Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

Lectures on analytical and numerical methods for pricing financial derivatives

Daniel Ševčovič

Vysokoškolský učebný text pre potreby zabezpečenia výučby predmetov Finančné deriváty a Analytické a numerické metódy oceňovania finančných derivátov

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- Explicit numerical scheme and the method of binomial trees
- Stable implicit numerical scheme using a linear algebra solver

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   Sensitivity analysis – dependence of the option price on parameters

   - Historical and implied volatility
   - Computation of implied volatility
   - Sensitivity with respect to parameters
   - Δ and Γ of an options. Other Greeks.

6. **Chapter**
   Path dependent exotic options – Asian and barrier options

   - Path dependent options, concepts and applications
   - Barrier options, formulation in terms of a solution to a partial differential equation on a time dependent domain
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   Pricing American type options – free boundary problems and numerical methods

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    - Density distribution function and the Fokker–Planck equation
    - Multidimensional extension of Itō's lemma

11. **References**

Analytical and numerical methods for pricing financial derivatives
Lectures on Computational Finance

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Lectures at Faculty of Mathematics, Physics and Informatics

Outline

1. Financial derivatives as tool for protecting volatile underlying assets
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9. Modeling of stochastic interest rates and interest rate derivatives
10. Appendix: Fokker–Planck equation and multidimensional Itō's lemma

- The content of these lectures is based on the textbooks:

1. D. Ševčovič, B. Stehlíková, K. Mikula:
   Analytical and numerical methods for pricing financial derivatives.
   Nova Science Publishers, Inc., Hauppauge, 2011. ISBN: 978-1-61728-780-0
2. D. Ševčovič, B. Stehlíková, K. Mikula:
   Analytické a numerické metódy oceňovania finančných derivátov,
   Nakladatelstvo STU, Bratislava 2009, ISBN 978-80-227-3014-3
3. P. Wilmott, J. Dewynne, J., S.D. Howison:
   Option Pricing: Mathematical Models and Computation,
   UK: Oxford Financial Press, 1995.
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   Options, Futures and Other Derivative Securities.
   Prentice Hall, 1989.

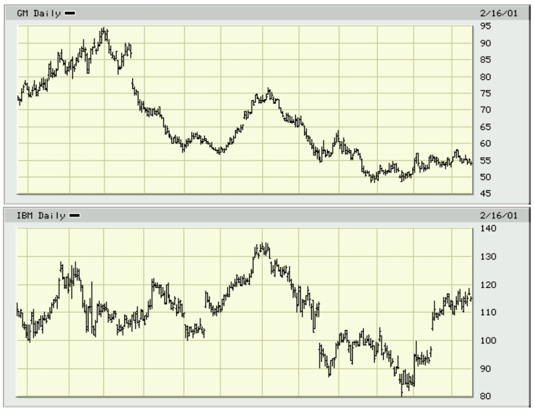
- The lecture slides are available for download from

`www.iam.fmph.uniba.sk/institute/sevcovic/derivaty`

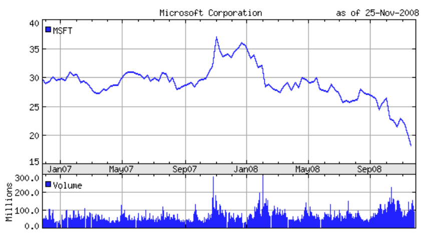
Chapter 1

- Stochastic character of assets (stocks, indices)
- Financial derivatives as tool for protecting volatile portfolios
- Examples of market data for Call and Put options

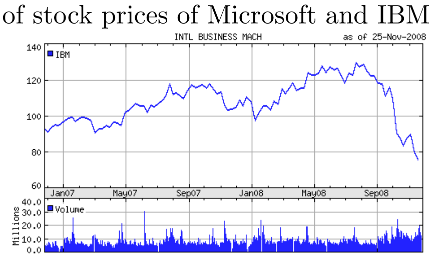
Stochastic character of stock prices



Daily behavior of stock prices of General Motors and IBM in 2001.



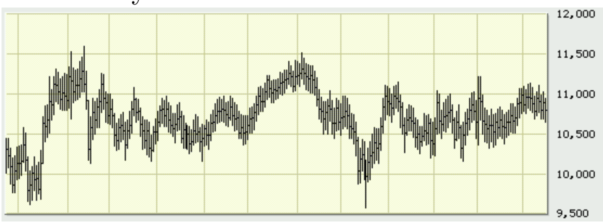
Daily behavior of stock prices of Microsoft and IBM in 2007 – 2008.



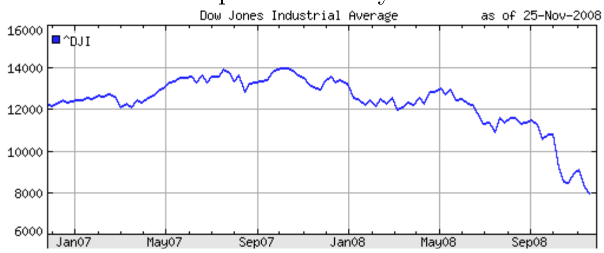
Volume of transactions is displayed in the bottom.

Stochastic character of indices

Daily behavior of Dow–Jones index



Precrisis period in the year 2000



Precrisis period 2007–2008.

- Forward
  is an agreement between a writer (issuer) and a holder representing the right and at the same time obligation to purchase assets at the specified time of maturity of a forward at predetermined price E

Pricing forwards is relatively simple as soon as we know the forward interest rate r measuring the rate of the decrease of the value of money

$$V_f = E \exp(-rT)$$

where E is the contracted expiration value of a forward at the expiration time T. Here V_f denotes the present value of a forward at the time when contract is signed

Financial derivatives as a tool for protecting volatile portfolios

- Option (Call option)
  is an agreement between a writer (issuer) and a holder representing the right BUT NOT the obligation to purchase assets at the prescribed exercise price E at the specified time of maturity T in the future

Pricing options is more involved as their price depends on:

$$V_c = \text{function of} \quad E, T, r, ..., ???$$

where E is the contracted expiration value of an options at the expiration time T, V_c is the present value of a Call option at the time when the contract is signed.

Call options							
Symbol	Last	Change	Bid	Ask	Volume	Open Int	Strike Price
MQFLE.X	15.20	0.00	15.10	15.20	42	34	5.00
MQFLB.X	10.15	0.00	10.10	10.20	74	2541	10.00
MQFLM.X	7.20	0.00	7.15	7.25	95	187	13.00
MQFLN.X	6.15	0.00	6.15	6.25	55	211	14.00
MQFLC.X	5.06	0.11	5.20	5.30	11	1348	15.00
MQFLO.X	4.35	0.00	4.25	4.35	263	368	16.00
MQFLQ.X	3.40	0.00	3.30	3.40	122	4157	17.00
MQFLS.X	1.83	0.05	1.89	1.92	36	7567	19.00
MQFLU.X	1.28	0.02	1.27	1.29	56	8886	20.00
MQFLU.X	0.78	0.09	0.75	0.78	105	72937	21.00
MSQLN.X	0.40	0.04	0.41	0.43	350	16913	22.00
MSQLQ.X	0.21	0.01	0.20	0.22	125	20801	23.00
MSQLD.X	0.09	0.02	0.09	0.11	92	12207	24.00
MSQLE.X	0.04	0.02	0.04	0.05	165	14193	25.00
MSQLR.X	0.02	0.00	0.02	0.03	161	9359	26.00
MSQLS.X	0.02	0.00	N/A	0.03	224	3643	27.00
MSQLT.X	0.02	0.00	N/A	0.02	59	2938	28.00
MSQLF.X	0.01	0.00	N/A	0.02	10	1330	30.00

Prices of Call options with different exercise (strike) prices E for Microsoft stocks from 26. 11. 2008. with expiration 8.12.2008.
The spot price $S = 20.12$
The Call option price $V_C \approx 1.28 > S - E = 20.12 - 20 = 0.12$

Microsoft Corporation (MSFT) At 9:41AM ET: **20.12** ↑ 0.13 (0.65%)

Options

View By Expiration: **Dec 08** | Jan 09 | Apr 09 | Jul 09 | Jan 10 | Jan 11

Options Expiring Fri, Dec 19, 2008

Calls							Strike Price	Puts						
Symbol	Last	Change	Bid	Ask	Volume	Open Int		Symbol	Last	Change	Bid	Ask	Volume	Open Int
MQFLE.X	**15.20**	**0.00**	15.10	15.20	42	34	**5.00**	MQFXE.X	N/A	**0.00**	N/A	N/A	0	0
MQFLB.X	**10.15**	**0.00**	10.10	10.20	74	2,541	**10.00**	MQFXB.X	**0.03**	**0.00**	0.02	0.04	97	3,473
MQFLM.X	**7.20**	**0.00**	7.15	7.25	95	187	**13.00**	MQFXM.X	**0.07**	**0.00**	0.05	0.07	459	2,994
MQFLN.X	**6.15**	**0.00**	6.15	6.25	55	211	**14.00**	MQFXN.X	**0.10**	**0.00**	0.07	0.10	204	2,147
MQFLC.X	**5.06**	↑ **0.11**	5.20	5.30	11	1,348	**15.00**	MQFXC.X	**0.14**	**0.00**	0.13	0.14	5	8,183
MQFLO.X	**4.35**	**0.00**	4.25	4.35	263	368	**16.00**	MQFXO.X	**0.20**	↓ **0.02**	0.19	0.21	2	337
MQFLQ.X	**3.40**	**0.00**	3.30	3.40	122	4,157	**17.00**	MQFXQ.X	**0.32**	↓ **0.02**	0.33	0.34	11	8,395
MQFLS.X	**1.83**	↓ **0.05**	1.89	1.92	36	7,567	**19.00**	MQFXS.X	**0.83**	↑ **0.06**	0.77	0.80	169	31,116
MQFLD.X	**1.28**	↓ **0.02**	1.27	1.29	56	8,886	**20.00**	MQFXD.X	**1.14**	↓ **0.06**	1.13	1.16	109	23,562
MQFLU.X	**0.78**	↓ **0.09**	0.75	0.78	105	72,937	**21.00**	MQFXU.X	**1.83**	↑ **0.23**	1.65	1.68	1	72,472
MSQLN.X	**0.40**	↓ **0.04**	0.41	0.43	350	16,913	**22.00**	MSQXN.X	**2.58**	↑ **0.23**	2.30	2.36	3	4,495
MSQLQ.X	**0.21**	↓ **0.01**	0.20	0.22	125	20,801	**23.00**	MSQXQ.X	**3.10**	**0.00**	3.05	3.15	30	3,840
MSQLD.X	**0.09**	↓ **0.02**	0.09	0.11	92	12,207	**24.00**	MSQXD.X	**3.80**	**0.00**	3.95	4.05	167	3,871
MSQLE.X	**0.04**	↓ **0.02**	0.04	0.05	165	14,193	**25.00**	MSQXE.X	**4.90**	**0.00**	4.85	4.95	157	2,075
MSQLR.X	**0.02**	**0.00**	0.02	0.03	161	9,359	**26.00**	MSQXR.X	**6.15**	**0.00**	5.85	5.95	210	1,795
MSQLS.X	**0.02**	**0.00**	N/A	0.03	224	3,643	**27.00**	MSQXS.X	**7.00**	**0.00**	6.85	6.95	45	1,156
MSQLT.X	**0.02**	**0.00**	N/A	0.02	59	2,938	**28.00**	MSQXT.X	**7.55**	**0.00**	7.80	7.95	24	874
MSQLF.X	**0.01**	**0.00**	N/A	0.02	10	1,330	**30.00**	MSQXF.X	**10.54**	**0.00**	9.85	10.00	26	124

Highlighted options are in-the-money.

Intraday behavior (Feb. 7, 2011) of MSFT (Microsoft Inc.) stock.
Source: Chicago Board Options Exchange: www.cboe.com

Name (MSFT) MICROSOFT CORP. Last $28.20 Change 0.43(1.54%) Options Chain Add to Portfolio

Chain Type: Calls and Puts | Chain Type: All | Expiration: Jul 2011 | View Chain

Calls								Puts					
Contract Name	Last Trade	Change	Bid	Ask	Volume	Interest	Strike Price	Contract Name	Last Trade	Change	Bid	Ask	Volume
MSFT\11G16\15.0	0.00	0.00	0.00	0.00	0	0	15.00	MSFT\11S16\15.0	0.00	0.00	0.00	0.00	0
MSFT\11G16\17.5	0.00	0.00	0.00	0.00	0	0	17.50	MSFT\11S16\17.5	0.00	0.00	0.00	0.00	0
MSFT\11G16\20.0	0.00	0.00	0.00	0.00	0	0	20.00	MSFT\11S16\20.0	0.00	0.00	0.00	0.00	0
MSFT\11G16\22.0	0.00	0.00	0.00	0.00	0	0	22.00	MSFT\11S16\22.0	0.00	0.00	0.00	0.00	0
MSFT\11G16\23.0	0.00	0.00	0.00	0.00	0	0	23.00	MSFT\11S16\23.0	0.00	0.00	0.00	0.00	0
MSFT\11G16\24.0	4.46	0.46	4.40	4.45	20	785	24.00	MSFT\11S16\24.0	0.40	-0.11	0.41	0.43	5
MSFT\11G16\25.0	3.50	0.16	3.55	3.60	14	3,851	25.00	MSFT\11S16\25.0	0.57	-0.15	0.57	0.59	249
MSFT\11G16\26.0	2.84	0.30	2.77	2.80	43	3,283	26.00	MSFT\11S16\26.0	0.80	-0.20	0.80	0.82	4
MSFT\11G16\27.0	2.13	0.22	2.08	2.11	266	127,259	27.00	MSFT\11S16\27.0	1.14	-0.23	1.12	1.14	138
MSFT\11G16\28.0	1.55	0.19	1.50	1.52	1,370	20,288	28.00	MSFT\11S16\28.0	1.54	-0.29	1.54	1.57	40
MSFT\11G16\29.0	1.08	0.16	1.03	1.05	276	19,675	29.00	MSFT\11S16\29.0	2.45	0.14	2.08	2.11	239
MSFT\11G16\30.0	0.70	0.10	0.68	0.70	271	39,363	30.00	MSFT\11S16\30.0	2.66	-0.34	2.73	2.76	5
MSFT\11G16\31.0	0.45	0.07	0.43	0.45	301	9,627	31.00	MSFT\11S16\31.0	3.55	-0.20	3.45	3.55	3
MSFT\11G16\32.0	0.29	0.06	0.26	0.28	734	4,502	32.00	MSFT\11S16\32.0	5.00	0.32	4.30	4.35	90
MSFT\11G16\33.0	0.16	0.01	0.15	0.17	100	2,098	33.00	MSFT\11S16\33.0	4.80	0.05	5.15	5.25	30
MSFT\11G16\35.0	0.06	0.00	0.05	0.06	10	9,857	35.00	MSFT\11S16\35.0	6.65	0.05	7.05	7.15	70
MSFT\11G16\40.0	0.00	0.00	0.00	0.00	0	0	40.00	MSFT\11S16\40.0	0.00	0.00	0.00	0.00	0

Call and Put option prices from Feb. 7, 2011, on MSFT (Microsoft Inc.) stock with expiration July 2011 for various exercise (strike) prices E.

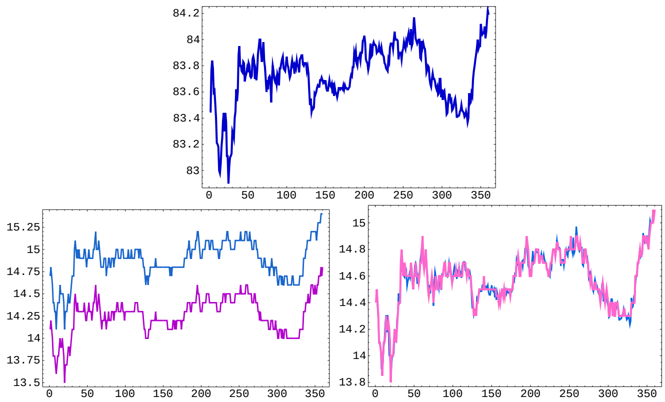


Figure: Top: Stock prices of IBM from 22. 5. 2002. Bottom: Bid and Ask prices of Call option for IBM stocks (left) and their arithmetic average value (right).

Financial derivatives as a tool for protecting volatile portfolios

- A natural question arises:
  Although the time evolution of the asset price S_t as well as its derivative (option) V_t is stochastic (volatile, unpredictable) CAN WE FIND A FUNCTIONAL DEPENDENCE
  $$V_t = V(S_t, t)$$
  relating the actual stock price S_t at time t and the price of its derivative (like e.g. a Call option) V_t?

- This was a long standing problem in financial mathematics until 1972. The answer is YES due to the pioneering work of M.Scholes, F.Black and R.Merton.
- M. Scholes and R. Merton were awarded the Price of the Swedish Bank for Economy in the memory of A. Nobel in 1997 (Nobel price for Economy).

Financial derivatives as a tool for protecting volatile portfolios

- The Black–Scholes formula

$$V = V(S, t; T, E, r, \sigma)$$

where $S = S_t$ is the spot (actual) price of an underlying asset, $V = V_t$ is a the spot price of the option (Call or put) at time $0 \le t \le T$. Here T is the time of maturity, E is the exercise price, $r > 0$ is the interest rate of a secure bond, $\sigma > 0$ is the volatility of underlying stochastic process of the asset price S_t.

