Existence and limiting behaviour for damped nonlinear evolution equations with nonlocal terms

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Abstract. In this paper we investigate both the existence and the long time behaviour of solutions to damped nonlinear evolution equations with nonlocal terms

$$u_{tt} + \beta A u_t + f(\|A^{1/2}u\|^2)Au + A^2u + g(u) = h$$

where A is a sectorial operator. If f and g satisfy certain regularity assumptions then a local existence of solutions is guaranteed. We will give a global existence result for the case where $A = -\Delta$. Furthermore we will establish that there exists a maximal compact attractor.

Keywords: nonlinear evolution equations with nonlocal terms, sectorial operator, analytic semigroup, dissipative semidynamical system, maximal attractor

Classification: Primary 35G25, Secondary 35B40

1. Introduction.

In the present paper we are interested in nonlinear damped evolution equations with nonlocal terms. We will investigate both the existence and the long time behaviour of solutions to

$$u_{tt} + \beta A u_t + f(\|A^{1/2}u\|^2)Au + A^2u + g(u) = h \tag{1}$$

subjected to the initial conditions $u(0) = u_0$, $u_t(0) = v_0$ where A is a sectorial operator in a Banach space X (with the norm $\|\ \|$), $h \in X$, β is a positive constant, g is a nonlinear operator from $D(A)$ into X satisfying certain regularity and growth assumptions and $f\colon \mathbf{R}^+ \longrightarrow \mathbf{R}$ is an increasing locally Lipschitz continuous function. The nonlocal character of (1) is described by the term $f(\|A^{1/2}u\|^2)Au$.

As an example for (1) we can consider an initial–boundary value problem

$$\frac{\partial^2 u}{\partial^2 t} - \beta \,\Delta \frac{\partial u}{\partial t} - f\left(\int_\Omega \nabla^2 u dx\right) \cdot \Delta u + \Delta^2 u + g(u) = h$$

$$u(0,x) = u_0(x),\ \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad \text{for a.e. } x \in \Omega$$

$$u(t,x) = 0, \quad x \in \partial\Omega,\ t \geq 0$$

$$\Delta u(t,x) = 0, \quad x \in \partial\Omega,\ t > 0 \tag{2}$$

Here Ω is a smooth bounded domain in $\mathbf{R}^n$, $n = 1,2,3$; and g is the Nemitzky's operator from $H_0^1(\Omega) \cap H^2(\Omega)$ into $L_2(\Omega)$. For the case where $f(0) \geq 0$, $g \equiv 0$,

$h \equiv 0$, the exponential decay of solutions to (2) has been studied by P. Biller [**3**]. A common example where this type of equations arises is in the mathematical study of structurally damped nonlinear vibrations of a string or a beam. For related problems, similar to ours, we refer to Ball [**2**], Fitzgibbon [**5**], Ghidaglia and Temam [**6**], Hale and Stavrakakis [**7**], Massat [**9**], Webb [**11**].

2. Preliminaries.

Let X be a Banach space with the norm $\| \ \|$. A linear operator A in X is called *a sectorial operator* if it is a closed densely defined operator such that for some constants $M \geq 1$, $\theta \in (0, \pi/2)$ and $\delta \in \mathbf{R}$ the sector $S_{\delta,\theta} = \{\lambda \in \mathbf{C};\, \theta < |\arg(\lambda - \delta)| \leq \pi;\, \lambda \neq \delta\}$ is in the resolvent set $\rho(A)$ and $\|(\lambda - A)^{-1}\| \leq M/|\lambda - \delta|$ for all $\lambda \in S_{\delta,\theta}$.

The assumptions $\operatorname{Re}\sigma(A) > \delta > 0$ (it means $\operatorname{Re}\lambda > \delta$ for all $\lambda \in \sigma(A)$) and A is sectorial imply that fractional powers A^α, $\alpha \in \mathbf{R}$, can be defined. Let X^α be a Banach space consisting of the domain $D(A^\alpha)$ with the graph norm $\| \ \|_\alpha$, i.e. $\|u\|_\alpha = \|A^\alpha u\|$ for all $u \in X^\alpha$. Furthermore X^α is continuously imbedded into X^β whenever $\alpha \geq \beta \geq 0$ and $\|u\|_\beta \leq \|A^{\beta-\alpha}\| \cdot \|u\|_\alpha$ for each $u \in X^\alpha$.

