It is also of interest to notice that in the finite-dimensional context a sufficient condition for generic transversality that is similar to ours (cf. condition (h5)(c) in Theorem 4.c.1) was developed, from another point of view in [Rob].

The Morse–Smale property of partial differential equations has been studied by other authors, but the results obtained up to now (and known to us) are in some way related to the one-dimensional reaction–diffusion equations, and to the theorem of Henry and Angenent mentioned above. For example, the Morse–Smale structure is preserved by some small singular perturbations of a 1D equation, as are thin domain problems (see [Hal-R]), 1D hyperbolic equations with large damping [Mo-S], and various discretizations of the equation (see [Fu-O, Ol-O-S, Ei-P]). Another result of this sort can be found in [Pol2]. As shown there, under some positiveness assumptions, one can prove transversality for equations on a ball using the fact that the solutions asymptotically behave like solutions of a 1D (radial) problem.

In our approach the nonlinearity plays the role of a parameter with respect to which we prove the genericity. A natural other parameter that could be considered is the domain Ω. In [He3], Henry proves that hyperbolicity of all equilibria is a generic property of (1.1), (1.2) with respect to domain perturbation (with the nonlinearity fixed). It is likely that the same is true of the transversality of stable and unstable manifolds. This ought to be the subject of a future study. There is certainly good motivation to look at this problem more closely. In particular, if equations with a special structure are to be dealt with, the genericity with respect to the nonlinearity many fail (this actually happens if Ω is the ball and f is radial in x: $f = f(|x|, u)$, see [Pol2]) and to retain it one has to allow for domain perturbations.

The paper is organized as follows. In Section 2, we give a version of abstract transversality theorem that we repeatedly use in the paper.

In Section 3, we study the equilibria of (1.1), (1.2). We prove that generically all the equilibria are hyperbolic and the linearization at each of them has only simple eigenvalues.

Section 4 is fully devoted to abstract parabolic equations and generic (with respect to a parameter) transversality of their stable and unstable manifolds.

In Section 5 we prove Theorem 1.1.

Frequently Used Notation. If X is a Banach space, we use $\|\cdot\|_X$ to denote its norm. We omit the subscript if there is no danger of confusion. For a C^1 map F, $DF(z)v$ denotes the derivative of F at a point z in the direction v. We use a subscript, for example $D_zF(z, \lambda)v$, to denote the partial derivative. $D^kF(z)$ denotes the kth order derivative of F at z.

If X, Y are Banach spaces, U is an open set in X, and $r \in \mathbb{N}$, we denote by $C_b^r(U, Y)$ ($C_b(U, Y)$ if $r = 0$) the space of C^r functions of U into Y whose derivatives up to order r are bounded on U. It is a Banach space with the norm

$$\|f\|_{C_b^r(U, Y)} = \sup_{u \in U} \max\{\|f(u)\|, D^1f(u)\|, ..., \|D^rf(u)\|\}. \tag{1.3}$$

By $C_b^r(\bar{U}, Y)$ we denote the closed subspace of $C_b^r(U, Y)$ consisting of functions whose derivatives up to order r have a continuous extension to $\bar{U}$. If X is finite-dimensional and U is bounded, we suppress the subscript b: $C^r(\bar{U}, Y) = C_b^r(\bar{U}, Y)$. We write $C^r(\bar{U})$ for $C^r(\bar{U}, \mathbb{R})$.

By $C^{r,\delta}(U, Y)$, $r \in \mathbb{N}$, $\delta \in (0, 1]$, we denote the subspace of $C_b^r(U, Y)$ consisting of the functions whose r-order derivative is δ-Hölder continuous on U. Its norm is given by

$$\|u(\cdot)\|_{C^{r,\delta}(U, Y)} = \|u(\cdot)\|_{C_b^r(U, Y)} + \sup_{x, y \in U, x \neq y} \frac{\|D^ru(x) - D^ru(y)\|_Y}{\|x - y\|_X}.$$

By $\mathscr{L}(X, Y)$ we denote the space of all bounded linear maps from X to Y; $\mathscr{L}(X) = \mathscr{L}(X, X)$.

Convention 1.2. When taking the intersection of Banach spaces X_1, X_2, we always assume that $X = X_1 \cap X_2$ is equipped with the norm

$$\|\cdot\|_X = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}.$$

2. REGULAR VALUES AND THE TRANSVERSALITY THEOREM

In this section we formulate an abstract transversality theorem that is applied at several places below.

We first recall a few definitions following [Ab-R]. A bounded linear map $L: X \to Z$ between Banach spaces X, Z is said to be *Fredholm* if its range $R(L)$ is closed and both dim ker(L) and codim $R(L)$ are finite; the *index* of L is then the integer

$$\mathrm{ind}(L) = \dim \ker(L) - \mathrm{codim}\, R(L).$$

Let $\mathscr{X}$, $\mathscr{Z}$ be Banach manifolds, and $\Phi: \mathscr{X} \to \mathscr{Z}$ be a C^1 map. A point z is a *regular value* of Φ if for any $x \in \Phi^{-1}(z)$ the derivative $D\Phi(x)$ is surjective (onto $T_{\Phi(x)}\mathscr{Z}$) and its kernel splits (has a closed complement in $T_x\mathscr{X}$). All other points in $\mathscr{Z}$ are *critical values* of Φ. We sometimes write $\Phi \pitchfork \{\xi\}$ to express that ξ is a regular value of Φ. A subset in a topological space is meager if it is contained in a countable intersection of closed nowhere dense sets. The complement of a meager set is a residual set.

THEOREM 2.1. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be smooth Banach manifolds, $\Phi: \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ a C^r map (r is a positive integer), and ξ a point in $\mathscr{Z}$. Assume that the following hypotheses are satisfied:*

(i) *For each $(x, y) \in \Phi^{-1}(\xi)$, $D_x\Phi(x, y): T_x\mathscr{X} \to T_\xi\mathscr{Z}$ is a Fredholm map of index less than r.*

(ii) *For each $(x, y) \in \Phi^{-1}(\xi)$, $D\Phi(x, y): T_x\mathscr{X} \times T_x\mathscr{Y} \to T_\xi\mathscr{Z}$ is surjective.*

(iii) *One of the following properties holds*

(a) *The map $(x, y) \mapsto y: \Phi^{-1}(\xi) \to \mathscr{Y}$ is σ-proper, that is, there is a countable system of subsets $V_n \subset \Phi^{-1}(\xi)$ such that $\bigcup_n V_n = \Phi^{-1}(\xi)$ and for each n the map $(x, y) \mapsto y: V_n \to \mathscr{Y}$ is proper (any sequence (or net) $(x_\nu, y_\nu) \in V_n$ such that y_ν is convergent in $\mathscr{Y}$, has a convergent subsequence (subnet) with the limit in V_n.*

(b) *$\mathscr{X}$, $\mathscr{Y}$ are separable metric spaces.*

Then the set of all $y \in \mathscr{Y}$ such that ξ is a regular value of $\Phi(\cdot, y)$ is residual (hence dense) in $\mathscr{Y}$.

The proof of this theorem and its stronger versions can be found in [He3]. See also [Quin, Sa-T] for similar theorems.

In applications below, $\mathscr{X}$ and $\mathscr{Y}$ are always local manifolds (open sets in Banach spaces) thus, in order to verify (a), one only needs to deal with sequences.

Let us add the following simple fact, which will be useful below.

LEMMA 2.2. *Let X, Y, Z be Banach spaces, $\mathscr{X} \times \mathscr{Y}$ an open set in $X \times Y$, and (x_0, y_0) a point in $\mathscr{X} \times \mathscr{Y}$. Let $\Phi: \mathscr{X} \times \mathscr{Y} \to Z$ be a C^1 map such that $D_x\Phi(x_0, y_0)$ is Fredholm. If $D\Phi(x, y): X \times Y \to Z$ is surjective for $(x, y) = (x_0, y_0)$ then the same is true for all (x, y) sufficiently close to (x_0, y_0).*

Proof. If X and Z are finite-dimensional, the result is elementary. One can reduce the proof of the lemma to this elementary case using the Lyapunov–Schmidt method (see [Cho-H] or [Sm, p. 175]). (The details are given in [He3, Proof of Theorem 5.4], although the result is not stated explicitly there.) ▮

3. GENERIC PROPERTIES OF EQUILIBRIA

In this section we prove that two properties of (1.1), (1.2), hyperbolicity of equilibria and simplicity of eigenvalues of the linearization of (1.1), (1.2) at any of its equilibria, are generic. At several places we use the following well-known properties of elliptic operators. For any $a(\cdot) \in C(\bar{\Omega})$ and $q \in (1, \infty)$, the operator $L: v \mapsto \Delta v + a(x)v : W^{2,q} \cap W_0^{1,q}(\Omega) \to L^q(\Omega)$ has finite-dimensional kernel (independent of q) and its range consists of precisely those functions h that satisfy

$$\int_\Omega v(x)\, h(x)\, dx = 0 \tag{3.1}$$

for any $v \in \ker L$. In particular, L is a Fredholm operator of index 0. Further, any equilibrium (stationary solution) u of (1.1), (1.2) is contained in $C^2(\bar{\Omega})$ and is a classical solution of the corresponding elliptic boundary-value problem.

For brevity, we often omit Ω in the notation of Sobolev spaces. For example, $W^{2,p} = W^{2,p}(\Omega)$.

3.a. *Hyperbolicity*

Recall that an equilibrium u of (1.1), (1.2) is hyperbolic if the operator $v \mapsto \Delta v + f_u(x, u(x))\, v : W^{2,q} \cap W_0^{1,q} \to L^q$ is an isomorphism (this property is independent of $q \in (1, \infty)$).

THEOREM 3.a.1. *There exists a residual set* $\mathfrak{G}^H$ *in* $\mathfrak{G}$ *such that for any* $f \in \mathfrak{G}^H$ *all equilibria of* (1.1), (1.2) *are hyperbolic.*

This theorem in different settings is proved in [Quit, Sa-T, Ry, Bab-V]. As the proof is rather simple, we include it here for completeness. We refer the reader to [Br-C, He2, Pol1, Roc] for generic hyperbolicity results for more specific classes of equations in one space dimension.

Proof. For $n = 1, 2, ...$ let $\mathfrak{G}_n^H$ denote the set of all functions $f \in \mathfrak{G}$ such that each equilibrium u of (1.1), (1.2) with $\|u(\cdot)\|_{L^\infty} \leqslant n$ is hyperbolic. We prove that $\mathfrak{G}_n^H$ is open and dense in $\mathfrak{G}$. The intersection of these sets yields a residual set with the required property.

It is obvious that for f to belong to $\mathfrak{G}_n^H$ the values of $f(x, u)$ with $|u| > n$ are irrelevant. Openness of $\mathfrak{G}_n^H$ therefore follows from the following claim:

(C) If $f_k \in \mathfrak{G} \setminus \mathfrak{G}_n^H$, $k = 1, 2, ...$ and the sequences f_k and $\partial_u f_k$ converge to $f_0, \partial_u f_0$ respectively, uniformly on $\bar{\Omega} \times [-n, n]$, then $f_0 \notin \mathfrak{G}_n^H$.

We prove (C). By definition of $\mathfrak{G}_n^H$, there are sequences of C^2 functions u_k, v_k such that $\|u_k\|_{L^\infty} \leqslant n$, $\|v_k\|_{L^2} = 1$,

$$\Delta u_k + f_k(x, u_k) = 0, \qquad x \in \Omega, \tag{3.a.1}$$

$$u_k = 0, \qquad x \in \partial\Omega, \tag{3.a.2}$$

and

$$\Delta v_k + \partial_u f_k(x, u_k(x))\, v_k = 0, \qquad x \in \Omega,$$

$$v_k = 0, \qquad x \in \partial\Omega.$$

Let $p > N$, so that $W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ by the Sobolev imbedding theorem. By standard a priori estimates, the sequence u_k is bounded in $W^{2,p}$. Therefore, passing to a subsequence, we may assume that u_k converges in $W_0^{1,p}$ to a function u_0 which satisfies $\|u_0\|_{L^\infty} \leqslant n$. Applying to (3.a.1) the inverse of the Laplacian (viewed as a bounded operator from L^p to $W^{2,p} \cap W_0^{1,p}$) and taking the limits, one next finds that u_0 is a solution of

$$\Delta u_0 + f_0(x, u_0) = 0, \qquad x \in \Omega,$$

$$u_0 = 0, \qquad x \in \partial\Omega.$$

Using similar arguments and passing to a further subsequence, one obtains that v_k converges in $W_0^{1,p}$ to a function v_0 that satisfies

$$\Delta v_0 + \partial_u f_0(x, u_0) v_0 = 0, \qquad x \in \Omega,$$
$$v_0 = 0, \qquad x \in \partial\Omega,$$

and $\|v_0\|_{L^2} = 1$. This readily implies $f_0 \notin \mathfrak{G}_n^H$.

We next prove the density of $\mathfrak{G}_n^H$. Fix an arbitrary $f \in \mathfrak{G}$. Choose an auxiliary smooth function $\eta: \mathbb{R} \to \mathbb{R}$ with compact support that is identical to 1 on the interval $[-n-1, n+1]$. We prove that in $C^s(\bar{\Omega})$ (for any integer $s \geqslant 2$) there exists a dense set of functions b for which the function $f(x, u) + b(x)\eta(u)$ belongs to $\mathfrak{G}_n^H$. Clearly, this function is arbitrarily close to f in $\mathfrak{G}$ if b is close to 0 and the density of $\mathfrak{G}_n^H$ follows.

We are going to apply the transversality theorem to a map Φ defined as follows. Let

$$\mathscr{X} = \{u \in W^{2,p} \cap W_0^{1,p} : \|u\|_{L^\infty} < n+1\}$$
$$\mathscr{Y} = C^s(\bar{\Omega}),$$
$$\mathscr{Z} = L^p.$$

(The topology on $\mathscr{X}$ is that induced from $W^{2,p}$.) For $(u, b) \in \mathscr{X} \times \mathscr{Y}$ let

$$\Phi(u, b)(x) = \Delta u(x) + f(x, u(x)) + b(x)\eta(u(x))$$
$$= \Delta u(x) + f(x, u(x)) + b(x) \qquad (x \in \Omega).$$

Observe that $\mathscr{X}, \mathscr{Y}$ are separable metric spaces and $\Phi: \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ is of class C^1. Moreover, $D_u\Phi(u, b)$ coincides with the operator

$$v \mapsto \Delta v + f_u(x, u(x))v : W^{2,p} \cap W_0^{1,p} \to L^p. \tag{3.a.3}$$

In particular, it is a Fredholm operator of index 0. It is further seen easily that $f(x, u) + b(x)\eta(u)$ belongs to $\mathfrak{G}_n^H$, provided the operator (3.a.3) is an isomorphism for any $u \in \mathscr{X}$ such that $\Phi(u, b) = 0$ (in other words, if 0 is a regular value of $\Phi(\cdot, b)$). To prove that this is the case for a dense (in fact residual) set of functions b, we show that Theorem 2.1 applies to Φ. As mentioned above, the hypotheses (i), (iii)(b) of that theorem are satisfied. We verify (ii). Let $(u, b) \in \Phi^{-1}(0)$ and $h \in \mathscr{Z}$ be arbitrary. We search for $v \in W^{2,p} \cap W_0^{1,p}$, $\bar{b} \in \mathscr{Y}$ such that

$$D_u\Phi(u, b)v + D_b\Phi(u, b)\bar{b} = h,$$

that is,

$$\Delta v + f_u(x, u(x))\, v + \bar{b}(x) = h(x), \qquad x \in \Omega. \tag{3.a.4}$$

This equation can be solved for v, provided $\bar{b}$ is chosen such that $h - \bar{b}$ satisfies

$$\int_\Omega (h(x) - \bar{b}(x))\, v_j(x)\, dx = 0, \qquad i = 1, ..., m, \tag{3.a.5}$$

where $v_1, ..., v_m$ is a basis of $\ker(\Delta + f_u(x, u)(x)))$ (cf. (3.1)). As the v_i are linearly independent, it is easy to see that the finite system of linear equations (3.a.5) has a solution $\bar{b} \in \mathcal{Y}$. This completes the proof. ∎

3.b. *Simplicity of the Eigenvalues of Linearization*

Given an equilibrium u of (1.1), (1.2), consider the eigenvalue problem

$$\Delta v + f_u(x, u(x))\, v + \mu v = 0, \qquad x \in \Omega, \tag{3.b.1}$$

$$v = 0, \qquad x \in \partial\Omega. \tag{3.b.2}$$

It is well known (see [Uh], that if $f_u(x, u(x))$ does not happen to fall to an exceptional (meager) set of potentials in $C(\bar{\Omega})$ then all eigenvalues μ are simple. Our aim here is to prove that, generically with respect to $f \in \mathfrak{G}$, $f_u(x, u(x))$ enjoys the latter property for any equilibrium u of (1.1), (1.2). Later we prove a useful consequence of this result to the effect that bounded solutions of (1.1), (1.2) approach their limit equilibria in the direction of an eigenfunction, provided the nonlinearity is generic.

Recall that the eigenvalues of (3.b.1), (3.b.2) are all real, have finite multiplicity and form a sequence $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$ approaching $+\infty$. Moreover, there is an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of (3.b.1), (3.b.2).

