

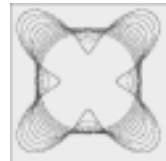
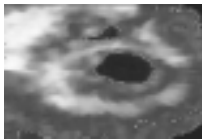
# Motion of planar curves: Geometric and analytical aspects



Daniel Ševčovič  
Habilitation lecture



June 25, 2001



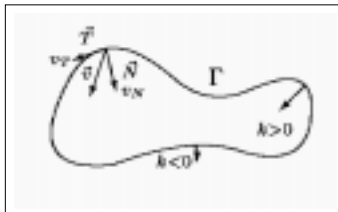
# Goals of the lecture

1. Motivation for studying mean curvature driven flows of planar curves
  - **Theory of phase interfaces** separating solid and liquid phases, Stefan problem, Gibbs-Thomson law governing the crystal growth.
  - **Image processing:** morphological image and shape multiscale analysis, analysis of image silhouettes, image segmentation.
  - **Differential geometry.** Evolution of planar curves driven by curvature.
2. Mathematical formulation and analysis of governing equations
  - **Methods for solving mean curvature flow.** Level set methods, phase field equations approach, intrinsic heat equation approach.
  - **Governing equations.** Formulation in the form of nonlinear parabolic partial differential equations. Qualitative analysis of solutions and geometric interpretation
3. Computational aspects and applications
  - **Level set method vs. direct approach.** Advantages of the direct method Computational results and important applications.

# Motivation

*Geometric equation*

$$V = \beta k, \nu$$



$V$  - normal velocity  
 $k$  - (mean) curvature  
 $\nu$  - tangential angle

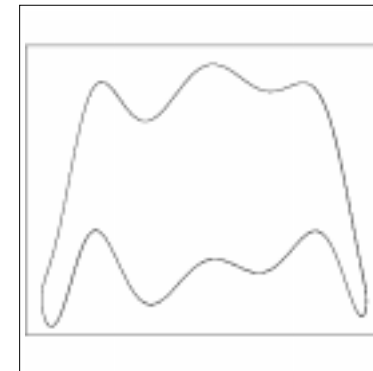
$\beta$  - function depending on the curvature and angle

# Motivation

*Example*

$$V = k$$

The mean curvature flow shrinks a planar Jordan curve to a circle rounded point in finite time.



K.Mikula, D.Ševčovič, 1999

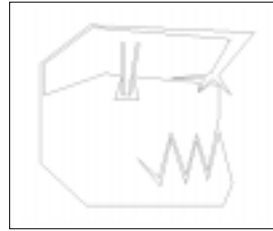
## Motivation

*Example*

$$V = k^{1/3}$$



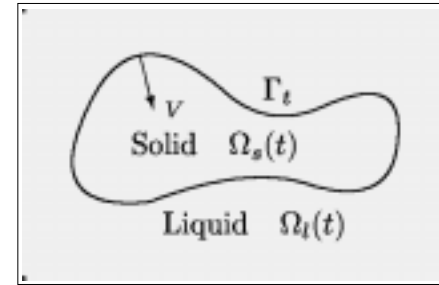
$$V = k^2$$



F.Cao, L.Moissan, 2000

## Motivation

*Stefan's theory of sharp phase interfaces*



$\Omega_s(t)$   
solid phase at time  $t$ ;

$\Omega_l(t)$   
liquid phase at time  $t$



Jozef Stefan, 1835-1893

## Motivation

*Stefan's theory of sharp phase interfaces*

$$\rho c \partial_t U = \lambda \Delta U$$

in  $\Omega_s(t)$  and  $\Omega_l(t)$

$$\lambda \partial_n U_s^l = -LV$$

$$\delta_\sigma^e U - U_m = -\gamma_2 \nu k + \gamma_1 \nu V \text{ on } \Gamma_t$$