(3)

It is known that if A is sectorial operator then $-A$ generates *an analytic semigroup* $\exp(-At)$. This family of continuous linear operators defined on X satisfies to

$$\begin{aligned} &\exp(-A(t+s)) = \exp(-At) \circ \exp(-As) \quad \text{for all } t,\, s \geq 0 \\ &\exp(-At)x \longrightarrow x \text{ as } t \longrightarrow 0^+, \text{for each } x \in X \end{aligned}$$

$$\text{the map } t \longrightarrow \exp(-At)x \text{ is real analytic on } (0,\infty), \quad \text{for each } x \in X \tag{4}$$

Moreover, $\exp(-At)x \in X^\alpha$ for all $x \in X$, $t > 0$ and $\alpha \in \mathbf{R}$. For each $\alpha \in (0,1]$ there is $C_\alpha > 0$ such that $\|A^\alpha \exp(-At)\| \leq C_\alpha t^{-\alpha} \exp(-\delta t)$ for all $t > 0$. If A^{-1} is a compact linear operator on X then $A^{-\alpha}$ is compact for each $\alpha > 0$. For the theory of analytic semigroups and fractional powers of sectorial operators see, for example, [**8**, Chapter 1].

In order to understand the results in section 4 and 5 we need following definitions each of which can be found in [**1**], [**7**], [**8**] and [**9**]. Let $\mathcal{X}$ be a Banach space. Let $\{S(t); t \geq 0\}$ be a *a semidynamical system* in $\mathcal{X}$ in the sense that

$$\begin{gathered} S(t) \text{ is a continuous mapping from } \mathcal{X} \text{ into } \mathcal{X} \text{ for each } t \geq 0 \\ S(\,.\,)x \text{ is continuous as a function from } [0,\infty) \text{ to } \mathcal{X}, \\ \text{for each fixed } x \in \mathcal{X} \\ S(0) = \mathrm{Id},\ S(t+s) = S(t) \circ S(s) \text{ for all } t,\, s \geq 0 \end{gathered}$$

A set $J \subseteq \mathcal{X}$ is called *invariant* if $S(t)J = J$ for all $t \geq 0$. An invariant set $\mathcal{U} \subseteq \mathcal{X}$ is called *a maximal attractor* for the semidynamical system $S(t)$ iff it is a closed bounded set in $\mathcal{X}$ and $\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{U}) = 0$ for any bounded set $B \subseteq \mathcal{X}$, where

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \sup_{x\in\mathcal{A}} \inf_{y\in\mathcal{B}} \|x - y\| .$$

A set B *dissipates* a set J if there exists $T = T(J) > 0$ such that $t \geq T$ implies $S(t)J \subseteq B$.

A semidynamical system $S(t)$ is called *point* (*compact*, *bounded*) *dissipative* if there exists a bounded set B which dissipates all points (compact sets, bounded sets).

The semiorbit of a set B is defined by $\gamma^+(B) = \bigcup_{t\geq 0} S(t)B$.

The omega-limit set is defined by

$$\Omega(B) = \bigcap_{t\geq 0} \operatorname{cl}\Big(\bigcup_{s\geq t} S(s)B\Big) \qquad \text{(the closure is taken in } \mathcal{X}\text{)}$$

Denote by $\mathcal{N}_\varepsilon(B) = \{y \in \mathcal{X}; \operatorname{dist}(y, B) < \varepsilon\}$

3. Local existence.

The problem (1) can be considered as an abstract first order ordinary differential equation in a Banach space $\mathcal{X}$. This is to do by letting $v = u_t$. Then we can rewrite (1) as

$$\frac{d}{dt}\Phi(t) + L\Phi(t) + \mathcal{F}(\Phi(t)) = 0; \quad \Phi(0) = \Phi_0 \tag{5}$$

where

$$\Phi(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}; \qquad L = \begin{pmatrix} 0, & -\mathrm{Id} \\ A^2, & \beta A \end{pmatrix}$$

and

$$\mathcal{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f(\|u\|_{1/2}^2)Au + g(u) - h \end{bmatrix} \tag{6}$$

The initial value problem (5) is considered in a Banach space $\mathcal{X} = X^1 \times X$ with the norm $\left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_x^2 = \|u\|_1^2 + \|v\|^2$. The domain $D(L)$ of the linear operator L is defined by $D(L) = D(A^2) \times D(A)$.

In this section we obtain a local existence of solutions to (5). Moreover, we will examine uniqueness, continuation and continuous dependence. We will prove these results by following the style of Henry's lecture notes [8].

Throughout this section we assume the following hypotheses

(H1)

(i) A is a sectorial operator in a Banach space X with a sector $S_{0,\theta}$ and $\operatorname{Re}\sigma(A) > 0$.