Chapter 2

- Stochastic differential calculus
- Wiener process, Brownian and geometric Brownian motion
- Itō's lemma, Itō's integral

Stochastic differential calculus, Itō's lemma

- Stochastic process is a t - parametric system of random variables $\{X(t), t \in I\}$, where I is an interval or a discrete set of indices
- Stochastic process $\{X(t), t \in I\}$ is a Markov process with the property: given a value $X(s)$, the subsequent values $X(t)$ for $t > s$ may depend on $X(s)$ but not on preceding values $X(u)$ for $u < s$. More precisely,

  If $t \geq s$, then for conditional probabilities we have:

$$P(X(t) < x | X(s)) = P(X(t) < x | X(s), X(u))$$

  for any $u \leq s$.

- a stochastic process $\{X(t), t \geq 0\}$ is called the Brownian motion if
  - i) all increments $X(t + \Delta) - X(t)$ are normally distributed with the mean value $\mu\Delta$ and dispersion (or variance) $\sigma^2\Delta$,
  - ii) for any division of times $t_0 = 0 < t_1 < t_2 < t_3 < ... < t_n$ the increments $X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$ are independent random variables
  - iii) $X(0) = 0$ and sample pathes are continuous almost surely
- Brownian motion $\{W(t), t \geq 0\}$ with the mean $\mu = 0$ and dispersion $\sigma^2 = 1$ is called Wiener process

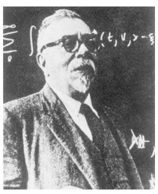



Figure: Norbert Wiener (1884-1964) and Robert Brown (1773-1858).

Stochastic differential calculus, Itō's lemma

- Additive (or semigroup) property of the Brownian motion (BM) $\{X(t), t \geq 0\}$ – Mean value

let $0 = t_0 < t_1 < ... < t_n = t$ be any division of the interval $[0, t]$. Then

$$X(t) - X(0) = \sum_{i=1}^{n} X(t_i) - X(t_{i-1}).$$

Therefore the mean value E and variance Var of the left and right hand side have to be equal. By definition of the BM we have

$$\mathbb{E}(X(t) - X(0)) = \mu(t - 0) = \mu t\,.$$

On the other side we have (due to the linearity of the mean value operator):

$\mathbb{E}\left(\sum_{i=1}^{n} X(t_i) - X(t_{i-1})\right) = \sum_{i=1}^{n} \mathbb{E}(X(t_i) - X(t_{i-1})) = \sum_{i=1}^{n} \mu(t_i - t_{i-1}) = \mu t$

- In order to verify the equality we had to require that increments $X(t_i) - X(t_{i-1})$ have the mean value $\mathbb{E}(X(t_i) - X(t_{i-1})) = \mu(t_i - t_{i-1})$

- Additive (or semigroup) property of the Brownian motion $\{X(t), t \geq 0\}$ – Variance

For dispersions of the random variables $X(t) - X(0)$ and $\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))$ we have, by definition,

$$Var(X(t) - X(0)) = \sigma^2(t - 0) = \sigma^2 t\,.$$

ReCall that for two random independent variables A, B it holds: $Var(A + B) = Var(A) + Var(B)$. Hence, assuming independence of increments $X(t_i) - X(t_{i-1})$ for different $i = 1, 2, ..., n$ we have

$Var\left(\sum_{i=1}^{n} X(t_i) - X(t_{i-1})\right) = \sum_{i=1}^{n} Var(X(t_i) - X(t_{i-1})) = \sum_{i=1}^{n} \sigma^2(t_i - t_{i-1}) = \sigma^2 t\,.$

- In order to verify the equality we had to require that increments $X(t_i) - X(t_{i-1})$ have the dispersion (variance) $V(X(t_i) - X(t_{i-1})) = \sigma^2(t_i - t_{i-1})$

Stochastic differential calculus, Itō's lemma

In summary:

- The Brownian motion $\{X(t), t \geq 0\}$ has the following stochastic distribution:

$$X(t) \sim N(\mu t, \sigma^2 t)$$

where $N(mean, variance)$ stands for a normal random variable with given mean and variance

- The Wiener process $\{W(t), t \geq 0\}$ (here $\mu = 0, \sigma^2 = 1$) has the following stochastic distribution:

$$W(t) \sim N(0, t).$$

Moreover, $dW(t) := W(t + dt) - W(t) \sim N(0, dt)$, i.e.

$$dW(t) := W(t + dt) - W(t) = \Phi\sqrt{dt}$$

where $\Phi \sim N(0, 1)$.

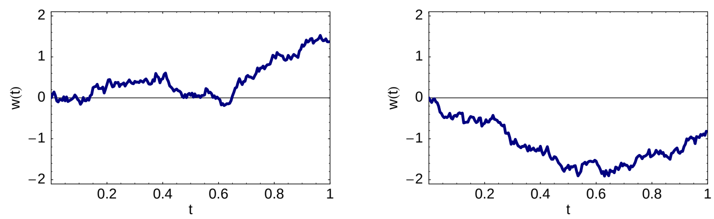


Figure: Two randomly generated samples of a Wiener process.

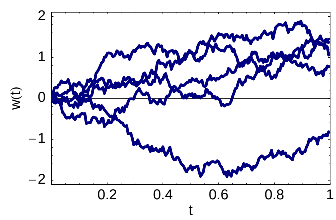


Figure: Five random realizations of a Wiener process.

Stochastic differential calculus, Itō's lemma

Since $W(t) \sim N(0,t)$ we have $Var(W(t)) = t$.

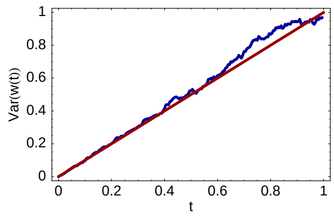


Figure: Time dependence of the variance $Var(W(t))$ for 1000 random realizations of a Wiener process $\{W(t), t \geq 0\}$.

Relation between Brownian and Wiener process:

- For a Brownian motion $\{X(t), t \geq 0\}$ with parameters μ and σ we have, by definition, $dX(t) = X(t+dt) - X(t) \sim N(\mu dt, \sigma^2 dt)$ Therefore, if we construct the process

$$W(t) = \frac{X(t) - \mu t}{\sigma}$$

we have

$$dW(t) = W(t+dt) - W(t) = \frac{dX(t) - \mu dt}{\sigma} \sim N(0, dt),$$

i.e. $\{W(t), t \geq 0\}$ is a Wiener process

Since $X(t) = \mu t + \sigma W(t)$ we may therefore write a Stochastic differential equation

$$dX(t) = \mu dt + \sigma dW(t),$$

Stochastic differential calculus, Itō's lemma

- Geometric Brownian motion

If $\{X(t), t \geq 0\}$ is a Brownian motion with parameters μ and σ we define a new stochastic process $\{Y(t), t \geq 0\}$ by taking

$$Y(t) = y_0 \exp(X(t))$$

where y_0 is a given constant. The process $\{Y(t), t \geq 0\}$ is called the Geometric Brownian motion.

- Statistical properties of the Geometric Brownian motion
- For simplicity, let us take $y_0 = 1$. Then

$$W(t) = \frac{\ln Y(t) - \mu t}{\sigma}$$

  is a Wiener process with $W(t) \sim N(0, t)$, i.e. we know its distribution function.

- Statistical properties of the Geometric Brownian motion:

For the distribution function $G(y, t) = P(Y(t) < y)$ it holds: $G(y, t) = 0$ for $y \leq 0$ (since $Y(t)$ is a positive random variable) and for $y > 0$

$$G(y,t) = P(Y(t) < y) = P\left(W(t) < \frac{-\mu t + \ln y}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{-\mu t + \ln y}{\sigma}} e^{-\xi^2/2t} d\xi.$$

Stochastic differential calculus, Itō's lemma

- Statistical properties of the Geometric Brownian motion:

Since $\mathbb{E}(Y(t)) = \int_{-\infty}^{\infty} yg(y,t)\,dy$ and $\mathbb{E}(Y(t)^2) = \int_{-\infty}^{\infty} y^2 g(y,t)\,dy$, where $g(y,t) = \frac{\partial}{\partial y}G(y,t)$, we can calculate

$$
\begin{aligned}
\mathbb{E}(Y(t)) \quad &= \int_{-\infty}^{\infty} yg(y,t)\,dy = \int_{0}^{\infty} yg(y,t)\,dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^\infty y e^{-\frac{(-\mu t+\ln y)^2}{2\sigma^2 t}} \frac{1}{\sigma y}\,dy \\
&\quad (\xi = (-\mu t + \ln y)/(\sigma\sqrt{t})) \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}+\sigma\sqrt{t}\xi}\,d\xi = \frac{e^{\mu t+\frac{\sigma^2}{2}t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\xi-\sigma\sqrt{t})^2}{2}}\,d\xi \\
&= e^{\mu t + \frac{\sigma^2}{2}t}.
\end{aligned}
$$

- Naive (and also wrong) formal calculation

Since $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$ we have

$$dY(t) = (\exp(X(t)))' dX(t) = \exp(X(t)) dX(t)$$

and therefore

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dW(t).$$

Hence by taking the mean value operator operator $\mathbb{E}(.)$ (it is a linear operator) we obtain

$$d\mathbb{E}(Y(t)) = \mathbb{E}(dY(t)) = \mu\mathbb{E}(Y(t))dt + \sigma\mathbb{E}(Y(t)dW(t)) = \mu\mathbb{E}(Y(t))dt$$

as the random variables $Y(t)$ and $dW(t)$ are independent and $\mathbb{E}(dW(t)) = 0$. Solving the differential equation $\frac{d}{dt}\mathbb{E}(Y(t)) = \mu\mathbb{E}(Y(t))$ yields

$$\mathbb{E}(Y(t)) = \exp(\mu t)$$

BUT according to our previous calculus $\mathbb{E}(Y(t)) = \exp(\mu t + \frac{\sigma^2}{2}t)$. Where is the mistake?

Stochastic differential calculus, Itō's lemma

- The correct answer is based on the famous Itō's lemma
- We cannot omit stochastic character of the process $\{X(t), t \geq 0\}$ when taking the differential of the COMPOSITE function $\exp(X(t))$!!!

Itō lemma

Let $f(x,t)$ be a C^2 smooth function of x, t variables. Suppose that the process $\{x(t), t \geq 0\}$ satisfies SDE:

$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$

Then the first differential of the process $f = f(x(t), t)$ is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt\,,$$

伊藤 清

Figure: Kiyoshi Itō (1915–2008).

- According to Wikipedia Itō's lemma is the most famous lemma in the world (citation 2009).

Stochastic differential calculus, Itō's lemma

- Meaning of the stochastic differential equation

$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$

  in the sense of Itō.
- Take a time discretization $0 < t_1 < t_2 < ... < t_n$. The above SDE is meant in the sense of a limit in probability when the norm $\nu = \max_i |t_{i+1} - t_i|$ of explicit in time discretization:

$$x(t_{i+1}) - x(t_i) = \mu(x(t_i), t_i)(t_{i+1} - t_i) + \sigma(x(t_i), t_i)(W(t_{i+1}) - W(t_i))$$

  tends to zero ($\nu \to 0$).
- Random variables $x(t_i)$ and $W(t_{i+1}) - W(t_i)$ are independent so does $\sigma(x(t_i), t_i)$ and $W(t_{i+1}) - W(t_i)$. Hence

$$\mathbb{E}(\sigma(x(t_i), t_i)(W(t_{i+1}) - W(t_i))) = 0$$

  because $\mathbb{E}(W(t_{i+1}) - W(t_i)) = 0$.

Intuitive (and not so rigorous) proof of Itō's lemma is based on Taylor series expansion of $f = f(x,t)$ of th 2nd order

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dx\,dt + \frac{\partial^2 f}{\partial t^2}(dt)^2\right) + \text{h.o.t.}$$

ReCall that $dw = \Phi\sqrt{dt}$, where $\Phi \approx N(0,1)$,

$$(dx)^2 = \sigma^2(dw)^2 + 2\mu\sigma dw\,dt + \mu^2(dt)^2 \approx \sigma^2 dt + O((dt)^{3/2}) + O((dt)^2)$$

because $\mathbb{E}(\Phi^2) = 1$ (dispersion of Φ is 1).
Analogously, the term $dx\,dt = O((dt)^{3/2}) + O((dt)^2)$ as $dt \to 0$.
Thus the differential df in the lowest order terms dt and dx can be expressed:

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt\,.$$

Stochastic differential calculus, Itō's lemma

- Example: Geometric Brownian motion
- $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$.
  Here $f(x,t) \equiv e^x$ and $Y(t) = f(X(t), t)$. Therefore

$$dY(t) = df = \frac{\partial f}{\partial x} dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt\,.$$

$$= e^{X(t)} dX(t) + \frac{1}{2}\sigma^2 e^{X(t)} dt = (\mu + \frac{1}{2}\sigma^2) Y(t) dt + \sigma Y(t) dW(t)$$

- As a consequence, we have for the mean value $\mathbb{E}(Y(t))$

$$d\mathbb{E}(Y(t)) = (\mu + \frac{1}{2}\sigma^2)\mathbb{E}(Y(t)) dt$$

  and so $\mathbb{E}(Y(t)) = e^{\mu t + \frac{1}{2}\sigma^2 t}$ provided that $Y(0) = 1$.

- Example: Dispersion of the Geometric Brownian motion
- $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$.
- Compute $\mathbb{E}(Y(t)^2)$. Solution: set
  $f(x,t) \equiv (e^x)^2 = e^{2x}$.Then

$$dY(t)^2 = df = \frac{\partial f}{\partial x} dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt\,.$$

$$= 2e^{2X(t)} dX(t) + \frac{1}{2}\sigma^2 4 e^{2X(t)} dt = 2(\mu + \sigma^2) Y(t)^2 dt + 2\sigma Y(t)^2 dW(t)$$

- As a consequence, for the mean value $\mathbb{E}(Y(t)^2)$ we have

$$d\mathbb{E}(Y(t)^2) = 2(\mu + \sigma^2)\mathbb{E}(Y(t)^2) dt$$

  and so $\mathbb{E}(Y(t)^2) = e^{2\mu t + 2\sigma^2 t}$. Hence

$$Var(Y(t)) = \mathbb{E}(Y(t)^2) - (\mathbb{E}(Y(t))^2 = e^{2\mu t + 2\sigma^2 t}(1 - e^{-\sigma^2 t}).$$

Black–Scholes model for pricing financial derivatives

Chapter 3

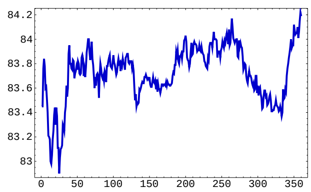
- Pricing European type of options - the Black–Scholes model
- Explicit solutions for European Call and Put options
- Put – Call parity
- Complex option strategies – straddles, butterfly

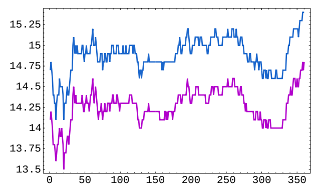
- Derivation of the Black–Scholes partial differential equation
- the case of Call (or Put) option
- Call option is an agreement (contract) between the writer (issuer) and the holder of an option. It represents the right BUT NOT the obligation to purchase assets at the prescribed exercise price E at the specified time of maturity $t = T$ in the future.
- The question is: What is the price of such an option (option premium) at the time $t = 0$ of contracting. In other words, how much money should the holder of the option pay the writer for such a derivative security

Black–Scholes model for pricing financial derivatives

Denote

- S - the underlying asset price
- V - the price of a financial derivative (a Call option)
- T - expiration time (time of maturity) of the option contract

S 

V 

Stock prices of IBM (2002/5/2)

Bid and Ask prices of a Call option

Idea

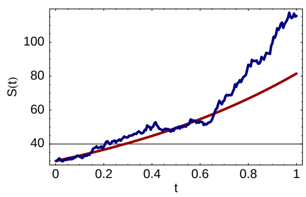
- Construct the price V as a function of S and time $t \in [0, T]$, i.e. $V = V(S, t)$

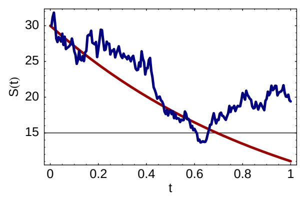
Assumption:

- the underlying asset price follows geometric Brownian motion

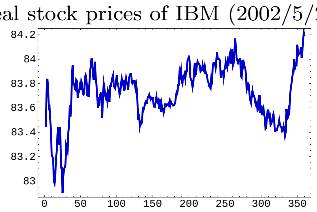
$$dS = \mu S dt + \sigma S dw.$$

Simulations of a geometric Brownian motion with $\mu > 0$ (left) and $\mu < 0$ (right)



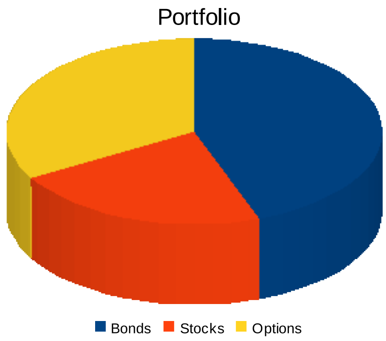


Real stock prices of IBM (2002/5/2)



Black–Scholes model for pricing financial derivatives

A financial portfolio consisting of stocks (underlying assets), options and bonds



- The aim is to dynamically (in time) rebalance the portfolio by buying/selling stocks/options/bonds in order to reduce volatile fluctuations and to preserve its value

Assumption:

- Fundamental economic balances:
  - conservation of the total value of the portfolio

$$S\,Q_S + V\,Q_V + B = 0$$

  - requirement of self-financing the portfolio

$$S\,dQ_S + V\,dQ_V + \delta B = 0$$

- Q_S is # of underlying assets with a unit price S in the portfolio
- Q_V is # of financial derivatives (options) with a unit price V
- B the cash money in the portfolio (e.g. bonds, T-bills, etc.)