THEOREM 3.b.1. *There exists a residual set* $\mathfrak{G}^{SE} \subset \mathfrak{G}$ *such that for any* $f \in \mathfrak{G}^{SE}$ *and for any equilibrium* u *of* (1.1), (1.2) *all eigenvalues of* (3.b.1), (3.b.2) *are simple.*

Proof. Let $\mathfrak{G}^{SE}_{nm}$ denote the set of all functions $f \in \mathfrak{G}$ such that for any equilibrium u of (1.1), (1.2) with $\|u\|_{L^\infty} \leqslant n$ each eigenvalue of (3.b.1), (3.b.2) with $|\mu| \leqslant m$ is simple. We prove that $\mathfrak{G}^{SE}_{nm}$ is open and dense in $\mathfrak{G}$. Then

$$\mathfrak{G}^{SE} := \bigcap_{n,m=1}^{\infty} \mathfrak{G}^{SE}_{nm}$$

is a residual set with the property as in the theorem.

To prove the openness of $\mathfrak{G}^{SE}_{nm}$, observe that for $f \in \mathfrak{G}$ one has $f \notin \mathfrak{G}^{SE}_{nm}$ if and only if there exist $u, v_1, v_2 \in W^{2,p} \cap W^{1,p}_0$ and $\mu \in \mathbb{R}$ that satisfy

$$\begin{aligned} \Delta u + f(x, u) &= 0, && x \in \Omega, \\ \Delta v_i + f_u(x, u(x)) v_i + \mu v_i &= 0, && x \in \Omega, \quad i = 1, 2, \\ \|v_i\|_{L^2} &= 1, && i = 1, 2, \\ \int_\Omega v_1 v_2 &= 0 \end{aligned}$$

and

$$\|u\|_{L^\infty} \leqslant n, \qquad |\mu| \leqslant m.$$

One can use these equations and arguments similar to those in the proof of Theorem 3.a.1 to prove the openness of $\mathfrak{G}^{SE}_{nm}$. We omit the details.

We next prove the density of $\mathfrak{G}^{SE}_{nm}$. Actually, we prove a stronger claim, namely that

$$\mathfrak{G}^{SE}_{n} = \bigcap_{m=1}^{\infty} \mathfrak{G}^{SE}_{nm}$$

is dense in $\mathfrak{G}$.

Fix any $f \in \mathfrak{G}$. We have to show that f can be approximated, arbitrarily closely, by a function in $\mathfrak{G}^{SE}_{n}$. Making an arbitrarily small perturbation of f, we achieve that $f \in \mathfrak{G}^{H}_{n+1}$. Recall that $\mathfrak{G}^{H}_{n+1}$ is the open and dense subset of $\mathfrak{G}$ consisting of all f's such that any equilibrium u of (1.1), (1.2) with $\|u\|_{L^\infty} \leqslant n+1$ is hyperbolic (see the previous subsection). Making another small perturbation in the open set $\mathfrak{G}^{H}_{n+1}$, we may in addition assume that the restriction of f to the set $\bar{\Omega} \times [-n-1, n+1]$ is of class C^s (henceforth we fix an integer $s \geqslant 3$). We choose two auxiliary smooth functions $\eta_1, \eta_2 \colon \mathbb{R} \to \mathbb{R}$ with compact supports such that $\eta_1(u) \equiv 1$ and $\eta_2(u) \equiv u$ on the interval $[-n-1, n+1]$. We prove that arbitrarily close to 0 in $C^s(\bar{\Omega}) \times C^s(\bar{\Omega})$, there exists a function $b = (b_1, b_2)$ for which $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u)$ belongs to $\mathfrak{G}^{SE}_{n}$.

Denote

$$U := \{u \in W^{2,p} \cap W^{1,p}_0 : \|u\|_{L^\infty} < n+1\}.$$

This is an open set in $W^{2,p} \cap W^{1,p}_0$ (we still assume $p > N$). As proved in the previous subsection, for $f \in \mathfrak{G}^{H}_{n+1}$ the map

$$u \mapsto \Delta u + f(x, u(x)) \colon U \to L^p$$

has 0 as a regular value.

We are going to apply the transversality theorem to a map Φ defined as follows. Set

$$\mathscr{X} = U \times (W^{2,p} \cap W_0^{1,p} \setminus \{0\}) \times \mathbb{R},$$

$$\mathscr{Z} = L^p \times L^p,$$

and let $\mathscr{Y}$ be the set of all functions $b = (b_1, b_2) \in C^s(\bar{\Omega}) \times C^s(\bar{\Omega})$ such that $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u)$ belongs to $\mathfrak{G}_{n+1}^H$. It is easy to see that $\mathscr{X}$ is open in $W^{2,p} \cap W_0^{1,p} \times W^{2,p} \cap W_0^{1,p} \times \mathbb{R}$ and $\mathscr{Y}$ is open in $C^s(\bar{\Omega}) \times C^s(\bar{\Omega})$. Moreover, $b = 0$ belongs to $\mathscr{Y}$ (because f belongs to $\mathfrak{G}_{n+1}^H$). Both $\mathscr{X}$ and $\mathscr{Y}$ are separable metric spaces (with the induced topologies). For $(u, v, \mu, b) \in \mathscr{X} \times \mathscr{Y}$ let $\Phi(u, v, \mu, b)$ be the element of $\mathscr{Z}$ given by

$$\Phi(u, v, \mu, b)(x) = \begin{pmatrix} \Delta u(x) + f(x, u(x)) + b_1(x) + b_2(x)\,u(x) \\ \Delta v(x) + (f_u(x, u(x)) + b_2(x) + \mu)\,v(x) \end{pmatrix}.$$

It is easy to check that $\Phi\colon \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ is of class C^2. We further claim that Φ has the following properties:

(a) If b is such that $\Phi(\cdot, b) \pitchfork \{0\}$ then the function $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u)$ belongs to $\mathfrak{G}_n^{SE}$.

(b) $D_{(u,v,\mu)}\Phi(u, v, \mu, b)$ is a Fredholm of index 1 for any $(u, v, \mu, b) \in \Phi^{-1}(0)$.

(c) $D\Phi(u, v, \mu, b)$ is surjective for any $(u, v, \mu, b) \in \Phi^{-1}(0)$.

Observe that these properties imply the density of $\mathfrak{G}_n^{SE}$. Indeed, (b) and (c) justify an application of Theorem 3.a.1, which yields a residual set in U consisting of functions b with the transversality property as in (a), thus belonging to $\mathfrak{G}_n^{SE}$. As $0 \in U$, this residual set contains an element arbitrarily close to 0, thus f can be approximated by elements of $\mathfrak{G}_n^{SE}$.

We prove the claim beginning with (b). Let $(u, v, \mu, b) \in \Phi^{-1}(0)$; hence

$$\Delta u + f(x,u) + b_1(x) + b_2(x)u = 0 \tag{3.b.3}$$

$$\Delta v + (f_u(x, u(x)) + b_2(x) + \mu)v = 0. \tag{3.b.4}$$

The kernel of the operator

$$L := D_{(u,v,\mu)}\Phi(u, v, \mu, b) \tag{3.b.5}$$

consists of those $(\bar{u}, \bar{v}, \bar{\mu}) \in W^{2,p} \cap W_0^{1,p} \times W^{2,p} \cap W_0^{1,p} \times \mathbb{R}$ that satisfy

$$\Delta\bar{u} + (f_u + b_2(x))\bar{u} = 0,$$

$$\Delta\bar{v} + (f_u + b_2(x) + \mu)\bar{v} + f_{uu} v\bar{u} + \bar{\mu} v = 0$$

(we have omitted the argument $(x, u(x))$ of f and its derivatives). By definition of $\mathscr{Y}$, u is a hyperbolic solution of (3.b.3). Therefore $\bar{u}=0$ and

$$\Delta\bar{v}+(f_u+b_2(x)+\mu)\bar{v}=-\bar{\mu}v.$$

As v is an eigenfunction corresponding to μ (see (3.b.4) and recall that $v\neq 0$ by the definition of $\mathscr{X}$), by (3.1), this forces $\bar{\mu}=0$ and $\bar{v}$ must be an eigenfunction corresponding to μ as well. On the other hand, if $\bar{v}$ is such an eigenfunction then $(0, \bar{v}, 0)$ obviously belongs to ker L. Thus

$$\dim\ker L=m,$$

where m is the multiplicity of the eigenvalue μ.

Next we find the range of L. Consider the following subspaces of $\mathscr{Z}$:

$$\mathscr{Z}_1=\{(g, h)\in\mathscr{Z} : h=0\}, \qquad \mathscr{Z}_2=\{(g, h)\in\mathscr{Z}: g=0\}.$$

As $\Delta+f_u+b_2(x)$ is surjective (u is hyperbolic), we have

$$\mathscr{Z}_1\subset R(L).$$

Now $R(L)\cap\mathscr{Z}_2$ consists of exactly those $(0, h)\in\mathscr{Z}$ for which there are $\bar{v}$ and $\bar{\mu}$ such that

$$\Delta\bar{v}+(f_u+b_2(x)+\mu)\bar{v}=-\bar{\mu}v+h.$$

A necessary and sufficient condition for h is that it be L^2-orthogonal to all solutions of

$$\begin{aligned} \Delta w+(f_u+b_2(x)+\mu)w&=0, && x\in\Omega,\\ w&=0, && x\in\partial\Omega, \\ \int_\Omega wv&=0. \end{aligned} \tag{3.b.6}$$

Thus

$$\begin{aligned} R(L)&=\mathscr{Z}_1\oplus(R(L)\cap\mathscr{Z}_2)\\ &=\left\{(g, h)\in\mathscr{Z} : \int_\Omega hw_i=0,\, i=1, ..., m-1\right\}, \end{aligned}$$

where $w_1, ..., w_{m-1}$ is a basis of the space of solutions of (3.b.6). We see that $R(L)$ is closed and has codimension $m-1$. Summarizing, L is Fredholm and ind $L=m-(m-1)=1$.

Now we prove (a). First observe, that the map Φ is defined in such a way that $\Phi(u, v, \mu, b)=0$ if and only if u is an equilibrium of (1.1), (1.2) with $f(x, u)$ is replaced by $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u))$, $\|u\|_{L^\infty} < n+1$, and μ is an eigenvalue of the corresponding linearized problem (3.b.1), (3.b.2) with the eigenfunction v.

Assume that $\Phi(\cdot, b) \pitchfork 0$. This means that for any $(u, v, \mu, b) \in \Phi^{-1}(0)$ the map L defined in (3.b.5) is surjective. As we have proved above, codim $R(L) = m-1$, where m is the multiplicity of the eigenvalue μ. Therefore we must have $m = 1$, that is, μ is simple. This implies that the function $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u)$ belongs to $\mathfrak{G}_n^{SE}$ as claimed.

Finally we prove (c). Fix any $(u, v, \mu, b) \in \Phi^{-1}(0)$. We have to prove that given $(h, g) \in \mathscr{X}$, there are $\bar{u}, \bar{v} \in W^{2,p} \cap W_0^{1,p}$, $\bar{\mu} \in \mathbb{R}$ and $\bar{b} = (\bar{b}_1, \bar{b}_2) \in C^s(\bar{\Omega}) \times C^s(\bar{\Omega})$ such that

$$\begin{aligned} \Delta\bar{u} + (f_u + b_2(x))\,\bar{u} + \bar{b}_1(x) + \bar{b}_2 u(x) &= g, \\ \Delta\bar{v} + (f_u + b_2(x) + \mu)\,\bar{v} + f_{uu} v\bar{u} + \bar{\mu} v + \bar{b}_2 v(x) &= h. \end{aligned} \tag{3.b.7}$$

Set $\bar{\mu} = 0$ and $\bar{b}(x) = (\bar{b}_1, \bar{b}_2) = (-a(x)\,u(x), a(x))$, where $a(x)$ is a function in $C^s(\bar{\Omega})$ to be determined later. (Note that $u \in C^s(\bar{\Omega})$, by elliptic regularity, since u is an equilibrium of an equation with a C^s nonlinearity.) Equations (3.b.7) now reduce to

$$\Delta\bar{u} + (f_u + b_2(x))\,\bar{u} = g, \tag{3.b.8}$$

$$\Delta\bar{v} + (f_u + b_2(x) + \mu)\,\bar{v} = -f_{uu} v\bar{u} - av + h. \tag{3.b.9}$$

Let $\bar{u}$ be the unique solution of (3.b.8) (remember that the equilibrium u is hyperbolic). We need to make a choice of $a(x)$ such that (3.b.9) has a solution $\bar{v} \in W^{2,p} \cap W_0^{1,p}$. Equivalently, $a(x)$ should be chosen in such a way that

$$\int_\Omega w_i va + \int_\Omega w_i (f_{uu} v\bar{u} - h) = 0, \qquad i = 1, ..., m, \tag{3.b.10}$$

where $w_1, ..., w_\nu$ is a basis of $\ker(\Delta + f_u + b_2(x) + \mu)$. Observe that the functions $vw_1, ..., vw_\nu$ are linearly independent because $w_1, ..., w_\nu$ is basis and $v \not\equiv 0$ on any open set. The latter follows from the unique continuation for elliptic equations (see [Mi]), as $v \neq 0$. It is therefore easy to see that the finite system of linear equations (3.b.10) has a solution $a \in C^s(\bar{\Omega})$ (for example, choose a as an appropriate linear combination of the C^s functions vw_i). This completes the proof of (c) and thereby the proof of the theorem. ∎

It will be useful to remember the specific way in which we perturbed the nonlinearity in the above proof:

LEMMA 3.b.2. *Given any* $f \in \mathfrak{G}^H_{n+1}$, *there exists* $(b_1, b_2) \in C^s(\bar{\Omega}) \times C^s(\bar{\Omega})$, *arbitrarily close to* 0, *such that the function* $f(x, u) + b_1(x) + b_2(x)u$ *belongs to* $\mathfrak{G}^{SE}_n$.

Note that $f(x, u) + b_1(x) + b_2(x)u$ has the same restriction to $\bar{\Omega} \times [-n, n]$ as (and therefore belongs to $\mathfrak{G}^{SE}_n$ together with) the function $f(x, u) + b_1(x)\,\eta_1(u) + b_2(x)\,\eta_2(u)$ in the above proof.

4. TRANSVERSALITY OF INVARIANT MANIFOLDS FOR ABSTRACT EQUATIONS

In this section we consider abstract parabolic equations involving a parameter and discuss transversality of stable and unstable manifolds of their equilibria. We derive sufficient conditions under which transversality is a generic property with respect to the parameter. On the way towards this goal, we first collect a few results on the properties of linear parabolic operators acting on spaces of functions on unbounded intervals (Subsection 4.a). Then we consider an abstract nonlinear parabolic equation (without parameter) and characterize the transversality of its stable and unstable manifolds in terms of the corresponding parabolic operator and its regular values (Subsection 4.b). Finally, in Subsection 4.c, we apply the transversality theorem to an operator associated with a nonlinear parabolic equation with parameter.

Throughout the section we assume the following hypothesis.

(A) A is a sectorial operator on a (real) Banach space X such that $\operatorname{Re} \lambda > 0$ for any $\lambda \in \sigma(A)$.

See [Am, Da-K, He1, Lun2, Paz] for a general background on sectorial operators and the corresponding analytic semigroups e^{-At}.

For $\alpha \geqslant 0$ we denote by X^α the domain of the fractional power A^α equipped with the norm

$$\|x\|_{X^\alpha} = \|A^\alpha x\|_X.$$

Recall that each X^α is a Banach space, $X^0 = X$, $X^1 = D(A)$, and X^β is continuously and densely imbedded in X^α if $\beta > \alpha$, the imbedding being compact if A^{-1} is compact.

By X^* we denote the dual space of X and by A^* the adjoint operator of A. We use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between X^* and X:

$$\langle v, \psi \rangle = \psi(v) \ (\psi \in X^*, v \in X).$$

Equations we consider in this section are all of the form

$$u_t + Au = f(t, u), \tag{4.1}$$

where f is a function from an open set $\mathbb{R}\times G$ in $\mathbb{R}\times X^\alpha$ into X, such that both f and f_u are continuous (actually, we only consider functions linear in u or independent of t). By a *mild solution* of (4.1) on an interval $[t_0, t_1)$ we mean a function $u\in C([t_0, t_1), X)\cap C((t_0, t_1), X^\alpha)$ such that $u(t)\in G$ $(t\in[t_0, t_1))$, $t\mapsto f(t, u(t))$ is in $L^1((t_0, t_1), X)$ and

$$u(t) = e^{-At}u(t_0) + \int_{t_0}^{t} e^{-A(t-s)}f(s, u(s))\,ds \qquad (t\in[t_0, t_1)).$$

It can be proved (see the above references) that for any $u_0\in G$ there is a unique mild solution $u(t, t_0, u_0)$ on an interval $[t_0, t_1)$ with $u(t_0)=u_0$ and this solution is in $C([t_0, t_1), X^\alpha)$.

If f is linear in u and $G=X^\alpha$ then $u(t, t_0, u_0)$ is defined for all $t\geqslant t_0$ and it is linear in u_0. Moreover, for $t>t_0$, $u_0\mapsto u(t, t_0, u_0)$ extends to a bounded linear operator from X to X^α. This gives a unique mild solution even for any $u_0\in X$. (See [Lun2, Theorem 7.1.5] for a more general discussion of such existence results.)

If f is locally Hölder continuous in t (in particular, if it is independent of t) then the mild solution u is classical: $u\in C^1((t_0, t_1), X)\cap C((t_0, t_1), X^1)$ and (4.1) is satisfied for any $t\in(t_0, t_1)$. If $u(t_0)=u_0\in X^1$, then $u\in C^1([t_0, t_1), X)\cap C([t_0, t_1), X^1)$ and the equation is satisfied at t_0 as well.