(ii) $\beta > 2 \cdot \sin\theta$

(iii) $f\colon [0,\infty) \longrightarrow \mathbf{R}$ is locally Lipschitz continuous, $g\colon X^1 \longrightarrow X$ is lipschitzian on bounded sets of X^1, $h \in X$

Theorem 3.1. *The assumptions are those made above, in particular for A and β. Then L is a sectorial operator in $\mathcal{X}$ and $-L$ generates an analytic semigroup $\exp(-Lt)$ on $\mathcal{X}$.*

PROOF : We will prove that L is the sectorial operator in $\mathcal{X}$. Clearly, L is the closed and densely defined operator in $\mathcal{X}$. Denote by β_1 and β_2 the roots of a quadratic equation

$$(8) \qquad r^2 - \beta r + 1 = 0, \text{ i.e. } \beta_{1,2} = (\beta \pm (\beta^2 - 4)^{1/2})/2$$

Formally we can compute the resolvent

$$(9) \quad (\lambda - L)^{-1} = \begin{pmatrix} \lambda, & \mathrm{Id} \\ -A^2, & \lambda - \beta A \end{pmatrix} = \begin{pmatrix} \lambda - \beta A, & \text{-Id} \\ A^2 & \lambda \end{pmatrix} \cdot [\lambda(\lambda - \beta A) + A^2]^{-1} =$$

$$= \begin{pmatrix} \lambda - \beta A, & \text{-Id} \\ A^2, & \lambda \end{pmatrix} \cdot (\lambda\beta_1 - A)^{-1} \cdot (\lambda\beta_2 - A)^{-1}$$

Assume that $\lambda\beta_1,\ \lambda\beta_2 \in \rho(A)$. Then the formal computation in (9) can be justified using the fact that $(\mu - A)^{-1}$ maps each X^α into $X^{\alpha+1}$ for all $\mu \in \rho(A)$. Since all of the operators commute in (9), it is a routine to show that (9) indeed is the resolvent. Furthermore, we see that

$$(10) \qquad \sigma(L) \subseteq \beta_1\sigma(A) \cup \beta_2\sigma(A)$$

Now we can easily find the sector $S_{0,\tau}$ for L. Let $\tau = \arg(\beta_1) + \theta$. According to (H1), part (ii), we have $0 < \tau < \pi/2$ and the sector $S_{0,\tau}$ is contained in the resolvent set $\rho(L)$. Moreover $\beta_i \cdot S_{0,\tau} \subset S_{0,\theta}$ for $i = 1, 2$.

Since A is sectorial then there exists $M \geq 1$ such that $\|(\mu - A)^{-1}\| \leq M/|\mu|$ for each $\mu \in S_{0,\theta}$. Let $u \in X^1$, $v \in X$ and $\lambda \in S_{0,\tau}$. Clearly, $\lambda\beta_i \in S_{0,\theta}$ for $i = 1, 2$. We will estimate the norm of the resolvent $(\lambda - L)^{-1}$ by computing term by term in (9). We start with the upper left term in (9).

$$\begin{aligned} &\|(\lambda - \beta A)(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}u\|_1 \leq \\ &\leq \|\beta(\lambda\beta_2 - A)^{-1} - \lambda\beta_1^2(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}\| \cdot \|u\|_1 \leq \\ &\leq \left\{ \frac{\beta M}{|\lambda\beta_2|} + \frac{|\lambda\beta_1^2| \cdot M^2}{|\lambda\beta_1| \cdot |\lambda\beta_2|} \right\} \cdot \|u\|_1 = \frac{M_1}{|\lambda|}\|u\|_1 \end{aligned}$$

Consider the upper right term. Then

$$\begin{aligned} &\|-(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}v\|_1 \leq \|A(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}\| \cdot \|v\| = \\ &\leq \|\lambda\beta_1(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1} - (\lambda\beta_2 - A)^{-1}\| \cdot \|v\|_1 \leq \\ &\leq \left\{ \frac{|\lambda\beta_1| \cdot M^2}{|\lambda\beta_1| \cdot |\lambda\beta_2|} + \frac{M}{|\lambda\beta_2|} \right\} \cdot \|v\| \leq \frac{M_2}{|\lambda|}\|v\| \end{aligned}$$

The next term is lower left one.

$$\|A^2(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}u\| \leq$$
$$\|A(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}\| \cdot \|u\|_1 \leq \frac{M_2}{|\lambda|}\|u\|_1$$