- dQ_S is the change in the number of assets
- dQ_V is the change in the number of options
- δB is the change in the cash due to buying/selling assets and options

Black–Scholes model for pricing financial derivatives

Assumption:

- Secure bonds are appreciated by the fixed interest rate $r > 0$
$$B(t) = B(0)e^{rt} \to dB = rB\,dt$$
- The change of the total value of bonds in the portfolio is therefore
$$dB = rB\,dt + \delta B$$
because we sell bonds ($\delta B < 0$) or buy bonds ($\delta B > 0$) when hedging (re-balancing) the portfolio in the time period $[t, t + dt]$.

- Differentiating the fundamental balance law: $S\,Q_S + V\,Q_V + B = 0$ in the time period $[t, t + dt]$ we obtain

$$\begin{aligned}
0 &= d\,(SQ_S + VQ_V + B) = d\,(SQ_S + VQ_V) + \overbrace{dB}^{rB\,dt+\delta B} \\
0 &= \overbrace{SdQ_S + VdQ_V + \delta B}^{=0} + Q_S dS + Q_V dV + rB\,dt \\
0 &= Q_S dS \;+\; Q_V dV \;\; \overbrace{-\,r(SQ_S + VQ_V)}^{rB}\; dt.
\end{aligned}$$

- Dividing the last equation by Q_V we obtain
$$dV - rV\,dt - \Delta(dS - rS\,dt) = 0\,, \quad \text{where } \Delta = -\frac{Q_S}{Q_V}.$$

Black–Scholes model for pricing financial derivatives

- ReCall that we have assumed the stock price S to follow the geometric Brownian motion

$$dS = \mu S dt + \sigma S dw.$$

- By Itō's lemma we obtain for a smooth function $V = V(S,t)$

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \, dt + \frac{\partial V}{\partial S} \, dS.$$

- Inserting the differential dV into the equation $dV - rV\,dt - \Delta(dS - rS\,dt) = 0$ we obtain

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV + \Delta r S \right) dt + \left(\frac{\partial V}{\partial S} - \Delta \right) dS = 0$$

Assumption:

- Holding a strategy in buying/selling stocks and options with the goal to eliminate possible volatile fluctuations. The only volatile (unpredictable) term in the equation

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV + \Delta r S \right) dt + \left(\frac{\partial V}{\partial S} - \Delta \right) dS = 0$$

  is $\left(\frac{\partial V}{\partial S} - \Delta\right) dS$ due to the stochastic differential dS
- Setting $\Delta = \frac{\partial V}{\partial S}$ (Delta hedging) we obtain, after dividing the equation by dt, the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black–Scholes model for pricing financial derivatives

- The parabolic partial differential equation for the option price $V = V(S,t)$ defined for $S > 0, t \in [0,T]$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

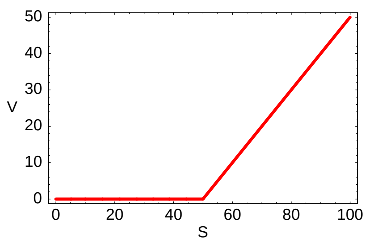
  is referred to as the Black–Scholes equation.

M. S. Scholes a R. C. Merton were awarded by the Price of the Swedish Bank for Economy in the memory of A. Nobel in 1997, Fisher Black died in 1995

Terminal conditions for the Black–Scholes equation:

- At the time $t = T$ of maturity (expiration) the price of a Call option is easy to determine.
  - If the actual (spot) price S of the underlying asset at $t = T$ is bigger then the exercise price E then it is worse to exercise the option, and the holder should price this option by the difference $V(S,T) = S - E$
  - If the actual (spot) price S of underlying asset at $t = T$ is less then the exercise price E then the Call option has no value, i.e. $V(S,T) = 0$
  - In both cases $V(S,T) = \max(S - E, 0)$.



Black–Scholes model for pricing financial derivatives

Mathematical formulation of the problem of pricing a Call option:

- Find a solution $V(S,t)$ of the Black–Scholes parabolic partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  defined for $S > 0, t \in [0,T]$, and satisfying the terminal condition

$$V(S,T) = \max(S-E, 0)$$

  at the time of maturity $t = T$.

Solution of the Black–Scholes equation.

- Using transformations $x = \ln(S/E)$ and $\tau = T - t$ transform the BS equation into the Cauchy problem

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(x,0) = u^0(x),$$

  for $-\infty < x < \infty\,, \tau \in [0,T]$.
- Solve this parabolic equation by means of the Green's function
- Transform back the solution and express $V(S,t)$ in the original variables S and t

Black–Scholes model for pricing financial derivatives

Solution of the Black–Scholes equation

- Transformation $x = \ln(S/E)$ and $\tau = T - t$ and introduction of an auxiliary function $Z(x,\tau)$ lead to

$$Z(x,\tau) = V(Ee^x, T-\tau)$$

- Then

$$\frac{\partial Z}{\partial x} = S\frac{\partial V}{\partial S}, \qquad \frac{\partial^2 Z}{\partial x^2} = S^2\frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S} = S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial Z}{\partial x}.$$

- The parabolic equation for Z reads as follows:

$$\frac{\partial Z}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 Z}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial Z}{\partial x} + rZ = 0,$$

$$Z(x,0) = \max(Ee^x - E, 0), \quad -\infty < x < \infty,\ \tau \in [0,T].$$

Solution of the Black–Scholes equation

- Using a new function $u(x,\tau)$

$$u(x,\tau) = e^{\alpha x + \beta\tau} Z(x,\tau)$$

where $\alpha, \beta \in \mathbb{R}$ are some constants leads to

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu = 0\,,$$

$$u(x,0) = Ee^{\alpha x}\max(e^x - 1, 0),$$

- Constants

$$A = \alpha\sigma^2 + \frac{\sigma^2}{2} - r\,, \quad \text{and} \quad B = (1+\alpha)r - \beta - \frac{\alpha^2\sigma^2 + \alpha\sigma^2}{2}.$$

can be eliminated (i.e. $A = 0, B = 0$) by setting

$$\alpha = \frac{r}{\sigma^2} - \frac{1}{2}, \qquad \beta = \frac{r}{2} + \frac{\sigma^2}{8} + \frac{r^2}{2\sigma^2}.$$

Black–Scholes model for pricing financial derivatives

Solution of the Black–Scholes equation

- A solution $u(x,\tau)$ to the Cauchy problem $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$ is given by Green's formula

$$u(x,\tau) = \frac{1}{\sqrt{2\sigma^2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma^2\tau}} u(s,0)\, ds\,.$$

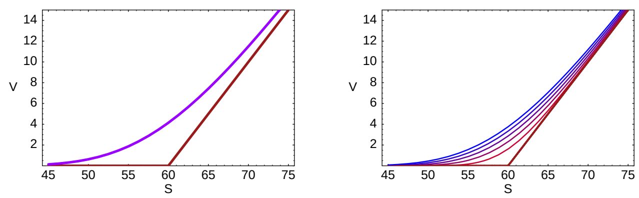
- Computing this integral and transforming back to the original variables S, t and $V(S,t)$, enables us to conclude

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)\,,$$

where $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\xi^2}{2}}\, d\xi$ is a distribution function of the normal distribution and

$$d_1 = \frac{\ln\frac{S}{E} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

Solution of the Black–Scholes equation



Graph of a solution $V(S,0)$ for a Call option together with the terminal condition $V(S,T)$ (left). Graphs of solutions $V(S,t)$ for different times $T-t$ to maturity (right).

Example:

- Present (spot) price of the IBM stock is $S = 58.5$ USD
- Historical volatility of the stock price was estimated to $\sigma = 29\%$ p.a., i.e. $\sigma = 0.29$.
- Interest rate for secure bonds $r = 4\%$ p.a., i.e. $r = 0.04$
- Call option for the exercise price $E = 60$ USD and exercise time $T = 0.3$-years
- Computed Call option price by Black–Scholes formula is: <u>V=V(58.5, 0) = 3.35</u> USD.
- Real market price was <u>V = 3.4</u> USD

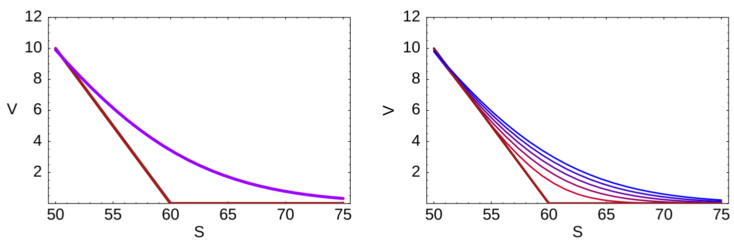
Black–Scholes model for pricing financial derivatives

- Put option
- Put option is an agreement (contract) between the writer (issuer) and the holder of an option. It represents the right BUT NOT the obligation to SELL the underlying asset at the prescribed exercise price E at the specified time of maturity $t = T$ in the future.
- If the actual (spot) price S of the underlying asset at $t = T$ is less then the exercise price E then it is worse to exercise the option, and the holder prices this option as the difference $V(S,T) = E - S$.
- If the actual (spot) price S of underlying asset at $t = T$ is higher then the exercise price E then it has no value for the holder, i.e. $V(S,T) = 0$.
- In both cases we have $V(S,T) = \max(E - S, 0)$.

- Put option
- explicit solution to the Black-Scholes equation with the terminal condition $V(S,T) = \max(E - S, 0)$

$$V_{ep}(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where $N(.), d_1, d_2$ are defined as in the case of a Call option.

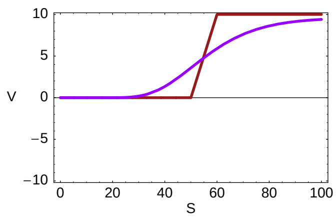


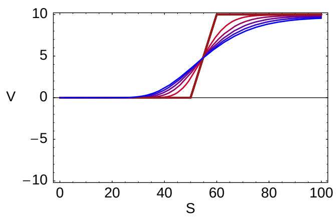
Graph of a solution $V(S,0)$ for a Put option and the terminal condition $V(S,T)$ (left). Graphs of solutions $V(S,t)$ for different times $T-t$ to maturity (right)

Black–Scholes model for pricing financial derivatives

- Put-Call parity
- Construct a portfolio of one long Call option and one short Put option: $V_f(S,T) = V_{ec}(S,T) - V_{ep}(S,T)$
- $$V_f(S,T) = \max(S-E,0) - \max(E-S,0) = S - E\,.$$
- The solution to the Black–Scholes equation with the terminal condition $V_f(S,T) = S - E$ can be found easily
$$V_f(S,t) = S - Ee^{-r(T-t)}$$
- According to the linearity of the Black–Scholes equation we obtain:
$$V_{ec}(S,t) - V_{ep}(S,t) = S - Ee^{-r(T-t)}$$
known as the Put–Call parity: Call - Put = Asset - Forward

- Bullish spread
Buy one Call option on the exercise price E_1 and sell one Call option on E_2 where $E_1 < E_2$. Therefore the Pay–off diagram: $V(S,T) = \max(S-E_1,0) \;-\; \max(S-E_2,0)$

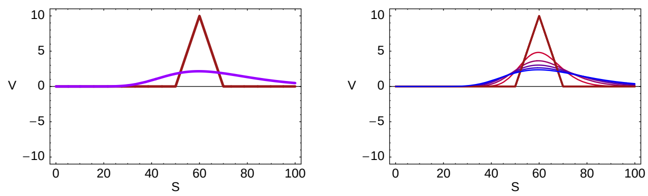




- The strategy has a limited profit and limited loss (pay-off diagram is bounded).
- It protects the holder for increase of the asset price in a short position (like a single Call option).
- Linearity of the Black–Scholes equation implies:
$$V(S,t) = V^c(S,t;E_1) - V^c(S,t;E_2), \qquad \text{for all } 0 \le t \le T$$

- Butterfly
  Buy two Call options - one with low exercise price E_1 and one with high E_4
  Sell two Call options with $E_2 = E_3$, where
  $E_1 < E_2 = E_3 < E_4$ and $E_1 + E_4 = E_2 + E_3 = 2E_2$.

$$V(S,T) = \max(S-E_1, 0) - \max(S-E_2, 0) - \max(S-E_3, 0) + \max(S-E_4, 0$$



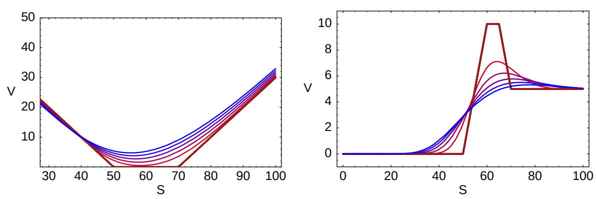
- The strategy has a limited profit and limited loss (pay-off diagram is bounded).
- It is profitable when the price of the asset is close to $E_2 = E_3$.
- Linearity of the Black–Scholes equation implies for $0 \le t \le T$:

$$V(S,t) = V^c(S,t;E_1) - V^c(S,t;E_2) - V^c(S,t;E_3) + V^c(S,t;E_4)$$

- Strangle is a combination of purchasing one Call on E_2, and one Put option on strike price $E_1 < E_2$

$$V(S,T) = (S - E_2)^+ + (E_1 - S)^+ .$$

- Condor is a strategy similar to butterfly, but the difference is that the strike prices of sold Call options need not be equal, $E_2 \neq E_3$, i.e., $E_1 < E_2 < E_3 < E_4$.



Left: Strangle option strategy for $E_1 = 50; E_2 = 70$ and prices $S \mapsto V(S,t)$
Right: Condor option strategy with
$E_1 = 50, E_2 = 60, E_3 = 65, E_4 = 70$

Black–Scholes equation for divedend paying assets

- the underlying asset is paying nontrivial continuous dividends with an annualized dividend yield $D \geq 0$
- holder of the underlying asset receives a dividend yield $DSdt$ over any time interval with a length dt
- paying dividends leads to the asset price decrease

$$dS = (\mu - D)S\,dt + \sigma S dw\,.$$

- on the other hand, it is added as an extra income to the money volume of secure bonds

$$dB = rB\,dt + \delta B + DSQ_S\,dt$$

- the portfolio balance equation then becomes

$$Q_V dV + Q_S dS + rB\,dt + DSQ_S\,dt = 0$$

- since $B = -Q_V V - Q_S S$ we obtain, after dividing by Q_V,

$$dV - rV\,dt - \Delta(dS - (r-D)S\,dt) = 0 \quad \text{where } \Delta = -Q_S/Q_V\,.$$

- repeating steps of derivation of the B-S equation, using Itō's lemma for dV we conclude with the equation

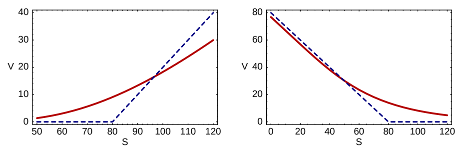
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0$$

- similarly as in the case $D = 0$ we obtain

$$V(S,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)\,,$$

$$d_1 = \frac{\ln\frac{S}{E} + (r - D + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

- Put option can be calculated from Put-Call parity: $V^c(S,t) - V^p(S,t) = Se^{-D(T-t)} - Ee^{-r(T-t)}$



Solutions $V(S,t), 0 \leq t < T$, for a European Call option (left) and Put option (right).

Chapter 4

- Transformation of the Black–Scholes equation to the heat equation
- Finite difference approximation
- Explicit numerical scheme and the method of binomial trees
- Stable implicit numerical scheme using a linear algebra solver

Numerical solution to the Black–Scholes equation

- using the transformation $V(S,t) = Ee^{-\alpha x-\beta\tau}u(x,\tau)$, where $\tau = T-t, x = \ln(S/E)$, leads to the heat equation
$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} = 0$$
for any $x \in \mathbb{R}, 0 < \tau < T$.
-
$$g(x,\tau) = \begin{cases} e^{\alpha x+\beta\tau}\max(e^x-1,0), & \text{for a Call option,}\\ e^{\alpha x+\beta\tau}\max(1-e^x,0), & \text{for a Put option.}\end{cases}$$
represents the transformed pay-off diagram of a Call (Put) option
- It satisfies the initial condition
$$u(x,0) = g(x,0), \quad \text{for any} \ \ x \in \mathbb{R}.$$

Here: $\alpha = \frac{r-D}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r+D}{2} + \frac{\sigma^2}{8} + \frac{(r-D)^2}{2\sigma^2}$

- spatial and time discretization yields the finite difference mesh

$$x_i = ih, \quad i = ..., -2, -1, 0, 1, 2, ..., \quad \tau_j = jk, \; j = 0, 1, ..., m.$$

$h = L/n, k = T/m.$

- approximation of the solution u at (x_i, τ_j) will be denoted by

$$u_i^j \approx u(x_i, \tau_j), \qquad \text{and also } \; g_i^j \approx g(x_i, \tau_j)$$

- using boundary conditions
  Call option: $V(0,t) = 0$ and $V(S,t)/S \to e^{-D(T-t)}$ for $S \to \infty$
  Put option: $V(0,t) = Ee^{-r(T-t)}$ and $V(S,t) \to 0$ as $S \to \infty$
  $\Rightarrow$ the boundary condition at $x = \pm L, L \gg 1$,

$$\begin{aligned} u_{-N}^j &= \phi^j := \begin{cases} 0, & \text{for a European Call option,} \\ e^{-\alpha N h + (\beta - r) jk}, & \text{for a European Put option,} \end{cases} \\ u_N^j &= \psi^j := \begin{cases} e^{(\alpha+1) N h + (\beta - D) jk}, & \text{for a European Call option,} \\ 0, & \text{for a European Put option.} \end{cases} \end{aligned}$$

- time derivative forward (explicit) and backward (implicit) finite difference approximation

$$\frac{\partial u}{\partial \tau}(x_i, \tau_j) \approx \underbrace{\frac{u_i^{j+1} - u_i^j}{k}}_{forward} \qquad \frac{\partial u}{\partial \tau}(x_i, \tau_j) \approx \underbrace{\frac{u_i^j - u_i^{j-1}}{k}}_{backward}$$

- central finite difference approximation of the spatial derivative

$$\frac{\partial^2 u}{\partial x^2}(x_i, \tau_j) \approx \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}$$

- Explicit and implicit finite difference approximation of the heat equation

$$\underbrace{\frac{u_i^{j+1} - u_i^j}{k} = \frac{\sigma^2}{2} \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}}_{explicit\ scheme}, \qquad \underbrace{\frac{u_i^j - u_i^{j-1}}{k} = \frac{\sigma^2}{2} \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}}_{implicit\ scheme}$$

Explicit scheme and binomial tree

- explicit scheme can be rewritten as:

$$u_i^{j+1} = \gamma u_{i-1}^j + (1-2\gamma)u_i^j + \gamma u_{i+1}^j, \qquad \text{where} \ \ \gamma = \frac{\sigma^2 k}{2h^2},$$

- in matrix form $u^{j+1} = \mathbf{A}u^j + b^j$ for $j = 0, 1, \dots, m-1$ where $\mathbf{A}$ is a tridiagonal matrix given by

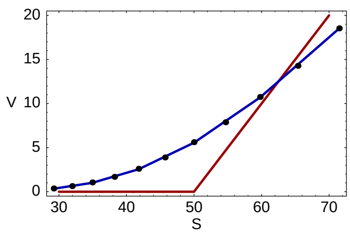
$$\mathbf{A} = \begin{pmatrix} 1-2\gamma & \gamma & 0 & \cdots & 0 \\ \gamma & 1-2\gamma & \gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \gamma & 1-2\gamma & \gamma \\ 0 & \cdots & 0 & \gamma & 1-2\gamma \end{pmatrix}, \quad b^j = \begin{pmatrix} \gamma\phi^j \\ 0 \\ \vdots \\ 0 \\ \gamma\psi^j \end{pmatrix}.$$

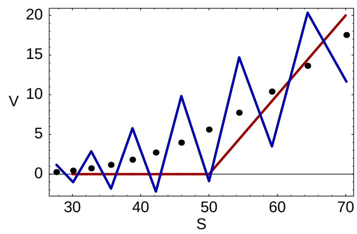
Under the so-called Courant–Fridrichs–Lewy (CFL) stability condition:

$$0 < \gamma \le \frac{1}{2}, \qquad \text{i.e.} \quad \frac{\sigma^2 k}{h^2} \le 1,$$

the explicit numerical discretization scheme is stable.