- Heat equations in solid and liquid phases

- Stefan's condition

- Gibbs-Thompson law

on a free boundary  $\Gamma_t$

$\gamma_1$  is a coefficient of attachment kinetics

$\gamma_2$  describes anisotropy of the interface

## Motivation

*Gibbs - Thompson condition*

$$V = \gamma \nu k + f(x, \nu)$$



Josiah Willard Gibbs  
1839-1903



William Thomson  
Lord Kelvin, 1816-1900

*Gibbs - Thompson contact angle condition*

$$\mu \nu, V \quad V = \gamma \nu k + f$$

$\mu$  is the mobility coefficient

$$V = \beta(x, k, \nu)$$

$V$  - normal velocity

$k$  - (mean) curvature

$\nu$  - tangential angle

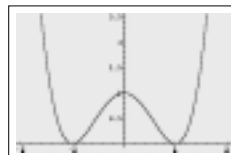
$x$  - position vector

## Motivation

### Allen - Cahn theory of phase interfaces

$$\partial_t U - \Delta U + \frac{1}{\epsilon^2} \Psi(U) = 0$$

$$\Psi(U) = \frac{\epsilon}{2} (1 - U^2)^2$$



Double well potential  $\Psi$

$0 < \epsilon \ll 1$  - thickness of the interface

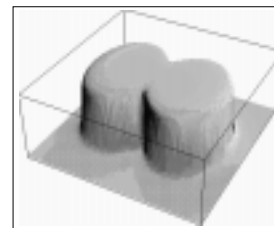
Liquid phase:  $U \sim 0$

Solid phase:  $U \sim 1$

S.Allen, J.Cahn, 1979

## Motivation

### Allen - Cahn theory of phase interfaces



Convergence as  $\epsilon \rightarrow 0$  of the Allen - Cahn equation to evolution by mean curvature has been studied by many authors (Brakke's motion by mean curvature)

$$V = k \quad \text{- Mean curvature evolution}$$

DeGiorgi 1990; Ilmanen 1993;  
Evans, Soner, Souganidis 1992

## Motivation

### Image processing

In the image processing the so-called morphological image and shape multiscale analysis is often used because of its contrast and affine invariance properties.

Analysis of image silhouettes (boundaries of distinguished shapes) leads to an equation of the form:

$$V = k^{1/3}$$

G.Sapiro, A.Tannenbaum, 1994

L.Alvarez, F.Guichard,  
P.Lions, J.Morel, 1993

## Motivation

### Image processing

In the morphological image multiscale analysis the so-called Perrona-Malik model is widely used. This analysis is represented by a viscosity solution to the following nonlinear degenerate parabolic equation in a two dimensional rectangular domain

$$\partial_t u = \nabla u \cdot \beta \operatorname{div} \frac{\nabla u}{|\nabla u|}$$

$$\beta k = k^{1/3}$$

Perrona - Malik model for 3D motion by mean curvature

K.Mikula, 1999

L.Alvarez, F.Guichard,  
P.Lions, J.Morel, 1993

## Motivation

### *Image processing*

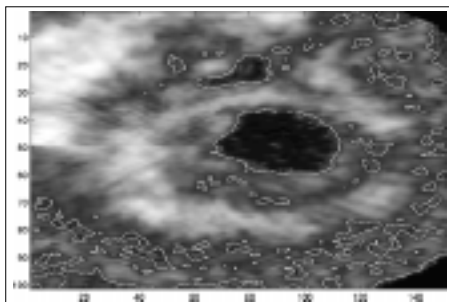
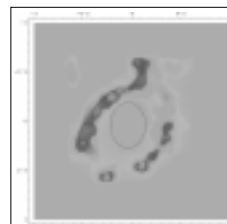


Image segmentation, pattern recognition in Echocardiography

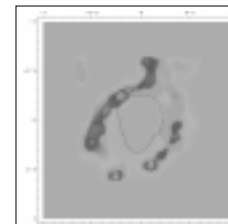
A.Sarti, K.Mikula, F.Sgallari, 2000

## Motivation

### *Image segmentation, pattern recognition*



Initial ellipse inside echocardiography



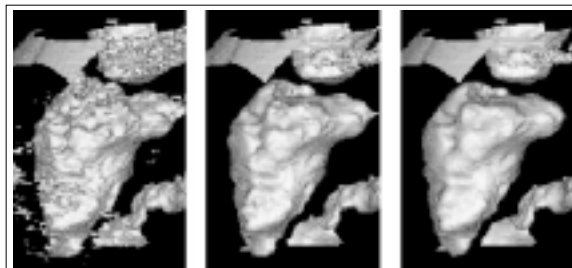
Pattern evolution inside echocardiography

$$V = \epsilon k - F$$

K.Mikula, D.Ševčovič, 2001 C.Lamberti, 2000

## Motivation

### *Image and video smoothing and filtering*



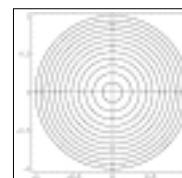
Original echocardiography coarse filtering fine filtering

