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Projective p-algebras

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1. Introduction

In this paper we give a characterization of projective p-algebras (for related problems see [5; Problems II.25 and II.28]). It is done in a similar manner as the description of projective lattices offered by R. Freese and J. B. Nation [4]. We use the description of free p-algebras from [8], which says, that a free p-algebra is a suitable free extension of some poset in the class of all lattices. We have also to acknowledge the influence of very stimulating papers by B. Jónsson and J. B. Nation [7] and R. McKenzie [15].

Let us note that there are some descriptions of projective distributive p-algebras: R. Balbes and G. Grätzer [1] characterized the finite projective Stone algebras, I. Düntsch [3] found a description of projective Stone algebras. Eventually, A. Urquhart [12] characterized those finite distributive p-algebras, which are projective in the classes $\mathbf{B}_n$, $1 \leq n \leq \omega$, where $\mathbf{B}_n$ denotes the n-th subvariety of the variety of all distributive p-algebras (see [5]).

The second section of our paper contains preliminaries. The subsequent section is concerned with the proof of the main theorem. The finitely generated projective p-algebras are characterized in Section 4. Examples and problems are discussed in the final section.

2. Preliminaries

A p-algebra P is *projective* if for all p-algebras A, B such that $g : A \to B$ is a homomorphism of A onto B, for every homomorphism $f : P \to B$, there is a homomorphism $h : P \to A$ with $g(h(a)) = f(a)$ for all $a \in P$. This is equivalent to the

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condition that there is a homomorphism g of a free p-algebra $\mathbf{FP}(X)$ onto P and a homorphism h of P into $\mathbf{FP}(X)$ such that $g(h(a)) = a$ for all $a \in P$.

A p-algebra is a universal algebra $(L; \vee, \wedge, {}^*, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation $*$ (of the pseudocomplementation) is defined by $a \wedge b = 0$ if and only if $a \leq b^*$. It is well known that the class of all p-algebras is equational (see [5]). We shall frequently use the following rules of computation in p-algebras:

(i) $a \leq b$ implies $a^* \geq b^*$.
(ii) $a \leq a^{**}$,
(iii) $a^* = a^{***}$
(iv) $(a \vee b)^* = a^* \wedge b^*$,
(v) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(vi) $(a \wedge b)^* \geq a^* \vee b^*$,
(viii) $0^* = 1$ and $1^* = 0$.

In any p-algebra L we can define the set of closed elements $\mathbf{B}(L) = \{x \in L; x = x^{**}\}$. It is known that $(\mathbf{B}(L); +, \wedge, {}^*, 0, 1)$ is a Boolean algebra, where $a + b = (a^* \wedge b^*)^*$.

In [8] the free p-algebras $\mathbf{FP}(X)$ in the class $\mathbf{P}$ of all p-algebras were characterized. (It is easy to verify that $\mathbf{FP}(\varnothing)$ is a two element Boolean algebra.) We recall the relevant parts of the construction for later applications. First we shall mention a result for R. Dean [2]. Let $(T, \leq)$ be a poset with families $\mathscr{L}$ and $\mathscr{U}$ of finite subsets of T such that

(i) $p \leq q$ in T implies $\{p, q\} \in \mathscr{L}$ and $\{p, q\} \in \mathscr{U}$
(ii) $S \in \mathscr{L}$ ($S \in \mathscr{U}$) implies that there exists $\inf_T S$ ($\sup_T S$).

Following [2; Theorems 5 and 6] there exists a free lattice $\mathbf{FL}(T; \mathscr{L}, \mathscr{U})$ generated by T and preserving bounds from $\mathscr{L}$ and $\mathscr{U}$.

Now let X be a set. Take a disjoint copy $\bar{X} = \{\bar{x}; x \in X\}$ and construct a free Boolean algebra $(\mathbf{FB}(\bar{X}); +, \wedge, {}^*, 0, 1)$. We can assume $X \cap \mathbf{FB}(\bar{X}) = \varnothing$. Define the poset $\mathscr{P}(X) = (\mathscr{P}(X); \leq) = (X \cup \mathbf{FB}(\bar{X}); \leq)$ associated with X as follows:

(i) $\mathscr{P}(X)$ is bounded, i.e. $0 \leq u \leq 1$ for every $u \in \mathscr{P}(X)$ and $0, 1 \in \mathbf{FB}(\bar{X})$;
(ii) $x \leq \bar{x}$ for every $x \in X$;
(iii) For $0 \neq a$, $a \in \mathbf{FB}(\bar{X})$, $a \leq u$ if and only if $u \in \mathbf{FB}(\bar{X})$ and $a \leq u$ in $\mathbf{FB}(\bar{X})$;
(iv) $x \leq u$ for $x \in X$ if and only if $\bar{x} \leq u$ or $x = u$.

The families $\mathscr{L}$ and $\mathscr{U}$ are described in the following manner:

(i) $A \in \mathscr{L}$ if and only if A is a finite subset of $\mathbf{FB}(\bar{X})$ or $A = \{a, b\} \subseteq \mathscr{P}(X)$ and $a \le b$ in $\mathscr{P}(X)$;
(ii) $A \in \mathscr{U}$ if and only if $A = \{a, b\} \subseteq \mathscr{P}(X)$ and $a \le b$ in $\mathscr{P}(X)$.

Now, the lattice $\mathbf{FL}(\mathscr{P}(X); \mathscr{L}, \mathscr{U})$ freely extends the poset $\mathscr{P}(X)$. By [8; Lemma 5] there exists a (unique) lattice epimorphism $\pi : \mathbf{FL}(\mathscr{P}(X); \mathscr{L}, \mathscr{U}) \to \mathbf{FB}(\bar{X})$ such that $\pi(x) = \bar{x}$ for every $x \in X$ and $\pi(a) = a$ for every $a \in \mathbf{FB}(\bar{X})$. In addition, we have

LEMMA A [8; Theorem 1, Lemma 7]). *Let $\mathscr{P}(X)$ denote the poset associated with the set X. Then $\mathbf{FL}(\mathscr{P}(X); \mathscr{L}, \mathscr{U})$ is a p-algebra if we set*

$$u^* = (\pi(u))'$$

for each $u \in \mathbf{FL}(\mathscr{P}(X); \mathscr{L}, \mathscr{U})$. Moreover, there exists a canonical isomorphism $w : \mathbf{FP}(X) \to \mathbf{FL}(\mathscr{P}(X); \mathscr{L}, \mathscr{U})$ such that $w(x) = x$ for all $x \in X$ and $w(u^*) = (\pi(w(u))'$ for every $u \in \mathbf{FP}(X)$.

LEMMA B ([8; Lemmas 2–4]). *Let L be a p-algebra generated by a subset X,* i.e. $[X] = L$. *Then*
(i) $\mathbf{B}(L) = [X^{**}]_{\mathrm{Bool}}$, *where* $X^{**} = \{x^{**}; x \in X\}$,
(ii) $L = [X \cup \mathbf{B}(L)]_{\mathrm{lat}}$,
(iii) *If* $L = \mathbf{FP}(X)$, *then* $\mathbf{B}(L) = \mathbf{FB}(X^{**})$.