Finally,

$$\|\lambda(\lambda\beta_1 - A)^{-1}(\lambda\beta_2 - A)^{-1}v\| \leq \frac{|\lambda| \cdot M^2}{|\lambda\beta_1| \cdot |\lambda\beta_2|}\|v\| = \frac{M^2}{|\lambda|}\|v\|$$

Therefore there exists a constant $M' \geq 1$, which does not depend on λ, such that $\|(\lambda - L)^{-1}\|_x \leq M'/|\lambda|$ for each $\lambda \in S_{0,\tau}$. Hence L is the sectorial operator in $\mathcal{X}$ and $-L$ generates the analytic semigroup $\exp(-Lt)$ on $\mathcal{X}$. ∎

Remark 3.1. Since $\operatorname{Re}\sigma(A) > 0$ then by looking at the spectrum $\sigma(L)$ we see that $\operatorname{Re}\sigma(L) > 0$. More precisely, by straightforward computations, we obtain that $\operatorname{Re}\sigma(L) > \delta \cdot \operatorname{Re}(\beta_2) \cdot \cos\tau$.

Remark 3.2. Let f, g and h be given. Thanks to the continuity of the imbedding $X^1 \subset X^{1/2}$, the assumptions of (H1), part (iii), imply that $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is lipschitzian on bounded sets of $\mathcal{X}$.

The main result of this section is the following theorem

Theorem 3.2. *Under assumptions* (H1), *for each* $\Phi_0 \in \mathcal{X}$ *there exists* $T = T(\Phi_0) > 0$ *and a unique function* $\Phi = \Phi(t, \Phi_0)$ *such that*

(i) $\Phi \in C([0,t_1))\colon \mathcal{X}) \cap C^1((t_0,t_1)\colon \mathcal{X}^\alpha)$ *for all* $0 \leq \alpha < 1$ *and* $0 < t_0 < t_1 < T$

(ii) $\Phi(t) \in D(L)$ *for each* $t \in (0,T)$

(iii) $\frac{d}{dt}\Phi(t) + L\Phi(t) + \mathcal{F}(\Phi(t)) = 0$ *on* $(0,T)$; $\Phi(0) = \Phi_0$

(iv) *If* $T(\Phi_0)$ *is maximal (in the sense that there exists no solution of (5) on* $(0,T_1)$ *where* $T_1 > T(\Phi_0)$ *) then either* $T(\Phi_0) = +\infty$ *or* $\|\Phi(t,\Phi_0)\|_x$ *is unbounded on* $[0,T)$

(v) *For each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\|\Phi_0 - \Psi_0\|_x < \delta$ *implies* $\|\Phi(t,\Phi_0) - \Phi(t,\Psi_0)\|_x < \varepsilon$ *uniformly on compact subintervals of* $[0, \min\{T(\Phi_0), T(\Psi_0)\})$.

PROOF : By Theorem 3.1 we know that $-L$ generates the analytic semigroup $\exp(-Lt)$ on $\mathcal{X}$. Moreover $\operatorname{Re}\sigma(L) > 0$. According to Remark 3.2 we have that $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is lipschitzian on bounded sets of $\mathcal{X}$. Hence our statement is a consequence of the general theory of semilinear parabolic equations which can be found in [8]. More precisely, it follows from [8, Theorem 3.3.3, 3.3.4, 3.4.1 and 3.5.2]. ∎

Remark 3.3. Define projections π_1 and π_2 from $\mathcal{X}$ into X^1 and X by $\pi_1 \begin{bmatrix} u \\ v \end{bmatrix} = u$ and $\pi_2 \begin{bmatrix} u \\ v \end{bmatrix} = v$. Let $\Phi(\,.\,)$ be a solution of (5) with $\Phi(0) = \Phi_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{X}$. Put $u(t,u_0,v_0) = \pi_1\Phi(t,\Phi_0)$ for each $t \in [0,T(\Phi_0))$. Then by Theorem 3.2 we see that $u_t(t) = \pi_2\Phi(t)$ and $u_{tt}(t) + \beta Au_t(t) + f(\|A^{1/2}u(t)\|^2)Au(t) + A^2u(t) + g(u(t)) = h$ on $(0,T(\Phi_0)$ and $u(0) = \pi_1\Phi(0) = u_0$; $u_t(0) := \lim_{h\to 0+} u_t(h) = \pi_2\Phi(0) = v_0$. Moreover $u \in C([0,t_1)\colon X^1) \cap C^1((t_0,t_1)\colon X^1) \cap C^2((t_0,t_1)\colon X)$ for each $0 < t_0 < t_1 < T(\Phi_0)$ and $u(t) \in D(A^2)$ for $t \in (0,T(\Phi_0))$.

