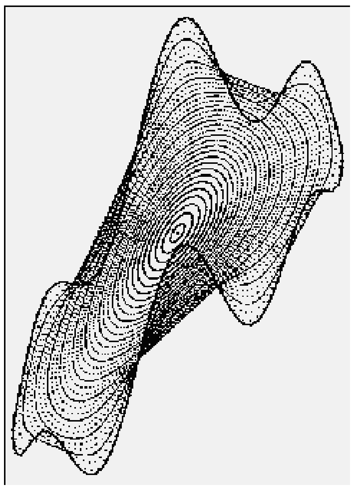
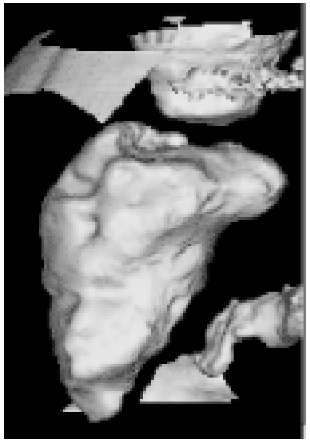
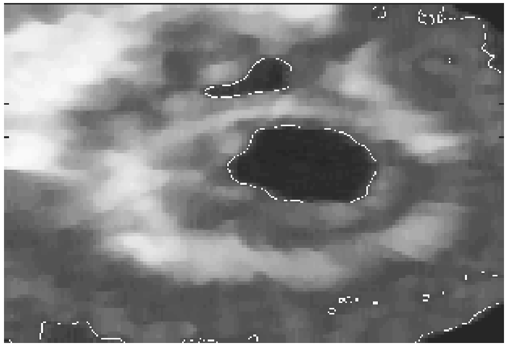
Motion of planar curves: Geometric and analytical aspects

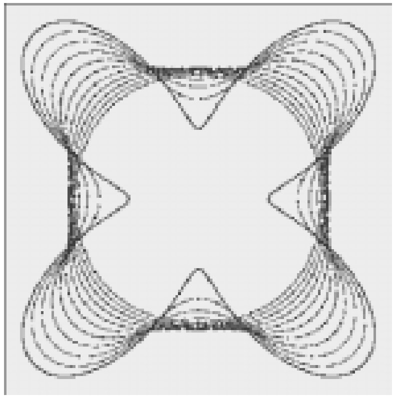




Daniel Ševčovič
Habilitation lecture

June 25, 2001





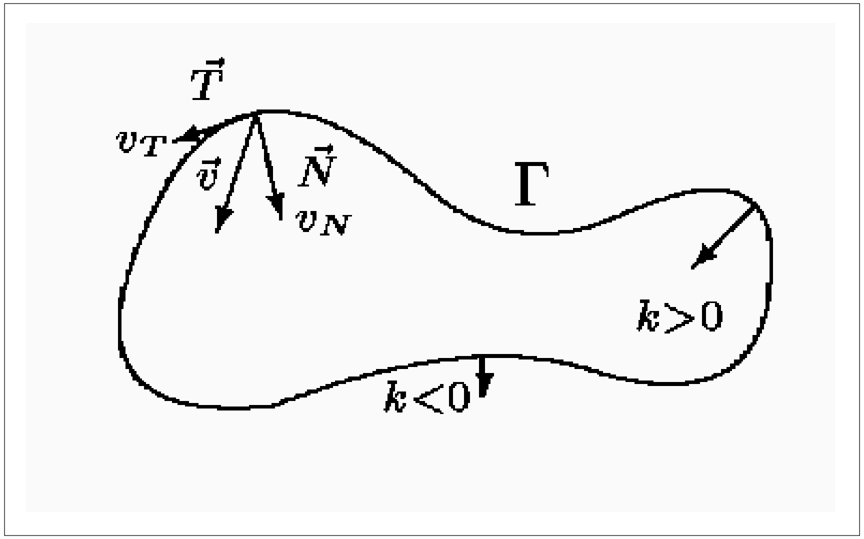
Goals of the lecture

1. Motivation for studying mean curvature driven flows of planar curves
   - ***Theory of phase interfaces*** separating solid and liquid phases, Stefan problem, Gibbs-Thomson law governing the crystal growth.
   - ***Image processing:*** morphological image and shape multiscale analysis, analysis of image silhouettes, image segmentation.
   - ***Differential geometry***. Evolution of planar curves driven by curvature.
2. Mathematical formulation and analysis of governing equations
   - ***Methods for solving mean curvature flow.*** Level set methods, phase field equations approach, intrinsic heat equation approach.
   - ***Governing equations***. Formulation in the form of nonlinear parabolic partial differential equations. Qualitative analysis of solutions and geometric interpretation
3. Computational aspects and applications
   - ***Level set method vs. direct approach.*** Advantages of the direct method Computational results and important applications.

Motivation

Geometric equation

$$V = \beta \ \ k, \nu$$



V - normal velocity
k - (mean) curvature
ν - tangential angle

β - function depending on the curvatute and angle

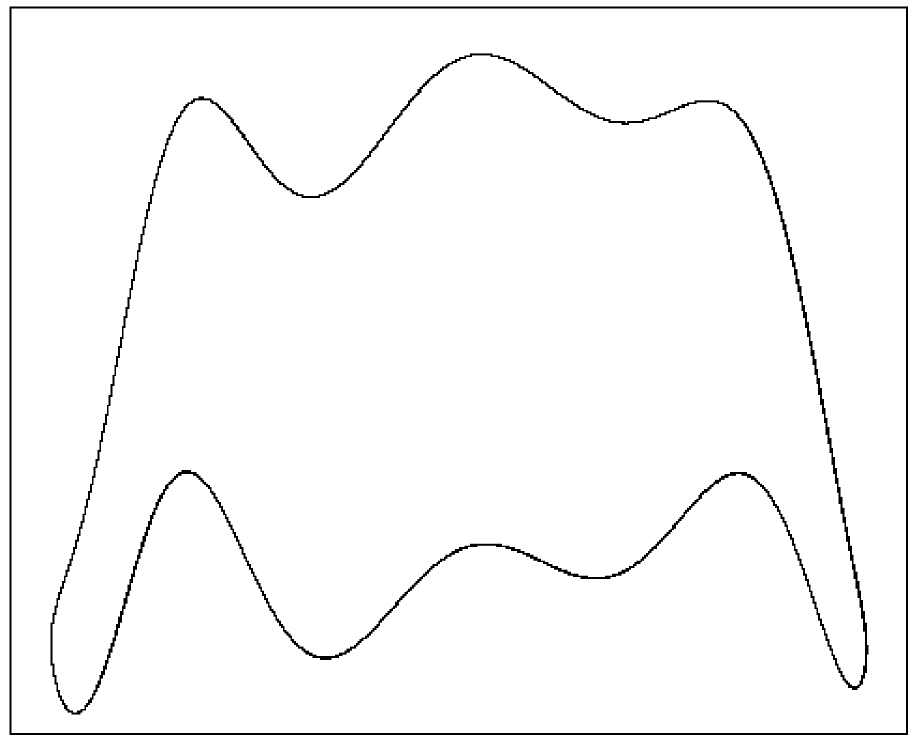
Motivation

Example

$$V = k$$

The mean curvature flow shrinks a planar Jordan curve to a circle rounded point in finite time.