LEMMA C ([8; Lemma 11], [9; Lemma 8]). *Let $\mathscr{P}(X)$ denote the subset $X \cup \mathbf{FB}(X^{**})$ of $\mathbf{FP}(X)$. Then*
(i) *every element $p \in \mathscr{P}(X)$ is join-prime,* i.e. *$p \le a \vee b$ implies $p \le a$ or $p \le b$;*
(ii) *$\mathbf{FP}(X)$ satisfies the Whitman condition*

$$a \wedge b \le c \vee d \text{ if and only if } \{a, b, c, d\} \cap [a \wedge b, c \vee d] \ne \varnothing. \tag{W}$$

LEMMA D ([8, Lemma 12]).

$$a \wedge b = 0 \text{ in } \mathbf{FP}(X) \text{ if and only if } a^{**} \wedge b^{**} = 0 \text{ in } \mathbf{FB}(X^{**}).$$

Moreover, in $\mathbf{FP}(X)$ holds
(i) *$0 \ne a \wedge b \le p$ for $p \in X$ if and only if $a \le p$ or $b \le p$,*
(ii) *$0 \ne a \wedge b \le p$ for $p \in \mathbf{FB}(X^{**})$ if and only if $a^{**} \wedge b^{**} \le p$ in $\mathbf{FB}(X^{**})$.*

Now, we can introduce the concept of a p-term in X. Let X be an arbitrary set. Let X^* and X^{**} denote two disjoint copies of X, i.e. X^* and X^{**} have the same cardinality as X, and $X \cap X^* = X \cap X^{**} = X^* \cap X^{**} = \varnothing$. The set $\bar{W}(X)$ of *special p-terms in X* is the smallest set satisfying

(i) $X^* \cup X^{**} \subseteq \bar{W}(X)$
(ii) if $p, q \in \bar{W}(X)$, then $(p \wedge q)$, $(p + q) \in \bar{W}(X)$.

The set $W(X)$ of *p-terms in X* is the smallest set satisfying

(iii) $X \cup \bar{W}(X) \subseteq W(X)$
(iv) if $p, q \in W(X)$, then $(p \wedge q)$, $(p \vee q) \in W(X)$.

Evidently, for every $a \in \mathbf{FP}(X)$ ($a \in \mathbf{FB}(X)$), there is a finite subset Y of X such that $a \in [Y]$.

LEMMA E. *Let $a \in \mathbf{FP}(X)$ ($a \in \mathbf{FB}(X)$). Let Y and Z be finite subsets of X such that*

$$a \in [Y] \quad \textit{and} \quad a \in [Z]. \tag{$*$}$$

Assume that Y and Z are minimal with respect to $(*)$. Then $Y = Z$.

Proof. Consider $a \in \mathbf{FP}(X)$. Since $0, 1 \in \mathbf{FP}(\varnothing)$, we can assume that $0 \neq a \neq 1$. Let $Y = \{y_0, \ldots, y_n\}$, $Z = \{z_0, \ldots, z_m\}$. Suppose to the contrary that $Y \neq Z$. Without lost of generality we can assume $z_0 \notin Y$. There are two p-terms $t(y_0, \ldots, y_n)$ and $q(z_0, \ldots, z_m)$ representing a. Let $\tau : \mathbf{FP}(X) \to \mathbf{FP}(X)$ be a homomorphisms determined by $f : X \to \mathbf{FP}(X)$ and satisfying

$$f(z_0) = 0, \qquad f(x) = x \quad \text{for every } x \in X \backslash \{z_0\}.$$

Therefore,

$$\tau t = t(y_0, \quad , y_n) = a = \tau q = q(0, z_1, \quad , z_m).$$

Replacing z_0, z_0^* and z_0^{**} in p-term $q(z_0, \ldots, z_m)$ by $z_1 \wedge z_1^*, z_1^* + z_1^{**}$ and $z_1^* \wedge z_1^{**}$, respectively, we get a new p-term $\bar{q}(z_1, \ldots, z_m)$ satisfying

$$\bar{q}(z_1, \ldots, z_m) = a = q(0, z_1, \ldots, z_m)$$

Therefore, $a \in [Z \backslash \{z_0\}]$, a contradiction.

For Boolean algebras it can be done in a similar way. $\square$

Now, for every $a \in \mathbf{FP}(X)$ ($a \in \mathbf{FB}(X)$), we can define a set $\operatorname{var}(a) \subseteq X$ as follows: $a \in [\operatorname{var}(a)]$ and $\operatorname{var}(a)$ is the smallest finite subset Y of X satisfying $a \in [Y]$. The set $\operatorname{var}(a)$ is uniquely determined by a (Lemma E).

Having a p-algebra $L = [Y]_{\text{lat}}$ for some $\varnothing \neq Y \subseteq L$, one can represent every $a \in L$ by a lattice term t, i.e. $a = t(b_0, \ldots, b_n)$, where $b_0, \ldots, b_n \in Y$. The rank (length) of a lattice term t is the number of times the symbols $\vee$ and $\wedge$ occur in t. (Notation $r(t)$.)

Now, we need some concepts (adapted for p-algebras) from [4] (see also [7]). Let L be a p-algebra. Assume that $U, V \subseteq L$ are finite nonempty sets. U is said to be *a cover* (*dual p-cover*) of $a \in L$ if $a \leq \bigvee U$ ($a \geq \bigwedge U$). U is *a trivial cover* of $a \in L$ if $a \leq \bigvee U$ implies $a \leq u$ for some $u \in U$ (i.e. a is join prime). U is a *trivial dual* p-cover of $a \in L$ if $a \geq \bigwedge U$ implies either $a \geq (\bigwedge U)^{**} = \bigwedge U^{**}$ or $a \geq u$ for some $u \in U$.

We can also define two relations $\ll$ and $\gg^*$ for finite nonempty subsets $U, V \subseteq L$. We write $U \ll V$ if for each $u \in U$ there is a $v \in V$ with $u \leq v$. Again, $U \gg^* V$ if for every $u \in U$ either $u \geq (\bigwedge V)^{**}$ or there is a $v \in V$ with $u \geq v$. (Note, that the dual p-cover and the relation $\gg^*$ are not dual concepts to the cover and $\ll$, respectively.) Observe, that both relations $\ll$ and $\gg^*$ are reflexive and transitive. Moreover,

$$U \ll V \text{ implies } \bigvee U \leq \bigvee V,$$

$$U \gg^* V \text{ implies } \bigwedge U \geq \bigwedge V.$$

A cover U of a is called *a minimal cover* of a, if whenever V is a cover of a such that $V \ll U$, then $U \subseteq V$. Similarly, a dual p-cover u of a is called *a minimal dual p-cover* of a, if whenever V is a dual p-cover of a such that $V \gg^* U$, then $U \gg^* V$.