Perrona - Malik model for 3+1D motion by mean curvature

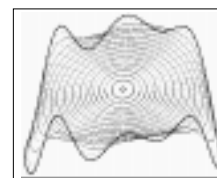
A.Sarti, K.Mikula, F.Sgallari, 2000; C.Lamberti, 2000

## Motivation

### *Differential geometry*



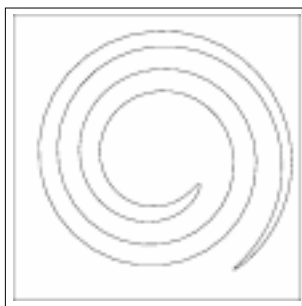
**Theorem** (M.Gage, R.S.Hamilton, 1986)  
*The mean curvature flow  $V = k$  shrinks any **convex** planar Jordan curve to a circular rounded point in finite time.*



**Theorem** (R.Grayson, 1987)  
*The mean curvature flow  $V = k$  shrinks **any** planar Jordan curve to a circular rounded point in finite time.*

## Motivation

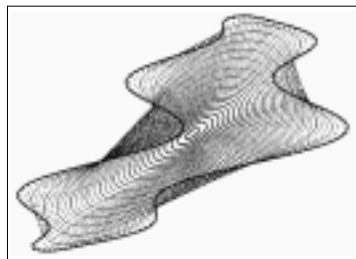
### Differential geometry



$$V = k^{1/3}$$

Affine invariant scaling

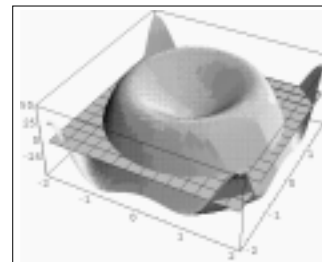
Ellipses are self-similar patterns



K.Mikula, D.Ševčovič, 1999

## Mathematical description

### Level set description of a mean curvature flow



$$\Gamma = x, \quad u|_{\Gamma} = 0$$

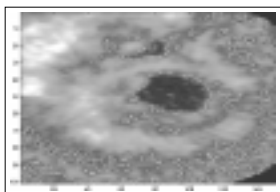
$$u: \mathbb{R}^2 \rightarrow \mathbb{R}$$

T.Ohta, D.Jasnow,  
K.Kawasaki, 1982

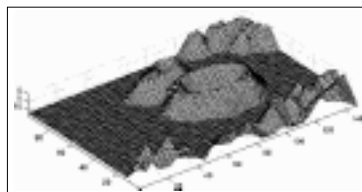
$$k_\Gamma = \operatorname{div} \frac{\nabla u}{|\nabla u|} \quad - \text{curvature of } \Gamma$$

## Mathematical description

### Level set description of a mean curvature flow



Density plot of image function



Surface plot of image function

$$u_0: \mathbb{R}^2 \rightarrow [0, 1]$$

Initial condition representing the image intensity function

A.Sarti, K.Mikula,  
F.Sgallari, 2000

## Mathematical description

### Level set description of a mean curvature flow

$$\partial_t u = \operatorname{tr} (I - \vec{N} \otimes \vec{N}) \nabla^2 u$$

$$u|_{\Gamma} = 0$$

$$\vec{N} = \frac{\nabla u}{|\nabla u|}$$

$$\Gamma_t = x, \quad u|_{\Gamma_t} = 0 \quad u: \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$$

L.Evans, H.Soner,  
P.Souganidis, 1992

S.Osher, J.Sethian, 1988

## Mathematical description

### Phase field description of a mean curvature flow

$$\partial_t u - \Delta u = \partial_t \Phi$$

$$\partial_t \Phi - \Delta \Phi + u - u_m \quad \nabla \Phi = \frac{1}{\epsilon^2} \Phi (1 - \Phi) \quad \Phi = 0.5$$

$0 < \epsilon \ll 1$  - thickness of the interface

$$\Gamma_t = x, \quad \Phi(x, t) = 0.5 \quad u, \Phi: R^2 \times [0, T] \rightarrow R$$