Below, a solution always refers to a classical solution; we stress if only mild solutions are to be considered.

A solution (mild solution) on an unbounded interval J is a continuous function that is a solution (mild solution) on each bounded subinterval.

4.a. *Fredholm Parabolic Operators and Exponential Dichotomy*

Consider a linear equation

$$v_t + C(t)v = 0, \tag{4.a.1}$$

where $C(t) = A + B(t)$ and $B(t)$ satisfies the following hypothesis:

(BC) $B(t)\in\mathscr{L}(X^\alpha, X)$ $(t\in\mathbb{R})$ and the map $t\mapsto B(t)$ belongs to $C_b(\mathbb{R}, \mathscr{L}(X^\alpha, X))$.

Often we shall need the following stronger hypothesis:

(BH) The map $t\mapsto B(t)$ belongs to $C^{0,\delta}(\mathbb{R}, \mathscr{L}(X^\alpha, X))$ for some $\delta\in(0, 1]$.

Our motivation to study (4.a.1) comes from nonlinear autonomous equations; (4.a.1) is obtained by linearization along a heteroclinic solution. Such a linear variational equation has the additional property that $C(t)$ has limits as $t \to \pm\infty$. If the limits are hyperbolic operators, this reflects in the fact that (4.a.1) has exponential dichotomies on each of the intervals $(-\infty, 0]$ $[0, \infty)$ (cf. Lemma 4.a.12 below). In this subsection, we collect some consequences of the presence of dichotomies that will be useful in the sequel. Most of these consequences are analogs or modifications of results existing in the literature (we give references below) and, except for some technicalities and rearrangements, we hardly make any contribution to them.

Let $T(t, s) \in \mathscr{L}(X)$, $t \geqslant s$, denote the evolution operator of (4.a.2), that is, for any $v_0 \in X$, $v(t) = T(t, s) v_0$ is the mild solution of (4.a.2) on (s, ∞) satisfying the initial condition $v(0) = v_0$. One has

$$T(t, \tau)\, T(\tau, s) = T(t, s), \qquad T(t, t) = I \qquad (t \geqslant \tau \geqslant s), \tag{4.a.2}$$

where I is the identity on X (below we also use I to denote the identity on X^*).

Definition 4.a.1. We say that $T(t, s)$, $t \geqslant s$, admits *exponential dichotomy* on an interval J with *exponent* $\gamma > 0$ and *constant* $M > 0$ if there is a family $P(t)$, $t \in J$, of continuous projections on X such that the following properties hold for any $t, s \in J$:

(i) $T(t, s)\, P(s) = P(t)\, T(t, s)$ $(t \geqslant s)$.

(ii) $T(t, s)|_{R(P(s))}$ is an isomorphism of $R(P(s))$ onto $R(P(t))$ $(t \geqslant s)$.

(iii) $$\|T(t, s)(I - P(s))\|_{\mathscr{L}(X)} \leqslant M e^{-\gamma(t-s)} \qquad (t \geqslant s).$$

(iv) $$\|T(t, s)\, P(s)\|_{\mathscr{L}(X)} \leqslant M e^{-\gamma(s-t)} \qquad (t < s).$$

Here $T(t, s)$: $R(P(s)) \to R(P(t))$ $(t < s)$ stands for the inverse of the isomorphism $T(s, t)|_{R(P(t))}$.

Now consider the adjoint equation of (4.a.2),

$$w_s - C(s)^*\, w = 0. \tag{4.a.3}$$

Here $C(s)^* = A^* + B(s)^*$ is the adjoint of $C(s)$.

For $\psi_0 \in X^*$ define the function ψ: $(-\infty, t] \to X^*$ by

$$\psi(s) = T(t, s)^*\, \psi_0. \tag{4.a.4}$$

We call the function $\psi(s)$ defined by (4.a.4) the *weak* solution of* (4.a.3) satisfying the initial condition $\psi(t)=\psi_0$. By a weak* solution of (4.a.3) on $\mathbb{R}$ we mean a function $\tilde{\psi}(s)$ that satisfies

$$\tilde{\psi}(s)=T(t,s)^*\,\tilde{\psi}(t)\qquad (t,s\in\mathbb{R},\ t\geqslant s).$$

These are the only solutions of (4.a.3) that we consider in this paper. Just for completeness we add the following remarks on weak* solutions (these remarks will not be needed below): If (BH) holds with $\delta>\alpha$ then $\psi(s)$ is a classical solution of (4.a.3) on $(-\infty,t)$; that is, $\psi(s)\in D(C(s)^*)$ $(s\in(-\infty,t))$, $\psi\colon(-\infty,t)\to X^*$ is of class C^1 and (4.a.3) is satisfied for $s<t$ (see [He1, Theorem 7.3.1]). In general, one has

$$\psi(s)\to\psi_0,\qquad \text{as}\quad s\to t-,\tag{4.a.5}$$

in the weak* topology only. However, if $D(A^*)$ is dense in X^* (which is the case if X^* is reflexive) then $\psi(s)\to\psi_0$, as $s\to t-$ in the norm of X^*. This follows from the fact that if A^* has dense domain then it is sectorial (see [He1, Paz]); hence the existence–uniqueness theory (of [Paz], e.g.) can be applied to (4.a.3). Finally, we note that even if A^* is not dense one can prove that ψ defined by (4.a.4) is the unique classical solution of (4.a.3) on $(-\infty,t)$ satisfying (4.a.5) in the weak* topology.

Denote

$$\tilde{T}(s,t)=T(t,s)^*\qquad (t\geqslant s).$$

DEFINITION 4.a.2. We say that the family $\tilde{T}(s,t)$, $t\geqslant s$, admits *reverse exponential dichotomy* on an interval J with *exponent* γ and *constant* M if there is a family $\tilde{P}(s)$, $s\in J$, of continuous projections on X^* such that the following properties hold for any $t,s\in J$:

(i) $\tilde{T}(s,t)\,\tilde{P}(t)=\tilde{P}(s)\,\tilde{T}(s,t)$ $(t\geqslant s)$.

(ii) $\tilde{T}(s,t)|_{R(\tilde{P}(t))}$ is an isomorphism of $R(\tilde{P}(t))$ onto $R(\tilde{P}(s))$ $(t\geqslant s)$.

(iii) $$\|\tilde{T}(s,t)(I-\tilde{P}(t))\|_{\mathscr{L}(X^*)}\leqslant Me^{-\gamma(t-s)}\qquad (t\geqslant s).$$

(iv) $$\|\tilde{T}(s,t)\,\tilde{P}(t)\|_{\mathscr{L}(X^*)}\leqslant Me^{-\gamma(s-t)}\qquad (t<s).$$

Here $\tilde{T}(s,t)\colon R(\tilde{P}(t))\to R(\tilde{P}(s))$ $(t<s)$ stands for the inverse of the isomorphism $\tilde{T}(t,s)|_{R(\tilde{P}(s))}$.

LEMMA 4.a.3. *If $T(t,s)$ admits exponential dichotomy on an interval J with projections $P(t)$, exponent γ and constant M, then $\tilde{T}(s,t)=T(t,s)^*$ admits reverse exponential dichotomy on J with the same exponent and constant and with the projections $\tilde{P}(t)=P(t)^*$.*

Proof. Clearly, $P(s)^*$ is a continuous projection on X^*.

We only prove that the property (ii) in Definition 4.a.2 is satisfied; (i), (iii), and (iv) are straightforward to verify using standard properties of adjoint operators.

First recall that for any $Q \in \mathscr{L}(X)$ one has

$$\begin{aligned} \ker Q^* &= R(Q)^{\perp} \\ \ker Q &= {}^{\perp}R(Q^*), \end{aligned} \tag{4.a.6}$$

where

$$\begin{aligned} X_0^{\perp} &= \{\psi \in X^* : \langle x, \psi \rangle = 0 \text{ for each } x \in X_0\} \qquad (X_0 \subset X), \\ {}^{\perp}Y_0 &= \{x \in X : \langle x, \psi \rangle = 0 \text{ for each } \psi \in Y_0\} \qquad (Y_0 \subset X^*) \end{aligned}$$

(see, e. g., [Ka, Sect. III.5.4.]). If Q is a projection then one has in addition

$$\begin{aligned} R(I-Q^*) &= \ker Q^* = R(Q)^{\perp} \\ R(I-Q) &= \ker Q = {}^{\perp}R(Q^*). \end{aligned} \tag{4.a.7}$$

By (4.a.6), (4.a.7), and property (ii) in Definition 4.a.1,

$$\ker(T(t,s)^* P(t)^*) = R(P(t)\, T(t,s))^{\perp} = R(P(t))^{\perp} = R(I - P(t)^*).$$

Hence $T(t,s)^*$ is 1–1 on $R(P(t)^*)$ $(t \geqslant s)$.

Now fix any $t \geqslant s$ and

$$\psi \in R(P(s)^*) = \ker(I - P(s)^*) = {}^{\perp}R(I - P(s)).$$

Set

$$\xi = (T(s,t)\, P(t))^* \psi,$$

where $T(s,t)$: $R(P(t)) \to R(P(s))$ $(s \leqslant t)$ is as in Definition 4.a.1(iv). Then for any $x \in R(I - P(t))$

$$\langle x, \xi \rangle = \langle T(s,t)\, P(t)\, x, \psi \rangle = 0;$$

hence

$$\xi \in R(I - P(t))^{\perp} = \ker(I - P(t)^*) = R(P(t)^*).$$

Furthermore, for any $x \in X$

$$\begin{aligned} \langle x, T(t,s)^* \xi \rangle &= \langle T(s,t)\, P(t)\, T(t,s) x, \psi \rangle \\ &= \langle P(s) x, \psi \rangle = \langle x, \psi \rangle \end{aligned}$$

(the last equality holds because $\psi \in {}^{\perp}R(I-P(s))$). It follows that

$$\psi = T(t, s)^* \xi.$$

This proves that $T(t, s)^*$ maps $R(P(t)^*)$ onto $R(P(s)^*)$; hence it is an isomorphism, as claimed. ∎

We are now going to examine a parabolic differential operator associated with Eq. (4.a.1). We choose appropriate Banach spaces to accommodate this operator. Recall Convention 1.2 according to which we take the norm of the intersection of Banach spaces.

Let

$$\mathscr{E} = C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1),$$
$$\mathscr{Z} = C^{0,\delta}(\mathbb{R}, X).$$

Consider the linear operator L defined for $u \in \mathscr{E}$ by

$$L(u(\cdot))(t) = u_t(t) + C(t)\, u(t), \qquad t \in \mathbb{R}. \tag{4.a.8}$$

If (BH) holds then obviously $L \in \mathscr{L}(\mathscr{E}, \mathscr{Z})$. In the next theorem, we show that, under additional conditions, L is a Fredholm operator. Similar theorems for ordinary and functional differential equations can be found in [Palm, Li1, Hal-L] and references therein. For parabolic PDEs, Peterhof [Pe] has proved a theorem that is essentially the same as ours except that the assumptions in [Pe] are slightly more restrictive. We remark that, as noted in [Pe], it is erroneous to take $\tilde{\mathscr{E}} = C_b(\mathbb{R}, X^1) \cap C_b^1(R, X)$, $\tilde{\mathscr{Z}} = C_b(R, X)$ for the domain and the target space of L as one could be tempted (cf. [Bl, Zh]). Indeed, $h \in \tilde{\mathscr{Z}}$ does not guarantee that a mild solution of $v_t + C(t) v = h(t)$ is in $\tilde{\mathscr{E}}$, unless X, A enjoy the property of maximal regularity [Lun 1].

THEOREM 4.a.4. *Let* (BH) *be satisfied. Assume that $T(t, s)$, $t \geqslant s$ admits exponential dichotomies on each of the intervals $(-\infty, 0], [0, \infty)$ and let $P^-(t)$, $P^+(t)$ and γ^-, γ^+ denote the corresponding projections and exponents, respectively. Assume that the ranges $R(P^-(0))$, $R(P^+(0))$ are finite-dimensional.*

Then (4.a.8) *defines a Fredholm operator $L: \mathscr{E} \to \mathscr{Z}$ with index*

$$\dim(R(P^-(0)) - \dim(R(P^+(0)).$$

A function $h \in \mathscr{Z}$ belongs to $R(L)$ if and only if

$$\int_{-\infty}^{+\infty} \langle h(s), \psi(s) \rangle \, ds = 0 \tag{4.a.9}$$

for any weak solution* $\psi(s)$, $s\in\mathbb{R}$, *of the adjoint equation* (4.a.3) *such that* $\|\psi(s)\|_{X^*}$, $s\in\mathbb{R}$, *is bounded.*

Remark 4.a.5. If $R(P^-(0))$ is finite dimensional then so is any $R(P^-(t))$ as these spaces are isomorphic by the definition of exponential dichotomy. Similarly, $R(P^+(t))$ is finite-dimensional if $R(P^+(0))$ is. These properties are automatically satisfied in A has compact resolvent. This follows from property (ii) in the definition of exponential dichotomy and the fact that for $t>s$, $T(t,s)$ is a compact operator on X (it is bounded from X to X^α).

We prepare the proof of Theorem 4.a.4 by several lemmas. For future purposes, some of them are stated in a more general form than needed for the proof.

We start with a basic time regularity.

LEMMA 4.a.6. *Let* $h\colon\mathbb{R}\to X$ *be continuous and let* v *be a mild solution of*

$$v_t+Av=h(t),\qquad t\in\mathbb{R}. \tag{4.a.10}$$

(i) *If* $h\in C^{0,\vartheta}(\mathbb{R},X)$ *for a* $\vartheta\in(0,1)$ *and* $v\in C_b(\mathbb{R},X)$ *then* $v\in C^{1,\vartheta}(\mathbb{R},X)\cap C^{0,\vartheta}(\mathbb{R},X^1)$ *and there is a constant* $c=c(A,\vartheta)$ *such that*

$$\|v\|_{C^{1,\vartheta}(\mathbb{R},X)\cap C^{0,\vartheta}(\mathbb{R},X^1)}\leqslant c(\|v\|_{C_b(\mathbb{R},X)}+\|h\|_{C^{0,\vartheta}(\mathbb{R},X)}).$$

(ii) *If* $h, v\in C_b(\mathbb{R},X)$ *then* $v\in C^{0,1-\vartheta}(\mathbb{R},X^\vartheta)$ *for* $\vartheta\in(0,1)$ *and there is a constant* $c=c(A,\vartheta)$ *such that*

$$\|v\|_{C^{0,1-\vartheta}(\mathbb{R},X^\vartheta)}\leqslant c(\|v\|_{C_b(\mathbb{R},X)}+\|h\|_{C_b(\mathbb{R},X)}).$$

Proof. (i) By Lemma IX.1.28 of [Ka] (see also [Lun2, Section 4.3.1]), the function $t\mapsto Av(t):\mathbb{R}\to X$ is locally ϑ-Hölder continuous. Furthermore, the proof of this lemma in [Ka] and standard semigroup estimates (cf. [Ka, Remark IX.1.20]) yield

$$\|Av(\cdot)\|_{C^{0,\vartheta}([\tau,\infty),X)}\leqslant c(\|v(\tau-1)\|_X+\|h\|_{C^{0,\vartheta}(\mathbb{R},X)})\qquad(\tau\in\mathbb{R}).$$

In view of the boundedness of $\|v(s)\|_X$, as assumed in the lemma, we thus obtain that

$$\|v\|_{C^{0,\vartheta}(\mathbb{R},X^1)}=\|Av\|_{C^{0,\vartheta}(\mathbb{R},X)}\leqslant c(\|v\|_{C_b(\mathbb{R},X)}+\|h\|_{C^{0,\vartheta}(\mathbb{R},X)}).$$

Using this and Eq. (4.a.10), we further obtain that $v\in C^{1,\vartheta}(\mathbb{R},X)$ and the estimate in (i) holds.

(ii) Due to (A), there are constants $C, \omega > 0$ such that

$$\|e^{-At}x\|_{X^{\vartheta}} \leqslant ce^{-\omega t}t^{-\vartheta}\,\|x\|.$$

One obtains (ii) from this estimate and the variation of constants formula as in [Lun2, proof of Proposition 4.2.1] (see also [Lun2, Theorem 4.4.7]). ∎

LEMMA 4.a.7. *Let* (BC) *be satisfied. Assume that* $T(t, s)$, $t \geqslant s$ *admits exponential dichotomies on each of the intervals* $(-\infty, 0]$, $[0, \infty)$ *and let* $P^{-}(t)$, $P^{+}(t)$ *and* γ^{-}, γ^{+} *denote the corresponding projections and exponents, respectively. Let* $h\colon \mathbb{R} \to X$ *be a continuous function such that for some numbers* $\theta^{\pm}$ *with* $|\theta^{\pm}| < \gamma^{\pm}$ *the values* $\limsup_{t \to \pm\infty} e^{\theta^{\pm}|t|}\,\|h(t)\|_X$ *are both finite. Then the following properties hold.*

(i) *A continuous function* $v\colon (-\infty, 0] \to X$ *is a mild solution of*

$$v_t + C(t)v = h(t), \qquad t < 0 \tag{4.a.11}$$

satisfying $e^{t\gamma^{-}}\,\|v(t)\|_X \to 0$ *as* $t \to -\infty$ *if and only if*

$$v(t) = T(t, 0)v^{-} + \int_0^t T(t, s)\,P^{-}(s)\,h(s)\,ds$$

$$+ \int_{-\infty}^t T(t, s)(I - P^{-}(s))\,h(s)\,ds, \qquad \textit{for} \quad t \leqslant 0, \tag{4.a.12}$$

where $v^{-} = P^{-}(0)\,v(0)$. *Such a* v *satisfies*

$$\sup_{t<0} e^{\theta^{-}|t|}\,\|v(t)\|_{X^{\alpha}} \leqslant c_1(\|v^{-}\|_X + \sup_{t<0} e^{\theta^{-}|t|}\,\|h(t)\|_X), \tag{4.a.13}$$

where c_1 *is a constant independent of* h *and* v^{-}.