The next concept originated with B. Jónsson. We let $\mathbf{K}_0(L)$ ($\mathbf{K}'_0(L)$) denote the set of all $a \in L$ having trivial covers (dual p-covers) only. (Note that $a \in \mathbf{K}_0(L)$ if and only if a is join-prime.) Inductively, we can define $\mathbf{K}_{n+1}(L)$ ($\mathbf{K}'_{n+1}(L)$) as follows:

$a \in \mathbf{K}_{n+1}(L)$ ($a \in \mathbf{K}'_{n+1}(L)$) if for every nontrivial cover (dual p-cover) U of a there exists $V \subseteq \mathbf{K}_n(L)$ ($V \subseteq \mathbf{K}'_n(L)$) such that V is a cover (dual p-cover) of a and $V \ll U$ ($V \gg^* U$).

Observe, that $m \leq n$ implies $\mathbf{K}_m(L) \subseteq \mathbf{K}_n(L)$ and $\mathbf{K}'_m(L) \subseteq \mathbf{K}'_n(L)$. Moreover, $\mathbf{B}(L) \subseteq \mathbf{K}'_0(L)$.

Put

$$\mathbf{K}(L) = \bigcup (\mathbf{K}_n(L) : n < \omega) \quad \text{and} \quad \mathbf{K}'(L) = \bigcup (\mathbf{K}'_n(L) : n < \omega).$$

3. The main result

THEOREM 1. *Let L be a p-algebra. Then L is projective in the class* $\mathbf{P}$ *of all p-algebras if and only if L satisfies the following conditions*:

(1) *every closed element $a \in \mathbf{B}(L)$ is join-prime,*

(2) (W),

(3) $\mathbf{B}(L)$ *is projective Boolean algebra,*

(4) $\mathbf{K}(L) = L = \mathbf{K}'(L)$,

(5) *for each $a \in L$, there is a finite set $S(a)$ of nontrivial covers of a, such that, if U is any nontrivial cover of a, then $V \ll U$ for some $V \in S(a)$,*

(6) *for each $a \in L$, there is a finite set $S'(a)$ of nontrivial dual p-covers of a, such that, if U is any nontrivial dual p-cover of a, then $V \gg^* U$ for some $V \in S'(a)$,*

(7) *for each $a \in L$ there are two finite sets $A(a) \subseteq \{c \in L : c \geq a\}$ and $B(a) \subseteq \{c \in L : c \leq a\}$ such that, if $a \leq b$, then $A(a) \cap B(b) \neq \varnothing$. In addition, $A(a) \subseteq \mathbf{B}(L)$ whenever $a \in \mathbf{B}(L)$, $c \in A(a)$ implies $c^{**} \in A(a)$ and, $c \in B(b)$ implies $c^{**} \in B(b)$ whenever $b \in \mathbf{B}(L)$.*

COROLLARY. *Let L be a countable p-algebra having a finite* $\mathbf{B}(L)$. *Then L is projective in* $\mathbf{P}$ *if and only if it satisfies* (1), (2) *and* (4)–(6).

For the proof of Theorem 1 we need more auxiliary results.

LEMMA 1. *Let L be a p-algebra projective in the class* $\mathbf{P}$. *Then L satisfies* (1)–(3).

Proof. (1) and (2) follow directly from Lemma C. As for (3), consider Boolean algebras A, B with Boolean homomorphisms $g : A \to B$ and $f : \mathbf{B}(L) \to B$, where g is onto. Since the class of all Boolean algebras is a subvariety of $\mathbf{P}$, there is $f_1 : L \to B$ defined as $f_1 = \gamma \circ f$, where $\gamma : x \mapsto x^{**}$. Evidently, $\gamma : L \to \mathbf{B}(L)$ is an onto homomorphism. Thus $f_1(x) = f(x^{**})$ for every $x \in L$. There exists $h_1 : L \to A$ such that $f_1(x) = g(h_1(x))$ for every $x \in L$. Take h the restriction $h_1 \mid \mathbf{B}(L)$. Since $f = f_1 \mid \mathbf{B}(L)$ and h is a Boolean homomorphism, we get $f = h \circ g$. □

We need a concept more (see [4] or [7]). Let $Y \subseteq L$ be such that $L = [Y]_{\text{lat}}$. Let $\mathbf{H}_0 = \mathbf{H}_0(Y)$ ($\mathbf{H}'_0 = \mathbf{H}'_0(Y)$) denote the set of all finite meets (joins) of elements of Y

and for $n < \omega$, let $\mathbf{H}_{n+1} = \mathbf{H}_{n+1}(Y)$ ($\mathbf{H}'_{n+1} = \mathbf{H}'_{n+1}(Y)$) be the set of all finite meets (joins) of finite joins (meets) of elements from $\mathbf{H}_n$ ($\mathbf{H}'_n$). Using the operators $\mathbf{S}$ and $\mathbf{P}$ for a join-closure and meet-closure, respectively, we can write $\mathbf{H}_n = (\mathbf{PS})^n\mathbf{P}(Y)$ and $\mathbf{H}'_n = (\mathbf{SP})^n\mathbf{S}(Y)$ for every $n < \omega$.

PROPOSITION 1. *Let L be a p-algebra with $[Y]_{\text{lat}} = L$ for some $\mathbf{B}(L) \subseteq Y \subseteq \mathbf{K}_0(L) \cap \mathbf{K}'_0(L)$. Let L satisfy* (1) *and* (2). *Then L satisfies* (4). *More precisely,*

(i) $\mathbf{K}_n(L) = \mathbf{H}_n(L)$

(ii) $\mathbf{K}'_n(L) = \mathbf{H}'_n(L)$

for all $n < \omega$.

Proof. It is enough to establish (i) and (ii). According to [7; Lemma 3.1] we have $\mathbf{K}_n(L) \subseteq \mathbf{H}_n(Y)$ for every $n < \omega$. Take $a \in \mathbf{H}_0$ and let U be a cover of a. Since $a = a_1 \wedge \cdots \wedge a_m$ for some $a \in Y$, we have $a_1 \wedge \cdots \wedge a_m \leq \bigvee U$. Using (W) and $Y \subseteq \mathbf{K}_0(L)$ we get $a \in \mathbf{K}_0(L)$. Hence $\mathbf{K}_0(L) = \mathbf{H}_0(Y)$. By an easy induction we can obtain the rest of (i).

As for the proof of (ii) we can use the same method as in [7]. First we show that $\mathbf{K}'_n \subseteq \mathbf{H}'_n$ for every $n < \omega$. We shall proceed by induction on n. Consider $a \in \mathbf{H}'_{m+1} \backslash \mathbf{H}'_m$. Evidently, $a = \bigvee (\bigwedge U_i)$ with $U_i \subseteq \mathbf{H}'_m$. Each U_i is a dual p-cover of a. If $a \in \mathbf{K}'_0$, then either $a \geq \bigwedge U_i^{**}$ or there exists $a_i \in U_i$ with $a \geq a_i$. Since $\bigwedge U_i^{**} \in \mathbf{B}(L)$ and $a_i \in \mathbf{H}'_m$, we see that $a = \bigvee v_i \in \mathbf{H}'_m$, where $v_i = \bigwedge U_i^{**}$ or $v_i = a_i$, a contradiction. Thus $\mathbf{K}'_0 \subseteq \mathbf{H}'_0$.