- transforming back to the original variables $S = Ee^x, t = T - \tau, V(S,t) = Ee^{-\alpha x - \beta\tau}u(x,\tau)$ we obtain the option price V





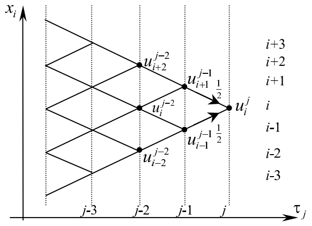
A solution $S \mapsto V(S,t)$ for the price of a European Call option obtained by means of the binomial tree method with $\gamma = 1/2$ (left) and comparison with the exact solution (dots). The oscillating solution $S \mapsto V(S,t)$ which does not converge to the exact solution for the parameter value $\gamma = 0.56 > 1/2$, where $\gamma > 1/2$, does not fulfill the CFL condition.

Explicit numerical scheme and binomial tree

- if we choose the ratio between the spatial and time discretization steps such that $h = \sigma\sqrt{k}$ then $\gamma = 1/2$

$$u_i^{j+1} = \frac{1}{2}u_{i-1}^j + \frac{1}{2}u_{i+1}^j.$$

- the solution u_i^{j+1} at the time τ_{j+1} is the arithmetic average between values u_{i-1}^j and u_{i+1}^j



A binomial tree as an illustration of the algorithm for solving a parabolic equation by an explicit method with $2\gamma = \sigma^2 k/h^2 = 1$.

The binomial pricing model can be also derived from the explicit numerical scheme.

$$V_i^j \approx V(S_i, T - \tau_j), \quad \text{where } S_i = Ee^{x_i} = Ee^{ih}.$$

- since $V(S,t) = Ee^{-\alpha x - \beta\tau}u(x,t)$, we obtain $V_i^j = Ee^{-\alpha ih - \beta jk}u_i^j$.
- in terms of the original variable V_i^j, the explicit numerical scheme can be expressed as follows:

$$V_i^{j+1} = e^{-rk}\left(q_- V_{i-1}^j + q_+ V_{i+1}^j\right), \quad \text{where } q_\pm = \frac{1}{2}e^{\pm\alpha h - (\beta - r)k}.$$

- for $k \to 0$ and $h = \sigma\sqrt{k} \to 0$ we have

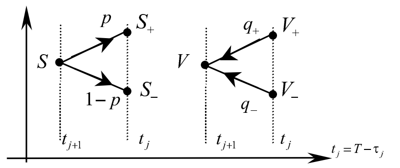
$$q_+ \doteq \frac{1+\alpha h}{2}, \quad q_- \doteq \frac{1-\alpha h}{2}, \qquad q_- + q_+ = 1.$$

and these constants are to as risk-neutral probabilities.

Explicit numerical scheme and binomial tree

- underlying stock price at t_{j+1} has a price S. Here $t_0 = T, \ldots, t_m = 0$
- at the time $t_j > t_{j+1}$ it attains a higher value $S_+ > S$ with a probability $p \in (0,1)$, and $S_- < S$ with probability $1 - p \in (0,1)$
- let V_+ and V_- be the option prices corresponding to the upward and downward movement of underlying prices
- the option price V at time t_{j+1} can be calculated as

$$V = e^{-rk}\left(q_+ V_+ + q_- V_-\right), \text{ where } q_+ = \frac{Se^{rk} - S_-}{S_+ - S_-},\ q_- = 1 - q_+$$



A binomial tree illustrating calculation of the option price by binomial tree

- implicit scheme can be rewritten as:

$$-\gamma u_{i-1}^{j} + (1+2\gamma) u_i^j - \gamma u_{i+1}^{j} = u_i^{j-1}, \qquad \text{where } \ \gamma = \frac{\sigma^2 k}{2h^2},$$

- in matrix form $\mathbf{A} u^j = u^{j-1} + b^{j-1}$ for $j = 1, 2, \ldots, m$ where $\mathbf{A}$ is a tridiagonal matrix given by

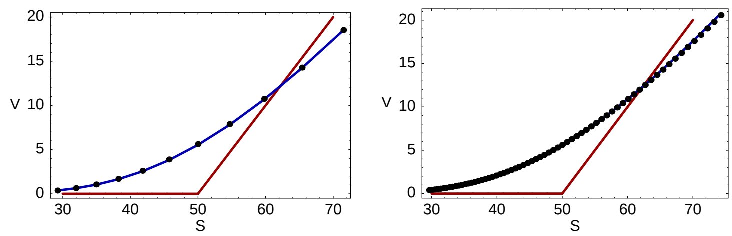
$$\mathbf{A} = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{pmatrix}, \quad b^j = \begin{pmatrix} \gamma\phi^{j+1} \\ 0 \\ \vdots \\ 0 \\ \gamma\psi^{j+1} \end{pmatrix}.$$

The implicit numerical discretization scheme is unconditionally stable for any

$$\gamma > 0$$

Implicit finite difference numerical scheme

- transforming back to the original variables $S = Ee^x, t = T - \tau, V(S,t) = Ee^{-\alpha x - \beta\tau} u(x,\tau)$ we obtain the option price V



A solution $S \mapsto V(S,t)$ for pricing a European Call option obtained by means of the implicit finite difference method with $\gamma = 1/2$ (left) and comparison with the exact analytic solution (dots). The numerical scheme is also stable for a large value of the parameter $\gamma = 20 > 1/2$ not satisfying the CFL condition (right).

Successive Over Relaxation method for solving $\mathbf{A}u = b$

- Decompose the matrix $\mathbf{A}$ as as sum of subdiagonal, diagonal and overdiagonal matrix $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$ where

$$\begin{array}{lll} \mathbf{L}_{ij} = \mathbf{A}_{ij} & \text{for } j < i, & \text{otherwise } \mathbf{L}_{ij} = 0, \\ \mathbf{D}_{ij} = \mathbf{A}_{ij} & \text{for } j = i, & \text{otherwise } \mathbf{D}_{ij} = 0, \\ \mathbf{U}_{ij} = \mathbf{A}_{ij} & \text{for } j > i, & \text{otherwise } \mathbf{U}_{ij} = 0. \end{array}$$

- We suppose that $\mathbf{D}$ is invertible. Let $\omega > 0$ be a relaxation parameter. A solution of $\mathbf{A}u = b$ is equivalent to

$$\mathbf{D}u = \mathbf{D}u + \omega(b - \mathbf{A}u).$$

or, equivalently,

$$(\mathbf{D} + \omega\mathbf{L})u = (1-\omega)\mathbf{D}u + \omega(c - \mathbf{U}u).$$

- Therefore u is a solution of

$$u = \mathbf{T}_\omega u + c_\omega, \qquad \text{where} \;\; \mathbf{T}_\omega = (\mathbf{D} + \omega\mathbf{L})^{-1}\left((1-\omega)\mathbf{D} - \omega\mathbf{U}\right)$$

a $c_\omega = \omega(\mathbf{D} + \omega\mathbf{L})^{-1} b$.

- Define a recurrent sequence of approximate solution

$$u^0 = 0, \qquad u^{p+1} = \mathbf{T}_\omega u^p + c_\omega \quad \text{for} \;\; p = 1, 2, \ldots$$

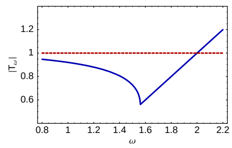
- the SOR algorithm reduces to successive calculation, for $p = 0, ..., p_{max}$ of

$$u_i^{p+1} = \frac{\omega}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p \right) + (1-\omega) u_i^p$$

for $i = 1, ..., N$

- where $\omega \in (1, 2)$ is a relaxation parameter

- if $\|\mathbf{T}_\omega\| < 1$ then the mapping $\mathbb{R}^n \ni u \mapsto \mathbf{T}_\omega u + c_\omega \in \mathbb{R}^n$ is contractive and the fixed point argument implies that u^p converges to u for $p \to \infty$ and $\mathbf{A}u = b$.



Graph of the spectral norm of the iteration operator $\|\mathbf{T}_\omega\|$ as a function of the relaxation parameter ω.

Black–Scholes model and sensitivity analysis

Chapter 5

- Historical and implied volatilities
- Computation of the implied volatility
- Sensitivity with respect to model parameters
- Delta and Gamma of an option. Other Greeks factors.

- Historical volatility
  How to estimate the historical volatility σ of the asset (a diffusion coefficient in the BS equation)
- $dS = \mu S dt + \sigma S dw$
- For the process of the underlying asset returns $X(t) = \ln S(t)$ we have, by Itō's lemma

$$dX = (\mu - \sigma^2/2)dt + \sigma dw.$$

- In the discrete form (for equidistant division $0 = t_0 < t_1 < ... < t_n = T$, $t_{i+1} - t_i = \tau$) we have

$$X(t_{i+1}) - X(t_i) = (\mu - \frac{1}{2}\sigma^2)\tau + \sigma(w(t_{i+1}) - w(t_i)).$$

- as $\sigma(w(t_{i+1}) - w(t_i)) = \sigma\Phi\sqrt{\tau}$, where $\Phi \sim N(0,1)$ we can use the estimator for the dispersion of the normally distributed random variable $\sigma\sqrt{\tau}\Phi \sim N(0, \sigma^2\tau)$

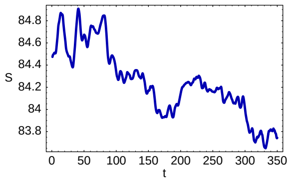
Black–Scholes model and sensitivity analysis

- The historical volatility $\sigma = \sigma_{hist}$ of the underlying asset price

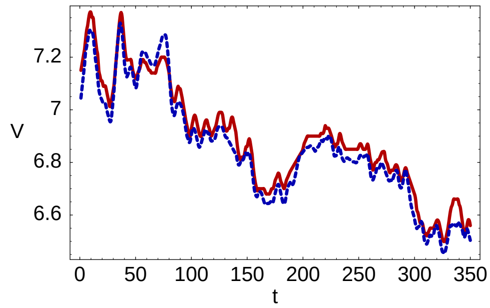
$$\sigma^2_{hist} = \frac{1}{\tau}\frac{1}{n-1}\sum_{i=0}^{n-1}\Big(\ln(S(t_{i+1})/S(t_i)) - \gamma\Big)^2$$

- where γ is the mean value of returns $X(t_i) = \ln(S(t_{i+1})/S(t_i))$

$$\gamma = \frac{1}{n}\sum_{i=0}^{n-1}\ln(S(t_{i+1})/S(t_i)).$$



IBM stock price evolution from 21.5.2002 with $\tau = 1$ minute. The computed historical volatility $\sigma_{hist} = 0.2306$ on the yearly basis, i.e. $\sigma_{hist} = 23\%$ p.a.



IBM Call option price from 21.5.2002 (red).
Computed $V^{ec}(S_{real}(t), t; \sigma_{hist})$ with $\sigma_{hist} = 0.2306$ (blue)

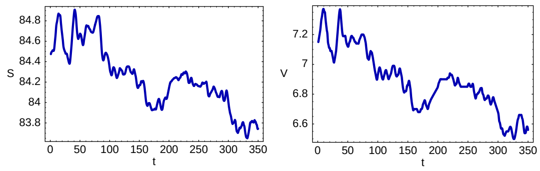
- In typical real market situations the historical volatility σ_{hist} produces lower option prices
- σ_{hist} is lower than the value that is needed for exact matching of market option prices

Black–Scholes model and sensitivity analysis

- Implied volatility
  The implied volatility is a solution of the following inverse problem: Find a diffusion coefficient of the Black-Scholes equation such that the computed option price is identical with the real market price.
- Denote the price of an option (Call or Put) as $V = V(S, t; \sigma)$ where σ - the volatility is considered as a parameter.
- Implied volatility σ_{impl} at the time t is a solution of the implicit equation

$$V_{real}(t) = V(S_{real}(t), t; \sigma_{impl}).$$

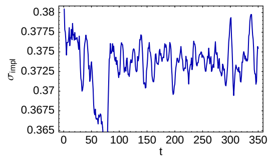
  where $V_{real}(t)$ is the market option price, $S_{real}(t)$ is the market underlying asset price at the time t.
- Solution σ exists and is unique due to monotonicity of the function $\sigma \mapsto V(S, t; \sigma)$ (it is an increasing function).



IBM stock price evolution from 21.5.2002 (left), the Call option for $E = 80$ and $T = 43/365$ (right)

$\Downarrow$

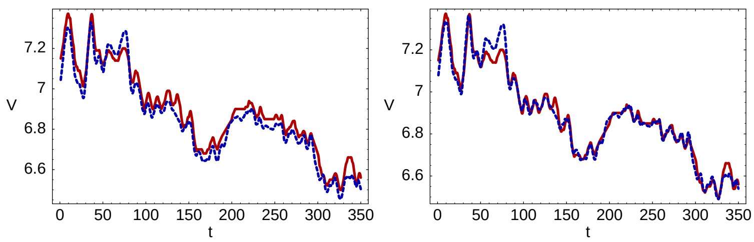
- The computed implied volatility $\sigma_{impl}(t)$



- The average value of the implied volatility is: $\bar{\sigma}_{impl} = 0.3733$ p.a.

Black–Scholes model and sensitivity analysis

- Comparison of market Call option data match for Historical and Implied volatilities



IBM Call option price from 21.5.2002 (red).
Computed $V_t = V^{ec}(S_{real}(t), t; \sigma_{hist})$ with $\sigma_{hist} = 0.2306$ (left).
Computed $V_t = V^{ec}(S_{real}(t), t; \sigma_{impl})$ with $\sigma_{impl} = 0.3733$ (right).

Sensitivity of the option price with respect to model parameters - Greeks

- Sensitivity with respect to the asset price: Delta - Δ,

$$\Delta = \frac{\partial V}{\partial S}$$

- It measures the rate of change of the option price V w.r. to the change in the asset price S
- It is used in the so-called Delta hedging because the risk-neutral portfolio is balanced according to the law:

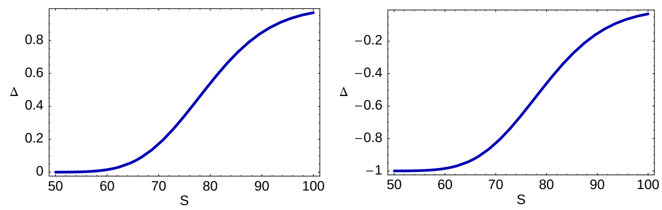
$$\frac{Q_S}{Q_V} = -\frac{\partial V}{\partial S} = -\Delta$$

where Q_V, Q_S is the number of options and stocks in the portfolio

Black–Scholes model and sensitivity analysis

- Delta for European Call and Put options:

$$\Delta^{ec} = \frac{\partial V^{ec}}{\partial S} = N(d_1), \qquad \Delta^{ep} = \frac{\partial V^{ep}}{\partial S} = -N(-d_1).$$



Δ^{ec} Δ^{ep}

Parameters: $E = 80, r = 0.04, T = 43/365$

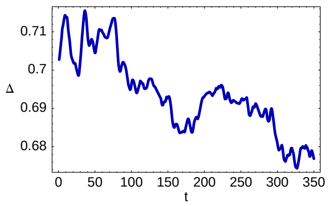
- Notice that $\Delta^{ec} \in (0, 1)$ and $\Delta^{ep} \in (-1, 0)$

Computation of Delta for market data time series

- Determine the implied volatility $\sigma_{impl}(t)$ from market data time series of the option price $V_{real}(t)$ and the underlying asset price $S_{real}(t)$. We solve

$$V_{real}(t) = V^{ec}(S_{real}(t), t; \sigma_{impl}(t)).$$

- Produce the graph of $\Delta^{ec}(t) = \frac{\partial V^{ec}}{\partial S}(S_{real}(t), t; \sigma_{impl}(t))$



- Observe that the decrease of Delta means that keeping one Call option we have to decrease the number Q_S of owed stocks in the portfolio.