G.Caginalp, 1988

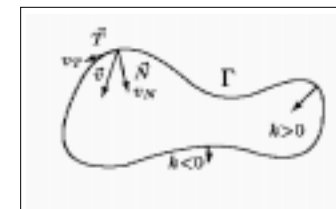
M.Beneš, K.Mikula, 1998

## Mathematical description

### Direct approach to the mean curvature flow

$$\partial_t x = k \vec{N}$$

$$x = x(u, t), \quad u \in S^1, t \in [0, T]$$



Position vector equation

Tangent vector -  $\partial_s x = \vec{T}$

Frenet's formula -  $\partial_s \vec{T} = k \vec{N}$

## Mathematical description

### Intrinsic heat equation

$$\partial_t x = \partial_s^2 x$$

- geometric intrinsic heat equation  
(in terms of the arc-length parameter  $s$ )

$$\partial_t x = \frac{1}{\partial_u x} \frac{\partial}{\partial u} \left( \frac{1}{\partial_u x} \frac{\partial x}{\partial u} \right)$$

- fixed domain Eulerian form  
of geometric heat equation

$$x(\cdot, 0) = x_0(\cdot)$$

- initial condition representing  
the initial curve

M.Gage, R.Hamilton, 1986

M.Grayson, 1987

## Mathematical description

### Intrinsic heat equation

$$V = \beta k$$

Normal velocity is a function of the  
curvature, e.g.  $\beta(k) = k^m, m > 0$ .

$$\partial_t x = \beta k \partial_s^2 x$$

$$x(\cdot, 0) = x_0(\cdot)$$

The curvature  $k$  itself depends  
on the second derivative of  $x$ .

$$k = \partial_s x \wedge \partial_s^2 x$$

The arc-length parameter  $s$   
depends on position vector  $x$ .

$$ds = \partial_u x \, du$$

## Mathematical description

### Curvature equation

$$V = \beta k$$

$$\partial_t k = \partial_s^2 \beta k + k^2 \beta k$$

$$k|_{t=0} = k_0$$

- heat equation for the curvature
- initial condition for curvature of an initial curve

M.Grayson, 1987

U.Abresh, J.Langer, 1986

## Mathematical description

### Curvature and local length equation

$$V = \beta k$$

$$\partial_t k = \partial_s^2 \beta k + k^2 \beta k$$

- heat equation for the curvature

$$\partial_t g = -g k \beta k$$

- ODE for local length element

$$ds = \partial_u x \quad du = g \quad du$$

$$k|_{t=0} = k_0, \quad g|_{t=0} = g_0$$

- initial conditions for curvature and local length

S.Angenent, 1987

## Mathematical description

### Curvature, tan. angle and local length eqs. $V = \beta k, \nu$

$$\partial_t k = \partial_s^2 \beta + k^2 \beta$$

- Heat equations for the curvature and tangent angle

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + \beta'_\nu k$$

- ODE for local length element

$$\partial_t g = -g k \beta$$

$$k|_{t=0} = k_0, \quad \nu|_{t=0} = \nu_0$$

- initial conditions for curvature, tangent angle and local length

$$g|_{t=0} = g_0$$

K.Mikula, D. Ševčovič, 2001

## Analysis of governing equations

### Local existence of a classical solution

Using the general theory due to Angenent one can prove local existence of a classical solution provided that the function  $\beta$  is regular in  $k$ .

$$k, \nu, g \in C[0, T], E_1 \cap C^1[0, T], E_0$$

where

$$E_k = C^{2k+\sigma} S^1 \times C^{2k+\sigma} S^1 \times C^{1+\sigma} S^1$$

K.Mikula, D. Ševčovič, 2001

S. Angenent, 1990

## Analysis of governing equations

### Local existence of a classical solution

In the case the velocity  $\beta$  is singular in  $k$  then one can prove local existence of a classical solution by mean of *Nash-Moser* iterative technique for obtaining maximal bounds for the modulus of the gradient of the velocity

$$V = \beta k, \nu := \gamma \nu k^m, \quad 0 < m < 2$$

In the case  $0 < m < 1$  - problem corresponds to a fast diffusion problem

In the case  $1 < m < 2$  - problem corresponds to a slow diffusion problem

K.Mikula,  
D. Ševčovič, 2001

B.Andrews,  
1999

S. Angenent, G.Sapiro,  
A.Tannenbaum, 1998

## Analysis of governing equations

### Nontrivial tangential velocity functional

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

In the case the velocity vector contains a nontrivial tangential component  $\alpha$  the resulting flow of planar curves **does not depend** on this tangential velocity.