(ii) *A continuous function* $v\colon [0, \infty) \to X$ *is a mild solution of*

$$v_t + C(t)v = h(t), \qquad t > 0 \tag{4.a.14}$$

satisfying $e^{-t\gamma^{+}}\,\|v(t)\|_X \to 0$ *as* $t \to \infty$ *if and only if*

$$v(t) = T(t, 0)v^{+} + \int_0^t T(t, s)(I - P^{+}(s))\,h(s)\,ds$$

$$- \int_t^{\infty} T(t, s)(P^{+}(s))\,h(s)\,ds, \qquad \textit{for} \quad t \geqslant 0, \tag{4.a.15}$$

where $v^+ = (I - P^+(0))\, v(0)$. *Such a* v *satisfies*

$$\sup_{t>0} e^{\theta^+ t} \|v(t)\|_{X^\alpha} \leqslant c_2(\|v^+\|_{X^\alpha} + \sup_{t>0} e^{\theta^+ t} \|h(t)\|_X), \qquad (4.a.16)$$

where c_2 *is a constant independent of* h *and* v^+.

Proof. An analogous statement for solutions on the whole real line (rather than a halfline) is proved in [He1, Theorem 7.6.3]. One easily modifies the arguments in [He1] to prove the above statements (i), (ii) (cf. [He1, Exercises 7, 8, Sect. 7.6]). We note that in the proof of Theorem 7.6.3, Henry actually works with mild solutions which happen to be classical due to his regularity assumptions. ∎

A result similar to Lemma 4.a.7 for spaces with maximal regularity can also be found in [Li2].

LEMMA 4.a.8. *Let the hypotheses of Lemma* 4.a.7 *be satisfied. If* v *is a mild solution of*

$$v_t + C(t)v = h(t), \qquad t \in \mathbb{R}, \qquad (4.a.17)$$

such that $e^{-|t|\gamma^\pm} \|v(t)\| \to 0$ *as* $t \to \pm\infty$ *then for* $v^- = P^-(0)\, v(0)$, $v^+ = (I - P^+(0))\, v(0)$ (4.a.12), (4.a.15) *are satisfied and*

$$v^+ - v^- = \int_{-\infty}^{0} T(0, s)(I - P^-(s))\, h(s)\, ds + \int_0^\infty T(0, s)\, P^+(s)\, h(s)\, ds. \qquad (4.a.18)$$

Conversely, if v *is defined by* (4.a.12), (4.a.15) *with* $v^- \in R(P^-(0))$, $v^+ \in R(I - P^+(0))$ *satisfying* (4.a.18) *then* v *is a mild solution of* (4.a.17) *and*

$$\sup_{t<0} e^{\theta^- |t|} \|v(t)\|_{X^\alpha} \leqslant c(\|v(0)\|_X + \sup_{t<0} e^{\theta^- |t|} \|h\|_X \qquad (4.a.19)$$

$$\sup_{t>0} e^{\theta^+ t} \|v(t)\|_{X^\alpha} \leqslant c(\|v(0)\|_{X^\alpha} + \sup_{t>0} e^{\theta^+ t} \|h\|_X), \qquad (4.a.20)$$

where c *is independent of* $h, v(0)$. *Moreover, if* (BH) *is satisfied and* $h \in \mathscr{Z}$ *then* $v \in \mathscr{E}$ *and it is a classical solution of* (4.a.17).

Proof. The first implication follows directly from Lemma 4.a.7 ((4.a.18) follows from the continuity of v at $t = 0$).

We prove the second implication. Assume that (4.a.12), (4.a.15), (4.a.18) are satisfied. Observe that these formulas imply that

$$\lim_{t\to 0^-} v(t) = \lim_{t\to 0^+} v(t),$$

hence v: $\mathbb{R} \to X$ is continuous. By Lemma 4.a.7, v is a mild solution of (4.a.11) and (4.a.14). By the variation of constants for these equations, we have

$$v(t) = T(t, 0)\, v(0) + \int_0^t T(t, s)\, h(s), \qquad t \geqslant 0,$$

$$v(0) = \lim_{t\to 0} v(t) = T(0, \tau)\, v(\tau) + \int_\tau^0 T(0, s)\, h(s), \qquad \tau < 0.$$

These formulas combined give

$$v(t) = T(t, \tau)\, v(\tau) + \int_r^t T(t, s)\, h(s), \qquad t \geqslant 0 > \tau;$$

hence v is a mild solution of (4.a.17). By (4.a.12), (4.a.15), $v^- = P^-(0)\, v(0)$, $v^+ = (I - P^+(0))\, v(0)$. Using this and (4.a.13), (4.a.16), we obtain (4.a.19), (4.a.20).

We prove the last statement. Assume (BH) holds and $h \in \mathscr{Z}$. Then v is a classical solution of (4.a.17) and, by (4.a.19), (4.a.20),

$$v \in C_b(\mathbb{R}, X^\alpha).$$

We next show that $v \in C^{0,\delta}(\mathbb{R}, X^\alpha)$. Fix any $s \in \mathbb{R}$. The function $\bar{v}(t) := v(t+s) - v(t)$ is a solution of

$$\bar{v}_t + C(t)\,\bar{v} = (B(t+s) - B(t))\, v(t) + h(t+s) - h(t) \tag{4.a.21}$$

and $\|\bar{v}(t)\|_{X^\alpha}$ is bounded. The right-hand side of (4.a.21) is locally Hölder continuous and contained in $C_b(\mathbb{R}, X)$. Applying the above results (specifically, we use (4.a.19), (4.a.20) with $\theta^+ = \theta^- = 0$), we obtain

$$\begin{aligned}\|\bar{v}(t)\|_{X^\alpha} &\leqslant c_4(\|B(\cdot)\|_{C^{0,\delta}(\mathbb{R},\, \mathscr{L}(X^\alpha, X))}\, \|v\|_{C_b(\mathbb{R}, X^\alpha)} + \|h\|_{C^{0,\delta}(\mathbb{R}, X)})\, |s|^\delta + c_4\, \|\bar{v}(0)\|_X \\ &= c_5\, |s|^\delta + c_4\, \|v(s) - v(0)\|_{X^\alpha},\end{aligned}$$

where the constants c_i are independent of s. Now, as v: $\mathbb{R} \to X^\alpha$ is bounded and locally Hölder continuous (actually, it is of class C^1),

$$\|v(s) - v(0)\|_{X^\alpha} \leqslant c_6\, |s|^\delta,$$

with c_6 independent of s. These estimates imply that $v \in C^{0,\delta}(\mathbb{R}, X^\alpha)$.

Using the latter property and (BH), we further see that the right-hand side of

$$v_t + Av = -B(t)v + h(t)$$

belongs to $C^{0,\delta}(\mathbb{R}, X)$. Therefore, by Lemma 4.a.6(i), $v \in \mathscr{E}$.

The proof is complete. ∎

COROLLARY 4.a.9. *Assume that the hypotheses on* $T(t, s)$ *of Lemma* 4.a.7 *and* (BH) *are satisfied. Let* $L \in \mathscr{L}(\mathscr{E}, \mathscr{Z})$ *be defined by* (4.a.8). *Then the map* $v(\cdot) \mapsto v(0)$ *is an isomorphism of* $\ker L$ *onto* $R(P^-(0)) \cap R(I - P^+(0))$. *In particular*, $\ker L$ *is finite-dimensional and*

$$\dim \ker L = \dim(R(P^-(0)) \cap R(I - P^+(0))). \tag{4.a.22}$$

Proof. If $v \in \ker L$ then, applying Lemma 4.a.8 with $h = 0$, we obtain

$$v(0) = v^- = v^+,$$

hence $v(0) \in R(P^-(0)) \cap R(I - P^+(0))$.

Formulas (4.a.12), (4.a.15) with $h \equiv 0$ imply that $v \equiv 0$ if $v(0) = 0$, thus

$$v \mapsto v(0) : \ker L \to R(P^-(0)) \cap R(I - P^+(0))$$

is 1–1. The second part of Lemma 4.a.8 implies that this map is surjective, hence it is an isomorphism, as claimed. ∎

LEMMA 4.a.10. *Let the hypotheses of Lemma* 4.a.7 *on* $T(t, s)$ *and* (BC) *be satisfied. Then for each* $\psi_0 \in R(I - P^-(0)^*) \cap R(P^+(0)^*)$ *there exists a unique weak* solution* $\psi(s)$, $s \in \mathbb{R}$, *of the adjoint equation* (4.a.3) *satisfying* $\psi(0) = \psi_0$, *namely*

$$\psi(s) = \tilde{T}(s, 0)\psi_0 = T(0, s)^* \psi_0 \qquad (s \leqslant 0), \tag{4.a.23}$$

$$\psi(s) = \tilde{T}(s, 0)\psi_0 \qquad (s > 0), \tag{4.a.24}$$

where $\tilde{T}(s, 0)$ $(s > 0)$ *is the inverse of* $\tilde{T}(0, s)|_{R(P^+(s)^*)} = T(s, 0)^*|_{R(P^+(s)^*)}$ (*see* (ii) *in Definition* 4.a.2 *and recall Lemma* 4.a.3). *This solution satisfies*

$$\begin{aligned} \|\psi(s)\|_{X^*} &\leqslant c e^{\gamma^- s} \|\psi_0\|_{X^*} \qquad (s \leqslant 0), \\ \|\psi(s)\|_{X^*} &\leqslant c e^{-\gamma^+ s} \|\psi_0\|_{X^*} \qquad (s \geqslant 0) \end{aligned} \tag{4.a.25}$$

for some constant c *independent of* ψ_0 *and* s. *Furthermore, if* $\psi(s)$, $s \in \mathbb{R}$ *is any solution of* (4.a.3) *such that* $\|\psi(s)\|_{X^*}$ *is bounded then necessarily* $\psi_0 = \psi(0) \in R(I - P^-(0)^*) \cap R(P^+(0)^*)$ (*hence* (4.a.25) *holds*).

Proof. Using $T(t, s)^* = T(\tau, s)^*\, T(t, \tau)^*$ $(t \geqslant \tau \geqslant s)$ it is easily seen that the ψ defined by (4.a.23), (4.a.24) satisfies

$$\psi(s) = T(t, s)^*\, \psi(t) \qquad (t \geqslant s), \tag{4.a.26}$$

hence it is a weak* solution of (4.a.3). Lemma 4.a.3 and properties (iii), (iv) in Definition 4.a.2 imply (4.a.25).

We next show that the weak* solution with $\psi(0) = \psi_0 \in R(I - P^-(0)^*) \cap R(P^+(0)^*)$ is unique. Due to linearity, we may assume $\psi(0) = \psi_0 = 0$. Then, by (4.a.26),

$$\psi(s) = 0 \qquad (s \leqslant 0).$$

From (4.a.6) and the definition of the reverse dichotomy it follows that for $s < t$, $\psi(s) \in R(P(s)^*)$ is possible only if $\psi(t) \in R(P(t)^*)$. Taking $s = 0 < t$ we thus have

$$0 = T(0, t)^*\, \psi(t) \qquad \text{and} \qquad \psi(t) \in R(P(t)^*).$$

By property (ii) of the reverse dichotomy, this implies

$$\psi(t) = 0.$$

The last statement of the lemma follows from the properties of the reverse dichotomy. ∎

Proof of Theorem 4.a.4. Let L be defined by (4.a.8). By Corollary 4.a.9, L has kernel of finite dimension given by (4.a.22). We now prove that $h \in R(L)$ if and only if the following property holds:

(NS) $K := \int_{-\infty}^{\infty} \langle h(s), \psi(s) \rangle\, ds = 0$ for any weak* solution $\psi(s)$, $s \in \mathbb{R}$, of (4.a.3) such that $\|\psi(s)\|_{X^*}$ is bounded.

By Lemma 4.a.10, any such ψ is given by (4.a.23), (4.a.24) with $\psi_0 = \psi(0) \in R(I - P^-(0)^*) \cap R(P^+(0)^*)$ and satisfies (4.a.25). In particular the integral is finite. Observe that

$$\begin{aligned} \psi(s) &\in R(I - P^-(s)^*) = R(P^-(s))^\perp \qquad && (s \leqslant 0) \\ \psi(s) &\in R(P^+(s)^*) = R(I - P^+(s))^\perp \qquad && (s \geqslant 0) \end{aligned}$$

(cf. (4.a.7)). Thus

$$K = \int_{-\infty}^{0} \langle (I - P^-(s))\, h(s), \psi(s) \rangle\, ds + \int_0^\infty \langle P^+(s)\, h(s), \psi(s) \rangle\, ds.$$

Substituting (4.a.23), (4.a.24), we further obtain

$$\begin{aligned} K &= \int_{-\infty}^{0} \langle T(0,s)(I-P^-(s))\,h(s), \psi_0\rangle\, ds \\ &\quad + \int_0^{\infty} \langle T(0,s)(P^+(s))\,h(s), \psi_0\rangle\, ds \\ &= \left\langle \int_{-\infty}^{0} T(0,s)(I-P^-(s))\,h(s)\,ds + \int_0^{\infty} T(0,s)\,P^+(s)\,h(s)\,ds, \psi_0 \right\rangle. \end{aligned}$$

(We have used $(T(0,s)\,P^+(s))^* = \tilde{T}(s,0)\,P^+(0)^*$ $(s>0)$ which can be proved by arguments as in the proof of Lemma 4.a.3.) It follows that (NS) holds if and only if

$$\begin{aligned} &\int_{-\infty}^{0} T(0,s)(I-P^-(s))\,h(s)\,ds \\ &\quad + \int_0^{\infty} T(0,s)(P^+(s))\,h(s)\,ds \in {}^{\perp}(R(I-P^-(0)^*) \cap R(P^+(0)^*). \end{aligned}$$

Now

$$^{\perp}(R(I-P^-(0)^*) \cap R(P^+(0)^*)) = R(P^-(0)) + R(I-P^+(0)) \tag{4.a.28}$$

(see [Ka, Theorem IV.4.8] and note that the sum in (4.a.28) is closed because $R(P^-(0))$ is finite-dimensional). It follows that (NS) holds if and only if

$$\int_{-\infty}^{0} T(0,s)(I-P^-(s))\,h(s)\,ds + \int_0^{\infty} T(0,s)(P^+(s))\,h(s)\,ds = v^+ - v^-$$

for some $v^- \in R(P^-(0))$, $v^+ \in R(I-P^+(0))$. Applying Lemma 4.a.8 we conclude that (NS) is equivalent to $h \in R(L)$.

The following standard arguments now complete the proof. For $\psi_0 \in R(I-P^-(0)^*) \cap R(P^+(0)^*)$ let $\Psi(s, \psi_0)$ denote the unique weak* solution ψ of (4.a.3) with $\psi(0) = \psi_0$. Define a functional l_{ψ_0} on $\mathscr{Z}$ by

$$l_{\psi_0}(h(\cdot)) = \int_{-\infty}^{\infty} \langle h(s), \Psi(s, \psi_0)\rangle\, ds.$$

Then $l_{\psi_0} \in \mathscr{Z}^*$, $\psi_0 \mapsto l_{\psi_0}$ is linear and $l_{\psi_0} = 0$ if and only if $\psi_0 = 0$. As we have proved,

$$R(L) = \{h \in \mathscr{Z} : l_{\psi_0}(h) = 0 \text{ for any } \psi_0 \in R(I-P^-(0)^*) \cap R(P^+(0)^*)\}.$$

It follows that $R(L)$ is closed and

$$\begin{aligned}\operatorname{codim} R(L) &= \dim(R(I-P^-(0)^*) \cap R(P^+(0)^*)) \\ &= \operatorname{codim}(R(P^-(0)) + R(I-P^+(0)))\end{aligned}$$

(cf. (4.a.28)). We conclude that L is Fredholm of index

$$\begin{aligned}\operatorname{index} L &= \dim\ker L - \operatorname{codim} R(L) \\ &= \dim(R(P^-(0)) \cap R(I-P^+(0))) \\ &\quad - \operatorname{codim}(R(P^-(0)) + R(I-P^+(0))) \\ &= \dim R(P^-(0)) - \operatorname{codim} R(I-P^+(0)) \\ &= \dim R(P^-(0)) - \dim R(P^+(0)). \quad \blacksquare\end{aligned}$$

To conclude the subsection, we recall two lemmas on roughness of exponential dichotomies.

LEMMA 4.a.11. *Assume* (BC) *holds. Let J be an interval of the form* $(-\infty, t_0]$ *or* $[t_0, \infty)$ *and assume that the evolution operator of* (4.a.1) *admits exponential dichotomy on J with projection $P(t)$ and exponent γ. Then, given any $\varepsilon > 0$, there is a $\rho = \rho(\varepsilon, B(\cdot))$ such that for any $\tilde{B} \in C_b(J, \mathscr{L}(X^\alpha, X)$ with $\|\tilde{B} - B\|_{C_b(J, \mathscr{L}(X^\alpha, X))} < \rho$ the evolution operator of the equation*

$$u_t + Au - \tilde{B}(t)u = 0$$

admits exponential dichotomy on J with projection $\tilde{P}(t)$ and exponent $\tilde{\gamma}$ satisfying

$$\sup_{t \in J} \|\tilde{P}(t) - P(t)\|_{\mathscr{L}(X)} < \varepsilon$$

$$\gamma - \varepsilon < \tilde{\gamma}.$$

We remark that the projection $\tilde{P}(t)$ is not unique, only its range (in case $J = (-\infty, t_0]$) or kernel (in case $J = [t_0, \infty)$) are uniquely determined. This, of course, is a general property of dichotomies on halflines; one has some freedom in choosing the projection.