Black–Scholes model and sensitivity analysis

- Sensitivity of Delta with respect to the asset price: Gamma - Γ

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$

- It measures the rate of change of the Delta of the option price V w.r. to the change in the asset price S

$$\Gamma^{ec} = \Gamma^{ep} = \frac{\partial \Delta^{ec}}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{\exp(-\frac{1}{2}d_1^2)}{\sigma\sqrt{2\pi(T-t)}S} > 0$$

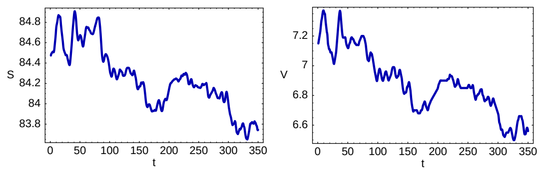
- It is used for generating signals for the owner of the option to rebalance his portfolio because change in the Delta factor means that the change in the ratio Q_S/Q_V should be done.
- High Gamma $\Rightarrow$ rebalance portfolio according to Delta hedging strategy

Computation of Gamma for market data time series

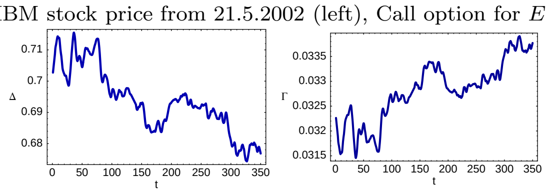
- Determine the implied volatility $\sigma_{impl}(t)$ from market data time series of the option price $V_{real}(t)$ and the underlying asset price $S_{real}(t)$. We solve

$$V_{real}(t) = V^{ec}(S_{real}(t), t; \sigma_{impl}(t)).$$

- Produce the graph of $\Gamma^{ec}(t) = \frac{\partial^2 V^{ec}}{\partial S^2}(S_{real}(t), t; \sigma_{impl}(t))$



IBM stock price from 21.5.2002 (left), Call option for $E = 80$ and $T = 43/365$ (right)



Delta (left)

Black–Scholes model and sensitivity analysis

Other Greeks - Sensitivity of the option price to model parameters

- Rho
  Sensitivity with respect to the interest rate r, $\quad P = \frac{\partial V}{\partial r}$
- Theta
  Sensitivity with respect to time t, $\quad \Theta = \frac{\partial V}{\partial t}$
- Vega
  Sensitivity with respect to volatility σ, $\quad \Upsilon = \frac{\partial V}{\partial \sigma}$

- Greek version of the Black–Scholes equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$\Downarrow$$

$$\Theta + \frac{\sigma^2}{2} S^2 \Gamma + rS\Delta - rV = 0$$

Chapter 6

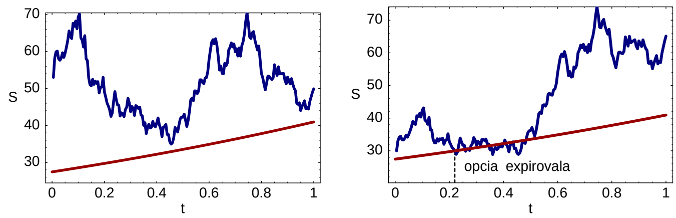
- Path dependent options, concepts and applications
- Barrier options, formulation in terms of a solution to a partial differential equation on a time dependent domain
- Asian options, formulation in terms of a solution to a partial differential equation in a higher dimension
- Numerical methods for solving barrier and Asian options

Exotic derivatives - Path dependent options

Path dependent options

- A path-dependent option = the option contract depends on the whole time evolution of the underlying asset in the time interval $[0, T]$
- Classical European options are not path dependent options, the contract depends only on the terminal pay-off $V(S, T)$ at the expiry T
- The path dependent options - Examples
  - Barrier options - the contract depends on whether the asset price jumped over/under prescribed barrier
  - Asian options - the contract depends on the average of the asset price over the time interval $[0, T]$
  - Many other like e.g. look-back options, Russian options, Israeli options, etc.
- Path dependent options are hard to price as the contract depends on the whole evolution of the asset price S_t in the future time interval $[0, T]$

- Example of an barrier options: Down–and–out Call option. This is a usual Call option with the terminal pay-off $V(S,T) = \max(S - E, 0)$ except of the fact that the option may expire before the maturity T at the time $t < T$ in the case when the underlying asset price S_t reaches the prescribed barrier $B(t)$ from above.



The option will expire at the maturity T (left) It will expire prematurely at $t < T$ (right)

- If the option expires prematurely at $t < T$ the writer pays the holder the prescribed rabat $R(t)$.

Exotic derivatives - Barrier options

- A typical exponential barrier function is: $B(t) = bEe^{-\alpha(T-t)}$ with $0 < b < 1$
- A typical exponential rabat function is: $R(t) = E\left(1 - e^{-\beta(T-t)}\right)$
- Mathematical formulation - the PDE on a time dependent domain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

for $t \in [0,T)$ and $B(t) < S < \infty$

$$V(B(t),t) = R(t), \qquad t \in [0,T)$$

at the left barrier boundary $S = B(t)$

$$V(S,T) = \max(S - E, 0), \qquad S > 0,$$

at $t = T$ (Barrier Call option).

- The fixed domain transformation

$$V(S,t) = W(x,t), \quad \text{where} \;\; x = \ln\left(S/B(t)\right), \quad x \in (0,\infty),$$

  transforms the problem from the time dependent domain $B(t) < S < \infty$ to the fixed domain $x \in (0,\infty)$.
- For an exponential barrier function $B(t) = bEe^{-\alpha(T-t)}$ we have $\dot{B}(t) = \alpha B(t)$.
- After performing necessary substitutions we obtain the PDE for the transformed function $W(x,t)$

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 W}{\partial x^2} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial W}{\partial x} - rW = 0.$$

- The terminal condition for the Call option case:

$$W(x,T) = E\max(be^x - 1, 0).$$

- The left side boundary condition

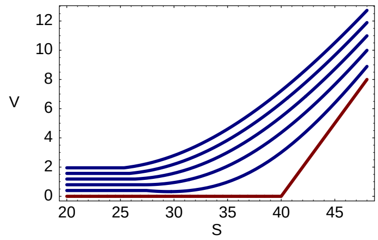
$$W(0,t) = R(t).$$

Exotic derivatives - Barrier options

A numerical solution - a simple code in the software Mathematica

```
b = 0.7; alfa = 0.1; beta = 0.05; X = 40; sigma = 0.4; r = 0.04; d = 0; T = 1;
xmax = 2;

Bariera[t_] := X b Exp[-alfa (T - t)]; Rabat[t_] := X (1 - Exp[-beta(T - t)]);
PayOff[x_] := X*If[b Exp[x] - 1 > 0, b Exp[x] - 1, 0];

riesenie = NDSolve[{
    D[w[x, tau], tau] == (sigma^2/2)D[w[x, tau], x, x]
       + (r - d - sigma^2/2 - alfa )* D[w[x, tau], x]
      - r *w[x, tau] ,

        w[x, 0] == PayOff[x],
        w[0, tau] == Rabat[T - tau],
        w[xmax, tau] == PayOff[xmax]},

      w, {tau, 0, T}, {x, 0, xmax}
      ];

w[x_, tau_] := Evaluate[w[x, tau] /. riesenie[[1]] ];
Plot3D[w[x, tau], {x, 0, xmax}, {tau, 0, T}];

V[S_, tau_] :=
    If[S > Bariera[T - tau],
          w[ Log[S/Bariera[T - tau]], tau],
          Rabat[T - tau]
      ];

Plot[ {V(S,0.2 T],V(S,0.4 T], V(S,0.6 T], V(S,0.8 T], V(S,T]}, {S,20,50}];
```

A numerical solution - an example of a solution to the Down-and-out barrier Call option



Graph of the solution of the barrier Call option for different times $t \in [0, T]$

Exotic derivatives - Asian options

- An example of an Asian option:
  This is a Call option with terminal pay-off $V(S,T) = \max(S - E, 0)$ except of the fact that the exercise price E is not prescribed but it is given as the arithmetic (or geometric) average of the underlying asset prices S_t within the time interval $[0, T]$, i.e. the terminal pay-off diagram is:

$$V(S,T) = \max(S - A_T, 0)$$

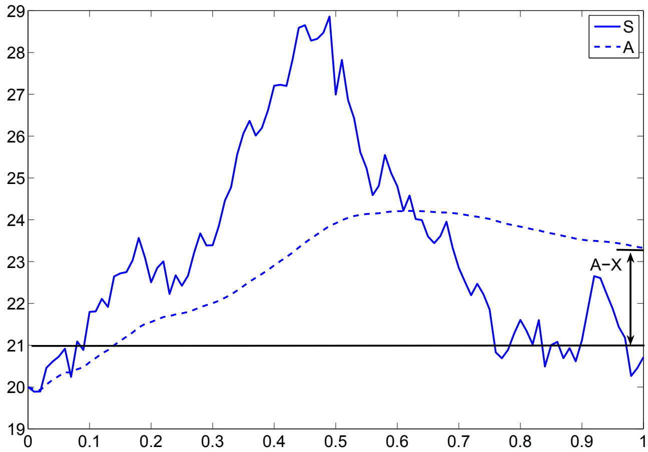
arithmetic average *geometric average*

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau, \qquad \ln A_t = \frac{1}{t}\int_0^t \ln S_\tau d\tau.$$

In the discrete form

$$A_{t_n} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}, \qquad \ln A_{t_n} = \frac{1}{n}\sum_{i=1}^{n} \ln S_{t_i},$$

where $t_1 < t_2 < ... < t_n$, and $t_{i+1} - t_i = 1/n$.



Simulated price of the underlying asset and the corresponding arithmetic average.

Exotic derivatives - Asian options

- Assume the asset price follows SDE: $dS = \mu S dt + \sigma S dw$
- The average A is the arithmetic average, i.e. $A_t = \frac{1}{t}\int_0^t S_\tau d\tau$
  Then
  $$\frac{dA}{dt} = -\frac{1}{t^2}\int_0^t S_\tau d\tau + \frac{1}{t}S_t = \frac{S_t - A_t}{t}$$
  an hence, in the differential form, $dA = \frac{S-A}{t}dt$.
- In general we may assume
  $$dA = A\, f\left(\frac{S}{A}, t\right) dt, \qquad f(x,t) = \frac{x-1}{t}, \quad f(x,t) = \frac{\ln x}{t}$$
  general form arithmetic average geometric average
- Construct the option price as a function
  $$V = V(S, A, t)$$
  It depends on a new variable: A - the average of the asset price

- Itō's lemma (extension to the function $V = V(S, A, t)$)

$$\begin{aligned} dV &= \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial A}dA + \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right) dt \\ &= \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial A}Af\left(\frac{S}{A}, t\right)\right) dt. \end{aligned}$$

$\Downarrow$ notice that $dA = Af(S/A, t)dt$ $\Downarrow$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + Af\left(\frac{S}{A}, t\right)\frac{\partial V}{\partial A} - rV = 0$$

- This is a two dimensional parabolic equation for pricing Asian type of average strike options

Exotic derivatives - Asian options

- The pay-off diagram $V(S, A, T) = \max(S - A, 0)$ can be rewritten as $V(S, A, T) = A\max(S/A - 1, 0)$

Use the change of variables $\Downarrow$

$$V(S, A, t) = A\,W(x, t), \quad \text{where } x = \frac{S}{A}, \quad x \in (0, \infty)$$

- The parabolic PDE for the transformed function $W(x, t)$ read as follows:

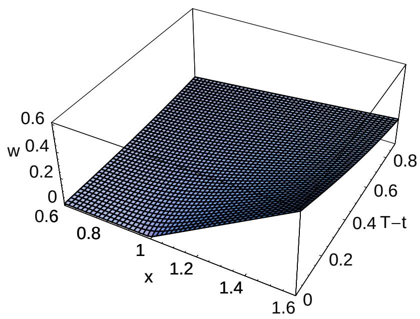
$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 W}{\partial x^2} + rx\frac{\partial W}{\partial x} + f(x, t)\left(W - x\frac{\partial W}{\partial x}\right) - rW = 0$$

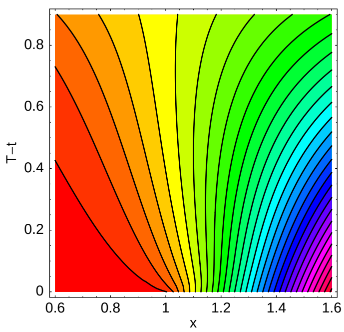
- The terminal condition $W(x, T) = \max(x - 1, 0)$ for an Asian Call option
- Although the solution can be found in a series expansion w.r.t. to Bessel functions it is more convenient to solve it numerically

A numerical solution - a simple code in the software Mathematica

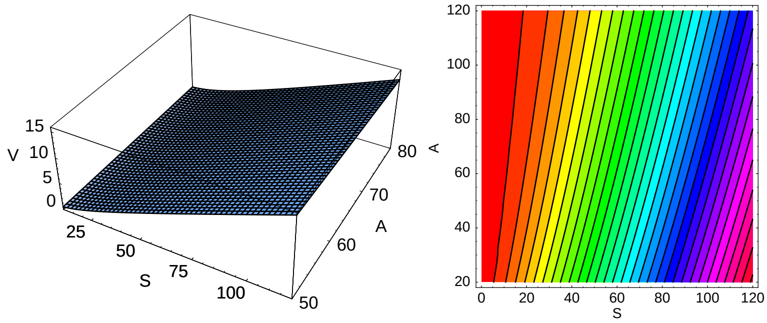
```
sigma=0.4; r=0.04; d=0; T=1; t=0.9; xmax=8;

PayOff[x_] := If[x - 1 > 0, x - 1, 0];

riesenie = NDSolve[{
 D[w[x, tau],tau] == (sigma^2/2) x^2 D[w[x, tau], x,x]
 + (r - d)*x * D[w[x, tau], x]
 + ((x - 1)/(T - tau))*(w[x, tau] - x*D[w[x, tau], x])
   - r*w[x, tau],
        w[x, 0] == PayOff[x],
        w[0, tau] == 0,
        w[xmax, tau] == PayOff[xmax]},
      w, {tau, 0, t}, {x, 0, xmax}
      ];

w[x_, tau_] := Evaluate[w[x, tau] /. riesenie[[1]]  ];
V[tau_, S_, A_] := A w[S/A, tau];
Plot3D[ V[t, S, A], {S, 10, 120}, {A, 50, 80}];
```

Exotic derivatives - Asian options





3D and countourplot graphs of the solution $W(x,t)$ of the transformed function $W(x,\tau)$ for parameters $\sigma = 0.4, r = 0.04, D = 0, T = 1$.



3D and countourplot graphs of the Asian average strike Call option $V(S, A, t) = AW(S/A, t)$ for the time $t = 0.1$ and $T = 1$ (i.e. $T - t = 0.9$)

American type of options

Chapter 7

- American options
- Early exercise boundary
- Formulation in the form of a variational inequality
- Projected successive over relaxation method (PSOR)

- American options are most traded types of options (more than 95% of option contracts are of the American type)
- The difference between European and American options consists in the possibility of early exercising the option contract within the whole time interval $[0, T]$, T is the maturity.
- the case of Call (or Put) option:
- American Call (Put) option is an agreement (contract) between the writer and the holder of an option. It represents the right BUT NOT the obligation to purchase (sell) the underlying asset at the prescribed exercise price E at ANYTIME in the forecoming interval $[0, T]$ with the specified time of maturity $t = T$.
- The question is: What is the price of such an option (the option premium) at the time $t = 0$ of contracting. In other words, how much should the holder of the option pay the writer for such a security.

American type of options

- American options gives the holder more flexibility in exercising
- An American option therefore has higher value compared to the European option

$$\Downarrow$$

$$V^{ac}(S,t) \geq V^{ec}(S,t), \quad V^{ap}(S,t) \geq V^{ep}(S,t)$$

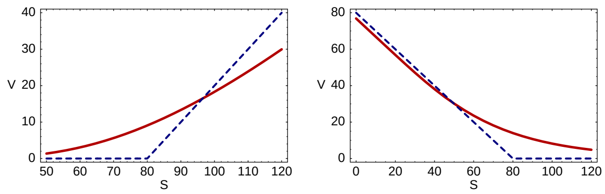
- An American option at time $t < T$ must always have higher value than the one in expiry, i.e.

$$\Downarrow$$

$$V^{ac}(S,t) \geq V^{ac}(S,T) = \max(S - E, 0),$$

$$V^{ap}(S,t) \geq V^{ap}(S,T) = \max(E - S, 0)$$

ec, ep indicates the European type of an option
ac, ap indicates the American type of an option



Solutions $V(S,t), 0 \le t < T$, for a European Call option (left) and Put option (right).