Suppose that $\mathbf{K}'_n \subseteq \mathbf{H}'_n$ for all $n \leq r$. Take $a \in \mathbf{K}'_{r+1}(L)$. Assume that $a \in \mathbf{H}'_{m+1} \backslash \mathbf{H}'_m$, where $m > r$. Evidently, $a = \bigvee (\bigwedge U_i)$ with $U \leq \mathbf{H}'_m$ and U_i is a dual p-cover of a. Pick $a_i \in U_i$ with $a \geq a_i$ or take $\bigwedge U_i^{**}$, if $a \geq \bigwedge U_i^{**}$, whenever U_i is a trivial dual p-cover of a. But if U_i is a nontrivial due p-cover of a, pick a dual p-cover $V_i \subseteq \mathbf{K}'_r$ of a with $V_i \gg^* U_i$. By the induction hypothesis, each $V_i \subseteq \mathbf{H}'_r$ and therefore $\bigwedge V_i \in \mathbf{H}'_{r+1} \subseteq \mathbf{H}'_m$. But a is the join of these elements $\bigwedge V_i$ and a_i or $\bigwedge U_i^{**}$, which also belong to $\mathbf{H}'_m$. Therefore $a \in \mathbf{H}'_m$, a contradiction.

Again (W) and $Y \subseteq \mathbf{K}'_0$ imply $\mathbf{K}'_0 = \mathbf{H}'_0$. Now, using (W) it is routine to establish the rest of (ii). □

PROPOSITION 2. *Let $L = \mathbf{FP}(X)$. Then $\mathbf{K}(L) = L = \mathbf{K}'(L)$.*

The proof follows from Proposition 1 and the fact that $L = [\mathscr{P}(X)]_{\text{lat}}$ (Lemmas B–D). □

PROPOSITION 3. *Under the assumptions of Proposition* 1 *L satisfies* (5) *and* (6). *In addition, if $U \in S(a)$ and $a \in \mathbf{K}_r(L)$ for $r \geq 1$, then $U \subseteq \mathbf{K}_{r-1}(L)$ (and the dual assertion for $S'(a)$).*

The proof is essentially the same as of [4; Lemma 2]. $\square$

PROPOSITION 4. *Let* $L = \mathbf{FP}(X)$. *Then* L *satisfies* (7).

First we establish this result for Boolean algebras.

LEMMA 2. *Any free Boolean algebra satisfies* (7). *More precisely,*

$$A_1(a) = \mathbf{FB}(\mathrm{var}(a)) \cap [a) \quad \text{and} \quad B_1(a) = \mathbf{FB}(\mathrm{var}(a)) \cap (a]$$

for every element a.

Proof. $L = (L; +, \wedge, ', 0, 1)$ denote a free Boolean algebra freely generated by a set Y, i.e. $L = \mathbf{FB}(Y)$. If $a \in L$, then $\mathbf{FB}(\mathrm{var}(a))$ is a finite subalgebra of L. Set

$$A_1(a) = \mathbf{FB}(\mathrm{var}(a)) \cap [a)$$

$$B_1(a) = \mathbf{FB}(\mathrm{var}(a)) \cap (a]$$

in L. Since $A_1(0) = \{0, 1\} = B_1(1)$, we have $A_1(0) \cap B_1(1) \neq \emptyset$ and $A_1(a) \cap B_1(1) \neq \emptyset$ for every $a \in L$. We can confine ourselves to elements $a \in L$ with $\mathrm{var}(a) \neq \emptyset$, i.e. $0 \neq a \neq 1$. Take such elements $a, b \in L$. There are expressions

$$a = a_1 + \cdots a_n, \qquad b = b_1 \wedge \cdots \wedge b_m$$

where

$$a_i = \bigwedge (y : y \in I_i) \wedge \bigwedge (y' : y' \in \bar{I}_i)$$

$$b_j = \sum (y : y \in J_j) + \sum (y' : y' \in \bar{J}_j)$$

and $I_i \cup \bar{I}_i$, $J_j \cup \bar{J}_j$ are nonempty finite subsets of Y satisfying $I_i \cap \bar{I}_i = \emptyset$, $J_j \cap \bar{J}_j = \emptyset$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. (More precisely, we can assume $I_i, \bar{I}_i \subseteq \mathrm{var}(a)$ and $J_j, \bar{J}_j \subseteq \mathrm{var}(b)$ for any i and j.)

We have to show that $a \leq b$ in L implies $A_1(a) \cap B_1(b) \neq \emptyset$. Assume $0 \neq a \leq b \neq 1$ in L. This is equivalent with the relations $0 \neq a_i \leq b_j \neq 1$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. But $0 \neq a_i \leq b_j \neq 1$ if and only if and $I_i \cap J_j \neq \emptyset$ or $\bar{I}_i \cap \bar{J}_j \neq \emptyset$ (see [5; Theorem II.3]). Therefore, $a \leq b$ implies that

$$Z = \mathrm{var}(a) \cap \mathrm{var}(b) \neq \emptyset.$$

Set

$$\bar{a}_i = \bigwedge (y : y = I_i \cap Z) \wedge \bigwedge (y' : y' \in \bar{I}_i \cap Z)$$
$$\bar{b}_j = \sum (y : y \in J_j \cap Z) + \sum (y' : y' \in \bar{J}_j \cap Z).$$

Clearly, $a_i \le \bar{a}_i \le \bar{b}_j \le b_j$ for any i and j. Put

$$\bar{a} = \bar{a}_1 + \quad + \bar{a}_n \quad \text{and} \quad \bar{b} = \bar{b}_1 \wedge \cdots \wedge \bar{b}_m.$$

Evidently $a \le \bar{a} \le \bar{b} \le b$ and $\bar{a}, \bar{b} \in A_i(a) \cap B_1(b)$.