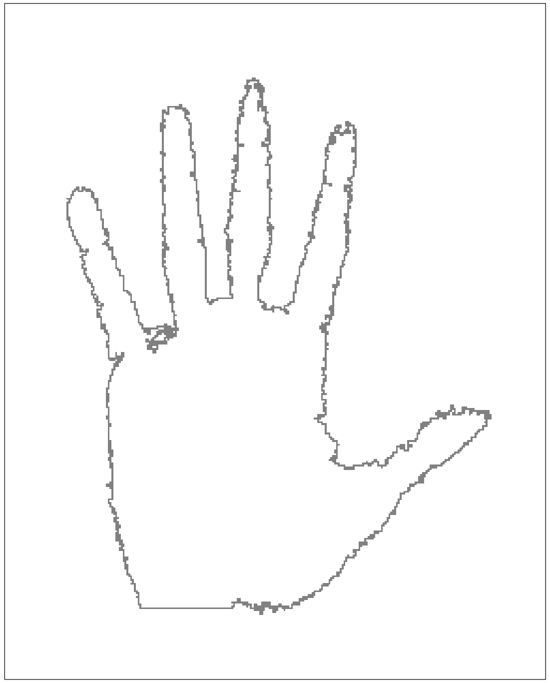
K.Mikula, D.Ševčovič, 1999



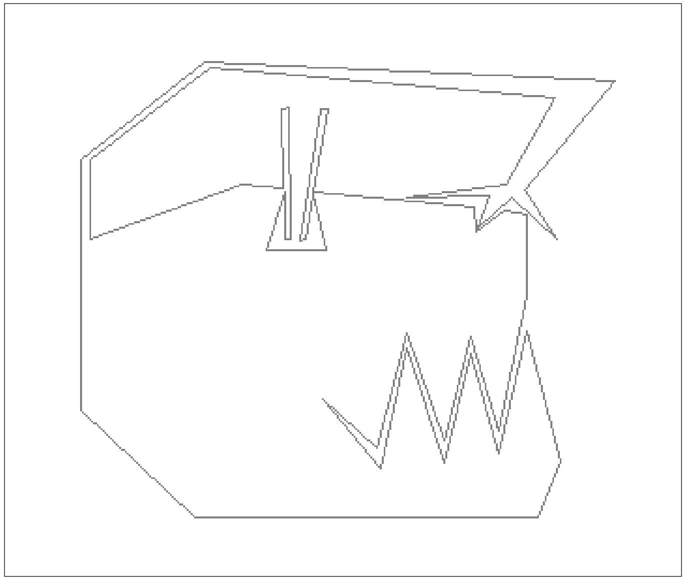
Motivation

Example

$$V = k^{1/3}$$



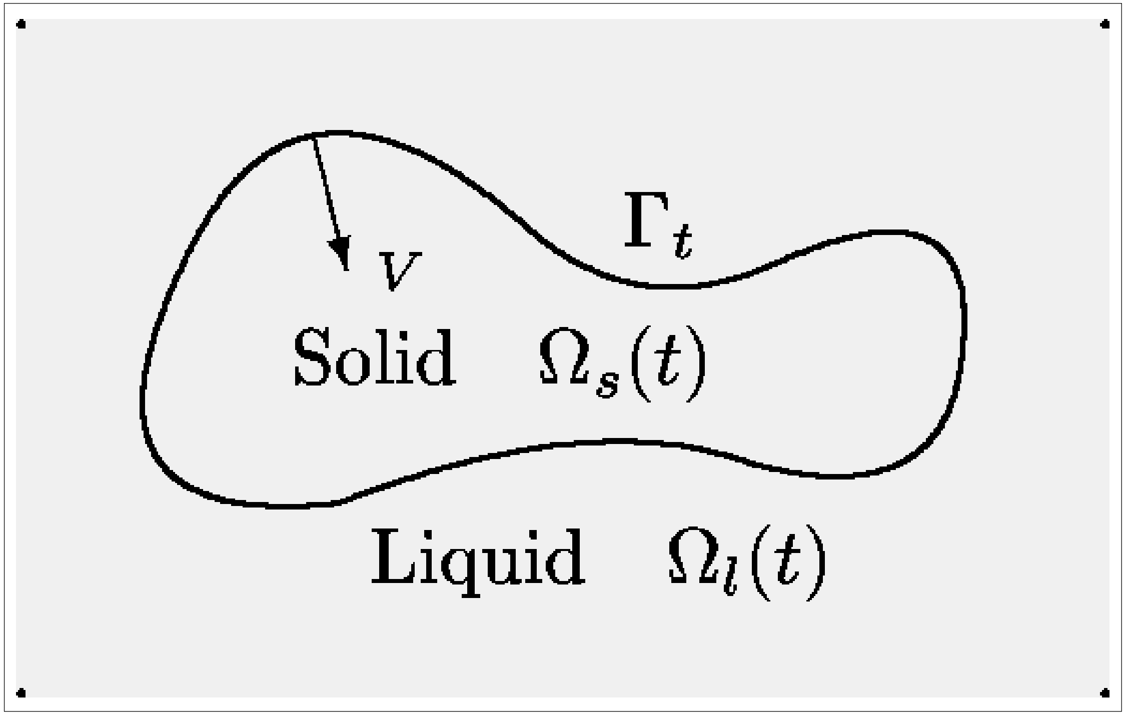
$$V = k^{2}$$



F.Cao, L.Moissan, 2000

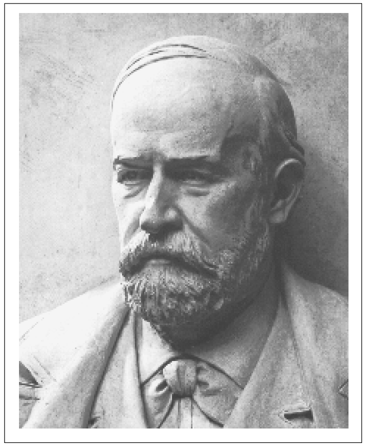
Motivation

Stefan's theory of sharp phase interfaces



$\Omega_s(t)$
solid phase at time t;

$\Omega_l(t)$
liquid phase at time t



Jozef Stefan, 1835-1893

Motivation

Stefan's theory of sharp phase interfaces

$$\rho c \partial_t U = \lambda \Delta U$$

in $\Omega_s\ t$ and $\Omega_l\ t$

$$\lambda \partial_n U\ {}_s^l = -L V$$

$$\delta \frac{e}{\sigma}\ U - U_m = -\gamma_2\ \nu\ k + \gamma_1\ \nu\ V \quad \text{on } \Gamma_t$$

- Heat equations in solid and liquid phases
- Stefan's condition
- Gibbs-Thompson law

on a free boundary Γ_t

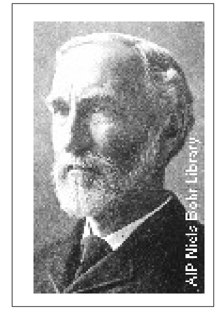
γ_1 is a coefficient of attachment kinetics