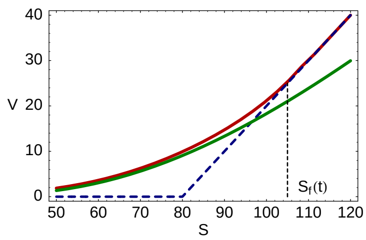
The solutions $V^{ec}(S,t), V^{ep}(S,t)$ always intersect their payoff diagrams $V(S,T) \Rightarrow$ these are not the graphs of $V^{ac}(S,t), V^{ap}(S,t)$

- In the left figure we plotted the price $V^{ec}(S,t)$ of a Call option on the asset paying dividends with a continuous dividend yield rate $D > 0$.
- The Black-Scholes equation for pricing the option is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0\,,$$

$$V(S,T) = \max(S-E,0), \quad S > 0,\ t \in [0,T]\,.$$

American type of options



Comparison of solutions $V^{ec}(S,t)$ and $V^{ac}(S,t)$ of European and American Call options at some time $0 \le t < T$.

- The problem is to find both the solution $V^{ac}(S,t)$ as well as the position of the free boundary $S_f(t)$ (the early exercise boundary).
- If $S < S_f(t)$, then $V^{ac}(S,t) > \max(S-E,0)$ and we keep the Call option
- If $S \ge S_f(t)$, then $V^{ac}(S,t) = \max(S-E,0)$ and we exercise the Call option

1. the function $V(S,t)$ is a solution to the Black–Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0$$

on a time dependent domain $0 < t < T$ and $0 < S < S_f(t)$.

2. The terminal pay–off diagram for the Call option

$$V(S,T) = \max(S-E, 0).$$

3. Boundary conditions for a solution $V(S,t)$ (case of an American Call option)

$$V(0,t) = 0, \quad V(S_f(t),t) = S_f(t) - E, \quad \frac{\partial V}{\partial S}(S_f(t),t) = 1,$$

at the boundary points $S = 0$ a $S = S_f(t)$ for $0 < t < T$

American type of options

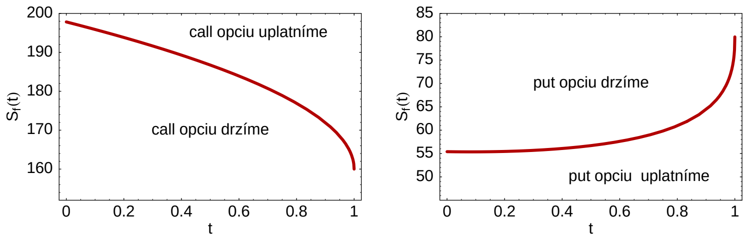
Smooth pasting principle

- boundary condition $V(S_f(t),t) = S_f(t) - E$ represents the continuity of the function $V^{ac}(S,t)$ across the free boundary $S_f(t)$
- boundary condition $\frac{\partial V}{\partial S}(S_f(t),t) = 1$ represents the C^1 continuity of the function $V^{ac}(S,t)$ across the free boundary $S_f(t)$

The C^1 continuity of a solution (smooth pasting principle) can be deduced from the optimization principle according to which the price of an American option is given by

$$V^{ac}(S,t) = \max_{\eta} V(S,t;\eta),$$

where the maximum is taken over the set of all positive smooth functions $\eta : [0,T] \to \mathbb{R}^+$ and $V(S,t;\eta)$ is the solution to the Black–Scholes equation on a time dependent domain $0 < t < T, 0 < S < \eta(t)$, and satisfying the terminal pay-off diagram and Dirichlet boundary conditions $V(0,t;\eta) = 0, V(\eta(t),t;\eta) = \eta(t) - E$.



Behavior of the free boundary $S_f(t)$ (early exercise boundary) for the American Call (left) and Put (right) option.

For the American Put option we must change:

- the time dependent domain to $0 < t < T$ and $S > S_f(t)$;
- the terminal pay-off diagram for the Put option $V(S,T) = \max(E - S, 0)$
- boundary conditions

$$V(+\infty, t) = 0, \quad V(S_f(t), t) = E - S_f(t), \quad \frac{\partial V}{\partial S}(S_f(t), t) = -1,$$

American type of options

Some recent and so so recent results on the early exercise behavior

- According to the paper by Dewynne et al. (1993) and Ševčovič (2001) the early exercise behavior of an American Call option close to the expiry T is given by
$$S_f(t) \approx K\left(1 + 0.638\,\sigma\sqrt{T-t}\right), \quad K = E\max(r/D, 1)$$
- According to the paper by Stamicar, Chadam, Ševčovič (1999) the early exercise behavior of an American Put option close to the expiry T is given by
$$S_f(t) = Ee^{-(r-\frac{\sigma^2}{2})(T-t)}e^{\sigma\sqrt{2(T-t)}\eta(t)} \quad \text{as} \;\; t \to T,$$
where $\eta(t) \approx -\sqrt{-\ln\left[\frac{2r}{\sigma}\sqrt{2\pi(T-t)}e^{r(T-t)}\right]}$
- Recently Zhu in papers from 2006, 2007 constructed an explicit approximation solution to the whole early exercise boundary problem obtained by the inverse Laplace transformation.

Valuation of American options by a variational inequality

- for an American Call option one can show that on the whole domain $0 < S < \infty$ and $0 \le t < T$ the following inequality holds:

$$\mathcal{L}[V] \equiv \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\, S \frac{\partial V}{\partial S} - rV \le 0.$$

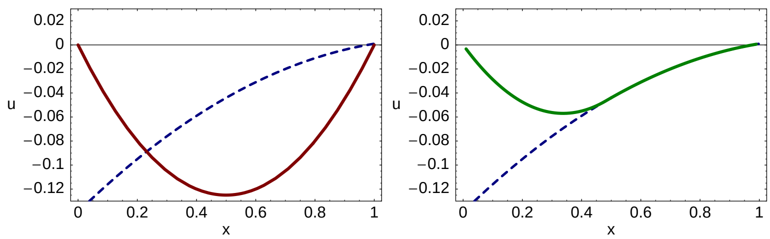
- Comparison with the terminal payoff diagram

$$V(S,t) \ge V(S,T) = \max(S - E, 0).$$

- A variational inequality for American Call option
  - If $V(S,t) > \max(S-E,0) \quad \Rightarrow \quad \mathcal{L}[V](S,t) = 0$
  - If $V(S,t) = \max(S-E,0) \quad \Rightarrow \quad \mathcal{L}[V](S,t) < 0$

American type of options

An analogy with the obstacle problem from the linear elasticity theory.



Left: a solution $\tilde{u}$ of the unconstrained problem $-\tilde{u}''(x) = b(x), \tilde{u}(0) = \tilde{u}(1) = 0$, and the obstacle (dashed line) $g(x)$.
Right: a solution u to the obstacle problem:

- $-u''(x) \ge b(x), u(x) \ge g(x),\ u(0) = u(1) = 0,$

and such that

- if $u(x) > g(x) \quad \Rightarrow \quad -u''(x) = b(x)$
- if $u(x) = g(x) \quad \Rightarrow \quad -u''(x) < b(x)$

Idea of the Project Successive Over Relaxation method

- using the transformation $V(S,t) = Ee^{-\alpha x - \beta\tau}u(x,\tau)$, where $\tau = T - t, x = \ln(S/E)$, leads to the variational inequality

$$\begin{aligned}\left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) &= 0,\\ \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) &\geq 0\end{aligned}$$

  for any $x \in \mathbb{R}, 0 < \tau < T$.
- $g(x,\tau) = e^{\alpha x + \beta\tau}\max(e^x - 1, 0)$ – the transformed pay-off diagram,
- It satisfies the initial condition

$$u(x,0) = g(x,0), \quad \text{for any} \quad x \in \mathbb{R}.$$

Here: $\alpha = \frac{r-D}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r+D}{2} + \frac{\sigma^2}{8} + \frac{(r-D)^2}{2\sigma^2}$

American type of options

Implicit finite difference approximation and transformation to the linear complementarity problem

- spatial and time discretization yields the finite difference mesh

$$x_i = ih, \quad i = ..., -2, -1, 0, 1, 2, ..., \quad \tau_j = jk, \; j = 0, 1, ..., m.$$

  $h = L/n, k = T/m$.
- approximation of the solution u at (x_i, τ_j) will be denoted by

$$u_i^j \approx u(x_i, \tau_j), \qquad \text{and also} \;\; g_i^j \approx g(x_i, \tau_j)$$

- transformation of the boundary condition at $x = \pm L, L \gg 1$,

$$u_{-N}^j = \phi^j := g(x_{-N}, \tau_j), \qquad u_N^j = \psi^j := g(x_N, \tau_j).$$

The linear complementarity problem for a solution of the discretized variational inequality can be rewritten as follows:

$$\begin{aligned}&\mathbf{A}u^{j+1} \geq u^j + b^j, \quad u^{j+1} \geq g^{j+1} \quad \text{for each} \quad j = 0, 1, ..., m-1,\\ &(\mathbf{A}u^{j+1} - u^j - b^j)_i (u^{j+1} - g^{j+1})_i = 0 \quad \text{for each } i,\end{aligned}$$

where $u^0 = g^0$. The matrix $\mathbf{A}$ is a tridiagonal matrix arising from the implicit in time discretization of the parabolic equation $\partial_\tau u = \frac{\sigma^2}{2}\partial_x^2 u$, i.e.

$$\mathbf{A} = \begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 \\ -\gamma & 1+2\gamma & -\gamma & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & -\gamma & 1+2\gamma & -\gamma \\ 0 & \cdots & 0 & -\gamma & 1+2\gamma \end{pmatrix}, \quad b^j = \begin{pmatrix} \gamma\phi^{j+1} \\ 0 \\ \vdots \\ 0 \\ \gamma\psi^{j+1} \end{pmatrix},$$

where $\gamma = \sigma^2 k/(2h^2)$.

American type of options

In each time level the goal is to solve linear complementarity

$$\begin{aligned}\mathbf{A}u \geq b, \quad u &\geq g,\\ (\mathbf{A}u - b)_i (u_i - g_i) &= 0 \quad \text{for each } i.\end{aligned}$$

- We define a recurrent sequence of approximative solution as

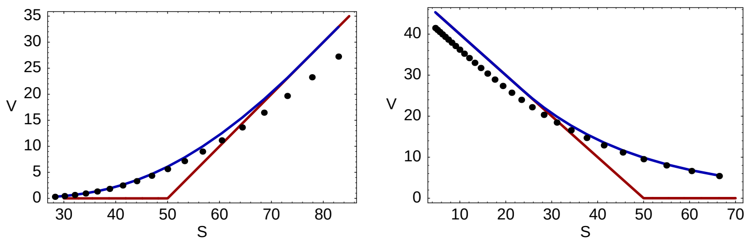
  $$u^0 = 0, \qquad u^{p+1} = \max\left(\mathbf{T}_\omega u^p + c_\omega,\ g\right) \quad \text{for} \quad p = 1, 2, ...,$$

  where the maximum is taken component-wise
- here $\mathbf{T}_\omega$ is the linear iteration operator arising from the classical SOR method for the linear problem $\mathbf{A}u = b$. Here $c_\omega = \omega(\mathbf{D} + \omega\mathbf{L})^{-1} b$
- in terms of vector components, the Projected SOR algorithm reduces to

  $$u_i^{p+1} = \max\left[\frac{\omega}{A_{ii}}\left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p\right) + (1-\omega) u_i^p,\ g_i\right]$$

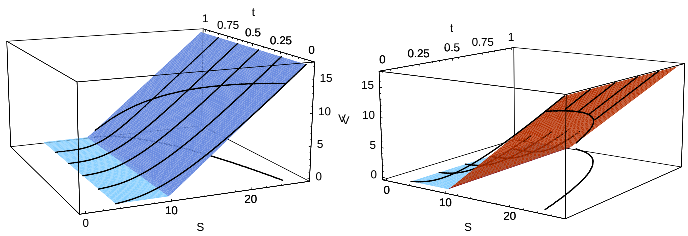
  where $\omega \in (1, 2)$ is a relaxation parameter, typically $\omega \approx 1.8$

A numerical solution to the problem of valuing American Call and Put options by the Projected Successive Over Relaxation method



A solution $S \mapsto V(S,t)$ of an American Call (left) and Put option (right) obtained by solving the variational inequality by means of the Projected SOR (PSOR) algorithm.
Dotted curves corresponds to European type of options

American type of options



Two 3D views on the graph of the solution $(S,t) \mapsto V(S,t)$ for the price of the American Call option. Five selected time profiles and comparison with the terminal pay-off function. One can see the effect of the smooth pasting of the solution to the pay-off function.

Chapter 8

- Modeling transaction costs
- Modeling investor's risk preferences
- Jumping volatility model
- Risk adjusted pricing methodology model
- Numerical approximation scheme

Nonlinear options pricing models

Nonlinear derivative pricing models

Classical Black-Scholes theory does not take into account

- Transaction costs (buying or selling assets, bid-ask spreads)
- Risk from unprotected (non hedged) portfolio
- Other effects
  - feedback effects on the asset price in the presence of a dominant investor
  - utility function effect of investor's preferences

Question: how to incorporate both transaction costs and risk arising from a volatile portfolio into the Black-Scholes equation framework?

- Leland model for pricing Call and Put options under the presence of transaction costs
- Hoggard, Whaley and Wilmott model - generalization to other options

Volatility $\sigma = \sigma(\partial_S^2 V)$ is given by

$$\sigma^2 = \hat{\sigma}^2(1 - \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V))$$

where $\hat{\sigma} > 0$ is a constant historical volatility and $\mathrm{Le} = \sqrt{2/\pi}C/(\hat{\sigma}\sqrt{\Delta t})$ is the Leland number where Δt is time lag between consecutive transactions

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{\sigma^2(\partial_S^2 V, S, t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Transaction costs – Leland model

Transaction costs are described following the Hoggard, Whalley and Wilmott approach (1994) (also referred to as Leland's model (1985))

$$d\Pi = dV + \delta dS - CSk$$

where

- C - the round trip transaction cost per unit dollar of transaction, $C = (S_{ask} - S_{bid})/S$
- k is the number of assets sold or bought during one time lag. Notice that

$$k \approx |\Delta\delta| = |\Delta\partial_S V| \approx |\partial_S^2 V||dS|, \qquad E(|dW|) = \sqrt{\frac{2}{\pi}}\sqrt{dt}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(1 - Le\, sgn(\partial_S^2 V)\right) \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $Le = \sqrt{\frac{2}{\pi}}\frac{C}{\sigma\sqrt{\Delta t}}$ is the so-called Leland number depending on

- C - the round trip transaction cost per unit dollar of transaction, $C = (S_{ask} - S_{bid})/S$
- Δt - the lag between two consecutive portfolio adjustments (re-hedging)

For a plain vanilla option (either Call or Put) the sign of $\partial_S^2 V$ is constant and therefore the above model is just the Black-Scholes equation with lowered volatility.

Frey - Stremme model for a large trader

- Frey and Stremme (1997) introduced directly the asset price dynamics in the case when the large trader chooses a given stock-trading strategy.

Volatility $\sigma = \sigma(\partial_S^2 V, S)$ is given by

$$\sigma^2 = \hat{\sigma}^2 \left(1 - \varrho S \partial_S^2 V\right)^{-2}$$

where $\hat{\sigma}^2, \varrho > 0$ are constants.

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{\sigma^2(\partial_S^2 V, S, t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

- If transaction costs are taken into account perfect replication of the contingent claim is no longer possible
- assuming that investor's preferences are characterized by an exponential utility function Barles and Soner (1998) derived a nonlinear Black-Scholes equation

Volatility $\sigma = \sigma(\partial_S^2 V, S, t)$ is given by

$$\sigma^2 = \hat{\sigma}^2 \left(1 + \Psi(a^2 e^{r(T-t)} S^2 \partial_S^2 V)\right)^2$$

where $\Psi(x) \approx (3/2)^{\frac{2}{3}} x^{\frac{1}{3}}$ for x close to the origin. $\hat{\sigma}^2, \kappa > 0$ are constants.

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{\sigma^2(\partial_S^2 V, S, t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Risk adjusted pricing methodology

- transaction costs are described following the Hoggard, Whalley and Wilmott approach (Leland's model)
- the risk from the unprotected volatile portfolio is described by the variance of the synthetised portfolio.

$$\Downarrow$$

1. Transaction costs as well as the volatile portfolio risk depend on the time-lag between two consecutive transactions.
2. Minimizing their sum yields the optimal length of the hedge interval - time-lag
3. It leads to a fully nonlinear parabolic PDE:

   RAPM model originally proposed by Kratka (1998) and further analyzed by Sevcovic and Jandacka (2005).

Transaction costs under δ - hedging

Transaction costs are described following the Hoggard, Whalley and Wilmott approach (1994)

- adopt $\delta = \frac{\partial V}{\partial S}$ hedging
- construct a portfolio $\Pi = V - \delta S$ donsisting of one option and δ underlying assets
- compare risk part of the portfolio to secure bonds

$$\begin{aligned} & d\Pi &= dV + \delta dS - CSk \\ r(V-\delta S)dt = r\Pi dt = & d\Pi & \end{aligned}$$

where

- C - the round trip transaction cost per unit dollar of transaction, $C = (S_{ask} - S_{bid})/S$
- k is the number of assets sold or bought during one time lag.

$$k \approx |\Delta\delta| = |\Delta\partial_S V| \approx |\partial_S^2 V||dS|, \qquad E(|dW|) = \sqrt{\frac{2}{\pi}}\sqrt{dt}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \left(1 - Le\,\mathrm{sgn}\,(\partial_S^2 V)\right)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $Le = \sqrt{\frac{2}{\pi}}\frac{C}{\hat{\sigma}\sqrt{\Delta t}}$ is the so-called Leland number depending on

- C - the round trip transaction cost per unit dollar of transaction, $C = (S_{ask} - S_{bid})/S$
- Δt - the lag between two consecutive portfolio adjustments (re-hedging)

For a plain vanilla option (either Call or Put) the sign of $\partial_S^2 V$ is constant and therefore the above model is just the Black-Scholes equation with lowered volatility.