4. Global existence.

From now we restrict X, A, β, f, g, and h by

(H2)

(i) $X = L_2(\Omega)$ where Ω is a smooth bounded domain in $\mathbf{R}^n$, $n = 1,2,3$; $\beta > 0$ and $h \in L_2(\Omega)$. The scalar product in X is denoted by $(\,.\,,\,.\,)$.

(ii) $Au = -\Delta u$ for each $u \in C_0^2(\Omega)$ and A is the selfadjoint closure in X of its restriction to $C_0^2(\Omega)$

(iii) $g\colon \mathbf{R} \longrightarrow \mathbf{R}$, $f\colon [0,+\infty) \longrightarrow \mathbf{R}$ are locally Lipschitz continuous functions such that f increases on $[0,+\infty)$,

$$\int_0^\infty f(s)ds > -\infty \text{ and } \liminf_{|s|\to+\infty} \frac{g(s)}{s} \geq 0$$

It is well known (cf. [8, Chapter 1]) that $\operatorname{Re}\sigma(A) > 0$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, A^{-1} is the compact operator on X and A is the sectorial operator in X with the sector $S_{0,\theta}$ for each $\theta \in (0, \pi/2)$.

Due to the Sobolev's imbedding $X^1 \subset L_\infty(\Omega)$ [8, Theorem 1.6.1] we see that $g\colon X^1 \longrightarrow X$ is well defined and it is the lipschitzian mapping on bounded sets of X^1. Here we have denoted by the same symbol the Nemitzky's operator g defined by $g(u)(x) := g(u(x))$ for $u \in X^1$ and for a.e. $x \in \Omega$. Hence A, β, f, g and h fulfill to the hypotheses (H1).

(11)

Define a functional $G\colon X^1 \longrightarrow \mathbf{R}$ by

$$G(u) = \int_\Omega \left(\int_0^{u(x)} g(s)ds \right) dx - (h,u) \text{ for each } u \in X^1 \tag{12}$$

Thanks to the continuity of the imbedding $X^1 \subset L_\infty(\Omega)$ we obtain that G is well defined and it is the continuous function from X^1 into $\mathbf{R}$. Moreover, if $u \in C^1((t_0,t_1)\colon X^1)$ then $G(u(t))$ is differentiable on (t_0,t_1) and

$$\frac{d}{dt}G(u(t)) = (g(u(t)), u_t(t)) - (h, u_t(t)) \tag{13}$$

Since $\liminf_{|s|\to+\infty} \frac{g(s)}{s} \geq 0$, it can easily be verified that for each $\varepsilon > 0$ there is $K_\varepsilon > 0$ (K_ε depends on ε, Ω, h, g and $\|A^{-1}\|$) such that

$$G(w) \geq -\varepsilon\|w\|_1^2 - K_\varepsilon \text{ for each } w \in X^1 \tag{14}$$

Again from the imbedding $X^1 \subset L_\infty(\Omega)$ we obtain that $G(B)$ is a bounded set in $\mathbf{R}$, provided B is a bounded set in X^1.

(15)

Now we denote by F the primitive of f, i.e.

(16) $$F(r) = \int_0^r f(s)ds$$

From the assumption $\int_0^\infty f(s)ds > -\infty$ the existence of c_0 with the property $F(r) \geq c_0$ for each $r \geq 0$ immediately follows.

(17)

Finally, if $u \in C^1((t_0, t_1) \colon X^1)$ then $F(\|u(t)\|_{1/2}^2)$ is differentiable on (t_0, t_1) and

$$\frac{d}{dt} F(\|u(t)\|_{1/2}^2) = 2 \cdot f(\|u(t)\|_{1/2}^2) \cdot (Au(t), u_t(t))$$

holds.

(18)

Theorem 4.1. *For each $\Phi_0 \in \mathcal{X}$ the unique solution $\Phi(\,.\,, \Phi_0)$ given by Theorem 3.2 exists and is bounded on $[0, \infty)$.*

PROOF : With regard to Theorem 3.2, part (iv), we will show that the maximal solution $\Phi(\,.\,, \Phi_0)$ of (5), defined on $[0, T(\Phi_0))$, stays bounded in $\mathcal{X}$.

From Remark 3.3 we know that $u(t) = \pi_1 \Phi(t)$ satisfies to (1) on $(0, T(\Phi_0))$. Moreover $u \in C^1((t_0, t_1) \colon X^1)$ for each $0 < t_0 < t_1 < T(\Phi_0)$ and $u(t) \in D(A^2)$ for $t \in (0, T(\Phi_0))$.