Proof of Lemma 4.a.11. Lemma 4.a.11 is a standard result in the theory of exponential dichotomy (see [Co, Palm, He1, Li1, Li2]) although in parabolic equations it is usually formulated for classical solutions rather than mild solutions. Dealing with mild solutions, one can easily adopt the arguments of [Cho-L], which are used to prove existence of invariant manifolds for nonautonomous equations. For linear equations the invariant manifolds are subspaces, $\tilde{X}^u(t)$, $t \in (-\infty, t_0]$. The projection $\tilde{P}(t)$ of X onto $\tilde{X}^u(t)$ with kernel equal to $\ker P(t)$ has the properties in the assertion for $(-\infty, t_0]$). Arguments for $[t_0, \infty)$ are analogous. We omit all further details. ∎

For the next lemma, we need the following preparation. A sectorial operator C on X is said to be *hyperbolic* if its spectrum $\sigma(C)$ does not intersect the imaginary axis. In this case, $\sigma(C) = \sigma_1 \cup \sigma_2$, where σ_1, σ_2 are closed subsets of the left and right open halfplanes in $\mathbb{C}$, respectively, and σ_1 is bounded. Let $P \in \mathscr{L}(X)$ be the spectral projection of L associated with this decomposition of the spectrum, that is, a projection commuting with C such that $\sigma(C|_{R(P)}) = \sigma_1$ and $\sigma(C|_{R(I-P)}) = \sigma_2$ (cf. [Ka]).

By standard semigroup estimates, the evolution operator $e^{-C(t-s)}$, $t \geq s$, admits exponential dichotomy on $\mathbb{R}$ with projections $P(t) \equiv P$ (cf. [He1, Example 1, Section 7.6]).

LEMMA 4.a.12. *Assume* (BC) *holds. Further, assume that the limit*

$$B(\infty) = \lim_{t \to \infty} B(t) \qquad \text{in} \quad \mathscr{L}(X^\alpha, X)$$

exists and $C(\infty) = A + B(\infty)$ *is a hyperbolic operator. Let* $P(\infty)$ *be the corresponding spectral projection, as introduced above, and assume that* $\dim R(P(\infty)) < \infty$. *Finally assume that for* $t > s$ *the evolution operator* $T(t, s)$ *is injective and its range is dense in* X. *Then the evolution operator* $(T(t, s)$ *of* (4.a.1) *admits exponential dichotomy on* $[0, \infty)$ *with a family of projections* $P(t)$, $t \geq 0$, *satisfying*

$$\dim R(P(t)) = \dim R(P(\infty)). \tag{4.a.29}$$

The same statement holds with ∞ *replaced by* $-\infty$ *(and* $[0, \infty)$ *replaced by* $(-\infty, 0]$).

Proof. Using the previous lemma, one obtains that $T(t, s)$ has exponential dichotomy on some interval $[t_0, \infty)$ with projection satisfying (4.a.29). This dichotomy can now be extended to $[0, \infty)$ as in [Li1, Lemma 2.3]. Similar arguments work for $(-\infty, 0]$ with [Li1, Lemma 2.3] replaced by [Li1, Lemma 2.4]. Note that the hypothesis of this lemma is satisfied due to $T(t, s)$ having dense range. ∎

4.b. *Characterization of the Transversality of Invariant Manifolds*

Consider the nonlinear equation

$$u_t + Au = f(u), \tag{4.b.1}$$

where A is as in (A) and the nonlinearity satisfies

(F) $f \in C_b^r(G, X)$, where G is an open convex set in X^α, for some $\alpha \in [0, 1)$, and $r \geqslant 1$.

In the whole subsection we assume that A has compact resolvent.

Recall a few definitions. An equilibrium e (stationary solution) of (4.b.1) is said to be *hyperbolic* if the linearized operator $A - f'(e)$ is hyperbolic. For such an equilibrium, the unstable manifold of e, $W^u(e)$, is the set all $u_0 \in G$ with the property that there exists a solution $u(t)$ of (4.b.1) defined for all $t \in (-\infty, 0]$ such that $u(0) = u_0$ and $u(t) \to e$ as $t \to -\infty$. The stable manifold, $W^s(e)$, is defined analogously with solutions on $[0, \infty)$ considered instead. Both $W^s(e)$ and $W^s(e)$ are immersed C^r submanifolds of $G \subset X^\alpha$, provided the following condition is satisfied.

(IS) For any solution $u(t)$ of (4.b.1) on an interval (t_1, t_2), the evolution operator $T(t, s) \in \mathscr{L}(X)$ of the linear variational equation

$$v_t + Av - f'(u(t))v = 0 \tag{4.b.2}$$

is injective and its range is dense in X for any $t_1 < s \leqslant t < t_2$.

We remark that the injectivity of $T(t, s)$ is equivalent to the backward uniqueness for (4.b.2), while the condition on the range of $T(t, s)$ is equivalent to the injectivity of $T(t, s)^*$, hence to the uniqueness for the adjoint equation to (4.b.2),

$$-v_t + (A - f'(u(t)))^* \, v = 0. \tag{4.b.3}$$

There are classes of equations (in particular Eqs. (1.1), (1.2)) where both these properties are known to be satisfied for any solution $u(t)$. See [He1] for a more detailed discussion and the proof of the fact that $W^u(e)$ and $W^s(e)$ are C^r manifolds.

Hypothesis (IS) is assumed, together with (A), (F), and compactness of A^{-1}, throughout the subsection.

By $m(e)$ we denote the *Morse index* of e:

$$m(e) = \dim W^u(e).$$

It coincides with the dimension of the subspace in the spectral decomposition for the operator $A - f'(e)$ corresponding to the spectral set $\{\mu \in \sigma(A - f'(e)) : \operatorname{Re} \mu < 0\}$. Note that $m(e) < \infty$ because A has compact resolvent.

A *heteroclinic solution* of (4.b.1) is a solution $u(t)$ defined for $t \in \mathbb{R}$ that converges, as $t \to \pm\infty$, to equilibria $e^\pm$. We also say that $u(t)$ is a hereoclinic solution from e^- to e^+, or that $u(t)$ connects e^- and e^+. Clearly, if e^- and e^+ are hyperbolic, then $u(t)$ is a heteroclinic solution from e^- to e^+ if and only if $u(t) \in W^u(e^-) \cap W^s(e^+)$ $(t \in \mathbb{R})$. We say that u is a *transverse heteroclinic solution* if

$$W^u(e^-) \pitchfork_{u(0)} W^s(e^+).$$

which means that $T_{u(0)} W^u(e^-)$ contains a closed complement of $T_{u(0)} W^s(e^+)$ in X^α. It is not difficult to see that this and condition (IS) imply

$$W^u(e^-) \pitchfork_{u(t)} W^s(e^+) \qquad (t \in \mathbb{R}).$$

(It follows, for example, from Corollary 4.b.4.)

Note that we do not insist on $e^- \neq e^+$; homoclinic solutions are treated simultaneously. In particular, if one proves that an (autonomous) equation has all hereoclinic solutions transverse then there cannot be any homoclinic solution.

We now prove several useful results leading to a sufficient condition for the transversality of stable and unstable manifolds formulated in terms of regular values for a parabolic operator. We start by a characterization of the tangent spaces of the stable and unstable manifolds.

LEMMA 4.b.1. *Let e be a hyperbolic equilibrium of* (4.b.1) *and let $\gamma > 0$ be such that*

$$|\mathrm{Re}\, \mu| > \gamma, \qquad \textit{for each} \qquad \mu \in \sigma(A - f'(e)). \tag{4.b.4}$$

Let $u(t)$ be a solution of (4.b.1) *on* $(-\infty, t_0]$ *such that $u(t_0) \in W^u(e)$. Then a vector $v_0 \in X^\alpha$ belongs to $T_{u(t_0)} W^u(e)$ if and only if there is a mild solution $v(t)$ of* (4.b.2) *on* $(-\infty, t_0]$ *such that $v(t_0) = v_0$ and*

$$\|v(t)\|_{X^\alpha} e^{-\gamma t} \to 0, \qquad \textit{as} \quad t \to -\infty. \tag{4.b.5}$$

Let $u(t)$ be a solution of (4.b.1) *on* $[t_0, \infty)$ *such that $u(t_0) \in W^s(e)$. Then a vector $v_0 \in X^\alpha$ belongs to $T_{u(t_0)} W^s(e)$ if and only if the mild solution $v(t)$ of* (4.b.2) *with $v(t_0) = v_0$ satisfies*

$$\|v(t)\|_{X^\alpha} e^{\gamma t} \to 0, \qquad \textit{as} \quad t \to \infty. \tag{4.b.6}$$

See [Che-C-H, Appendix C] for the proof. Although dynamical systems with discrete time are considered there, the present case is easily deduced from the results of [Che-C-H].

LEMMA 4.b.2. *Let u be a solution of* (4.b.1) *such that $u \in C_b(\mathbb{R}, X^\alpha)$* (*in particular this applies to heteroclinic solutions*). *Then*

$$u \in C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1)$$

for any $\delta \in [0, 1 - \alpha)$.

Proof. By (F) and the assumption on u we have $f(u(\cdot)) \in C_b(\mathbb{R}, X)$. Lemma 4.a.6(ii) applied to (4.b.1) gives $u \in C^{0,\delta}(\mathbb{R}, X^\alpha)$ for $\delta < 1-\alpha$. It follows that $f(u(\cdot)) \in C^{0,\delta}(\mathbb{R}, X)$ and therefore, by Lemma 4.a.6(i), $u(\cdot) \in C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1)$. (We remark, that one can now use bootstrapping arguments to show that the conclusion holds for any $\delta \in (0, 1)$ but this is not needed below.) ▮

LEMMA 4.b.3. *Let u be a heteroclinic solution of* (4.b.1) *with* $\lim_{t\to\pm\infty} = e^{\pm}$, *where* e^-, e^+ *are hyperbolic equilibria. Then the evolution operator for the linear variational equation* (4.b.2) *admits exponential dichotomies on each of the intervals* $(-\infty, 0]$, $[0, \infty)$. *The projections* $P^-(t), t \leqslant 0$, *and* $P^+(t)$, $t \geqslant 0$, *of these dichotomies satisfy*

$$R(P^-(t)) = T_{u(t)} W^u(e^-), \qquad R(I-(P^+(t)) \cap X^\alpha = T_{u(t)} W^s(e^+). \quad (4.b.7)$$

Proof. We have $f'(u(\cdot)) \in C_b(\mathbb{R}, \mathscr{L}(X^\alpha, X)$ and $f'(u(t)) \to f'(e^{\pm})$ as $t \to \pm\infty$. Thus, by Lemma 4.a.12, (4.b.1) admits exponential dichotomies on each of the intervals $(-\infty, 0]$, $[0, \infty)$. Equalities (4.b.7) now follow easily from properties of the dichotomies (see Definition 4.a.1) and Lemma 4.b.1. ▮

COROLLARY 4.b.4. *Let u be as in Lemma* 4.b.3. *Then u is transverse if and only if the adjoint Eq.* (4.b.3) *has no nontrivial weak* solution defined and bounded* (*in the norm of* X^*) *on* $\mathbb{R}$.

Proof. By Lemma 4.b.3, the heteroclinic solution $u(t)$ is transverse if and only if $R(P^-(0)) + R(I-P^+(0)) \cap X^\alpha = X^\alpha$ which holds if and only if $R(P^-(0)) + R(I-P^+(0)) = X$. As this sum is closed, it equals X if and only if

$$(R(P^-(0)) + R(I-P^+(0)))^\perp = \{0\}.$$

By (4.a.28), this is the same as

$$R(I-P^-(0)^*) \cap R(P^+(0)^*) = \{0\},$$

and, by Lemma 4.a.10, this is equivalent to (4.b.3) not having any bounded nontrivial weak* solution. ▮

For the remaining part of this subsection we fix a $\delta \in (0, 1-\alpha)$. As above, we denote

$$\mathscr{E} = C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1),$$

$$\mathscr{Z} = C^{0,\delta}(\mathbb{R}, X).$$

LEMMA 4.b.5. *Let u be as in Lemma* 4.b.3 *and let the integer r in* (F) *be greater than* 1. *Let L be the operator defined on $\mathscr{E}$ by*

$$L(v(\cdot))(t) = v_t(t) + Av(t) - f'(u(t))\, v(t). \tag{4.b.8}$$

Then $L \in \mathscr{L}(\mathscr{E}, \mathscr{Z})$ and it is a Fredholm operator of index $m(e^-) - m(e^+)$. Moreover, the hereroclinic solution u is transverse if and only if L is surjective.

Proof. By Lemma 4.b.3, the evolution operator of (4.b.2) has exponential dichotomies on the intervals $(-\infty, 0]$, $[0, \infty)$ whose projections satisfy

$$\dim R(P^-(t)) = m(e^-), \qquad \operatorname{codim} R(I - (P^+(t)) = m(e^+).$$

The assumption on f implies that $f'(u(\cdot)) \in C^{0,\delta}(\mathbb{R}, \mathscr{L}(X^\alpha, X))$. The fact that L is a Fredholm operator with the given index now follows directly from Theorem 4.a.4. The last statement follows from the same theorem combined with Corollary 4.b.4. ∎

COROLLARY 4.b.6. *Assume that the integer r in* (F) *is greater than* 1. *Further assume that all equilibria of* (4.b.1) *are hyperbolic. Let U be an open set in $\mathscr{E}$ such that $u(t) \in G$ $(t \in \mathbb{R})$ for any $u \in U$. Define a map Φ on U by*

$$\Phi(u(\cdot))(t) = u_t(t) + Au(t) - f(u(t)).$$

Then $\Phi\colon U \to \mathscr{Z}$ is of class C^1 and if $\Phi \pitchfork 0$ then all heteroclinic solutions $u(t)$ of (4.b.1) *such that $u(\cdot) \in U$ are transverse.*

Proof. The fact that Φ is of class C^1 follows from a more general result that we prove in the next subsection (see Lemma 4.c.4). There we also show that $D\Phi(u(\cdot))$ coincides with the operator L defined by 4.b.8. The conclusion now follows directly from Lemma 4.b.5. ∎

4.c. *Generic Transversality of Invariant Manifolds*

In this subsection we consider nonlinear equations with a parameter λ,

$$u_t + Au = f(u, \lambda). \tag{4.c.1}$$

We assume the following hypotheses.

(FP) $f\colon G \times \Upsilon \to X$ is of class C^1, where G is an open convex set in X^α, for some $\alpha \in [0, 1)$, and Υ is open in a Banach space Λ.

(ISP) If $\lambda \in \Upsilon$ and $u(t)$ is a solution of (4.c.1) on an interval (t_1, t_2) then the evolution operator $T(t, s) \in \mathscr{L}(X)$ of the linear variational equation

$$v_t - Av - f_u(u(t), \lambda) v = 0 \tag{4.c.2}$$

is injective and its range is dense in X for any $t_1 < s \leqslant t < t_2$.

The following theorem is the main result of Section 4.

THEOREM 4.c.1. *Assume that* (A), (FP), *and* (ISP), *together with the following additional hypotheses, are satisfied.*

(h1) *The Banach space Λ is separable.*

(h2) *A has compact resolvent.*

(h3) *For any bounded set $\Upsilon_0 \subset \Upsilon$ one has*

$$f|_{G \times \Upsilon_0} \in C_b^1(G \times \Upsilon_0, X).$$

(h4) *For any $\lambda \in \Upsilon$ all equilibria of* (4.c.1) *are hyperbolic.*

(h5) *Given any $\lambda_0 \in \Upsilon$ and any neighborhood $\mathscr{V}$ of λ_0 in Υ, there exist $\hat{\lambda} \in \mathscr{V}$ and a Banach space $\hat{\Lambda}$ with the following properties*:

(a) *$\hat{\lambda} \in \hat{\Lambda}$ and $\hat{\Lambda}$ is continuously imbedded in Λ.*

(b) *$\hat{\lambda}$ has an open neighborhood $\hat{\mathscr{V}}$ in $\hat{\Lambda}$ such that $\hat{\mathscr{V}} \subset \mathscr{V}$ and*

$$f|_{G \times \hat{\mathscr{V}}} \in C_b^r(G \times \hat{\mathscr{V}}, X), \tag{4.c.3}$$

where $r > m(e) + 1$ for any equilibrium e of (4.c.1) *with $\lambda = \hat{\lambda}$.*

(c) *If u is a heteroclinic solution of* (4.c.1) *with $\lambda = \hat{\lambda}$ and $\psi(t)$ ($t \in \mathbb{R}$) is a nontrivial bounded weak* solution of*

$$-w_t + (A - f_u(u(t), \hat{\lambda}))^* \, w = 0 \tag{4.c.4}$$

then there exists a $\lambda \in \hat{\Lambda}$ such that

$$\int_{-\infty}^{+\infty} \langle D_\lambda f(u(t), \hat{\lambda}) \, \lambda, \psi(t) \rangle \neq 0.$$

Under these assumptions, there is a residual subset $\Gamma \subset \Upsilon$ such that for any $\lambda \in \Gamma$ any heteroclinic solution u of (4.c.1) *is transverse.*

We prepare the proof of this theorem by the following lemmas. Throughout the subsection, we assume the hypotheses of the theorem to be satisfied.