Risk adjusted pricing methodology model

- a portfolio Π consists of options and assets $\qquad \Pi = V + \delta S$
- is the portfolio Π is highly volatile an investor usually asks for a price compensation.

Volatility of a fluctuating portfolio can be measured by the variance of relative increments of the replicating portfolio

$$\Downarrow$$

introduce the measure r_{VP} of the portfolio volatility risk as follows:

$$r_{VP} = R\frac{Var\left(\frac{\Delta\Pi}{S}\right)}{\Delta t}\,.$$

- Using Itô's formula the variance of $\Delta\Pi$ can be computed as follows:

$$Var(\Delta\Pi) = \mathbb{E}\left[(\Delta\Pi - E(\Delta\Pi))^2\right]$$
$$= \mathbb{E}\left[\left((\partial_S V + \delta)\,\hat\sigma S\phi\sqrt{\Delta t} + \frac{1}{2}\hat\sigma^2 S^2\Gamma\left(\phi^2 - 1\right)\Delta t\right)^2\right]\,.$$

  where $\phi \approx N(0,1)$ and $\Gamma = \partial_S^2 V$.
- assuming the δ-hedging of portfolio adjustments, i.e. we choose $\delta = -\partial_S V$. For the risk premium r_{VP} we have

$$r_{VP} = \frac{1}{2}R\hat\sigma^4 S^2\Gamma^2\Delta t\,.$$

Balance equation for $\Pi = V + \delta S$

- $d\Pi = dV + \delta dS$
- $d\Pi = r\Pi dt + (r_{TC} + r_{VP})Sdt$

Using Itô's formula applied to $V = V(S,t)$ and eliminating stochastic term by taking $\delta = -\partial_S V$ hedge we obtain

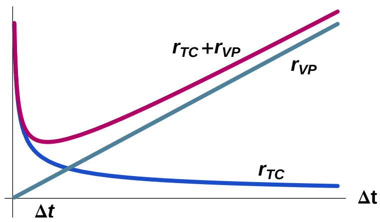
$$\partial_t V + \frac{\hat{\sigma}^2}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = (r_{TC} + r_{VP})S$$

where

- $r_{TC} = \frac{C|\Gamma|\hat{\sigma}S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure
- $r_{VP} = \frac{1}{2} R\hat{\sigma}^4 S^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure

and $\Gamma = \partial_S^2 V$.

Total risk $r_{TC} + r_{VP}$



Tran. costs risk r_{TC} Volatile portfolio risk r_{VP} Total risk $r_{TC} + r_{VP}$

Both r_{TC} and r_{VP} depend on the time lag Δt

$\Downarrow$

Minimizing the total risk with respect to the time lag Δt yields

$$\min_{\Delta t} (r_{TC} + r_{VP}) = \frac{3}{2} \left(\frac{C^2 R}{2\pi} \right)^{\frac{1}{3}} \hat{\sigma}^2 |S\partial_S^2 V|^{\frac{4}{3}}$$

Nonlinear PDE equation for RAPM

$$\partial_t V + \frac{1}{2}\hat{\sigma}^2 S^2 \left(1 - \mu(S\partial_S^2 V)^{1/3}\right)\partial_S^2 V + rS\partial_S V - rV = 0$$

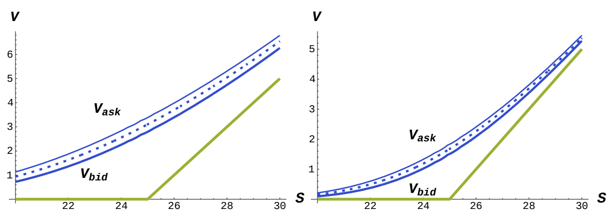
$S > 0, t \in (0, T)$ where
$$\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$$

fully nonlinear parabolic equation

- If $\mu = 0$ (i.e. either $R = 0$ or $C = 0$) the equation reduces to the classical Black-Scholes equation
- minus sign in front of $\mu > 0$ corresponds to Bid option price V_{bid} (price for selling option).

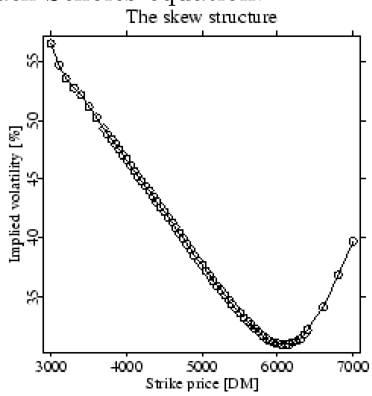
$$\partial_t V + \frac{1}{2}\hat{\sigma}^2 S^2 \left(1 \pm \mu(S\partial_S^2 V)^{1/3}\right)\partial_S^2 V + rS\partial_S V - rV = 0$$

A comparison of Bid ($-$ sign) and Ask (+ sign) option prices computed by means of the RAPM model. The middle dotted line is the option price computed from the Black-Scholes equation.



RAPM and explanation of volatility smile

Volatility smile phenomenon is non-constant, convex behavior (near expiration price E) of the implied volatility computed from classical Black-Scholes equation.



Volatility smile for DAX index

By RAPM model we can explain the volatility smile analytically.

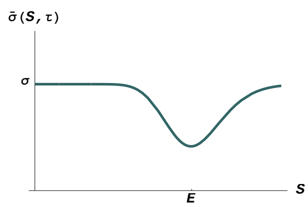
The Risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

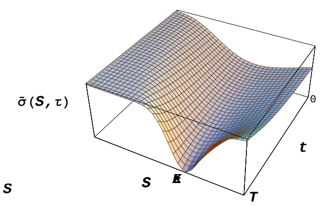
$$\partial_t V + \frac{\sigma^2(S,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = 0$$

where $\sigma^2(S,t)$ depends on a solution $V = V(S,t)$ as follows:

$$\sigma^2(S,t) = \hat{\sigma}^2 \left(1 - \mu(S\partial_S^2 V(S,t))^{1/3}\right) .$$

Dependence of $\sigma(S,t)$ on S is depicted in the left for t close to T. The mapping $(S,t) \mapsto \sigma(S,t)$ is shown in the right.





Numerical scheme for quasilinear equation

- denote $\beta(H) = \frac{\sigma^2}{2}(1 - \mu H^{\frac{1}{3}})H$
- reverse time $\tau = T - t$ (time to maturity)
- use logarithmic scale $x = \ln(S/E)$ $(x \in R \leftrightarrow S > 0)$
- introduce new variable $H(x, \tau) = S\partial_S^2 V(S, t)$

Then the RAPM equation can be transformed into quasilinear equation

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H \qquad \tau \in (0, T), x \in R$$

- Boundary conditions: $H(-\infty, \tau) = H(\infty, \tau) = 0$
- Initial condition: $H(x, 0) = \frac{PDF(d_1)}{\sigma\sqrt{\tau^*}}$ $\quad d_1 = \frac{x + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau^*}}$ where $0 < \tau^* << 1$ is the switching time.

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H \qquad \tau \in (0, T), x \in R$$

$$H_i^j \approx H(ih, jk) \qquad \Downarrow \qquad k = \frac{T}{m}, \quad h = \frac{L}{n}$$

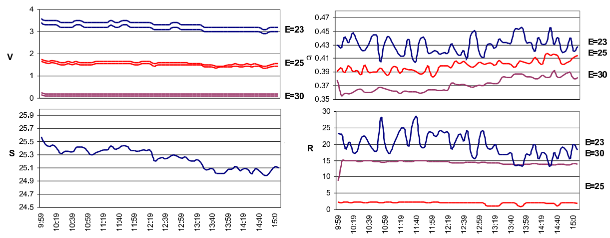
$$a_i^j H_{i-1}^j + b_i^j H_i^j + c_i^j H_{i+1}^j = d_i^j, \quad H_{-n}^j = 0, \quad H_n^j = 0\,,$$

for $i = -n + 1, ..., n - 1$, and $j = 1, ..., m$, where $H_i^0 = H(x_i, 0)$

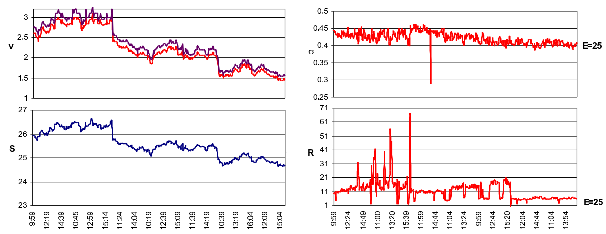
$$\begin{aligned} a_i^j &= -\frac{k}{h^2}\beta'(H_{i-1}^{j-1}) + \frac{k}{h}r\,, \qquad b_i^j = 1 - (a_i^j + c_i^j)\,, \\ c_i^j &= -\frac{k}{h^2}\beta'(H_i^{j-1})\,, \quad d_i^j = H_i^{j-1} + \frac{k}{h}\left(\beta(H_i^{j-1}) - \beta(H_{i-1}^{j-1})\right)\,. \end{aligned}$$

Calibration of RAPM model

Intra-day behavior of Microsoft stocks (April 4, 2003) and shortly expiring Call options with expiry date April 19, 2003. Computed implied volatilities σ_{RAPM} and risk premium coefficients R.



One week behavior of Microsoft stocks (March 20 - 27, 2003) and Call options with expiration date April 19, 2003. Computed implied volatilities σ_{RAPM} and risk premiums R.



Jumping volatility nonlinear model

Avellaneda, Levy and Paras proposed a model is to describe option pricing in incomplete markets where the volatility σ of the underlying stock process is uncertain but bounded from bellow and above by given constants $\sigma_1 < \sigma_2$.

- Avellaneda, Levy and Paras nonlinear extension of the Black–Scholes equation

$$\frac{\partial V}{\partial t} + (r-D)S\frac{\partial V}{\partial S} + \frac{\sigma^2(\partial_S^2 V)}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

- where the volatility depends on the sign of $\Gamma = \partial_S^2 V$

$$\sigma^2(S^2\partial_S^2 V) = \begin{cases} \hat{\sigma}_1^2, & \text{if } \partial_S^2 V < 0, \\ \hat{\sigma}_2^2, & \text{if } \partial_S^2 V > 0. \end{cases}$$

Similarly as in previously studied nonlinear Black–Scholes models, we can introduce the new variable $H(x,\tau) = S\partial_S^2 V$, where $x = \ln(S/E)$ and $\tau = T - t$. We obtain

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial \beta}{\partial x} + r\frac{\partial H}{\partial x}\,,$$

where $\beta = \beta(H(x,\tau))$ is given by

$$\beta(H) = \begin{cases} \frac{\hat{\sigma}_1^2}{2}H & \text{if } H < 0, \\ \\ \frac{\hat{\sigma}_2^2}{2}H & \text{if } H > 0. \end{cases}$$

We have to impose the boundary conditions corresponding to the limits $S \to 0$ ($x \to -\infty$) and $S \to \infty$ ($x \to +\infty$) for $H(x,\tau) = S\partial_S^2 V$,

$$H(-\infty,\tau) = H(\infty,\tau) = 0\,, \quad \tau \in (0,T)\,.$$

Results of numerical approximation of the jumping volatility model for the case of the bullish spread.

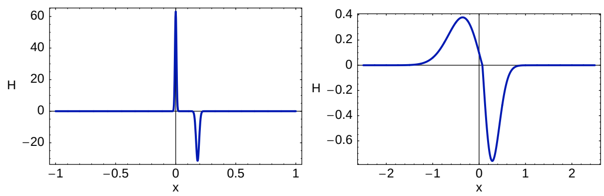
- bullish spread strategy = buying one Call option with exercise price $E = E_1$ and selling one Call option with $E_2 > E_1$

$$V(S,T) = (S - E_1)^+ - (S - E_2)^+.$$

- in terms of the transformed variable H we have As for the initial condition we have

$$H(x,0) = \delta(x - x_0) - \delta(x - x_1), \qquad x \in \mathbb{R},$$

where $x_0 = 0, x_1 = \ln(E2/E1)$.



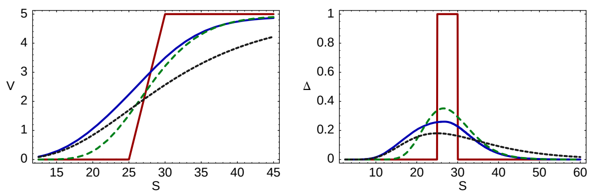
Plots of the initial approximation of the function $H(x,0)$ (left) and the solution profile $H(x,T)$ at $\tau = T$ (right).

Jumping volatility nonlinear model

Transforming back to the original variable $V(S,t)$ we obtain from $S\partial_S^2 V = H(x,\tau)$ where $x = \ln(S/E)$ and $\tau = T - t$ that

$$V(S,t) = \int_{-\infty}^{\infty} (S - Ee^x)^+ H(x, T - t) dx,$$

where $E = E_1$.



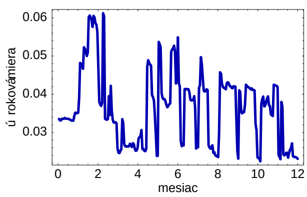
A comparison of the Call option price $V(S,0)$ (left) and its delta (right) computed from the jumping volatility model (solid line) by the linear Black–Scholes. Option prices obtained from the linear Black–Scholes equation are depicted by dashed curved (for volatility σ_1) and fine-dashed curve (for volatility σ_2).

Chapter 9

- A stochastic differential equation for modeling the short interest rate process
- Vašíček and Cox-Ingersoll–Ross models for the short rate process
- Interest rate derivatives – zero coupons bonds
- Pricing interest rate derivatives by means of a solution to the parabolic partial differential equation

Interest rate derivatives derivatives

Modeling the short rate (overnight) stochastic process



Daily behavior of the overnight interest rate of BRIBOR in 2007.

- modeling the short rate $r = r(t)$ by a solution to a one factor stochastic differential equation

$$dr = \mu(t, r)dt + \sigma(t, r)dw.$$

  - $\mu(t, r)dt$ represents a trend or drift of the process
  - $\sigma(t, r)$ represents a stochastic fluctuation part of the process

Modeling the short rate (overnight) stochastic process

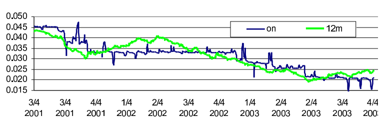
- Among short rate models the dominant position have the mean-reversion processes in which $\mu(t, r) = \kappa(\theta - r)$. The solution (if $\sigma = 0$) is therefore attracted to the stable equilibrium θ as $t \to \infty$.
- A short overview of one factor interest rate models

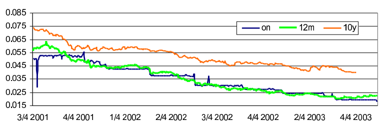
Model	Stochastic equation for r
Vašíček	$dr = \kappa(\theta - r)dt + \sigma dw$
Cox–Ingersoll–Ross	$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$
Dothan	$dr = \sigma r dw$
Brennan–Schwarz	$dr = \kappa(\theta - r)dt + \sigma r dw$
Cox–Ross	$dr = \beta r dt + \sigma r^{\gamma} dw$

Interest rate derivatives derivatives

Modeling the short rate (overnight) stochastic process

Oldřich Alfons Vašíček, graduated from FJFI and Charles University in Prague





EUROLIBOR Short-rate (overnight) and 1 year interest rates PRIBOR

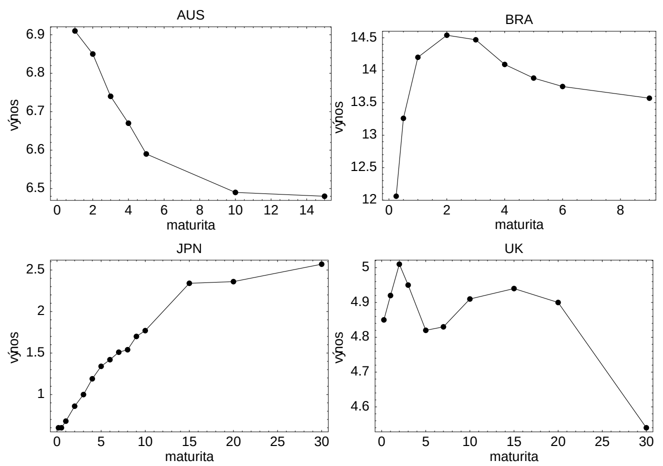
Bond – a derivative of the underlying short rate process

- Term structure models describe a functional dependence between the time to maturity of a discount bond and its present price
- Yield of bonds, as a function of maturity, forms the so-called term structure of interest rates
- If we denote by $P = P(t,T)$ the price of a bond paying no coupons at time t with maturity at T then the term structure of yields $R(t,T)$ is given by

$$P(t,T) = e^{-R(t,T)(T-t)}, \quad \text{i.e. } R(t,T) = -\frac{\log P(t,T)}{T-t}$$

Interest rate derivatives derivatives

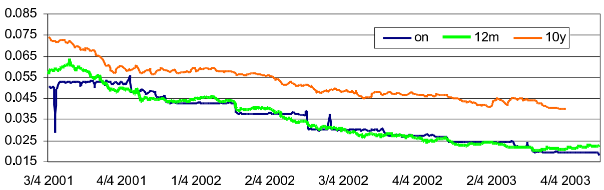
The yield curves $R(t,T)$



The term structure (the yield curve) $R(t,T)$ of governmental bonds in % p.a. from t =27.5.2008 as a function of the yield R with respect to the time to maturity $T-t$.

Australia, Brazil, Japan United Kingdom.