Proof of Proposition 4. If $a \in L$, then $\mathcal{P}(\mathrm{var}(a)) \subseteq \mathcal{P}(X)$ and $\mathcal{P}(\mathrm{var}(a))$ is finite. If $a \in \mathbf{K}_r(L)$ then $a = \bigwedge (\bigvee V_i)$ for some $V_i \subseteq \mathbf{K}_{r-1}(L)$. Using the procedure from the proof of Lemma E we can assume that the variables occurring in the representation of a are from $\mathrm{var}(a)$. Moreover, any variable occurring in $u \in V_i$ is from $\mathrm{var}(u) \subseteq \mathrm{var}(a)$. Similarly for $b \in \mathbf{K}'_s(L)$. Now, let $a \in \mathbf{K}_r(L) \backslash \mathbf{K}_{r-1}(L)$ and $b \in \mathbf{K}'_s(L) \backslash \mathbf{K}'_{s-1}(L)$. Set

$$A(a) = (\mathbf{PS})^r\mathbf{P}(\mathcal{P}(\mathrm{var}(a))) \cap [a)$$

$$B(b) = (\mathbf{SP})^s\mathbf{S}(\mathcal{P}(\mathrm{var}\ (b))) \cap (b]$$

in L. Clearly,

$$A(a) = \mathbf{H}_r(\mathcal{P}(\mathrm{var}(a))) \cap [a) \quad \text{and} \quad B(b) = \mathbf{H}'_s(\mathcal{P}(\mathrm{var}(b))) \cap (b].$$

It is easy to see that $A(a)$ and $B(b)$ are finite. Moreover, $A(a) \supseteq A(a^{**}) \supseteq A_1(a^{**})$ and $B(a) \supseteq B(a^{**}) \supseteq B_1(a^{**})$ (see Lemma 2).

We shall proceed by induction on the pair (r, s) that if $a \le b$ in L then $A(a) \cap B(b) \neq \emptyset$.

(1) Assume $r = s = 0$. Therefore $a = a_1 \wedge \cdots \wedge a_n$ and $b = b_1 \wedge \cdots \wedge b_m$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{P}(X)$. First we shall consider $n = m = 1$. Hence, $a, b \in \mathcal{P}(X)$. Four cases can occur: $a, b \in \mathbf{B}(L)$; $a, b \in X$; $a \in \mathbf{B}(L)$, $b \in X$ and $a \in X$, $b \in \mathbf{B}(L)$. In the first event the result follows from Lemma 2. In the second one $a \le b$ in X is equivalent with $a = b$. Now, $a \in \mathbf{B}(L)$ and $b \in X$ implies $a = 0$. Eventually, $b \in \mathbf{B}(L)$ implies $a \le a^{**} \le b$ in L. It can readily be verified that $A(a) \cap B(b) \supseteq A(a^{**}) \cap B(b) \neq \emptyset$.

Suppose that $n \ge 1$, $m \ge 1$. According to (W) we get $a \le a_i \le b$ or $a \le b_j \le b$ for some i, j. Since the elements from $\mathcal{P}(X)$ are join prime in L (Lemma C), we obtain $a_i \le b_k$ for some k in the first case. Similarly in the second event $a_i \le b_j$ for some i

if $b_j \in X$. It remains to analyze $b_j \in \mathbf{B}(L)$. But we have again $a \le a^{**} \le b_j$. Straightforward calculations show that $A(a) \cap B(b) \neq \emptyset$ in any case.

(2) $r = 0$ and $s > 0$ can be handled in the same way as [4; Lemma 5].

(3) $r > 0$ and $s = 0$. Then $a = \bigwedge(\bigvee V_i) \in \mathbf{K}_r(L) \backslash \mathbf{K}_{r-1}(L)$ and $b = b_1 \vee \cdots \vee b_m$; $b_1, \ldots, b_m \in \mathscr{P}(X)$. Now, $a \le b$ and (W) imply $a \le b_j$ for some j or $\bigvee V_i \le b$ for some i.

Two cases can occur in the first event: $b_j \in \mathbf{B}(L)$ or $b_j \in X$. If $b_j \in \mathbf{B}(L)$, then $a \le a^{**} \le b_j$. Since $A(a^{**}) \subseteq A(a)$ and $A(a^{**}) \cap B(b_j) \neq \emptyset$ by the first part of this proof, we have the desired relation. Assume $b_j \in X$. Since $a \neq 0$, b_j is meet-prime (Lemma D). Therefore $\bigvee V_i \le b_j$ for some i. This is equivalent with $v \le b_j$ for every $v \in V_i$. However, $V_i \subseteq \mathbf{K}_{r-1}(L)$. Hence by induction there is $c_v \in A(v) \cap B(b_j)$ for every $v \in V_i$. Set $c = \bigvee (c_v : v \in V_i)$. Similarly as in [4; Lemma 5] we get $c \in A(a)$. Since $c \le b_j$ and $B(b_j)$ is join-closed, we have $c \in B(b_j)$. Thus $c \in A(a) \cap B(b_j)$. In the same way we can solve the case $\bigvee V_i \le b$.

(4) $r > 0$ and $s > 0$ is straightforward (see [4; Lemma 5]).

The last part of condition (7) follows immediately from the definitions of sets $A(a)$ and $B(b)$. $\square$

LEMMA 3. *Let L be a p-algebra and let $f : \mathbf{FP}(X) \to L$ be an onto homomorphism. Let $g : L \to \mathbf{FP}(X)$ be a map such that $f(g(a)) = a$ for all $a \in L$. Then*

(i) *if g is join-preserving, then L satisfies* (5).

(ii) *if g is meet-preserving, then L satisfies* (6).

The proof is essentially the same as [4; Lemma 3]. We need to observe that $\mathbf{FP}(X)$ satisfies (5) and (6), which follows from Proposition 3.

LEMMA 4. *Under the assumptions of Lemma* 3 *we have*:

(i) *if g is join-preserving, then $\mathbf{K}(L) = L$*

(ii) *if g is meet-preserving and $g(a^{**}) = g(a)^{**}$ for all $a \in L$, then $\mathbf{K}'(L) = L$.*

Proof. We can proceed as in [4; Lemma 4]. We leave it for the reader to show that $\mathbf{K}(L) = L$. As for $\mathbf{K}'(L) = L$, we sketch the details of the proof. By induction on n, we shall show that for all $b \in \mathbf{F} = \mathbf{FP}(X)$ (see Proposition 2)

$$b \in \mathbf{K}'_n(\mathbf{F}) \quad \text{and} \quad g(f(b)) \le b \quad \text{imply} \quad f(b) \in \mathbf{K}'_n(L).$$

The case $n = 0$ is straightforward. Suppose that U is a nontrivial dual p-cover of $f(b)$. It follows from Lemma 3 that there is a minimal, nontrivial dual p-cover U' of $f(b)$ with $U' \gg^* U$. Then $g(U')$ is a nontrivial dual p-cover of b and by Proposition 3 there is a $U_0 \in S'(b)$ with $U_0 \gg^* g(U')$. Now $f(U_0)$ is a nontrivial dual

p-cover of $f(b)$ and $f(U_0) \gg^* U'$. Since U' is minimal, we have $U' \gg^* f(U_0)$. Thus $U_0 \gg^* g(f(U_0))$.