γ_2 describes anisotropy of the interface

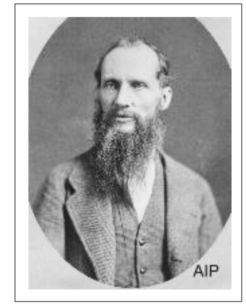
Motivation

Gibbs - Thompson condition

$$V=\gamma\ \ \nu\ \ \ k+f\ \ \ x,\nu$$



Josiah Willar Gibbs
1839-1903



William Thomson
lord Kelvin, 1816-1900

Gibbs - Thompson contact angle condition

$$\mu\ \ \nu,V\ \ V=\gamma\ \ \nu\ \ \ k+f$$

μ is the mobility coefficient

$$V=\beta\ \ x,k,\nu$$

V - normal velocity

k - (mean) curvature

ν - tangential angle

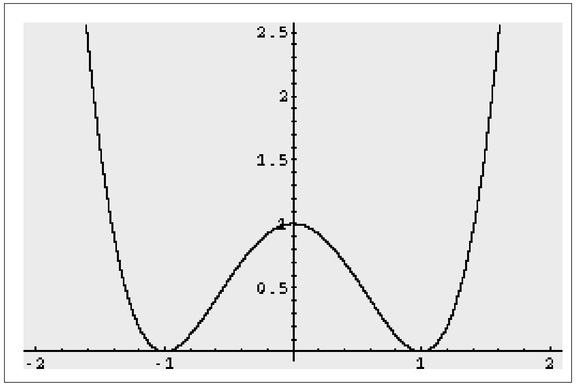
x - position vector

Motivation

Allen - Cahn theory of phase interfaces

$$\partial_t U - \Delta U + \frac{1}{\epsilon^2}\psi(U) = 0$$

$$\psi = \Psi'/2 \qquad \Psi(U) = (1 - U^2)^2$$



Double well potential Ψ

$0 < \varepsilon << 1$ - thickness of the interface

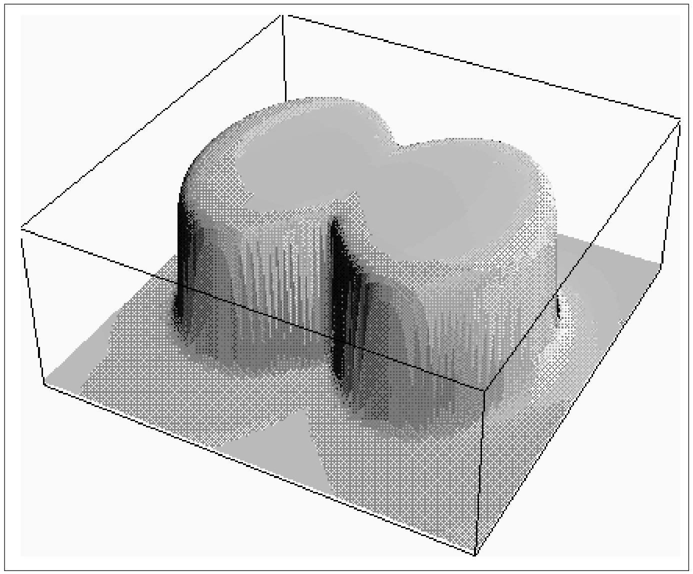
Liquid phase: $U \sim 0$

Solid phase: $U \sim 1$

S.Allen, J.Cahn, 1979

Motivation

Allen - Cahn theory of phase interfaces



Convergence as $\varepsilon \to 0$ of the Allen - Cahn equation to evolution by mean curvature has been studied by many authors (Brakke's motion by mean curvature)

$$V = k$$ - Mean curvature evolution

DeGiorgi 1990; Ilmanen 1993;
Evans, Soner, Souganidis 1992

Motivation

Image processing

In the image processing the so-called morphological image and shape multiscale analysis is often used because of its contrast and affine invariance properties.

Analysis of image silhouettes (boundaries of distinguished shapes) leads to an equation of the form:

$$V = k^{1/3}$$

G.Sapiro, A.Tannenbaum, 1994

L.Alvarez, F.Guichard, P.Lions, J.Morel, 1993

Motivation

Image processing

In the morphological image multiscale analysis the so-called Perrona-Malik model is widely used. This analysis is represented by a viscosity solution to the following nonlinear degenerate parabolic equation in a two dimensional rectangular domain

$$\partial_t u = \nabla u \ \beta \ \ \mathrm{div} \ \ \nabla u / \nabla u$$

$$\beta \ k = k^{1/3}$$

Perrona - Malik model for 3D motion by mean curvature

K.Mikula, 1999

L.Alvarez, F.Guichard, P.Lions, J.Morel, 1993

Motivation

Image processing

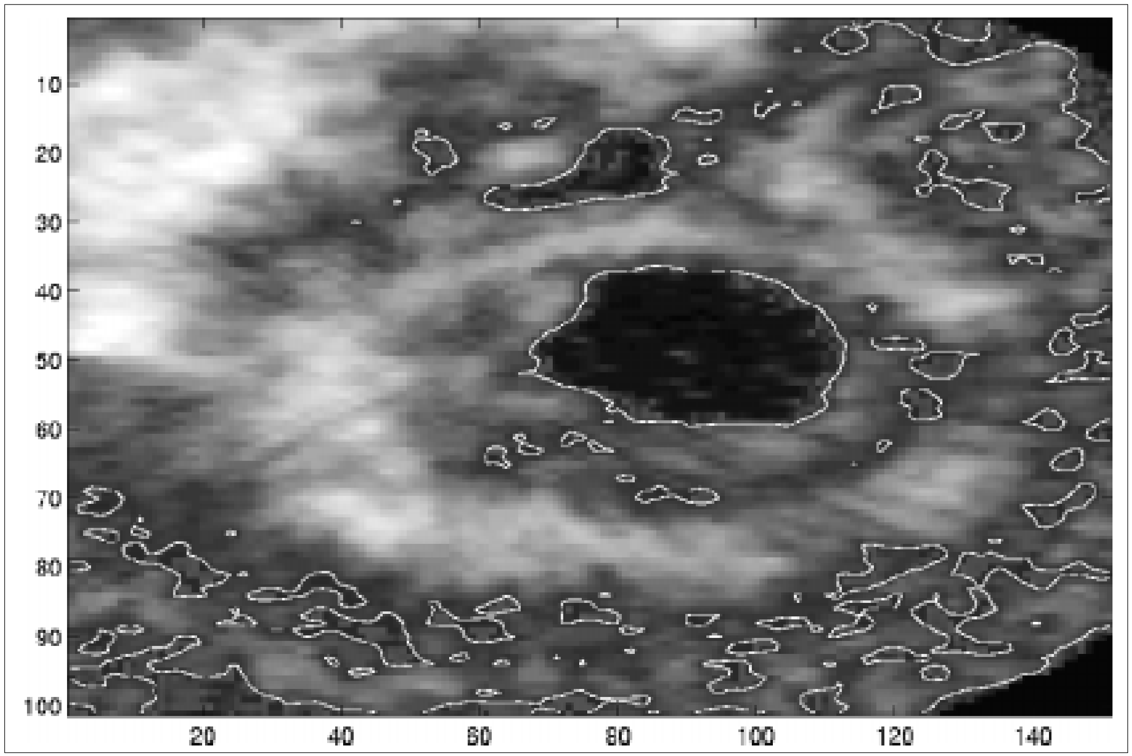
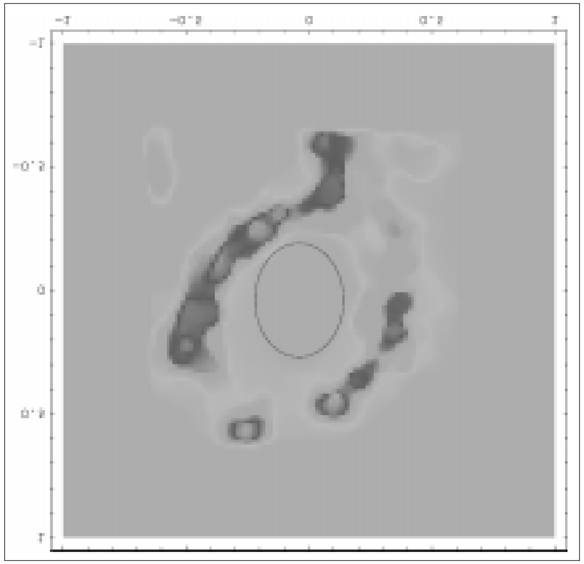