LEMMA 4.c.2 (A Uniform Saddle-Point Neighborhood). *For a $\lambda_0 \in \Upsilon$ let $e \in G$ be an equilibrium of* (4.c.1) *with $\lambda = \lambda_0$. There exist open bounded neighborhoods $\mathscr{U}_0 \subset G$ and $\mathscr{V} \subset \Upsilon$ of e and λ_0, respectively, such that the* *following properties are satisfied. For each $\lambda \in \mathscr{V}$ there is a unique equilibrium $e(\lambda)$ of* (4.c.1) *in* $\mathscr{U}_0$. *This equilibrium has the same Morse index as* e, $\lambda \mapsto e(\lambda)$ *is of class* C^1 (*in the norm of* X^α) *and if* $u(t)$ *is a solution of* (4.c.1) *such that* $u(t) \in \mathscr{U}_0$ *for all sufficiently large negative* (*respectively, large positive*) t, *then* $u(t) \to e(\lambda)$, *in* X^α, *as* $t \to -\infty$ (*respectively,* $t \to \infty$). *Moreover, there is a constant* c *with the following property. If* $\lambda_1, \lambda_2 \in \mathscr{V}$, u_1, u_2 *are solutions of* (4.c.1) *with* $\lambda = \lambda_1$, $\lambda = \lambda_2$, *respectively, and* $u_1(t), u_2(t) \in \mathscr{U}_0$ *for all* $t \in J$, *where* $J = (-\infty, t_0]$ *or* $J = [t_0, \infty)$ *for some* $t_0 \in \mathbb{R}$, *then*

$$\sup_{t \in J} \|u_1(t) - u_2(t)\|_{X^\alpha} \leqslant c(\|\lambda_1 - \lambda_2\|_\Lambda + \|u_1(t_0) - u_2(t_0)\|_{X^\alpha}). \tag{4.c.5}$$

Proof. The existence of the equilibrium $e(\lambda)$ for λ near λ_0 and smoothness of the function $\lambda \mapsto e(\lambda)$ follow immediately from the implicit function theorem. Continuity properties of the spectrum of $A - f'(e(\lambda))$ (cf. [Ka, Theorems IX.2.4, IV.3.1, IV.3.18]) imply that $e(\lambda)$ is hyperbolic and has the same Morse index as e. Due to the saddle-point property of hyperbolic equilibria (see [He1, Theorem 5.2.1]), $e(\lambda)$ has a neighborhood $\mathscr{U}_0$ such for any solution $u(t)$ of (4.c.1) on $(-\infty, t_0)$ one has $u(t) \in \mathscr{U}_0$ ($t \in (-\infty, t_0]$) if and only if $u(t)$ is contained in the local unstable manifold of $e(\lambda)$: $u(t) \in W^u_{\mathrm{loc}}(e(\lambda)$ ($t \leqslant t_0$). The fact that $\mathscr{U}_0$ can be chosen independently of λ, for λ near λ_0, follows from the construction of the local unstable manifold (see the proof of Theorem 5.2.1 in [He1]). Indeed, any $u_0 = u(t_0) \in W^u_{\mathrm{loc}}(e(\lambda))$ is found as the fixed point of a certain integral operator. The integral operator is a contraction of a neighborhood of the constant function $u \equiv e \in C_b((-\infty, t_0), X^\alpha)$, uniformly with respect to λ, u_0 from sufficiently small neighborhoods V, $\mathscr{U}_0$ of λ_0 and e, respectively. This yields the λ-independent neighborhood $\mathscr{U}_0$ as needed. Moreover, the integral operator is Lipschitz continuous in λ and u_0. By the uniform contraction principle (see [He1, Sect. 1.2.6]), the fixed point $u(\cdot)$ depends continuously on λ, u_0 which gives (4.c.5). Similar arguments apply to $[t_0, \infty)$ with $W^s_{\mathrm{loc}}\, e(\lambda))$, the local stable manifold, replacing $W^u_{\mathrm{loc}}(e(\lambda))$. ▮

In the next lemma we consider the following situation. For a $\lambda_0 \in \Upsilon$ we are given two equilibria e_1, e_2 of (4.c.1) with $\lambda = \lambda_0$. We choose saddle point neighborhoods $\mathscr{U}_1, \mathscr{U}_2$ of e_1, e_2 as in Lemma 4.c.2 and let $\mathscr{V}_1, \mathscr{V}_2$ be the corresponding neighborhoods of λ_0. Replacing the $\mathscr{V}_i$ by $\mathscr{V}_1 \cap \mathscr{V}_2$ we may assume, without loss of generality, that $\mathscr{V}_1 = \mathscr{V}_2$.

LEMMA 4.c.3. *Let* e_1, e_2, $\mathscr{U}_1$, $\mathscr{U}_2$, $\mathscr{V}_1$ *be as above and let* B, $\mathscr{D}_1$, $\mathscr{D}_2$ *be any bounded sets in* X^α *such that* $\bar{B} \subset G$, $\bar{\mathscr{D}}_i \subset \mathscr{U}_i$, $i = 1, 2$. *Let* $\lambda^\nu \in \mathscr{V}_1$ *be a convergent sequence with* $\lambda^\infty = \lim \lambda^\nu \in \mathscr{V}_1$. *Assume that for* $\nu = 1, 2, .., u^\nu$ *is a heteroclinic solution of* (4.c.1) *with* $\lambda = \lambda^\nu$ *such that*

$$
\begin{aligned}
&u^v(t)\in \bar{B} && (t\in \mathbb{R}),\\
&u^v(t)\in \bar{\mathscr{D}}_1 && (t\in(-\infty, -m]),\\
&u^v(t)\in \bar{\mathscr{D}}_2 && (t\in[m, \infty)),
\end{aligned}
\tag{4.c.6}
$$

for some fixed $m\geqslant 0$.

Then u^v *has a subsequence that converges in* $C_b(\mathbb{R}, X^\alpha)$ *to a heteroclinic solution* u *of* (4.c.1) *with* $\lambda=\lambda^\infty$. *Moreover, if* f *is of class* C^2 *and* $D^2 f(u, \lambda)$ *is bounded uniformly for* $(u, \lambda)\in G\times \mathscr{V}_1$ *then this subsequence converges to* u *in* $C^{1,\delta}(\mathbb{R}, X)\cap C^{0,\delta}(\mathbb{R}, X^1)$ *for any* $\delta\in[0, 1-\alpha)$.

Proof. We are going to prove that a subsequence of u^v is convergent uniformly on $[-m, m]$ and then combine this with the properties of the saddle-point neighborhoods $\mathscr{U}_1, \mathscr{U}_2$ to obtain the uniform convergence on $\mathbb{R}$. Below c_1, c_2, etc. denote positive constants independent of v.

As $\bar{B}$ is bounded in X^α, we have

$$\|u^v(\cdot)\|_{C_b(\mathbb{R}, X^\alpha)}\leqslant c_1.$$

By (h3),

$$\|f(u^v(\cdot), \lambda^v)\|_{C_b(\mathbb{R}, X)}\leqslant c_2. \tag{4.c.7}$$

We can thus apply Lemma 4.a.6(ii) to the equation

$$u_t^v + Au^v = f(u^v(t), \lambda^v), \tag{4.c.8}$$

to obtain

$$\|u^v(\cdot)\|_{C^{0,\vartheta}(\mathbb{R}, X^\alpha)}\leqslant c_3$$

for a $\vartheta>0$. From (h3) it now follows that the right-hand side of (4.c.8) is uniformly bounded in $C^{0,\vartheta}(\mathbb{R}, X)$. Therefore, by Lemma 4.a.6(i),

$$\|u(\cdot)\|_{C^{1,\vartheta}(\mathbb{R}, X)\cap C^{0,\vartheta}(\mathbb{R}, X^1)}\leqslant c_4.$$

Now, as X^1 is compactly imbedded in X^α, the last estimate and the Arzela–Ascoli theorem imply that u^v has a subsequence (denoted again by u^v) such that $u^v|_{[-m,m]}$ is convergent in $C([-m, m], X^\alpha)$. Using this and the properties of $\mathscr{U}_1, \mathscr{U}_2$ (cf. 4.c.5), we see that

$$u^v|_{[m,\infty)},\ u^v|_{[-\infty,m]},$$

form Cauchy sequences in $C_b([m, \infty), X^\alpha)$, $C_b((-\infty, m], X^\alpha)$, respectively. Combining these properties, we obtain that the subsequence u^ν is convergent in $C_b(\mathbb{R}, X^\alpha)$. Let u denote the limit function:

$$u = \lim_{\nu \to \infty} u^\nu.$$

One clearly has

$$\begin{aligned} &u(t) \in \bar{B} && (t \in \mathbb{R}), \\ &u(t) \in \bar{\mathscr{D}}_1 \subset \mathscr{U}_1 && (t \leqslant -m), \\ &u(t) \in \bar{\mathscr{D}}_2 \subset \mathscr{U}_2 && (t \geqslant m) \end{aligned} \tag{4.c.9}$$

and

$$f(u^\nu(\cdot), \lambda^\nu) \to f(u(\cdot), \lambda^\infty) \qquad \text{in } C_b(\mathbb{R}, X).$$

Taking the limits in the variation of constants formula,

$$u^\nu(t) = e^{-A(t-\tau)} u^\nu(\tau) + \int_\tau^t e^{-A(t-s)} f(u^\nu(s), \lambda^\nu)\, ds,$$

we obtain that u is a solution of (4.c.1) with $\lambda = \lambda^\infty$. By (4.c.9), and the properties of $\mathscr{U}_1$, $\mathscr{U}_2$ (cf. Lemma 4.c.2), we obtain that u is a heteroclinic solution.

Now assume that $D^2 f$ is bounded on $G \times \mathscr{V}_1$. The function

$$w^\nu = u^\nu - u$$

satisfies

$$w^\nu_t + A w^\nu = f(u^\nu, \lambda^\nu) - f(u, \lambda^\infty). \tag{4.c.10}$$

For $\nu \to \infty$, both w^ν and the right-hand side of (4.c.10) converge to 0 in $C_b(\mathbb{R}, X)$. Lemma 4.a.6(ii) therefore implies that for $\delta \in (0, 1-\alpha)$

$$w^\nu \to 0 \qquad \text{in } C^{0,\delta}(\mathbb{R}, X^\alpha).$$

In view of the boundedness of $D^2 f$, we now see that the right-hand side of (4.c.10) converges to 0 in $C^{0,\delta}(\mathbb{R}, X)$. Therefore, by Lemma 4.a.6(i),

$$w^\nu \to 0 \qquad \text{in} \quad C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1).$$

This completes the proof of the lemma. ∎

Proof of Theorem 4.c.1. We choose a sequence of open bounded sets $B_\imath$, $\imath = 1, 2, ...$ such that

$$\bar{B}_\imath \subset G, \qquad \inf_{x \in \bar{B}_\imath} \operatorname{dist}(x, \partial G) > 0,$$

and

$$\bigcup_\imath B_\imath = G.$$

For example, take

$$B_\imath = \{x \in G : \|x\|_{X^\alpha} < \imath \text{ and } \operatorname{dist}(x, \partial G) > \imath^{-1}\}.$$

Recall that for $E \subset X^\alpha$

$$\operatorname{dist}(x, E) = \inf_{y \in E} \|x - y\|_\alpha.$$

The theorem will be proved, provided we show that for each $\imath$ there is a residual set in Υ consisting of elements λ such that any heteroclinic solution u of (4.c.1) satisfying $u(t) \in \bar{B}_\imath$ $(t \in \mathbb{R})$ is transverse. Indeed, the intersection of these residual sets then gives Γ as needed in the conclusion of the theorem. In fact, it is sufficient to prove a yet simpler property, as stated in the following claim.

Claim. *Let B be any bounded set in X^α such that*

$$\bar{B} \subset G, \qquad \inf_{x \in \bar{B}} \operatorname{dist}(x, \partial G) > 0. \tag{4.c.11}$$

Given any $\lambda_0 \in \Upsilon$, there are a neighborhood $\mathscr{V}_0$ of λ_0 in Υ and a residual set Γ_0 in $\mathscr{V}_0$ such that for any $\lambda \in \Gamma_0$ any heteroclinic solution u of (4.c.1) *with $u(t) \in \bar{B}$ is transverse.*

Suppose for a while that this claim holds true. Take a countable cover $\mathscr{V}_1, \mathscr{V}_2, ...$ of Υ by such neighborhoods (which exists as Υ is separable, hence Lindelöf, see [En]) and let $\Gamma_1 \subset \mathscr{V}_1$, $\Gamma_2 \subset \mathscr{V}_2$, ... be the corresponding residual sets. Set

$$\Gamma = \Upsilon \setminus \bigcup_{i=1}^{\infty} (\mathscr{V}_i \setminus \Gamma_i).$$

As each $\mathscr{V}_i \setminus \Gamma_i$ is meager in Υ, Γ is residual in Υ. Obviously, $\Gamma \subset \bigcup_i \Gamma_i$. Hence, for each $\lambda \in \Gamma$, any heteroclinic solution u of (4.c.1) with $u(t) \in \bar{B}$ is transverse. Applying this to each of the above sets $B_\imath$, we obtain the desired property that implies the conclusion of the theorem.

We now prove the claim. Fix an arbitrary $\lambda_0 \in \Upsilon$. Since the equilibria of (4.c.1) satisfy

$$u = A^{-1} f(u, \lambda),$$

hypotheses (h2), (h3) imply that the set of all equilibria in $\bar{B}$ is compact in X^α. Since all the equilibria are hyperbolic, thus isolated, for each λ there are only finitely many of them. Let e_i, $i = 1, ..., l$ be all the equilibria of (4.c.1) with $\lambda = \lambda_0$ in $\bar{B}$. For each of these equilibria, we choose a saddle-point neighborhood $\mathscr{U}_i$ as in Lemma 4.c.2 and let $\mathscr{V}_0$ be a corresponding neighborhood of λ_0 (which can be assumed independent of $i = 1, ..., l$, otherwise we take the intersection). Thus, for every $\lambda \in \mathscr{V}_0$ there is a unique equilibrium $e_i(\lambda)$ of (4.c.1) in $\mathscr{U}_i$ and this equilibrium depends continuously on λ. Due to (h2), (h3), we may assume, making $\mathscr{V}_0$ smaller if necessary, that for $\lambda \in \mathscr{V}_0$ there is no other equilibrium of (4.c.1) in $\bar{B}$. Making $\mathscr{V}_0$ yet smaller, we may further assume that for $i = 1, ..., l$, all the equilibria $e_i(\lambda)$, $\lambda \in \mathscr{V}_0$, are contained in an open set $\mathscr{D}_i$ satisfying

$$\bar{\mathscr{D}}_i \subset \mathscr{U}_i, \qquad \inf_{x \in \mathscr{D}_i} \operatorname{dist}(x, \partial \mathscr{U}_i) > 0. \tag{4.c.12}$$

Set

$$\mathscr{D} := \bigcup_{i=1}^{l} \mathscr{D}_i.$$

Denote by Γ^m the set of all $\lambda \in \mathscr{V}_0$ that have the following property: any heteroclinic solution u of (4.c.1) satisfying

$$\begin{aligned} &u(t) \in \bar{B} && (t \in \mathbb{R}), \\ &u(t) \in \bar{\mathscr{D}} && (t \in (-\infty, -m] \cup [m, \infty)) \end{aligned} \tag{4.c.13}$$

is transverse. Note that due to the choice of $\mathscr{V}_0$ the set

$$\Gamma_0 := \bigcap_m \Gamma^m$$

satisfies the transversality condition as in the claim. It is therefore sufficient to prove that each Γ^m is open and dense in $\mathscr{V}_0$.

Proof of Openness. Assume we are given a sequence $\lambda^\nu \in \mathscr{V}_0 \backslash \Gamma^m$ such that $\lambda^\nu \to \lambda^\infty \in \mathscr{V}_0$. We need to show that $\lambda^\infty \notin \Gamma^m$.

By definition of Γ^m, for each ν there is a heteroclinic solution u^ν of (4.c.1) with $\lambda = \lambda^\nu$ that satisfies (4.c.13) and that is not transverse. As there are only finitely many sets $\mathscr{D}_i$, passing to a subsequence we may assume that

$$u^\nu(t) \in \overline{\mathscr{D}}_i \qquad (t \in (-\infty, -m],$$
$$u^\nu(t) \in \overline{\mathscr{D}}_j \qquad (t \in [m, \infty)),$$

for some fixed indices i, j. By Lemma 4.c.3, replacing u^ν by a further subsequence, we obtain that

$$u^\nu \to u \qquad \text{in } C_b(\mathbb{R}, X^\alpha),$$

where u is a heteroclinic solution of (4.c.1) with $\lambda = \lambda^\infty$. Of course, u also satisfies (4.c.13). We prove that u is not transverse, hence $\lambda^\infty \notin \Gamma^m$.