Let V_1 denote the set of all $u \in U_0$ satisfying $u \geq g(f(v))$ for some $v \in U_0$. Moreover, let V_2 be the set of all $u \in U_0 \backslash V_1$ satisfying $u \geq \bigwedge g(f(U_0))^{**}$. Since $U_0 \gg^* g(f(U_0))$, we have $U_0 = V_1 \cup V_2$. Consider $V_1' = \{u \in V_1 : u \geq g(f(u))\}$. It can readily be shown, similarly as in the proof of [4; Lemma 4], that $\bigwedge f(V_1) = \bigwedge f(V_1')$. Now, set

$$U_0' = V_1' \cup \{w\}, \qquad \text{where } w = \bigwedge g(f(U_0))^{**}.$$

Evidently, $w \in \mathbf{B}(\mathbf{F}) \subseteq \mathbf{K}_0'(\mathbf{F})$ and $V_1' \subseteq \mathbf{K}_{n-1}'(\mathbf{F})$ (Proposition 3). By induction is $f(U_0) \subseteq \mathbf{K}_{n-1}'(L)$. Consequently,

$$\bigwedge f(U) \geq \bigwedge f(U_0) = \bigwedge f(V_1) \wedge \bigwedge f(V_2) \geq \bigwedge f(V_1') \wedge f(w) = \bigwedge f(U_0').$$

It is easy to check that $f(U_0') \gg^* f(U_0)$. Since $f(U_0) \gg^* U$, $f(U_0') \gg^* U$. Hence $f(b) \in K_n'(L)$. $\square$

LEMMA 5. *Let L be a p-algebra with a projective Boolean algebra $\mathbf{B}(L)$. Then condition* (7) *holds in L if and only if*

there is a homomorphism f from a free p-algebra $\mathbf{FP}(X) = \mathbf{F}$ onto L and an order preserving map $g : L \to \mathbf{F}$ such that $f(g(a)) = a$ and $g(a^{**}) = g(a)^{**}$ for all $a \in L$. Moreover, the restriction $h = g \mid \mathbf{B}(L)$ is a Boolean homomorphism from $\mathbf{B}(L)$ into $\mathbf{B}(\mathbf{F})$. $(*)$

Proof. It can be readily shown that if $(*)$ holds for some homomorphism of a free p-algebra onto L, it holds for all such homomorphisms. Thus proving the lemma we shall take $X = \{x(a) : a \in L\}$ and $f : \mathbf{FP}(X) \to L$ be the extension of $f(x(a)) = a$.

Suppose L satisfies $(*)$. For all $a \in L$ set $A(a) = f(A(g(a)))$ and $B(a) = f(B(g(a)))$. If $a \leq b$ in L then $g(a) \leq g(b)$. Therefore, $A(a) \cap B(b) \neq \varnothing$. If $a \in \mathbf{B}(L)$ then $g(a) \in \mathbf{B}(\mathbf{F})$. Hence $A(g(a)) \subseteq \mathbf{B}(\mathbf{F})$ by Proposition 4. Since f is a homomorphism, we have $A(a) \subseteq \mathbf{B}(L)$. A similar argument yields the rest of condition (7).

Now suppose that L satisfies condition (7). We arrange L into a well-ordered doubly indexed sequence as follows (see [4; Theorem 2]). $L = \{a_{\alpha,i} : \alpha < \gamma, i < \mu_\alpha\}$ for some ordinals γ and $\mu_\alpha \leq \omega$. We can assume that $\mathbf{B}(L) = \{a_{\alpha,0} : \alpha < \tau\}$ for some $\tau \leq \gamma$. Therefore, $a_{\tau,0} \in L \backslash \mathbf{B}(L)$. Let $a_{\tau,0}, a_{\tau,1}, \ldots, a_{\tau,n}$ be $A(a_{\tau,0}) \cup B(a_{\tau,0}) \backslash \mathbf{B}(L)$. Let

$a_{\tau,n+1}, \ldots, a_{\tau,n+m}$ be $A(a_{\tau,1}) \cup B(a_{\tau,1}) \backslash (\mathbf{B}(L) \cup \{a_{\tau,0}, a_{\tau,1}, \ldots, a_{\tau,n}\})$. Continuing in this way we obtain a countable or finite sequence $a_{\tau,0}, a_{\tau,1}, \ldots$. If this does not exhaust L, we choose again $a_{\tau+1,0} \in L \backslash (\mathbf{B}(L) \cup \{a_{\tau,0}, \ldots\})$. This sequence has the property that if $b \in A(a) \cup B(a)$ and $a \in L \backslash \mathbf{B}(L)$, then either b comes before a in the sequence, or at worst, only finitely many places after a. By assumption there is a Boolean homomorphism $h : \mathbf{B}(L) \to \mathbf{B}(\mathbf{F})$ such that $f(h(a)) = a$ for every $a \in \mathbf{B}(L)$. Define inductively $g : L \to \mathbf{FP}(X)$ by

$$\begin{aligned}
g(a_{\alpha,0}) &= h(a_{\alpha,0}) \qquad \text{for } \alpha < \tau,\\
g(a_{\alpha,i}) &= [t(a_{\alpha,i}) \wedge \bigwedge (g(a_{\alpha,j}) : j < i, a_{\alpha,j} > a_{\alpha,i})\\
&\qquad \wedge \bigwedge (g(a_{\beta,j}) : \beta < \alpha, j < \mu_\beta, a_{\beta,j} \in A(a_{\alpha,i}))]\\
&\qquad \vee [\bigvee (g(a_{\alpha,j}) : j < i, a_{\alpha,j} < a_{\alpha,i})\\
&\qquad \vee \bigvee (g(a_{\beta,j}) : \beta < \alpha, j < \mu_\beta, a_{\beta,j} \in B(a_{\alpha,i}))] \qquad \text{for } a \le \tau
\end{aligned} \tag{8}$$

where $t(a_{\alpha,i}) = x(a_{\alpha,i}) \vee ((x(a_{\alpha,i}))^* \wedge g(a_{\alpha,i}^{**}))$.

Evidently, $f(t(a_{\alpha,i})) = a_{\alpha,i}$ for any $a \ge \tau$ and $i < \mu_\alpha$. (Empty joins or meets are simply omitted.) A straightforward induction shows that $f(g(a)) = a$ for all $a \in L$. Again, by induction, we shall show that g is order-preserving. Let $T(\alpha, i)$ denote the sentence: g is order-preserving on the set $\{a_{\beta,j} : \beta = \alpha \text{ and } j < i \text{ or } \beta < \alpha\}$. Assume that (α, i) holds true for some (α, i). Evidently $\alpha \ge \tau$, as $T(\tau, 0)$ is a true sentence. Take $a = a_{\beta,j}$, $b = a_{\alpha,i}$ with $\beta = \alpha$ and $j < i$ or $\beta < \alpha$. A straightforward induction shows that if a and b are comparable and $\beta \ge \tau$, then g is order-preserving on the subset $\{a, b\}$. Consider $\beta < \tau$, i.e. $a \in \mathbf{B}(L)$. If $a < b$, then $A(a) \subseteq \mathbf{B}(L)$ and $c \in A(a) \cap B(b)$ for some $c = a_{\delta,0}$, $\delta < \tau$. Now, by (8) and induction $g(a) \le g(b)$. If $a > b$, then there exists with $c \in \mathbf{B}(L)$ with $c \in A(b) \cap B(a)$. Again, by induction, $g(a) \ge g(b)$. Hence $T(\alpha, i+1)$ or $T(\alpha+1, 0)$ (if $a_{\alpha+1,0}$ is a successor of $a_{\alpha,i}$) holds true and g is order-preserving.