Image segmentation, pattern recognition in Echocardiography

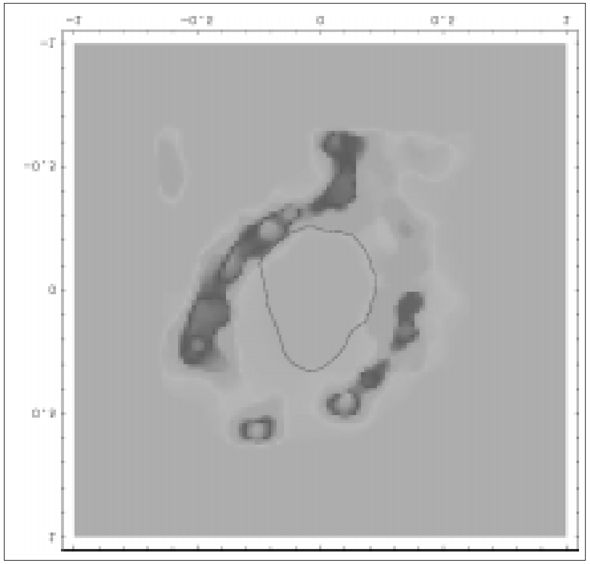
A.Sarti, K.Mikula, F.Sgallari, 2000

Motivation

Image segmentation, pattern recognition



Initial ellipse inside echocardiography



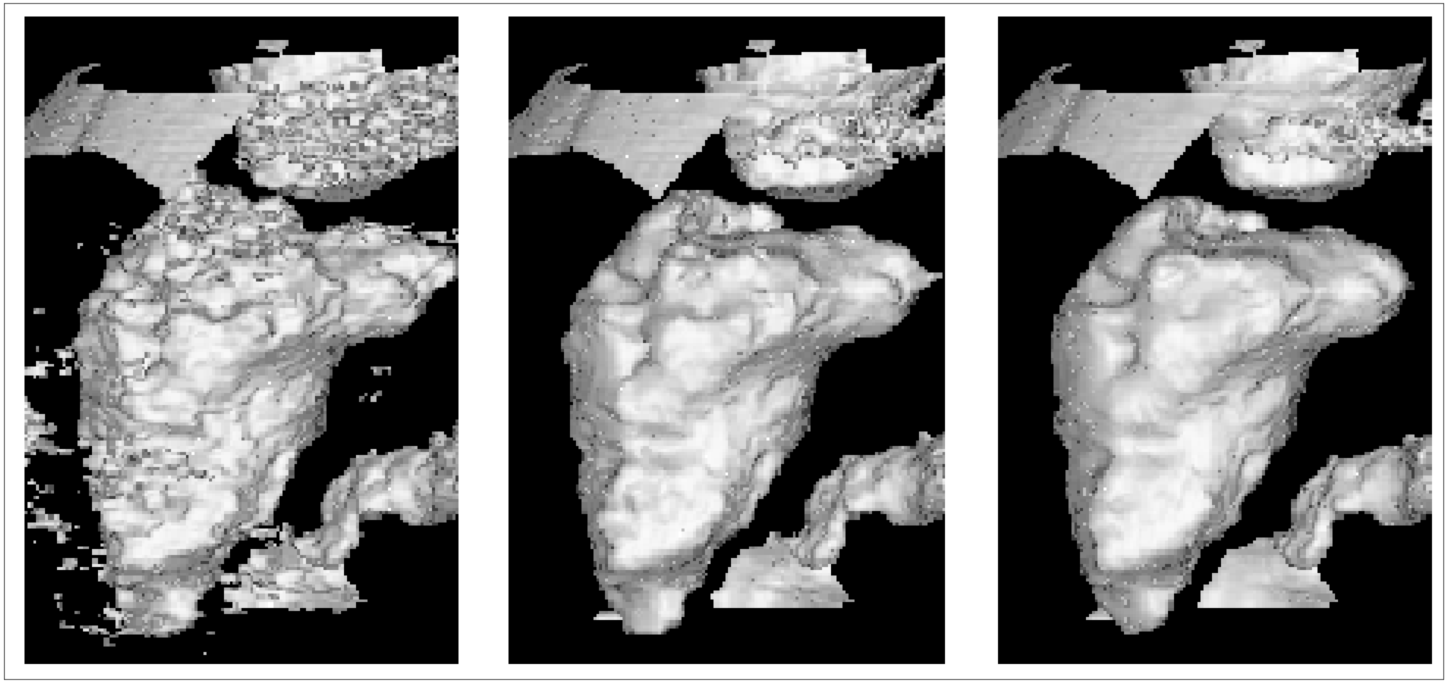
Pattern evolution inside echocardiography

$$V = \epsilon\, k - F$$

K.Mikula, D.Ševčovič, 2001 C.Lamberti, 2000

Motivation

Image and video smoothing and filtering



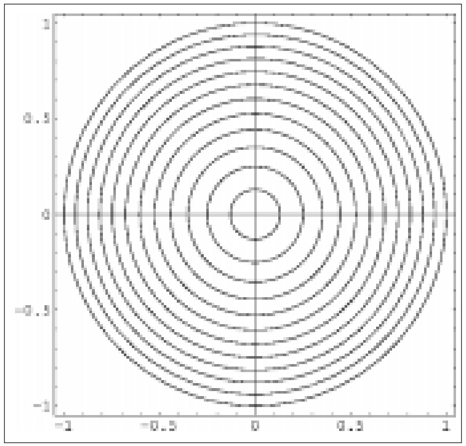
Original echocardiography coarse filtering fine filtering

Perrona - Malik model for 3+1D motion by mean curvature

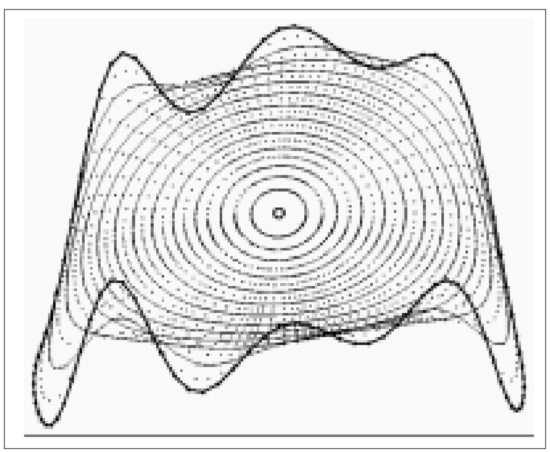
A.Sarti, K.Mikula, F.Sgallari, 2000; C.Lamberti, 2000