We take the scalar product in X of (1) with $u_t(t)$. Then for each $t \in (0, T(\Phi_0))$ we obtain

$$(u_{tt}(t), u_t(t)) + \beta(Au_t(t), u_t(t)) + f(\|A^{1/2}u(t)\|^2) \cdot (Au(t), u_t(t)) + \\ (A^2 u(t), u_t(t)) + (g(u(t)), u_t(t)) - (h, u_t(t)) = 0$$

Then we can deduce from (13) and (18) that

(19) $$\frac{1}{2}\frac{d}{dt}\left\{\|\Phi(t)\|_x^2 + F\left(\|\pi_1\Phi(t)\|_{1/2}^2\right) + 2 \cdot G(\pi_1\Phi(t))\right\} + \beta\|\pi_2\Phi(t)\|_{1/2}^2 = 0$$

Since $\beta > 0$ we see that

$$\|\Phi(t)\|_x^2 + F\left(\|\pi_1\Phi(t)\|_{1/2}^2\right) + 2 \cdot G(\pi_1\Phi(t)) \leq \\ \leq \|\Phi_0\|_x^2 + F\left(\|\pi_1\Phi_0\|_{1/2}^2\right) + 2 \cdot G(\pi_1\Phi_0)$$

Put $\varepsilon = 1/4$. According to (14) and (17) we obtain

(20) $$\frac{1}{2}\|\Phi(t)\|_x^2 \leq \|\Phi_0\|_x^2 + F\left(\|\pi_1\Phi_0\|_{1/2}^2\right) + 2 \cdot G(\pi_1\Phi_0) - c_0 + K_{1/4}$$

Therefore $\Phi(\,.\,, \Phi_0)$ remains bounded on $[0, T(\Phi_0))$. By Theorem 3.2 we have that $T(\Phi_0) = +\infty$. Hence $\Phi(\,.\,, \Phi_0)$ exists on $[0, +\infty)$. ∎

5. Limiting behaviour.

In this section we will consider solutions of (5) as a semidynamical system $\{S(t); t \geq 0\}$ in the Hilbert space $\mathcal{X}$.

Define

$$S(t)\Phi_0 := \Phi(t, \Phi_0) \text{ for all } \Phi_0 \in \mathcal{X} \text{ and } t \geq 0 \tag{21}$$

According to Theorem 3.2 and 4.1 $\{S(t); t \geq 0\}$ is the semidynamical system in $\mathcal{X}$.

(22)

Remark 5.1. It readily follows from (15), (20) and assumptions on f that $\gamma^+(B)$ is bounded in $\mathcal{X}$ for any bounded set $B \subseteq \mathcal{X}$.

Theorem 5.1. *Assume the hypotheses (H2). Then there exists a maximal compact attractor $\mathcal{U}$ for the semidynamical system $\{S(t); t \geq 0\}$.*

Before proving this theorem we need four auxiliary lemmas each of which is under hypotheses (H2).

Lemma 5.1. *$L^{-\alpha}$ is a compact linear operator on $\mathcal{X}$ for each $\alpha > 0$.*

PROOF : Let $\{\Phi_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathcal{X} = X^1 \times X$. Since A^{-1} is compact in X, there exists a subsequence (again denoted by $\{\Phi_n\}_{n=1}^{\infty}$) such that $\{\pi_1\Phi_n\}_{n=1}^{\infty}$ and $\{A^{-1}\pi_2\Phi_n\}_{n=1}^{\infty}$ converge in X. Then $\{A^{-1}\pi_1\Phi_n\}_{n=1}^{\infty}$ and $\{A^{-2}\pi_2\Phi_n\}_{n=1}^{\infty}$ converge in X^1. From (9) we see that

$$L^{-1} = \begin{pmatrix} \beta A^{-1}, & A^{-2} \\ \text{-Id}, & 0 \end{pmatrix}$$

Therefore $\{L^{-1}\Phi_n\}_{n=1}^{\infty}$ converges in $\mathcal{X}$. Thus L^{-1} is the compact linear operator on $\mathcal{X}$. Hence $L^{-\alpha}$ is compact on $\mathcal{X}$ for each $\alpha > 0$. ∎

Lemma 5.2. *For each $\Phi_0 \in \mathcal{X}$ the semiorbit $\gamma^+(\{\Phi_0\})$ is precompact in $\mathcal{X}$. For each fixed $t > 0$, $S(t)\colon \mathcal{X} \longrightarrow \mathcal{X}$ is the compact operator on $\mathcal{X}$.*

PROOF : Since $\mathcal{F}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is lipschitzian on bounded sets of $\mathcal{X}$ we have that $\mathcal{F}(B)$ is bounded in $\mathcal{X}$ for any bounded set $B \subseteq \mathcal{X}$. Furthermore L has the compact resolvent L^{-1}. Hence our first statement is a consequence of [8, Theorem 3.3.6].