As the closure of $\{u(t)\colon t \in \mathbb{R}\}$ is compact in G, it follows from (h3) that

$$f_u(u^\nu(\cdot), \lambda^\nu) \to f_u(u(\cdot), \lambda^\infty) \qquad \text{in } C_b(\mathbb{R}, X). \tag{4.c.14}$$

Now, by Lemma 4.b.3, the variational equations

$$v_t + (A - f_u(u^\nu(t), \lambda^\nu))v = 0$$

and

$$v_t + (A - f_u(u(t), \lambda^\infty))v = 0$$

have exponential dichotomies on each of the intervals $(-\infty, 0]$, $[0, \infty)$. Moreover, if $P^{\nu-}(t)$, $P^{\nu+}(t)$ and $P^-(t)$, $P^+(t)$ are the projections of these dichotomies then

$$R(P^{\nu-}(0)) = T_{u^\nu(0)} W^u(e^{\nu-}), \qquad R(I - P^{\nu+}(0)) \cap X^\alpha = T_{u^\nu(0)} W^s(e^{\nu+}),$$
$$R(P^-(0)) = T_{u(0)} W^u(e^-), \qquad R(I - P^+(0)) \cap X^\alpha = T_{u(0)} W^s(e^+),$$

where $e^{\nu\pm}$ and $e^\pm$ are the limits, as $t \to \pm\infty$, of the heteroclinic solutions u^ν and u, respectively. By (4.c.14) and Lemma 4.a.11, the projections $P^{\nu-}(t)$ and $P^{\nu+}(t)$ can be chosen in such a way that, as $\nu \to \infty$,

$$P^{\nu-}(0) \to P^-(0), \qquad P^{\nu+}(0) \to P^+(0) \qquad \text{in } \mathscr{L}(X). \tag{4.c.15}$$

Now suppose that u is transverse, that is,

$$R(P^-(0)) + (R(I - P^+(0)) \cap X^\alpha) = X^\alpha.$$

As $R(P^-(0))$ is finite-dimensional, one checks easily, using (4.c.15), that this equality remains valid if P^-, P^+ are replaced by $P^{\nu-}$, $P^{\nu+}$ with sufficiently large ν. But this means that u^ν is transverse, in contradiction to our assumption. We see that u cannot be transverse, hence $\lambda^\infty \notin \Gamma^m$. This completes the proof of the openness of Γ^m.

Proof of Density. Fix any $\lambda_0 \in \mathscr{V}_0$ and any neighborhood $\mathscr{V}$ of λ_0. We have to show that $\mathscr{V}$ contains an element of Γ^m.

Let $\hat{\lambda}$, $\hat{\Lambda}$ and $\hat{\mathscr{V}} \subset \mathscr{V}$ be as in (h5). We are going to apply the transversality Theorem 2.1 to a map Φ defined as follows. Fix a $\delta \in (0, 1-\alpha)$ and let

$$\mathscr{E} = C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}(\mathbb{R}, X^1),$$

$$\mathscr{Z} = C^{0,\delta}(\mathbb{R}, X).$$

Choose open bounded sets B^*, D_i^* such that

$$\begin{aligned} &\bar{B} \subset B^* \subset \overline{B^*} \subset G, \\ &\bar{\mathscr{D}}_i \subset \mathscr{D}_i^* \subset \overline{\mathscr{D}_i^*} \subset \mathscr{U}_i, \qquad i = 1, ..., l. \end{aligned} \tag{4.c.16}$$

(This choice is possible due to (4.c.11), (4.c.12)). Let

$$\begin{aligned} \mathscr{X} = \Big\{ u \in \mathscr{E} : \mathrm{cl}_{X^\alpha}\{u(t) : t \in \mathbb{R}\} \subset B^* \qquad &\text{and} \\ \mathrm{cl}_{X^\alpha}\{u(t) : t \in (-\infty, -m] \cup [m, \infty)\} \subset \mathscr{D}^* \Big\}&, \end{aligned}$$

where

$$\mathscr{D}^* = \bigcup_{i=1}^{l} \mathscr{D}_i^*.$$

Note that $\mathscr{X}$ is open in $\mathscr{E}$. Indeed, as X^1 is compactly imbedded in X^α, for any $u \in \mathscr{X}$, the set $\mathrm{cl}_{X^\alpha}\{u(t)\colon t \in \mathbb{R}\}$ is a compact subset of B^* and therefore the distances of its points to ∂B^* are uniformly bounded away from zero. Similar arguments can be used for $\mathscr{D}^*$, which gives that a neighborhood of u in $\mathscr{E}$ is contained in $\mathscr{X}$.

By Lemma 4.b.2 and (4.c.16), $\mathscr{X}$ contains all heteroclinic solutions of (4.c.1) (for any $\lambda \in \mathscr{V}_0$) satisfying (4.c.13). For $u \in \mathscr{X}$ and $\lambda \in \hat{\mathscr{V}}$ let $\Phi(u, \lambda)$ be the element of $\mathscr{Z}$ defined by

$$\Phi(u, \lambda)(t) = u_t(t) + Au(t) - f(u(t), \lambda) \qquad (t \in \mathbb{R}). \tag{4.c.17}$$

It follows from the properties of saddle-point neighborhoods that for $(u, \lambda) \in \mathscr{X} \times \hat{\mathscr{V}}$ one has $\Phi(u, \lambda) = 0$ if and only if u is a heteroclinic solution of (4.c.1). We prove in a moment (see Lemma 4.c.4 below) that

$\Phi\colon \mathscr{X} \times \hat{\mathscr{V}} \to \mathscr{Z}$ is of class C^{r-1} (with r as in (h5)). It follows from Corollary 4.b.6 that Γ^m certainly contains all $\lambda \in \hat{\mathscr{V}}$ for which $\Phi(\cdot, \lambda) \pitchfork \{0\}$. We next show, applying Theorem 2.1, that the set of such λ's is residual in $\hat{\mathscr{V}}$. This way we prove that $\hat{\mathscr{V}} \subset \mathscr{V}$ contains an element of Γ^m, as desired.

Making the neighborhood $\hat{\mathscr{V}}$ of $\hat{\lambda}$ smaller, if necessary, we may assume that it is convex. In the following four steps we verify that Φ satisfies the hypotheses of Theorem 2.1 (in the last step, $\hat{\mathscr{V}}$ may have to be made yet smaller).

Step 1. $\Phi\colon \mathscr{X} \times \hat{\mathscr{V}} \to \mathscr{Z}$ is of class C^{r-1}, with r as in (h5).

This is a consequence of the following more general statement.

LEMMA 4.c.4. *Let U be any open set in $\mathscr{E}$ such that* $\mathrm{cl}_{X^\alpha}\{u(t)\colon t \in \mathbb{R}\} \subset G$ *for any $u \in U$. Then the map $\Phi\colon U \times \hat{\mathscr{V}} \to \mathscr{Z}$ defined by* (4.c.17) *is of class C^{r-1} and its derivatives are given by*

$$[D\Phi(u, \lambda)(\bar{u}, \bar{\lambda})](t) = \bar{u}_t(t) + A\bar{u}(t) - Df(u(t), \lambda)(\bar{u}(t), \bar{\lambda}), \tag{4.c.18}$$

$$[D^j\Phi(u, \lambda)(\bar{u}, \bar{\lambda})^j](t) = D^jf(u(t), \lambda)(\bar{u}(t), \bar{\lambda})^j, \qquad j = 1, ..., r-1, \tag{4.c.19}$$

for any $(u, \lambda) \in U \times \hat{\mathscr{V}}$, $(\bar{u}, \bar{\lambda}) \in \mathscr{E} \times \hat{\Lambda}$.

Proof. First observe that Φ is continuous on $U \times \hat{\mathscr{V}}$. Further, for any fixed $(u, \lambda) \in U \times \hat{\mathscr{V}}$, the right-hand sides of (4.c.18), (4.c.19) define continuous homogeneous polynomials from $\mathscr{E} \times \hat{\Lambda}$ into $\mathscr{Z}$, and these polynomials depend continuously (in the standard norms on the spaces of homogeneous polynomials) on $(u, \lambda) \in \mathscr{E} \times \hat{\mathscr{V}}$. The proof of this involves estimates of the supremum and Hölder norms. The estimates are somewhat tedious but straightforward and are left to the reader. It may be useful to note, however, that in these estimates all the relevant functions $u(\cdot)$ are contained in a bounded set in $\mathscr{E}$ and therefore, by (h2), their orbits $\{u(t)\colon t \in \mathbb{R}\}$ are all contained in a compact subset of G. Thus $D^rf(\cdot, \lambda)$ is uniformly continuous there.

Now for $a, b \in G$, $\lambda, \mu \in \hat{\Lambda}$ denote

$$H(a, \lambda; b, \mu) = \int_0^1 [D^{r-1}f(sa + (1-s)b, s\lambda + (1-s)\mu) - D^{r-1}f(a, \lambda)]\, ds.$$

For any $u, v \in U$, $\lambda, \mu \in \hat{\mathscr{V}}$, the map

$$(\bar{u}, \bar{\lambda}) \mapsto \tilde{H}(u, \lambda; v, \mu)(\bar{u}, \bar{\lambda})^{r-1} : \mathscr{E} \times \hat{\Lambda} \to \mathscr{E},$$

where

$$\tilde{H}(u, \lambda; v, s)(\bar{u}, \bar{\lambda})^{r-1}(t) = H(u(t), \lambda; v(t), \mu)(\bar{u}(t), \bar{\lambda})^{r-1} \qquad (t \in \mathbb{R}),$$

is also a homogeneous polynomial (of degree $r-1$) and it depends continuously on $(u, \lambda; v, \mu) \in U \times \hat{\mathscr{V}} \times U \times \hat{\mathscr{V}}$. Obviously, $\tilde{H}(u, \lambda; u, \lambda) = 0$. Using the Taylor formula for f and then the converse Taylor theorem ([Ab-M-R, Ab-R]), we obtain that Φ is of class C^{r-1}. ▮

Step 2. *The map* Π: $(u, \lambda) \mapsto \lambda$: $\Phi^{-1}(0) \to \hat{\mathscr{V}}$ is σ-proper.

To see this, choose closed sets $\bar{B}_n^* \subset B^*$, $\bar{\mathscr{D}}_{in}^* \subset \mathscr{D}_i^*$, $i = 1, ..., l$, $n = 1, 2, ...$ such that

$$\bigcup_n \bar{B}_n^* = B^* \qquad \text{and} \qquad \bigcup_n \bar{\mathscr{D}}_{in}^* = \mathscr{D}_i^*.$$

Let V_n be the set of all $(u, \lambda) \in \Phi^{-1}(0)$ satisfying

$$\begin{aligned} &u(t) \in \bar{B}_n^* \ (t \in \mathbb{R}), \\ &u(t) \in \bigcup_{i=1}^{l} \bar{\mathscr{D}}_{in}^* \ (t \in (-\infty, -m] \cup [m, \infty)). \end{aligned} \tag{4.c.20}$$

Clearly

$$\bigcup_n V_n = \Phi^{-1}(0).$$

We next show that for each n $\Pi|_{V_n}$ is proper. Let (u^ν, λ^ν) be sequence in V_n such that $\lambda^\nu \to \lambda^\infty \in V_n$. Using Lemma 4.c.3, similarly as in the proof of the openness of Γ^m, one shows that a subsequence of u^ν converges in $\mathscr{E}$ to a heteroclinic solution u of (4.c.1) with $\lambda = \lambda^\infty$. This solution u satisfies (4.c.20) and $\Phi(u, \lambda^\infty) = 0$, hence $(u, \lambda^\infty) \in V_n$. This proves that $\Pi|_{V_n}$ is proper and completes step 2.

Step 3. $D_u\Phi(u, \lambda)$ is a Fredholm operator of index less than $r-1$ for any $(u, \lambda) \in \Phi^{-1}(0)$.

This follows directly from (4.c.18), Lemma 4.b.5 and property (h5)(b). We only need to note that for any $\lambda \in \hat{\mathscr{V}} \subset \mathscr{V}_0$ the unique equilibrium of (4.c.1) in $\mathscr{U}_i$ has the same Morse index as e_i (the equilibrium for $\lambda = \lambda_0$).

Step 4. Making the neighborhood $\hat{\mathscr{V}}$ of $\hat{\lambda}$ smaller, if necessary, one achieves that for any $(u, \lambda) \in \Phi^{-1}(0)$ the operator $D\Phi(u, \lambda) \in \mathscr{L}(\mathscr{E} \times \hat{\Lambda}, \mathscr{Z})$ is surjective.

If this is not true then there is a sequence $(u^\nu, \lambda^\nu) \in \Phi^{-1}(0)$ such that $\lambda^\nu \to \hat{\lambda}$ and $D\Phi(u^\nu, \lambda^\nu)$ is not surjective. By Lemma 4.c.3, we may assume that u^ν converges in $\mathscr{E}$ to a heteroclinic solution u of (4.c.1) with $\lambda = \hat{\lambda}$.

As $\mathrm{cl}_{X^\alpha}\{u(t)\colon t \in \mathbb{R}\}$ is a compact subset of G, there is neighborhood U of u in $\mathscr{E}$ such that $\{y(t)\colon t \subset \mathbb{R}\} \subset G$ for any $y \in U$. By Lemma 4.c.4, (4.c.17) defines a C^1 map on $U \times \hat{\mathscr{V}}$. By Lemma 4.b.5, $D_u\Phi(u, \hat{\lambda})$ is a Fredholm operator. If we prove that $D\Phi(u, \hat{\lambda})$ is surjective then we have a contradiction because, by Lemma 2.1, for ν large enough $D\Phi(u^\nu, \lambda^\nu)$ would have to be surjective, as well. To complete Step 4, it is therefore sufficient to prove that $D\Phi(u, \hat{\lambda})$ is surjective.

Using the formulas from Lemma 4.c.4, we see that the surjectivity is equivalent to the following property: given any $h \in \mathscr{Z}$ there are $v \in \mathscr{E}$ and $\bar{\lambda} \in \hat{\Lambda}$ such that

$$v_t + Av - f_u(u(t), \hat{\lambda})v = h + f_\lambda(u(t), \hat{\lambda})\bar{\lambda}.$$

By Theorem 4.a.4 (which is applicable because of Lemma 4.b.3), this equation can be solved for $v \in \mathscr{E}$ if $\bar{\lambda}$ is chosen such that

$$\int_{-\infty}^{+\infty} \langle D_\lambda f(u(t), \hat{\lambda})\lambda, \psi(t)\rangle \, dt = -\int_{-\infty}^{+\infty} \langle h(t), \psi(t)\rangle \, dt \tag{4.c.21}$$

for any bounded weak* solution of the homogeneous adjoint equation (4.c.4). Such a choice of $\bar{\lambda}$ is readily possible if for a basis $\psi_1(t), ..., \psi_q(t)$ of the (finite-dimensional) space of bounded weak* solutions of (4.c.4) the linear map

$$\bar{\lambda} \mapsto \left(\int_{-\infty}^{+\infty} \langle D^\lambda f(u(t), \hat{\lambda})\lambda, \psi_l(t)\rangle \, dt\right)_{l=1}^{q} : \hat{\Lambda} \to \mathbb{R}^q \tag{4.c.22}$$

is surjective. We prove the surjectivity by contradiction. Suppose the range of (4.c.22) is not the whole R^q, so that there is a nonzero vector orthogonal to the range. But this means that there is a nontrivial linear combination ψ of the ψ_l such that

$$\int_{-\infty}^{+\infty} \langle D_\lambda f(u(t), \hat{\lambda})\bar{\lambda}, \psi(t)\rangle \, dt = 0$$

for any $\bar{\lambda} \in \hat{\Lambda}$, contradicting (h5)(c). This contradiction proves the surjectivity of (4.c.22) and completes Step 4.

We have now verified that Theorem 2.1 indeed applies to the map Φ. As demonstrated above, this implies the density of Γ^m. The proof of Theorem 4.c.1 is now complete. ∎

5. THE MORSE–SMALE PROPERTY FOR REACTION–DIFFUSION EQUATIONS

In this section we prove Theorem 1.1. Recall that $\mathfrak{G}$ is the set of all C^k functions $f: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ equipped with the C^k Whitney topology ($k \geqslant 1$ is a fixed integer) and $\mathfrak{G}_n^H$ is the set of those $f \in \mathfrak{G}$ for which any equilibrium u of (1.1), (1.2) with $\|u\|_{L^\infty} \leqslant n$ is hyperbolic.

Let $\mathfrak{G}_n^{MS}$ be the set of all $f \in \mathfrak{G}_n^H$ such that any heteroclinic solution $u(x, t)$ of (1.1), (1.2) with $\sup_{t \in \mathbb{R}} \|u(\cdot, t)\|_{L^\infty} < n$ is transverse. We prove that $\mathfrak{G}_n^{MS}$ is residual in $\mathfrak{G}$. Once we have done this, Theorem 1.1 is proved since

$$\mathfrak{G}^{MS} = \bigcap_{n=1}^{\infty} \mathfrak{G}_n^{MS}$$

is a residual set such that for any $f \in \mathfrak{G}^{MS}$ all heteroclinic solutions of (1.1), (1.2) are transverse. From Section 2 we recall that $\mathfrak{G}_n^H$ is open and dense in $\mathfrak{G}$. It is therefore sufficient to prove the following.

LEMMA 5.1. *$\mathfrak{G}_n^{MS}$ is residual in $\mathfrak{G}_n^H$.*

The remaining part of this section is devoted to the proof of this lemma. To this aim we want to apply Theorem 4.c.1. In order to bring (1.1), (1.2) into the context of abstract equations considered in Section 4, we choose a $p > N$ and define an operator A on $X = L^p = L^p(\Omega)$ by

$$D(A) = W^{2,p} \cap W_0^{1,p},$$

$$Au = -\Delta u, \qquad u \in D(A).$$

Note that this operator satisfies hypotheses (A) of Section 4 and it has compact resolvent. The fractional power space $X^{1/2}$ is continuously imbedded in $C(\bar{\Omega})$ and therefore

$$G = \{u \in X^{1/2} : \|u\|_{L^\infty} < n\}.$$

is an open set in $X^{1/2}$.