Motivation

Differential geometry



Theorem (M.Gage, R.S.Hamilton, 1986)
*The mean curvature flow V =k shrinks any **convex** planar Jordan curve to a circular rounded point in finite time.*



Theorem (R.Grayson, 1987)
*The mean curvature flow V =k shrinks **any** planar Jordan curve to a circular rounded point in finite time.*

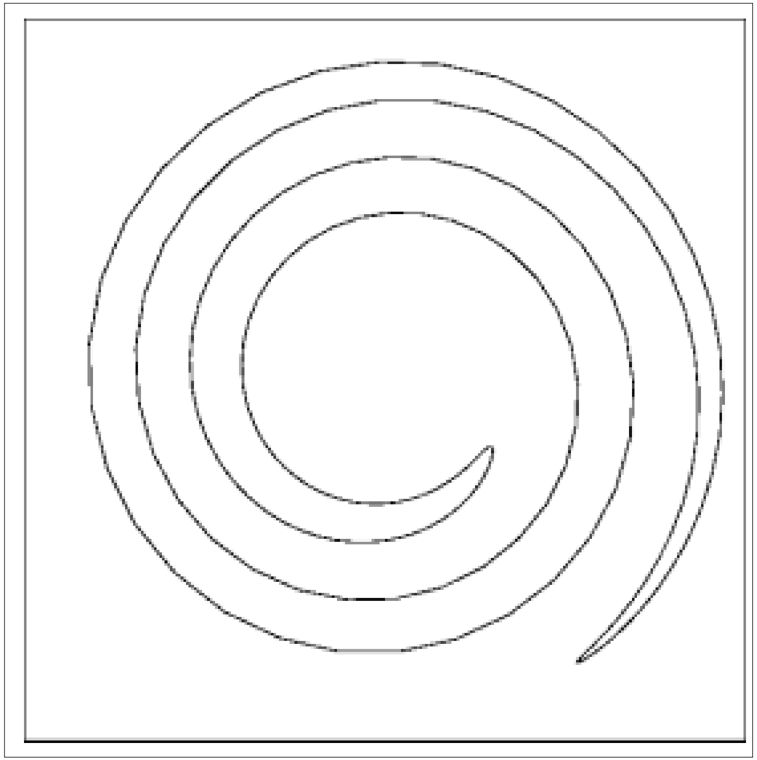
Motivation

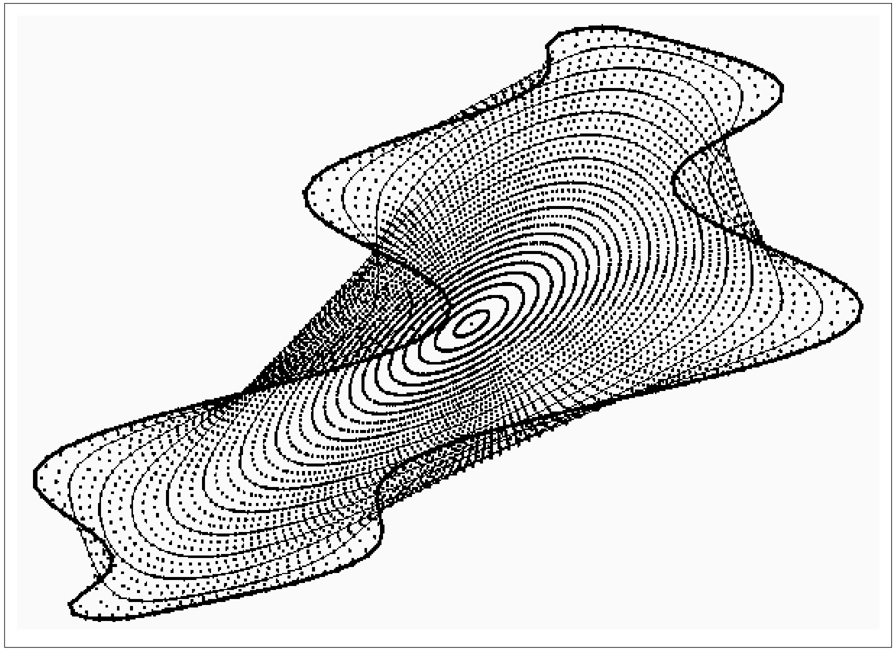
Differential geometry

Affine invariant scaling

$$V = k^{1/3}$$

Ellipses are self-similar patterns

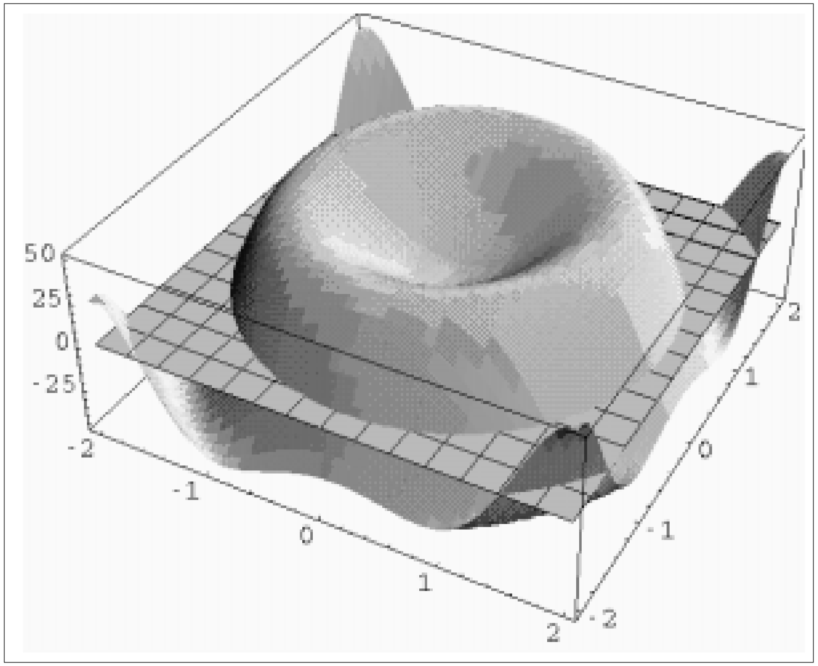




K.Mikula, D.Ševčovič, 1999

Mathematical description

Level set description of a mean curvature flow



$$\Gamma = \{ x, \; u(x) = 0 \}$$

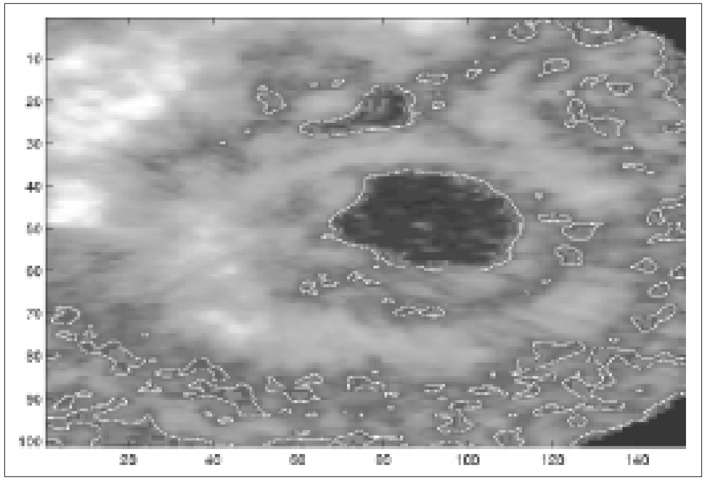
$$u : R^2 \to R$$

T.Ohta, D.Jasnow, K.Kawasaki, 1982

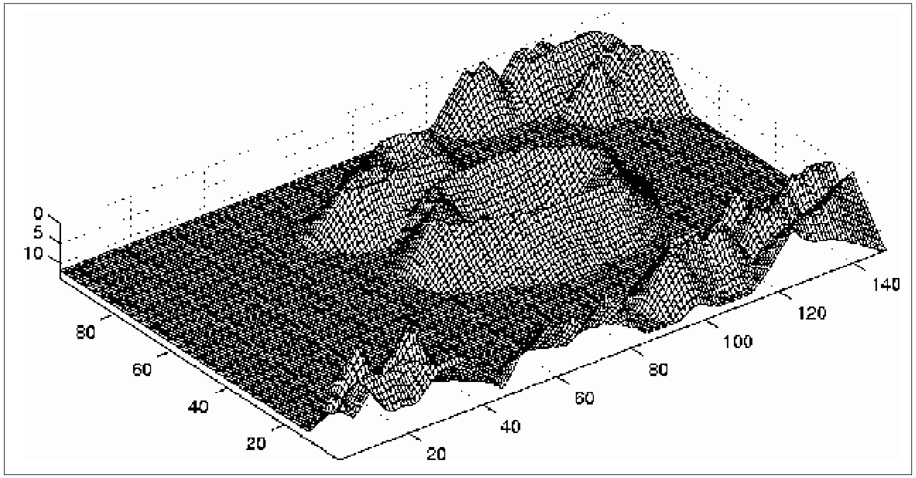
$$k_\Gamma = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|} \right)$$ - curvature of Γ

Mathematical description

Level set description of a mean curvature flow



Density plot of image function



Surface plot of image function

$$u_0 : R^2 \rightarrow \ 0,1$$

Initial condition representing the image intensity function

A.Sarti, K.Mikula, F.Sgallari, 2000

Mathematical description

Level set description of a mean curvature flow

$$\partial_t u = \mathrm{tr}\ \ I - \ \vec{N} \otimes \vec{N}\ \ \nabla^2 u$$

$$u\ \ .,0\ = u_0\ \ .$$

$$\vec{N} = \frac{\nabla u}{\nabla u}$$

$$\Gamma_t = \ x, \quad u\ \ x,t\ = 0 \qquad u: R^n \times\ 0,T\ \to R$$

L.Evans, H.Soner,
P.Souganidis, 1992

S.Osher, J.Sethian, 1988

Mathematical description

Phase field description of a mean curvature flow

$$\partial_t u - \Delta u = \partial_t \Phi$$

$$\partial_t \Phi - \Delta \Phi + \; u - u_m \;\; \nabla \Phi = \frac{1}{\epsilon^2} \Phi \;\; 1 - \Phi \;\;\; \Phi - 0.5$$

$0 < \varepsilon << 1$ - thickness of the interface

$$\Gamma_t = \;\; x, \quad \Phi \;\; x,t \;\; = 0.5 \qquad\qquad u, \Phi : R^2 \times \;\; 0,T \;\; \rightarrow R$$