Let $t > 0$ be fixed. To show that $S(t)$ is the compact operator it suffices to show that $L^{1/2}S(t)B$ is bounded in $\mathcal{X}$ for any bounded set $B \subseteq \mathcal{X}$. Let B be a bounded set in $\mathcal{X}$, i.e. $\|\Phi_0\|_x \leq c_1$ for each $\Phi_0 \in B$. By Remark 5.1; $\gamma^+(B)$ is bounded in $\mathcal{X}$. Therefore $\mathcal{F}(\gamma^+(B))$ is bounded in $\mathcal{X}$, i.e. $\|\mathcal{F}(S(s)B)\|_x \leq c_2$ for each $s \geq 0$. It is well known (cf. [8, Lemma 3.3.2]) that $S(t)\Phi_0$ satisfies to an integral equation

$$S(t)\Phi_0 = \exp(-Lt)\Phi_0 + \int_0^t \exp(-L(t-s))\mathcal{F}(S(s)\Phi_0)ds$$

Using the fact that $L^{1/2}$ is closed (see [8, p. 25]) we obtain

$$L^{1/2}(S(t))\Phi_0 = L^{1/2}\exp(-Lt)\Phi_0 + \int_0^t L^{1/2}\exp(-L(t-s))\mathcal{F}(S(s)\Phi_0)ds$$

Therefore

$$\|L^{1/2}S(t)B\|_x \le C_{1/2}\cdot\{c_1 t^{-1/2} + 2c_2 t^{1/2}\}$$

Thus $L^{1/2}S(t)B$ is bounded in $\mathcal{X}$. Hence $S(t)$ is the compact mapping in $\mathcal{X}$ for each fixed $t > 0$. ∎

The following statement is an easy consequence of the previous lemma and the general result of [8, Theorem 4.3.3].

Lemma 5.3. *For each $\Phi_0 \in \mathcal{X}$ the omega–limit set, $\Omega(\{\Phi_0\})$, is nonempty compact, connected and*

$$\lim_{t\to\infty} \operatorname{dist}(S(t)\Phi_0, \Omega(\{\Phi_0\})) = 0$$

Denote by E a set of the stationary states of (5), i.e.

3) $$E = \{\Phi \in D(L); L\Phi + \mathcal{F}(\Phi) = 0\}$$

Clearly, $\begin{bmatrix} u \\ v \end{bmatrix} \in E$ iff $v = 0$ and $u \in D(A^2)$ satisfies to a stationary equation

(24) $$f(\|A^{1/2}u\|^2)Au + A^2u + g(u) = h$$

Lemma 5.4. *For each $\Phi_0 \in \mathcal{X}$, $\Omega(\{\Phi_0\}) \subseteq E$.*

PROOF : Define a Liapunov functional $V\colon \mathcal{X} \longrightarrow \mathbf{R}$ by

$$V(\Phi) = \frac{1}{2}\left\{\|\Phi\|_x^2 + F\left(\|\pi_1\Phi\|_{1/2}^2\right) + 2\cdot G(\pi_1\Phi)\right\}$$

According to (19) we know that

(25) $$\frac{d}{dt}V(S(t)\Phi) + \beta\|\pi_2 S(t)\Phi\|_{1/2}^2 = 0 \text{ for each } t > 0 \text{ and } \Phi \in \mathcal{X}$$

Thus a real valued function $t \longrightarrow V(S(t)\Phi)$ is nonincreasing on $[0, +\infty)$. Moreover, by (14) and (17), $V(S(t)\Phi)$ is bounded below for $t \ge 0$. Now, the rest of the proof is essentially the same as of [11, Theorem 4.1].