The time dependence yields and short (overnight) rates



PRIBOR: Short-rate (overnight) and 1 year interest rates

PRIBOR = PRague Interbank Offering Rate

- The goal is to find a functional dependence of the yield R and the underlying short rate r
- $$P = P(r,t,T) = P(r,T-t)$$
  where
  $$R(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

Interest rate derivatives derivatives

Modeling the bond price by a solution to a PDE

- Suppose that the underlying short rate process follows the SDE:
  $$dr = \tilde{\mu}(t,r)dt + \tilde{\sigma}(t,r)dw.$$
  - for the Vašíček model: $dr = \kappa(\theta - r)dt + \sigma dw$
  - for the Cox–Ingersoll–Ross model: $dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$
- Suppose that the price of a zero coupon bond P is a smooth function $P = P(r,t,T)$ of the short rate r, actual time t and the maturity time T $(t < T)$.
- by Itō's lemma we have
  $$dP = \underbrace{\left(\frac{\partial P}{\partial t} + \tilde{\mu}\frac{\partial P}{\partial r} + \frac{\tilde{\sigma}^2}{2}\frac{\partial^2 P}{\partial r^2}\right)}_{\mu_B(t,r)} dt + \underbrace{\tilde{\sigma}\frac{\partial P}{\partial r}}_{\sigma_B(t,r)} dw$$
  where $\mu_B(r,t)$ and $\sigma_B(r,t)$ stand for the drift and volatility of the bond price

Modeling the bond price by a solution to a PDE

- Construct a portfolio from two bonds with two different maturities T_1 and T_2
- It consits of one bond with maturity T_1 and Δ − bonds with maturity T_2
- Its value is therefore $\pi = P(r, t, T_1) + \Delta P(r, t, T_2)$
- the change of the portfolio $d\pi$ is equal to:

$$\begin{aligned} d\pi &= dP(r, t, T_1) + \Delta dP(r, t, T_2) \\ &= (\mu_B(r, t, T_1) + \Delta\mu_B(r, t, T_2))\, dt \\ &\quad +(\sigma_B(r, t, T_1) + \Delta\sigma_B(r, t, T_2))\, dw. \end{aligned}$$

Interest rate derivatives derivatives

Modeling the bond price by a solution to a PDE

- similarly as in the case of options our goal is to eliminate the volatile (fluctuating) part of the portfolio of bonds (tenor)
- it can be accomplished by taking

$$\Delta = -\frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)}$$

- then the differential of the risk-neutral portfolio of bonds (tenor)

$$d\pi = \left(\mu_B(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)}\mu_B(t, r, T_2)\right) dt.$$

- to avoid the possibility of arbitrage the yield of the portfolio should be equal to the risk-less short interest rate r, i.e. $d\pi = r\pi dt$. Therefore

$$\mu_B(t, r, T_1) - \frac{\sigma_B(t, r, T_1)}{\sigma_B(t, r, T_2)}\mu_B(t, r, T_2) = r\pi\,.$$

Modeling the bond price by a solution to a PDE

- inserting the value of the portfolio π we obtain

$$\mu_B(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}\mu_B(t,r,T_2)$$
$$= r\left(P(t,r,T_1) - \frac{\sigma_B(t,r,T_1)}{\sigma_B(t,r,T_2)}P(t,r,T_2)\right).$$

- Since maturities T_1 and T_2 were arbitrary we may conclude that there is a common value $\tilde{\lambda}$ such that

$$\tilde{\lambda}(r,t) = \frac{\mu_B(r,t,T) - rP(r,t,T)}{\sigma_B(r,t,T)} \quad \text{for any } T > t.$$

- $\tilde{\lambda}$ may depend on r but not on the maturity T, i.e. $\tilde{\lambda} = \tilde{\lambda}(r)$.

Interest rate derivatives derivatives

Modeling the bond price by a solution to a PDE

- ReCall that

$$\begin{aligned} \mu_B(t,r) &= \frac{\partial P}{\partial t} + \tilde{\mu}\frac{\partial P}{\partial r} + \frac{\tilde{\sigma}^2}{2}\frac{\partial^2 P}{\partial r^2} \\ \sigma_B(t,r) &= \tilde{\sigma}\frac{\partial P}{\partial r} \end{aligned}$$

  where we supposed that the underlying short rate process follows the SDE: $dr = \tilde{\mu}(t,r)dt + \tilde{\sigma}(t,r)dw$.

- In summary, we can deduce the parabolic PDE for the zero coupon bond price

$$\frac{\partial P}{\partial t} + (\tilde{\mu}(r,t) - \tilde{\lambda}(r,t)\tilde{\sigma}(r,t))\frac{\partial P}{\partial r} + \frac{\tilde{\sigma}^2(r,t)}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0.$$

- At the maturity $t = T$ the price of the bond is prescribed and it is independent of the short rate r, i.e.

$$P(r,T,T) = 1 \quad \text{for any} \quad r > 0.$$

Modeling the bond price by a solution to a PDE

- for the Vašíček model where $dr = \kappa(\theta - r)dt + \sigma dw$ we take $\tilde{\lambda}(r,t) \equiv \lambda$ and we obtain the PDE:

$$-\frac{\partial P}{\partial \tau} + (\kappa(\theta - r) - \lambda\sigma)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial P}{\partial r^2} - rP = 0$$

- for the Cox–Ingersoll–Ross model where $dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$ we take $\tilde{\lambda}(r,t) = \lambda\sqrt{r}$ and we obtain the PDE:

$$-\frac{\partial P}{\partial \tau} + (\kappa(\theta - r) - \lambda\sigma r)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2} r \frac{\partial^2 P}{\partial r^2} - rP = 0,$$

- In both models $\tau = T - t$ stands for the time remaining to maturity of the bond

Interest rate derivatives derivatives

An explicit solution for the Cox–Ingersoll–Ross model

- construct a solution in the form $P(r,\tau) = A(\tau)e^{-B(\tau)r}$
- inserting this ansatz into the CIR equation and comparing the terms of the order 1 and r we obtain

$$\begin{aligned} \dot{A} + \kappa\theta AB &= 0, \\ \dot{B} + (\kappa + \lambda\sigma)B + \frac{\sigma^2}{2}B^2 - 1 &= 0, \end{aligned}$$

- functions A, B satisfy initial conditions $A(0) = 1$, $B(0) = 0$
- the explicit solution to the system of ODEs for A, B is:

$$B(\tau) = \frac{2\left(e^{\phi\tau} - 1\right)}{(\psi + \phi)\left(e^{\phi\tau} - 1\right) + 2\phi},$$

$$A(\tau) = \left(\frac{2\phi e^{(\phi+\psi)\tau/2}}{(\phi + \psi)(e^{\phi\tau} - 1) + 2\phi}\right)^{\frac{2\kappa\theta}{\sigma^2}},$$

where $\psi = \kappa + \lambda\sigma$, $\phi = \sqrt{\psi^2 + 2\sigma^2} = \sqrt{(\kappa + \lambda\sigma)^2 + 2\sigma^2}$.

An explicit solution for the Vašíček model

- construct a solution in the form $P(r,\tau) = A(\tau)e^{-B(\tau)r}$
- one can construct an analogous system of ODEs for functions A, B
- the explicit solution of the system of ODEs yields:

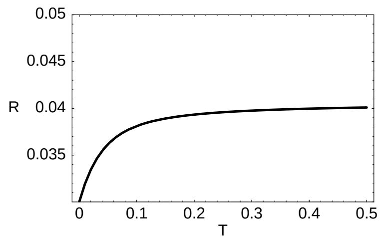
$$B(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa},$$

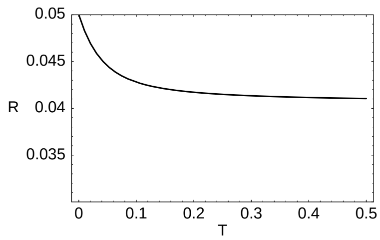
$$\ln A(\tau) = \left[\frac{1}{\kappa}(1 - e^{-\kappa\tau}) - \tau\right] R_\infty - \frac{\sigma^2}{4\kappa^3}(1 - e^{-\kappa\tau})^2,$$

where $R_\infty = \theta - \frac{\lambda\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}$.

Interest rate derivatives derivatives

An explicit solution for the Vašíček model





The term structure of interest rates $R(r,t,T)$ on bonds computed by the Vašíček model for two different values of the short rate r ($r = 0.03$ and $r = 0.05$) at given time $t < T$.

Appendix

- Stochastic differential calculus
- Density distribution function and the Fokker–Planck equation
- Multidimensional extension of Itō's lemma

Itō's lemma and Fokker–Planck equation

- Suppose that a process $\{x(t), t \geq 0\}$ follows a SDE (It$\bar{0}$'s process)
$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$
where μ a drift function and σ is a volatility of the process.
- Denote by
$$G = G(x,t) = P(x(t) < x \quad | \quad x(0) = x_0)$$
the conditional probability distribution function of the process $\{x(t), t \geq 0\}$ starting almost surely from the initial condition x_0.
- Then the cumulative distribution function G can be computed from its density function $g = \partial G/\partial x$ where $g(x,t)$ is a solution to the Fokker–Planck equation:
$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right), \quad g(x,0) = \delta(x - x_0).$$

Here $\delta(x - x_0)$ is the Dirac function with support at x_0. It means:

$$\delta(x - x_0) = \begin{cases} 0 & \text{if } x \neq x_0, \\ +\infty & \text{if } x = x_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x - x_0) dx = 1.$$

In our case we have, at the origin $t = 0$,

$$G(x, 0) = \int_{-\infty}^{x} \delta(\xi - x_0) d\xi = \begin{cases} 0 & \text{if } x < x_0, \\ 1 & \text{if } x \geq x_0, \end{cases}$$

so the process $\{x(t), t \geq 0\}$ at $t = 0$ is almost surely equal to x_0.

Itō's lemma and Fokker–Planck equation

Intuitive proof of the Fokker-Planck equation:

- Let $V = V(x, t)$ be any smooth function with a compact support, i.e. $V \in C_0^\infty(\mathbb{R} \times (0, T))$
- By Itō's lemma we have

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) dt + \sigma \frac{\partial V}{\partial x} dW.$$

- Let E_t be the mean value operator with respect to the random variable having the density function $g(., t)$, i.e.

$$E_t(V(., t)) = \int_{\mathbb{R}} V(x, t)\, g(x, t)\, dx$$

Then

$$dE_t(V(.,t)) = E_t(dV(.,t)) = E_t\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right) dt.$$

because random variables $\sigma(.,t)\frac{\partial V}{\partial x}(.,t)$ and $dW(t)$ are independent and $\mathbb{E}(dW(t)) = 0$. Therefore

$$E_t\left(\sigma(.,t)\frac{\partial V}{\partial x}(.,t)dW(t)\right) = 0$$

Itō's lemma and Fokker–Planck equation

- Since $V \in C_0^\infty$ we have $V(x,0) = V(x,T) = 0$ and $V(x,t) = 0$ for $|x| > R$, where $R > 0$ is sufficiently large.
- By integration by parts we obtain

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}E_t(V(.,t))dt = \int_0^T E_t\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right) dt \\ &= \int_0^T \int_{\mathbb{R}} \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right) g(x,t)\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}} V(x,t)\left(-\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}(\mu g)\right) dx\, dt. \end{aligned}$$

- Since $V \in C_0^\infty(\mathbb{R} \times (0,T))$ is an arbitrary function we obtain the Fokker–Planck equation for the density $g = g(x,t)$:

$$-\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}(\mu g) = 0, \quad x \in \mathbb{R}, t > 0,$$
$$g(x,0) = \delta(x - x_0), \quad x \in \mathbb{R}.$$

- Example: $dx = dW$ and $x(0) = 0$ a.s.
  It means $x(t)$ is a Wiener process
- The Fokker–Planck (diffusion) equation reads as follows:

$$\frac{\partial g}{\partial t} - \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0,$$

- Its solution (normalized to be a probabilistic density)

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

is indeed a density function of the normal random variable $W(t) \sim N(0,t)$

Itō's lemma and Fokker–Planck equation

- Example: $dr = \kappa(\theta - r)dt + \sigma dW$ and and $r(0) = r_0$.
  This is the so-called Ornstein-Uhlenbeck mean reversion process used arising the modeling of the the rate interest rate stochastic process $\{r(t), t \geq 0\}$.
- The Fokker–Planck equation reads as follows:

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}\left(\kappa(\theta - r)f\right)$$

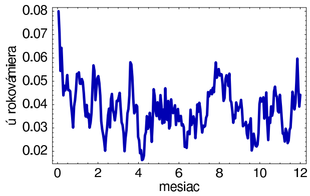
- Its solution (normalized to be a probabilistic density function)

$$f(r,t) = \frac{1}{\sqrt{2\pi\bar{\sigma}_t^2}} e^{-\frac{(r-\bar{r}_t)^2}{2\bar{\sigma}_t^2}}$$

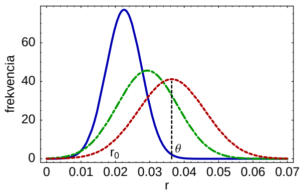
is the density function for the normal random variable $r(t) \sim N(\bar{r}_t, \bar{\sigma}_t^2)$ satisfying the above SDE. Here

$$\bar{r}_t = \theta(1 - e^{-\kappa t}) + r_0 e^{-\kappa t}, \quad \bar{\sigma}_t^2 = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}).$$

- Simulation of the process $r(t)$ satisfying $dr = \kappa(\theta - r)dt + \sigma dW$ and $r(0) = r_0 = 0.08$. Here $\theta = 0.04$.



- Time steps of the evolution of the density function $f(r,t)$ for various times t. The process $r(t)$ started from $r_0 = 0.02$. The limiting value $\theta = 0.04$.



Shift of the density function $f(r,t)$ is due to the drift in the F-P equation

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}(\kappa(\theta - r)f)$$

Multidimensional Itō's lemma

- Multidimensional stochastic processes

$$dx_i = \mu_i(\vec{x}, t)dt + \sum_{k=1}^{n} \sigma_{ik}(\vec{x}, t)dw_k ,$$

where $\vec{w} = (w_1, w_2, ..., w_n)^T$ is a vector of Wiener processes having mutually independent increments

$$\mathbb{E}(dw_i\, dw_j) = 0 \text{ for } i \neq j , \quad \mathbb{E}((dw_i)^2) = dt .$$

- It can be rewritten in a vector form

$$d\vec{x} = \vec{\mu}(\vec{x}, t)dt + K(\vec{x}, t)d\vec{w} ,$$

where $\vec{x} = (x_1, x_2, ..., x_n)^T$ and K is an $n \times n$ matrix

$$K(\vec{x}, t) = (\sigma_{ij}(\vec{x}, t))_{i,j=1,...,n}.$$

- Expanding a smooth function $f = f(\vec{x}, t) = f(x_1, x_2, ..., x_n, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ into the second order Taylor series yields:

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \nabla_x f . d\vec{x} \\ &+ \frac{1}{2} \left((d\vec{x})^T \nabla_x^2 f \, d\vec{x} + 2 \frac{\partial f}{\partial t} . \nabla_x f d\vec{x} \, dt + \frac{\partial^2 f}{\partial t^2} (dt)^2 \right) + \text{ h.o.t.} \end{aligned}$$

- The term $(d\vec{x})^T \nabla_x^2 f \, d\vec{x} = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \, dx_j$ can be expanded using the relation between processes x_i and x_j

$$dx_i \, dx_j = \sum_{k,l=1}^n \sigma_{ik} \sigma_{jl} dw_k \, dw_l + O((dt)^{3/2}) + O((dt)^2)$$

$$\approx (\sum_{k=1}^n \sigma_{ik} \sigma_{jk}) dt + O((dt)^{3/2}) + O((dt)^2) \quad \text{as} \quad dt \to 0.$$

Multidimensional Itō's lemma

- The multidimensional Itō's lemma gives the SDE for the composite function $f = f(\vec{x}, t)$ in the form:

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2} K : \nabla_x^2 f \, K \right) dt + \nabla_x f \, d\vec{x}$$

  where

$$K : \nabla_x^2 f \, K = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \sum_{k=1}^n \sigma_{ik} \sigma_{jk}$$

- By following the same procedure of as in the scalar case we obtain, for the joint density distribution function $g(x_1, x_2, ..., x_n, t)$,

  $g(x_1, x_2, ..., x_n, t) = P(x_1(t) = x_1, x_2(t) = x_2, ..., x_n(t) = x_n, t)$

  conditioned to the initial condition state $x_1(0) = x_1^0, x_2(0) = x_2^0, ..., x_n(0) = x_n^0$ that:

$$\frac{\partial g}{\partial t} + \mathrm{div}(\vec{\mu} g) = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} \frac{\partial^2 g}{\partial x_i \partial x_j}$$

$$g(\vec{x}, 0) = \delta(\vec{x} - \vec{x}^0),$$

Fokker–Planck equation in the multidimensional case

Multidimensional Itō's lemma and Fokker-Planck equation

- Example: The multidimensional Fokker–Planck equation for a system of uncorrelated SDE's

$$\begin{aligned} dx_1 &= \mu_1(\vec{x}, t)dt + \bar{\sigma}_1 dw_1 \\ dx_2 &= \mu_2(\vec{x}, t)dt + \bar{\sigma}_2 dw_2 \\ \vdots \quad & \vdots \qquad \vdots \\ dx_n &= \mu_n(\vec{x}, t)dt + \bar{\sigma}_n dw_n \end{aligned}$$

  with mutually independent increments of Wiener processes

$$\mathbb{E}(dw_i\, dw_j) = 0 \text{ for } i \neq j\,, \quad \mathbb{E}((dw_i)^2) = dt\,.$$

- The Fokker–Planck equations reads as follows:

$$\frac{\partial g}{\partial t} + \mathrm{div}(\vec{\mu} g) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \left(\bar{\sigma}_i^2 g\right)$$

  This is a scalar parabolic reaction–diffusion equation for g

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