G.Caginalp, 1988

M.Beneš, K.Mikula, 1998

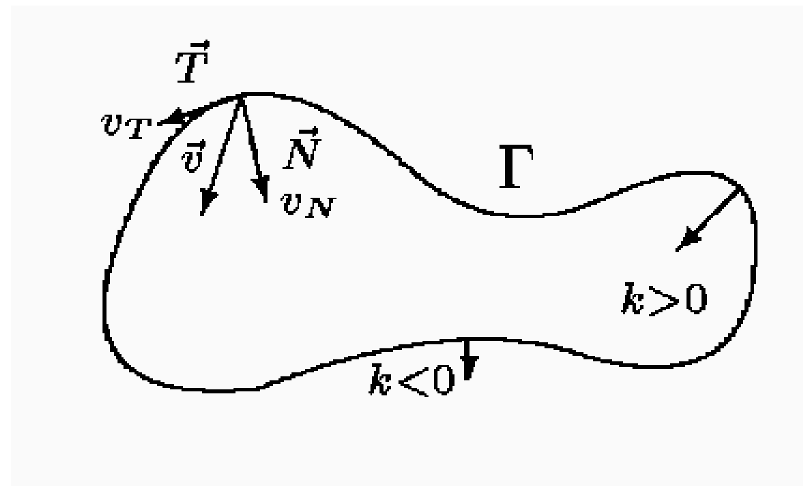
Mathematical description

Direct approach to the mean curvature flow

$$\partial_t x = k\vec{N}$$

$$x = x(u,t),\quad u \in S^1, t \in [0,T]$$

Position vector equation



Tangent vector - $\partial_s x = \vec{T}$

Frenet's formula - $\partial_s T = k\vec{N}$

Mathematical description

Intrinsic heat equation

$$\partial_t x = \partial_s^2 x$$

- geometric intrinsic heat equation (in terms of the arc-length parameter *s*)

$$\partial_t x = \frac{1}{\partial_u x} \frac{\partial}{\partial u} \frac{1}{\partial_u x} \frac{\partial x}{\partial u}$$

- fixed domain Eulerian form of geometric heat equation

$$x \;\; .,0 \; = x_0 \;\; .$$

- initial condition representing the initial curve

M.Gage,R..Hamilton, 1986

M.Grayson, 1987

Mathematical description

Intrinsic heat equation

$$V = \beta\ \ k$$

Normal velocity is a function of the curvature, e.g. $\beta(k) = k^m$, $m>0$.

$$\partial_t x = \beta\ \ k\ \ \partial_s^2 x$$

$$x\ \ .,0\ = x_0\ \ .$$

The curvature k itself depends on the second derivative of x.

$$k = \partial_s x \wedge \partial_s^2 x$$

The arc-length parameter s depends on position vector x.

$$ds = \ \partial_u x\ \ du$$

Mathematical description

Curvature equation

$$V = \beta(k)$$

$$\partial_t k = \partial_s^2 \beta(k) + k^2 \beta(k)$$

$$k(.,0) = k_0(.)$$

- heat equation for the curvature
- initial condition for curvature of an initial curve

M.Grayson, 1987

U.Abresh, J.Langer, 1986

Mathematical description

Curvature and local length equation

$$V=\beta\ \ k$$

$$\partial_t k=\partial_s^2 \beta\ \ k\ +k^2 \beta\ \ k$$

- heat equation for the curvature

$$\partial_t g=-g\,k\,\beta\ \ k$$

- ODE for local length element

$$ds=\ \partial_u x\ \ du=g\,du$$

$$k\ \ .,0\ =k_0\ \ .\ ,\quad g\ \ .,0\ =g_0\ \ .$$

- initial conditions for curvature and local length

S.Angenent, 1987

Mathematical description

Curvature, tan. angle and local length eqs.