Denote

$$\Lambda^r = C^r(\bar{\Omega} \times [-n, n]), \qquad r = 1, 2, \ldots.$$

Let $R: \mathfrak{G} \to \Lambda^k$ be the restriction operator

$$Rf = f|_{\bar{\Omega} \times [-n,n]}.$$

Then R is a continuous, open, and surjective linear map. We set

$$\Upsilon = R\mathfrak{G}_n^H \subset \Lambda^k,$$

which is an open set in Λ^k, and assume the induced topology on Υ.

Problem (1.1), (1.2) can now be recast in the form of an abstract equation as studied in subsection 4.c,

$$u_t + Au = F(u, f),$$

where

$$F\colon (u, f) \mapsto F(u, f) = f(\cdot, u(\cdot))\colon G \times \Upsilon \to X$$

and $f \in \Upsilon$ plays the role of the parameter. Note that F is of class C^k and is linear in f.

We show that the hypotheses of Theorem 4.c.1 are satisfied for this equation. This will complete the proof of Theorem 1.1, for taking the residual set Γ as in Theorem 4.c.1, one checks easily that $R^{-1}\Gamma$ is a residual set in $\mathfrak{G}_n^H$ and it is contained in $\mathfrak{G}_n^{MS}$. Therefore $\mathfrak{G}_n^{MS}$ is also residual.

By definition of A and F, hypotheses (A), (h2), and (FP) are satisfied. (ISP) holds by backward uniqueness for linear second order parabolic equations and their adjoint equations (cf. [He1, Section 7.3]). Further, the space Λ^k is separable (so that (h1) holds), (h4) is satisfied by the definition of G and Υ, and (h3) is obvious. It remains to verify (h5). Fix any $f_0 \in \Upsilon$ and any neighborhood $\mathscr{V}$ of f_0 in Υ.

In this neighborhood, we find an $\hat{f}$ with the following properties:

- $\hat{f} \in R\mathfrak{G}_n^H \cap \Lambda^r$ where $r > N = \dim \Omega$ and $r > m(e) + 1$ for any equilibrium e of (1.1), (1.2) with $f = \hat{f}$ satisfying $\|e\|_{L^\infty} \leqslant n$.
- $\hat{f}$ is analytic in u, uniformly with respect to x.
- $\hat{f} \in R\mathfrak{G}^{SE}$ where $\mathfrak{G}^{SE}$ is as in Subsection 3.b.

We find such an $\hat{f}$ making a few successive perturbations of f_0. Fix an $r \geqslant \max k, N$ such that $r > m(e) + 1$ for any equilibrium e of (1.1), (1.2) satisfying

$$\|e\|_{L^\infty} \leqslant n \tag{5.1}$$

(there are only finitely many such hyperbolic equilibria, as follows by a simple compactness argument, as in the proof of Theorem 4.c.1). As $R\mathfrak{G}_n^H$ is open in Λ^k and Λ^r is dense in Λ^k, perturbing f slightly we may assume that it is contained in $\mathscr{V} \cap R\mathfrak{G}_n^H \cap \Lambda^r$. Two further forthcoming perturbations will be done in the space Λ^r. With no further notice we shall assume that the perturbations are so small that the perturbed function f will still be contained in $\mathscr{V} \cap R\mathfrak{G}_n^H$ and, moreover, the maximal Morse index of equilibria satisfying (5.1) will still be less than $r - 1$ (to guarantee the latter we use continuous dependence of the Morse indices on f and compactness of the set of equilibria; see the proof of Theorem 4.c.1). As (the restrictions of) analytic functions are dense in Λ^r, perturbing f, we first achieve that it is analytic (in (x, u) jointly). Next, making one more perturbation, we find an $\hat{f}$ that is analytic in u and contained in $\mathfrak{G}^{SE}$. Here we invoke Lemma 3.b.2. This $\hat{f}$ satisfies all the above properties.

Now let

$$\hat{\Lambda} = \Lambda^r.$$

Choose a bounded neighborhood $\hat{\mathscr{V}}$ of $\hat{f}$ in Λ^r such that

$$\hat{\mathscr{V}} \subset \mathscr{V}.$$

With this choice, properties (a), (b) of (h5) are satisfied. In order to prove (c), we assume that $u(x, t)$ is a heteroclinic solution of (1.1), (1.2) with $f = \hat{f}$ such that $u(\cdot, t) \in G$ $(t \in \mathbb{R})$, and $\psi(\cdot, t) \in L^q$ $(1/p + 1/q = 1)$ is a solution of the adjoint linearized problem

$$-w_t - \Delta w - \hat{f}_u(x, u(x, t))\, w = 0, \qquad t \in \mathbb{R},\ x \in \Omega, \tag{5.2}$$

$$w = 0, \qquad t \in \mathbb{R},\ x \in \partial\Omega, \tag{5.3}$$

with $\|\psi(\cdot, t)\|_{L^q}$ bounded. (We remark that as $\hat{f}$ is of class C^r with $r \geqslant 2$ each weak* solution of (5.2), (5.3) is classical.) Using the duality between L^p, L^q and the linearity of F in f, we see that condition (c) requires that there be a function $g \in \Lambda^r$ such that

$$\int_{-\infty}^{\infty} \int_{\Omega} \psi(x, t)\, g(x, u(x, t))\, dx\, dt \neq 0. \tag{5.4}$$

For the proof of existence of such a g we prepare the following two lemmas.

LEMMA 5.2. *For each $x \in \Omega$, $u(x, t)$ and $\psi(x, t)$ are real analytic functions of t.*

Proof. Since $\hat{f}$ is real analytic in u, one proves, using abstract regularity results (see [He1, Section 3.4]), that

$$t \mapsto u(\cdot, t) : \mathbb{R} \to X^{1/2}$$

is analytic. Then

$$t \mapsto \hat{f}_u(\cdot, u(\cdot, t)) : \mathbb{R} \to C(\bar{\Omega})$$

is analytic. Using the abstract regularity results again, one obtains that $\psi(x, -t)$ (and $\psi(x, t)$ along) is analytic in t. ∎

LEMMA 5.3. *There exists a subset $D \subset \Omega$, residual in Ω, such that for any $x \in \Omega$ the following properties are satisfied*:

(i) $e^-(x) \neq e^+(x)$, *where* $e^\pm(x) := \lim_{t \to \pm\infty} u(x, t)$ (*both limits exist as u is a heteroclinic solution and the convergence in $X^{1/2}$ implies* (*uniform*) *pointwise convergence*).

(ii) $u_t(x, t) \neq 0$ *for all t with $|t|$ sufficiently large.*

(iii) $e^-(x)$ *is a regular value of the function* $t \mapsto u(x, t) : \mathbb{R} \to \mathbb{R}$.

Proof. Let $e^\pm$, be the limit equilibria of $u(\cdot, t)$ as $t \to \pm\infty$. Then $v := e^+ - e^-$ solves the equation

$$\begin{aligned} \Delta v + a(x) v = 0, \qquad & x \in \Omega, \\ v = 0, \qquad & x \in \partial\Omega, \end{aligned}$$

where

$$a(x) = \int_0^1 f_u(x, e^-(x) + s(e^+(x) - e^-(x)))\, ds.$$

As $e^- \neq e^+$ ((1.1), (1.2) has a Lyapunov function, see Section 1, so it has no homoclinic solutions), by unique continuation for elliptic equations (see [Mi]), v cannot be identical to zero on any nonempty open set. Thus (i) holds for all x in an open dense subset $\tilde{\Omega}$ of Ω.

Next, the function $z = u_t$ solves the equation

$$\begin{aligned} z_t = \Delta z + f_u(x, u(x, t)) z, \qquad & t \in \mathbb{R}, x \in \Omega \\ z = 0, \qquad & t \in \mathbb{R}, x \in \partial\Omega, \end{aligned}$$

where the coefficient of z converges, as $t \to \pm\infty$, to $f_u(x, e^\pm(x))$. As the eigenvalues of $L^\pm := (\Delta + f_u(x, e^\pm(x))$ are all simple ($\hat{f}$ belongs to $R\mathfrak{G}^{SE}$), it follows that, as $t \to \pm\infty$, $z/\|z\|_{L^2}$ converges (uniformly in x) to $\phi^\pm$, an eigenfunction of $L^\pm$. This convergence property is obtained by a direct application of Theorems B.4, B.5 in [Che-C-H]. By unique continuation, each of the eigenfunctions $\phi^\pm$ is nonzero on an open dense set in Ω. Clearly, for each x in this set property (ii) is satisfied.

It remains to consider (iii). Let

$$y(x, t) = u(x, t) - e^-(x).$$

This function solves

$$\begin{aligned} y_t = \Delta y + a(x, t)\, y, \qquad & t \in \mathbb{R}, x \in \Omega \\ y = 0, \qquad & t \in \mathbb{R}, x \in \partial\Omega, \end{aligned}$$

with

$$a(x,t)=\int_0^1 f_u(x, e^-(x)+s(u(x,t)-e^-(x)))\,ds.$$

For $j=1, 2, ...$ denote

$$D_j=\{x\in\Omega: 0 \text{ is a regular value of } t\mapsto y(x,t):(-j,j)\to\mathbb{R}\}.$$

We claim that D_j is residual in Ω. This implies that for any x in the residual set $\bigcap_j D_j$ property (iii) is satisfied. We prove our claim using the parameterized transversality theorem and the following property.

(H) The Hausdorff dimension of the singular nodal set of y,

$$S=\{(x,t)\in\tilde{\Omega}\times[-j,j]: y(x,t)=0 \text{ and } \nabla_x y(x,t)=0\},$$

does not exceed $N-1$. Here $\tilde{\Omega}$ is as above: (i) holds for $x\in\tilde{\Omega}$.

This property has been proved by Han and Lin (see [Han-L, Proposition 1.2]) under an extra assumption, the so-called doubling condition, on y. The doubling condition is satisfied, provided y enjoys a unique continuation property such as the one in Theorem 1.1 (II) of [Al-V]. As Alessandrini and Vessella prove, y does enjoy this property, provided it has no zero of infinite order. This is the case here, as y is analytic in t (see Lemma 5.2) and it is not identical to zero for any fixed $x\in\tilde{\Omega}$. Let us remark that for equations on R^N or for the Dirichlet problem on a convex domain (but not yet for equations on any bounded domain), a general unique continuation theorem, which implies (H) for solutions of such equations, has recently been proved by Chen (see [Che1, Che2]; see also [Poo] for an earlier weaker result).

From (H) one obtains easily that the set

$$K:=\{x\in\tilde{\Omega}: y(x,t)=0 \text{ and } \nabla_x y(x,t)=0 \text{ for some } t\in[-j,j]\},$$

which is the projection of S in Ω, has empty interior. (In fact, the N-dimensional Hausdorff measure, that is, the Lebesgue measure, of K must be zero.) As this set is relatively closed,

$$Q=\tilde{\Omega}\backslash K$$

is open and dense in $\tilde{\Omega}$.

Consider the map

$$(x,t)\mapsto y(x,t): Q\times(-j,j)\to\mathbb{R}.$$

It is of class C^r, where r was chosen greater than N, and, by definition of Q, it has 0 as a regular value. Therefore, by Theorem 2.1, D_j contains a set that is residual in Q (hence also in Ω). This proves the claim and completes the proof of the lemma. ∎

We are now ready to complete the verification of (h5)(c). Suppose it fails; that is, for any $g \in \Lambda^r$, one has

$$\int_{-\infty}^{\infty} \int_{\Omega} \psi(x, t)\, g(x, u(x, t))\, dx\, dt = 0 \tag{5.5}$$

We show this leads to a contradiction. First taking g in the form $b(x)\, g(u)$, where $b \in C^r(\bar{\Omega})$, $g \in C^r([-n, n])$ are arbitrary, and changing the order of integration, we see that (5.5) implies

$$\int_{-\infty}^{\infty} \psi(x, t)\, g(u(x, t))\, dt = 0 \qquad \text{for any} \quad x \in \Omega,\ g \in C^r([-n, n]). \tag{5.6}$$

Let D be as in Lemma 5.3. Fix any $x \in D$. For definiteness, we assume the following relations (cf. Fig. 1):

(1) $u_t(x, t) > 0$ for each sufficiently large negative t.

(2) $e^-(x) > e^+(x)$.

All other sign combinations can be treated in a similar way.

Denote

$$\zeta_0 := e^-(x).$$

By (1), (2), there are positive constants T_1, δ such that

$$\begin{aligned} u_t(x, t) &> 0 \qquad && (t \leqslant -T_1),\\ u(x, t) &< \zeta_0 - \delta \qquad && (t > T_1). \end{aligned}$$

As ζ_0 is a regular value of $u(x, \cdot)$, making δ smaller, if necessary, we may in addition assume that the interval $(\zeta_0 - \delta,\ \zeta_0 + \delta)$ consists of regular values of $u(x, \cdot)$. Let m be the number of preimages of ζ_0 under the map $u(x, \cdot)$. By the inverse function theorem, there are real analytic strictly monotone functions $t_1(\zeta), ..., t_m(\zeta)$ defined on $(\zeta_0 - \delta, \zeta_0 + \delta)$ such that

$$\begin{gathered} -T_1 < t_1(\zeta) < t_2(\zeta) < \cdots < t_m(\zeta),\\ u(x, t_i(\zeta)) = \zeta, \qquad i = 1, ..., m, \end{gathered}$$

and $t_1(\zeta), ..., t_m(\zeta)$ are all the preimages of ζ under $u(x, \cdot)$ in $[-T_1, \infty)$. Furthermore, for $\zeta \in (\zeta_0 - \delta, \zeta_0)$ there is no other preimage of ζ in $\mathbb{R}$, whereas for $\zeta \in (\zeta_0, \zeta_0 + \delta)$ there is exactly one more preimage $t_0(\zeta)$ which is contained in $(-\infty, -T_1)$ (cf. Fig. 1). Again, $t_0(\zeta)$ is a real analytic strictly monotone function.

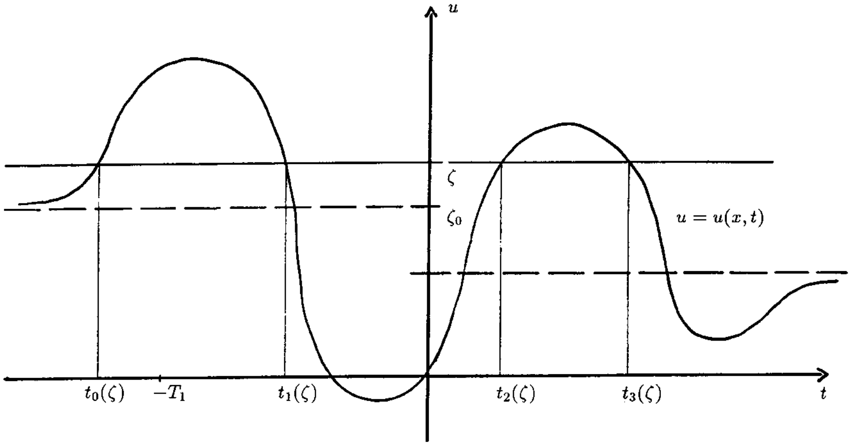

FIG. 1. $\zeta \in (\zeta_0 - \delta, \zeta_0) : u(x, t_j) = \zeta$ for $j > 0$; $\zeta \in (\zeta_0, \zeta_0 + \delta) : u(x, t_j) = \zeta$ for $j \geqslant 0$.

Let us now reconsider (5.6). We claim that it implies that for any $\zeta \in (\zeta_0 - \delta, \zeta_0)$ one has

$$\sum_{i=1}^{m} \psi(x, t_i(\zeta))\, t_i'(\zeta) = 0. \tag{5.7}$$

To prove that (5.7) indeed follows from (5.6), take

$$g(u) = \frac{1}{\varepsilon}\, b\left(\frac{u - \zeta}{\varepsilon}\right),$$

where b is a smooth bump function, for example,

$$b(s) = \begin{cases} e^{-1/(1-s^2)}, & |s| \leqslant 1, \\ 0, & |s| > 1. \end{cases}$$

Using the properties of the local preimages of ζ under $u(x, \cdot)$, we see that for small $\varepsilon > 0$ the integral in (5.6) equals the sum of m integrals in which it is justified to perform the change of variables $t = t_i(\xi)$. Letting $\varepsilon \to 0$ in the transformed integrals, we obtain (5.7).

Similar arguments prove that for any $\zeta \in (\zeta_0, \zeta_0 + \delta)$ one has

$$\sum_{i=0}^{m} \psi(x, t_i(\zeta))\, t_i'(\zeta) = 0. \tag{5.8}$$

Now, by analyticity, (5.7) continues to hold for $\zeta \in [\zeta_0, \zeta_0 + \delta)$. Therefore (5.8) implies

$$\psi(x, t_0(\zeta))\, t_0'(\zeta) = 0 \qquad (\zeta \in (\zeta_0, \zeta_0 + \delta)).$$

Since $t_0'(\zeta) \neq 0$, we obtain that

$$\psi(x, t) \equiv 0$$

on some time interval and therefore, due to analyticity of $\psi(x, \cdot)$, on the whole real line. We have derived this conclusion for each x in the residual set D. Hence $\psi \equiv 0$, a contradiction. This contradiction completes the verification of (h5)(c) and thereby the proof of the theorem.

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