Since $a \le a^{**}$ and g is order-preserving, we have $g(a) \le g(a^{**})$. The equality $(g(a))^{**} = g(a^{**})$ can be shown by induction. Clearly, $(g(a_{\alpha,i}))^{**} = g(a_{\alpha,i}^{**})$ for any $\alpha < \tau$. Assume that this equality holds for every $a_{\beta,j}$ coming before $a_{\alpha,i}$ in the sequence. Now, by (8),

$$\begin{aligned}
(g(a_{\alpha,i}))^{**} &= [(t(a_{\alpha,i}))^{**} \wedge \bigwedge (g(a_{\alpha,j}^{**}) : j < i, a_{\alpha,j} > a_{\alpha,i})\\
&\qquad \wedge \bigwedge (g(a_{\beta,j}^{**}) : \beta < \alpha, j < \mu_\beta, a_{\beta,j} \in A(a_{\alpha,i}))] + \quad \cdot]^{**}\\
&= ((x(a_{\alpha,i}))^{**} + ((x(a_{\alpha,i}))^* \wedge g(a_{\alpha,i}^{**}))) \wedge g(a_{\alpha,i}^{**})\\
&\qquad + [\cdots]^{**} = g(a_{\alpha,i}^{**}).
\end{aligned}$$

as $a_{\alpha,i}^{**} = a_{\beta,0}$ for some $\beta < \tau$ and $a_{\alpha,i}^{**} \in A(a_{\alpha,i})$. $\square$

Proof of Theorem 1. The proof of necessity follows from Lemmas 1, 3, 4 and 5. The proof of sufficiency is essentially the same as [4] or [7]. For the sake of completeness, let us sketch the details here. Let $f: \mathbf{FP}(X) \to L$ be an homomorphism onto a p-algebra L which satisfies (1)–(7). Define

$$S_m(a) = \{U : U \in S(a) \text{ and } U \text{ is minimal (with respect to } \ll)\}.$$

Clearly, $a \in \mathbf{K}_r(L)$ and $U \in S_m(a)$ implies $U \subseteq \mathbf{K}_{r-1}(L)$. Similarly, set for $a \in \mathbf{K}'_r(L)$,

$$S'_m(a) = \{U : U \in S'(a) \text{ and } U \text{ is minimal (with respect to } \gg^*)\}.$$

It can be easily verified that for every $U \in S'(a)$ there exists $V \in S'_m(a)$ such that $V \gg^* U$. By (3), there is a Boolean homomorphism $h: \mathbf{B}(L) \to \mathbf{B}(\mathbf{FP}(X))$ such that $f(h(a)) = a$ for all $a \in \mathbf{B}(L)$. According to Lemma 5 f has an order-preserving transversal g satisfying $(g(a))^{**} = g(a^{**})$ for all $a \in L$ (see (8)). Define $g_0 = g^0 = g$ and

$$g_{k+1}(a) = g(a) \wedge \bigwedge(\bigvee (g_k(U) \colon U \in S_m(a))) \tag{9}$$

$$g^{k+1}(a) = g(a) \vee \bigvee(\bigwedge (g^k(U) \colon U \in S'_m(a))). \tag{10}$$

We leave it for the reader to show that if $a \in \mathbf{K}_r(L)$, then $g_r(a) = g_m(a)$ for all $m \geq r$. Hence the sequence $g_0(a), g_1(a), \ldots$ is eventually constant, Let $g_-(a)$ denote this final value. It is easy to show that g_- is order-preserving, and moreover, $g_-(a \vee b) = g_-(a) \vee g_-(b)$ for any $a, b \in L$. In addition, g_- preserves meets whenever g has this property (see [4]).

It is easy to check by induction that $g^r(a) = g^m(a)$ for all $m \geq r$, whenever $a \in \mathbf{K}'_r(L)$. This implies that the sequence $g^0(a), g^1(a), \ldots$ is eventually constant. Let $g^-(a)$ denote this final value. Again, by induction on k we will show that g^k is order-preserving for every $k \geq 0$. For g^0 it is true by Lemma 5. Suppose that g^k is order-preserving. Assume $a \leq b$ and $U \in S'_m(a)$. Therefore, $\bigwedge U \leq b$. Two cases can occur:

(a) U is a trivial dual p-cover of b, i.e. either $\bigwedge U^{**} \leq b$ or $u \leq b$ for some $u \in U$. Therefore

$$\begin{aligned}\bigwedge g^k(U) &\leq \bigwedge g^k(U^{**}) = \bigwedge g(U^{**}) = \bigwedge h(U^{**}) = h(\bigwedge U^{**})\\ &= g(\bigwedge U^{**}) \leq g(b) \leq g^{k+1}(b) \quad \text{or}\\ \bigwedge g^k(U) &\leq g^k(u) \leq g^k(b) \leq g^{k+1}(b).\end{aligned}$$

(b) U is a nontrivial dual p-cover of b. Then there is $V \in S'_m(b)$ such that $V \gg^* U$. Similarly as in part (a) we can show that

$$\bigwedge g^k(U) \leq \bigwedge g^k(V) \leq g^{k+1}(b)$$

(see also (10)).

Now, taking into account (10) we get $g^{k+1}(a) \leq g^{k+1}(b)$. It follows that g^- is also order-preserving.

Next we will show that g^- preserves meets (and joins as well, whenever g does it). Clearly, $g^-(a \wedge b) \leq g^-(a) \wedge g^-(b)$, as g is order-preserving. Let a and b incomparable elements of L and choose k large enough so that $g^k(a) = g^-(a)$, $g^k(b) = g^-(b)$ and $g^k(a \wedge b) = g^-(a \wedge b)$. $U = \{a, b\}$ is a dual p-cover of $a \wedge b$. If $a \wedge b \geq (a \wedge b)^{**}$, then $a \wedge b = (a \wedge b)^{**} = a^{**} \wedge b^{**}$. But

$$\begin{aligned} g^-(a \wedge b) &= g^-((a \wedge b)^{**}) = h((a \wedge b^{**}) = h(a^{**}) \wedge h(b^{**}) \\ &= g^-(^{**}) \wedge g^-(b^{**}) \geq g^-(a) \wedge g^-(b). \end{aligned}$$

Assume that U is a nontrivial dual p-cover of $a \wedge b$. Then there exists $V \in S'_m(a \wedge b)$ with $V \gg^* U$. Therefore,

$$\bigwedge g^k(V) \geq \bigwedge g^k(U) = g^k(a) \wedge g^k(b).$$

By (10), $g^{k+1}(a \wedge b) \geq \bigwedge g^k(V) \geq g^{k+1}(a) \wedge g^{k+1}(b)$. Hence g^- preserves meets. We leave it for the reader to show that if $g(a \vee b) = g(a) \vee g(b)$ for some $a, b \in L$, then $g^-(a \vee b) = g^-(a) \vee g^-(b)$ as well (see [4]).