$$V=\beta \;\; k,\nu$$

$$\partial_t k=\partial_s^2 \beta+k^2 \beta$$

$$\partial_t \nu=\beta_k^{'} \partial_s^2 \nu+\beta_\nu^{'} k$$

$$\partial_t g=-g\, k\, \beta$$

$$k \;\; .,0 \; =k_0 \;\; . \; , \quad \nu \;\; .,0 \; =\nu_0 \;\; .$$

$$g \;\; .,0 \; =g_0 \;\; .$$

- Heat equations for the curvature and
- tangent angle
- ODE for local length element
- initial conditions for curvature, tangent angle and local length

K.Mikula, D. Ševčovič, 2001

Analysis of governing equations

Local existence of a classical solution

Using the general theory due to Angenent one can prove local existence of a classical solution provided that the function β is regular in k .

$$k, \nu, g \in C\left(0,T\right), E_1 \cap C^1\left(0,T\right), E_0$$

where

$$E_k = c^{2k+\sigma}\left(S^1\right) \times c^{2k+\sigma}\left(S^1\right) \times c^{1+\sigma}\left(S^1\right)$$

K.Mikula, D. Ševčovič, 2001

S. Angenent, 1990

Analysis of governing equations

Local existence of a classical solution

In the case the velocity β is singular in *k* then one can prove local existence of a classical solution by mean of ***Nash-Moser*** iterative technique for obtaining maximal bounds for the modulus of the gardient of the velocity

$$V=\beta\ \ k,\nu\ \ :=\gamma\ \ \nu\ \ k^m,\ \ 0<m<2$$

In the case $0<m<1$ - problem corresponds to a fast diffusion problem

In the case $1<m<2$ - problem corresponds to a slow diffusion problem

K.Mikula,
D. Ševčovič, 2001

B.Andrews,
1999

S. Angenent, G.Sapiro,
A.Tannenbaum, 1998

Analysis of governing equations

Nontrivial tangential velocity functional

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

In the case the velocity vector contains a nontrivial tangential component α the resulting flow of planar curves **<u>does not depend</u>** on this tangential velocity.

However, presence of a nontrivial tangential velocity functional can prevent the numerically computed solution of from forming numerical singularities like e.g. collapsing of grid points or formation of the so-called swallow tails

K.Mikula,
D. Ševčovič, 1999

M.Kimura, 1997

K. Deckelnik, 1997

Analysis of governing equations

Governing equations with a nontrivial tangential velocity

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta$$

$$\partial_t \nu = \beta_k^{'} \partial_s^2 \nu + \beta_\nu^{'} k + \alpha k$$

$$\partial_t g = -g k \beta + \partial_u \alpha$$

$$k \;\; .,0 \; = k_0 \;\; . \;, \quad \nu \;\; .,0 \; = \nu_0 \;\; .$$

$$g \;\; .,0 \; = g_0 \;\; .$$

Governing equations for the curvature, tangential angle and local length element contains a nontrivial tangential velocity functional α

K.Mikula,
D. Ševčovič, 2001

Analysis of governing equations

Reasonable choice of a tangential velocity functional

$$\frac{g(u,t)}{L_t} = \frac{g(u,0)}{L_0}$$

Relative local length is preserved during evolution of a curve

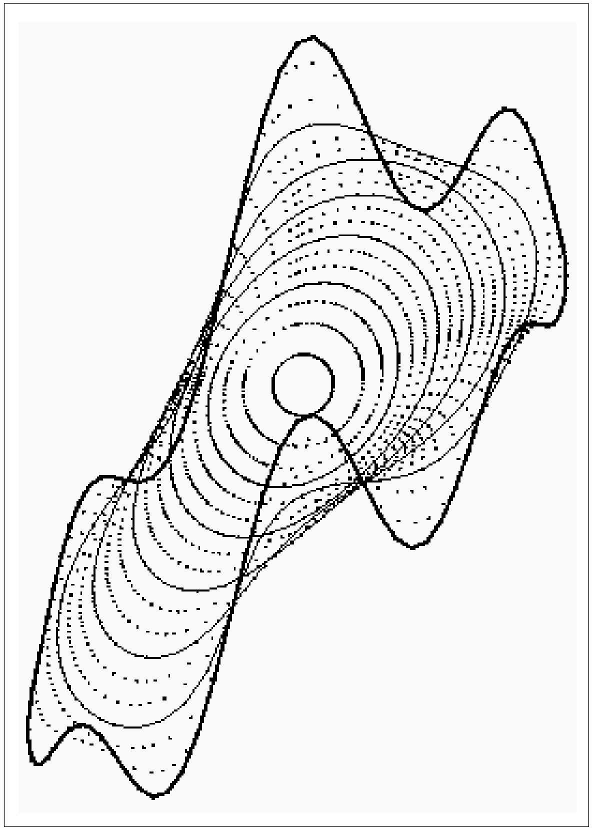
$$\partial_s \alpha = k\beta - \oint_\Gamma k\beta\, ds$$