Indeed, if $\Phi \in \Omega(\{\Phi_0\})$ then $\Phi = \lim_{n\to\infty} S(t_n)\Phi_0$ for some sequence $t_n \longrightarrow +\infty$. Since $V(S(t)\Phi_0)$ is continuous (see (13)) then we have

$$V(\Phi) = \lim_{n\to\infty} V(S(t_n)\Phi_0) = \inf_{s\ge 0} V(S(s)\Phi_0) = \lim_{n\to\infty} V(S(t+t_n)\Phi_0) =$$
$$= V(S(t)\Phi) \text{ for each } t \ge 0$$

Then, from (25), we obtain that $\pi_2 S(t)\Phi = 0$ for each $t > 0$. By Remark 3.3 we know that $\frac{d}{dt}\pi_1 S(t)\Phi = 0$ for $t > 0$. Thus $LS(t)\Phi + \mathcal{F}(S(t)\Phi) = 0$ for each $t > 0$. Since L is closed and $\mathcal{F}$ is continuous we obtain (by letting $t \longrightarrow 0^+$) that $\Phi \in D(L)$ and $L\Phi + \mathcal{F}(\Phi) = 0$, i.e. $\Phi \in E$. Hence $\Omega(\{\Phi_0\}) \subseteq E$. ∎

PROOF of Theorem 5.1: First we will show that E is the bounded set in $\mathcal{X}$. Let $\begin{bmatrix} u \\ 0 \end{bmatrix} \in E$. Then we multiply (24) with u to obtain

$$\|u\|_1^2 + f(\|u\|_{1/2}^2) \cdot \|u\|_{1/2}^2 + (g(u), u) = (h, u)$$

Since f increases on $[0, +\infty)$ and F is lower bounded by c_0 (see (17)) we have $c_0 \le F(\|u\|_{1/2}^2) \le f(\|u\|_{1/2}^2) \cdot \|u\|_{1/2}^2$. According to the assumption $\lim_{|s| \to +\infty} \inf \frac{g(s)}{s} \ge 0$ it is as routine to show that there is $K' > 0$ such that $(g(u), u) \ge -\frac{1}{2}\|u\|_1^2 - K'$. ($K'$ depends only on g, Ω and $\|A^{-1}\|$).

We now combine the previous result to obtain

$$\frac{1}{2}\|u\|_1^2 + c_0 - K' \le (h, u) < \|A^{-1}\|^2 \cdot \|h\|^2 + \frac{1}{4}\|A^{-1}\|^{-2} \cdot \|u\|_{1/2}^2 \le$$

$$\le \|A^{-1}\|^2 \cdot \|h\|^2 + \frac{1}{4}\|u\|_1^2$$

(here we have used the inequality $a \cdot b \le \frac{1}{2}\{(\varepsilon a)^2 + (b/\varepsilon)^2\}$). Thus $\|u\|_1^2 \le 4 \cdot \{K' - c_0 + \|A^{-1}\|^2 \|h\|^2\}$. Hence E is bounded in $\mathcal{X}$.

Using similar ideas as of [**9**, Theorem 5] the rest of the proof comes very quickly. Let $B_1 = \mathcal{N}_1(E)$. Clearly B_1 is bounded on $\mathcal{X}$. With regard to Lemma 5.3 we see that for each $\Phi_0 \in \mathcal{X}$ there exists $T(\Phi_0) > 0$ such that $S(t)\Phi_0 \in B_1$ whenever $t \ge T(\Phi_0)$. Hence B_1 dissipates all points. Let $B_2 = \gamma^+(\mathcal{N}_1(B_1))$. By Remark 5.1 we have that B_2 is bounded in $\mathcal{X}$. Let $\Phi_0 \in \mathcal{X}$. Then $S(t)\Phi_0 \in B_1$ whenever $t \ge T(\Phi_0)$. From the continuity of $S(T(\Phi_0))$ we obtain that there exists a neighbourhood $\mathcal{N}_\delta(\Phi_0)$ with $S(T(\Phi_0))\mathcal{N}_\delta(\Phi_0) \subseteq \mathcal{N}_1(B_1)$. Thus $S(t)\mathcal{N}_\delta(\Phi_0) \subseteq \gamma^+(\mathcal{N}_1(B_1)) = B_2$ for each $t \ge T(\Phi_0)$. Therefore B_2 dissipates all compact sets. Since $S(1)B$ is a compact set in $\mathcal{X}$ for any bounded set $B \subseteq \mathcal{X}$ we have that $\{S(t); t \ge 0\}$ is bounded dissipative. More precisely, for any bounded set $B \subseteq \mathcal{X}$ there is $T(B) > 0$ such that $S(t)B \subseteq B_2$, whenever $t \ge T(B)$.

Now, by [**1**, Theorem 1.2 and Remark 1.0] we have that there exists a maximal compact attractor $\mathcal{U}$ for the semidynamical system $S(t)$. More precisely, $\mathcal{U} = \Omega(B_2)$, which completes the proof of Theorem 5.1. ∎

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