However, presence of a nontrivial tangential velocity functional can prevent the numerically computed solution of from forming numerical singularities like e.g. collapsing of grid points or formation of the so-called swallow tails

K.Mikula,  
D. Ševčovič, 1999

M.Kimura, 1997

K. Deckelnik, 1997

## Analysis of governing equations

### Governing equations with a nontrivial tangential velocity

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta$$

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + \beta'_\nu k + \alpha k$$

$$\partial_t g = -g k \beta + \partial_u \alpha$$

$$k_{.,0} = k_0 ., \quad \nu_{.,0} = \nu_0 .$$

$$g_{.,0} = g_0 .$$

Governing equations for the curvature, tangential angle and local length element contains a nontrivial tangential velocity functional  $\alpha$

K.Mikula,  
D. Ševčovič, 2001

## Analysis of governing equations

### Reasonable choice of a tangential velocity functional

$$g_{u,t} = g_{u,0} \\ L_t = L_0$$

Relative local length is preserved during evolution of a curve

$$\partial_s \alpha = k \beta - \oint_\Gamma k \beta ds$$

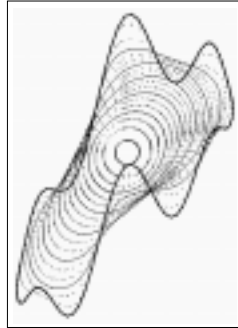
This geometric requirement results to a constraint for  $\alpha$

K.Mikula,  
D. Ševčovič, 2001

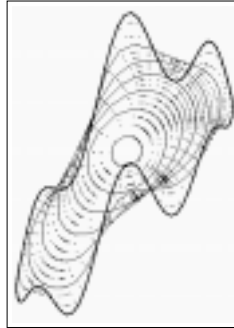


## Computational aspects

*A role of a tangential velocity functional in computations*



with tangential velocity

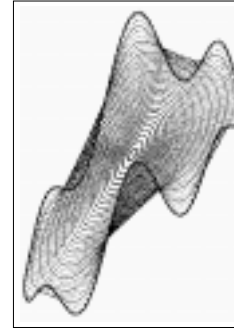


without tangential velocity

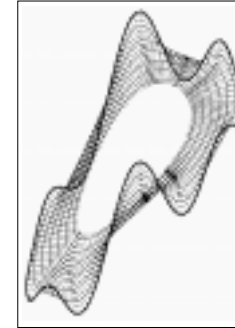
$$V = k$$

## Computational aspects

*A role of a tangential velocity functional in computations*



with tangential velocity



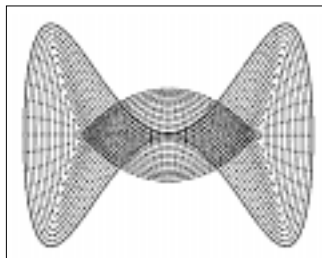
without tangential velocity

$$V = k^{1/3}$$

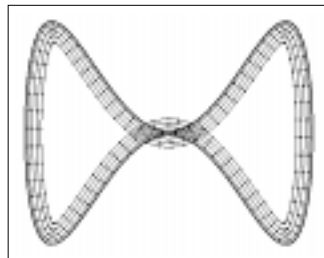
## Computational aspects

*A role of a tangential velocity functional in computations*

$$V = k + 100$$



with tangential velocity



without tangential velocity

K. Mikula, D.Ševčovič, 2001

## Conclusion

1. In many applied fields including, in particular, theory of phase interfaces, image processing, differential geometry, evolution of planar curves driven by curvature plays an important role
2. There are several different approaches for solving mean curvature and generalized mean curvature flows:
  - Level set methods, phase field equations approaches can handle mean curvature flow by pushing the problem into higher dimension.
  - Intrinsic heat equation (or direct) one space dimensional approach seems to be more promising, at least from numerical point of view
3. In numerical realization one has to take into account a role of a suitable tangential redistribution preventing thus numerically computed solution from forming various instabilities

The document and papers are available at: [www.iam.fmph.uniba.sk/institute/sevcovic](http://www.iam.fmph.uniba.sk/institute/sevcovic)

## Silhouettes ...



## ... and images