This geometric requirement results to a contraint for α

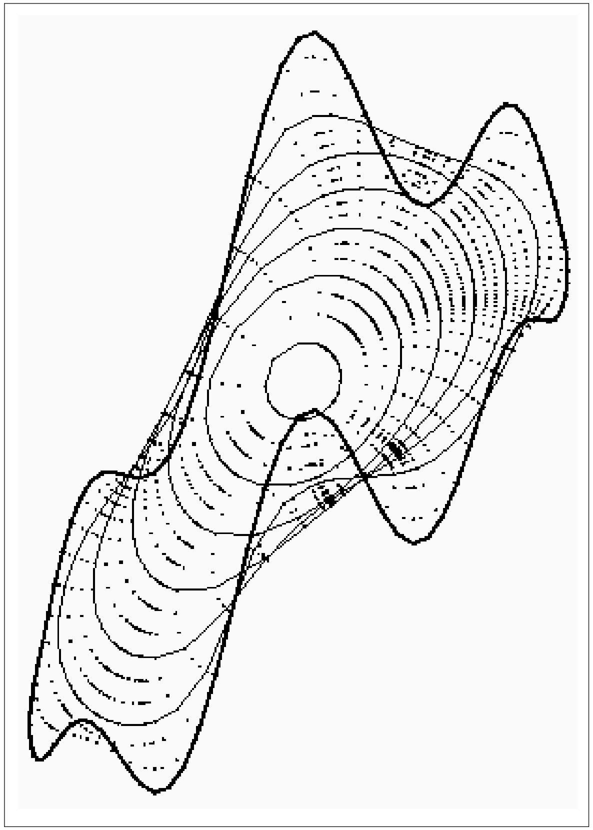
K.Mikula,
D. Ševčovič, 2001

Computational aspects

A role of a tangential velocity functional in computations



with tangential velocity

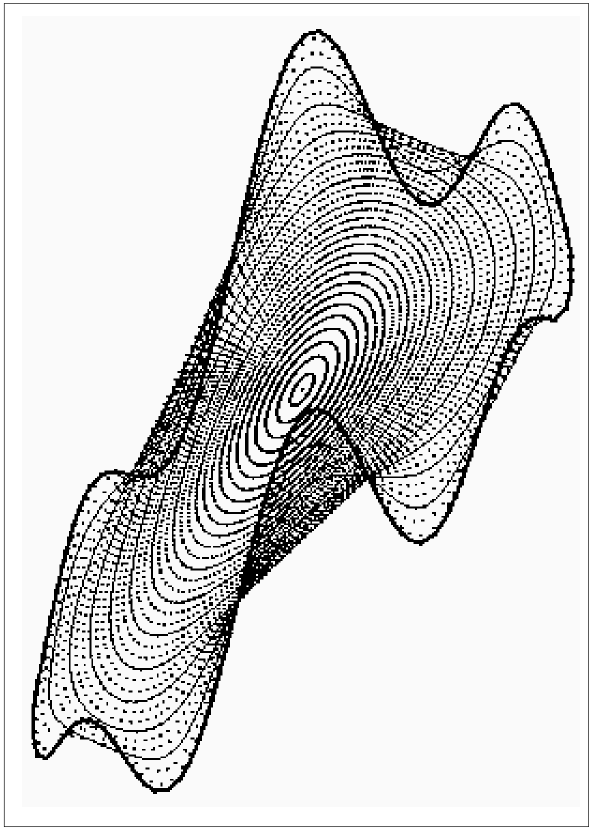


without tangential velocity

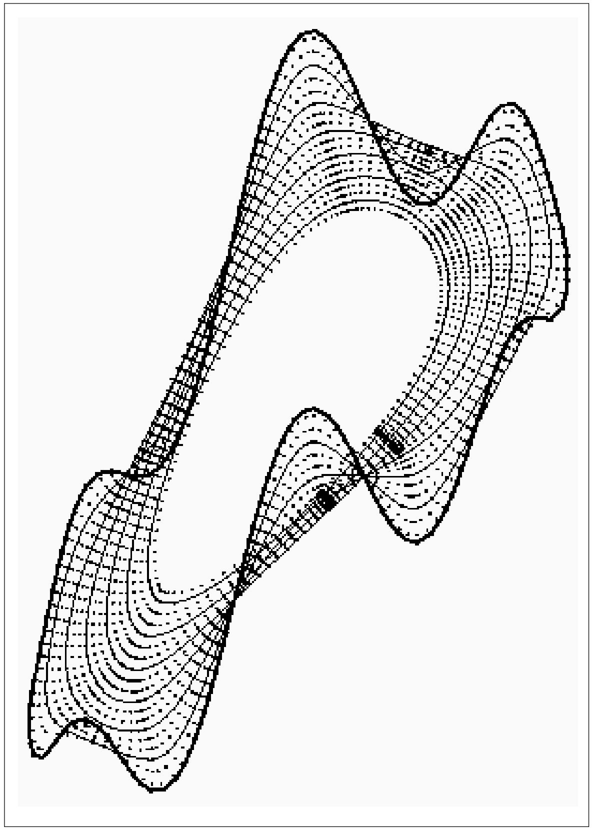
$$V = k$$

Computational aspects

A role of a tangential velocity functional in computations



with tangential velocity



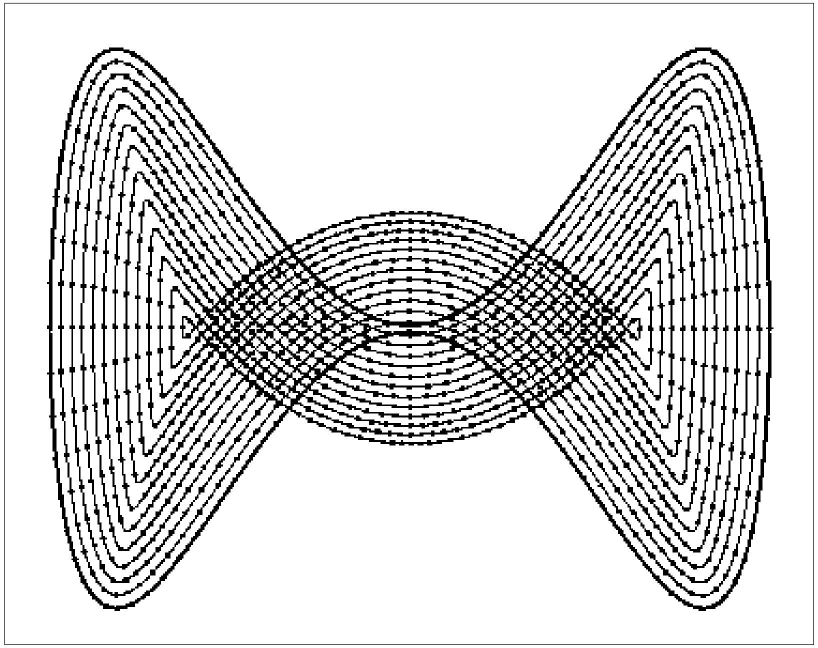
without tangential velocity

$$V = k^{1/3}$$

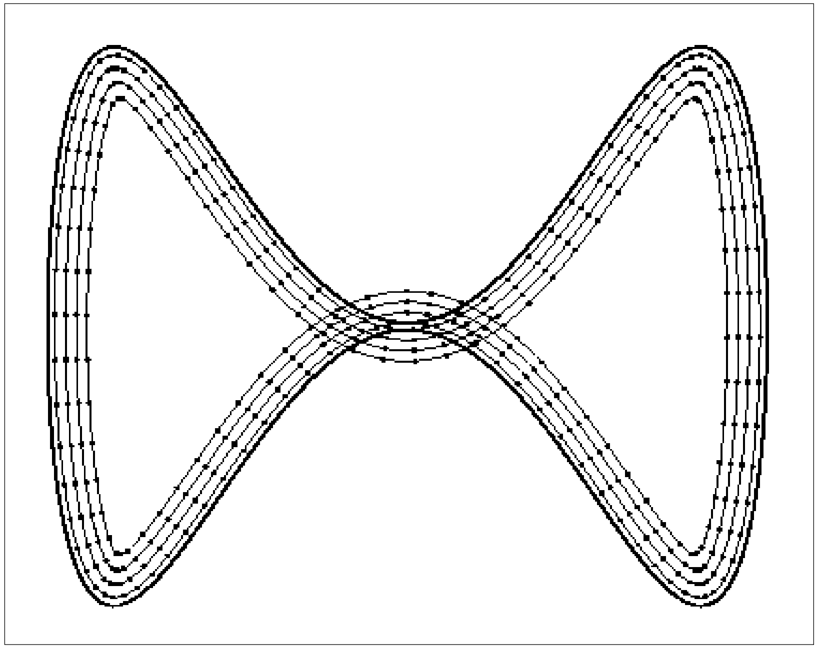
Computational aspects

A role of a tangential velocity functional in computations

$$V = k + 100$$



with tangential velocity



without tangential velocity

K. Mikula, D.Ševčovič, 2001

Conclusion

1. In many applied fields including, in particular, theory of phase interfaces, image processing, differential geometry. evolution of planar curves driven by curvature plays an important role

2. There are several different approaches for solving mean curvature and generalized mean curvature flows:
   - Level set methods, phase field equations approaches can handle mean curvature flow by pushing the problem into higher dimension.
   - Intrinsic heat equation (or direct) one space dimensional approach seems to be more promising, at least from numerical point of view

3. In numerical realization one has to take into account a role of a suitable tangential redistribution preventing thus numerically computed solution from forming various instabilitiest

The document and papers are available at: www.iam.fmph.uniba.sk/institute/sevcovic

Silhouettes ...

... and images