Summarizing we see that the map $\bar{g} = (g^-)_- : L \to \mathbf{FP}(X)$ is a lattice homomorphism such that $\bar{g} \mid \mathbf{B}(L) = h$ and $f(\bar{g}(a)) = a$ for all $a \in L$. It remains only to show that $\bar{g}$ preserves pseudocomplements. First we shall verify that $(g_k(a))^{**} = g_k(a^{**})$ and $(g^k(a))^{**} = g^k(a^{**})$ for all $a \in L$ and $k \geq 0$ (see (9) and (10)). We can proceed by induction on k. For $k = 0$ it is true (Lemma 5). Suppose that this assertion is true for some $k > 0$. Therefore.

$$\begin{aligned} (g_{k+1}(a))^{**} &= (g(a))^{**} \wedge \bigwedge (\bigvee (g_k(U) : U \in S_m(a)))^{**} \\ &= g(a^{**}) \wedge \bigwedge (\sum ((g_k u))^{**} : u \in U,\, U \in S_m(a))) \\ &= g(a^{**}) \wedge \bigwedge (\sum (g_k(U^{**}) : U \in S_m(a))) = g(a^{**}) = g_{k+1}(a^{**}) \end{aligned}$$

using the induction hypothesis and rules of computation. Similar calculations yield the desired result for the maps g^k. Now, we can conclude that $(\bar{g}(a))^{**} = \bar{g}(a^{**})$ for all $a \in L$. Finally, $(\bar{g}(a))^* = (\bar{g}(a))^{***} = (\bar{g}(a^{**}))^* = (h(a^{**}))^* = h(a^*) = \bar{g}(a^*)$ for all $a \in L$. □

Proof of Corollary. Only the sufficiency needs a proof. $\mathbf{B}(L)$ is finite, and therefore $\mathbf{B}(L)$ is a projective Boolean algebra (see [6]). Assume $L = \{a_0, a_1, \ldots\}$. Define

$$A(a_n) = \{b \in \mathbf{B}(L) : b \geq a_n\} \qquad \text{for } a_n \in \mathbf{B}(L)$$

and

$$A(a_n) = \{a_i : i \leq n \text{ and } a_i \geq a_n\} \cup \{b \in \mathbf{B}(L) : b \geq a_n\} \quad \text{otherwise;}$$

$$B(a_n) = \{a_i : i \leq n \text{ and } a_i \leq a_n\} \cup \{b \in \mathbf{B}(L) : b \leq a_n\}.$$

A straightforward verification shows that L satisfies (7). □

THEOREM 2. *Let L be a p-algebra satisfying $L = [\mathbf{B}(L)]_{\mathrm{lat}}$. Then L is projective in* $\mathbf{P}$ *if and only if L satisfies* (1)–(3).

Proof. With respect to Theorem 1, Proposition 1 and Proposition 3 we have only to show that L satisfies (7). Assume that $f : \mathbf{FP}(X) \to L$ is an onto homomorphism. The restriction $f_1 = f \mid \mathbf{B}(\mathbf{FP}(X))$ is a Boolean homomorphism onto $\mathbf{B}(L)$. By (3), there is a Boolean homomorphism $h : \mathbf{B}(L) \to \mathbf{B}(\mathbf{F})$ such that $f_1(h(a)) = a$ for all $a \in \mathbf{B}(L)$.

Now, we shall proceed by Lemma 5. By hypothesis every element $x \in L$ can be represented by a lattice term over $\mathbf{B}(L)$, that means, there is a lattice term t such that

$$x = t(a_1, \ldots, a_n),\ a_1, \ldots, a_n \in \mathbf{B}(L).$$

Define a map $g : L \to \mathbf{FP}(X)$ as follows

$$g(x) = t(h(a_1), \quad , h(a_n)).$$

First we show that g is order-preserving. Assume that we have $y \in L$ satisfying $x \leq y$. Let q be a lattice term over $\mathbf{B}(L)$ representing y, i.e. $y = q(b_1, \ldots, b_m)$. Now using the properties (1) and (2) as well as lattice identities $x \leq y$ is equivalent to a finite set of inequalities:

$$a_i \leq b_j \quad \text{and} \quad u_i \wedge \cdots \wedge u_s \leq b_i$$

where $a_i, b_i, b_j \in \mathbf{B}(L)$ and $u_1, \ldots, u_s$ are subterms of t. Since

$$u_1 \wedge \cdots \wedge u_s \leq u_1^{**} \wedge \cdots \wedge u_s^{**} \leq b_i$$

and $\mathbf{B}(L)$ is a meet-subsemilattice of L, $x \leq y$ is equivalent to a finite set of inequalities:

$$a_i \leq b_j \quad \text{and} \quad u_1^{**} \wedge \cdots \wedge u_s^{**} \leq b_i$$

in $\mathbf{B}(L)$. The last relations imply inequalities

$$h(a_i) \leq h(b_j) \quad \text{and} \quad h(u_1^{**}) \wedge \cdots \wedge h(u_s^{**}) \leq h(b_i)$$

in $\mathbf{B}(\mathbf{F})$, which in turn yields

$$g(x) = t(h(a_1), \ldots, h(a_n)) \leq g(y) = q(hb_1), \ldots, h(b_m))$$

in $\mathbf{FP}(X)$. Hence, g is well defined and order-preserving. Evidently, g is an extension of h and $f(g(x)) = x$ for all $x \in L$.

It remains to show that $(g(x))^{**} = g(x^{**})$ for all $x \in L$. Let t be again a lattice term over $\mathbf{B}(L)$ representing $x \in L$. It is easy to check that x^{**} can be represented by a Boolean term $\bar{t}$ over $\mathbf{B}(L)$, where $\bar{t}$ can be obtained from t by replacing every symbol $\vee$ with $+$. More precisely, if $x = t(a_1, \ldots, a_n)$, then $x^{**} = \bar{t}(a_1, \ldots, a_n)$. The same is true for $\mathbf{FP}(X)$ as well. Finally, $(g(x))^{**} = (t(h(a_1), \ldots, h(a_n))^{**} = \bar{t}(h(a_1), \ldots, h(a(a_n)) = h(x^{**}) = g(x^{**})$. $\square$

4. The finitely generated case

The concept of a bounded homomorphic image introduced by R. McKenzie [11] can be extended for p-algebras as well (see [9]):

A p-algebra L is *a bounded homomorphic image* of a free p-algebra $\mathbf{FP}(X)$ if there is a homomorphism $f : \mathbf{FP}(X) \to L$, mapping $\mathbf{FP}(X)$ onto L, such that $\{u \in \mathbf{FP}(X) : f(u) = a\}$ has a least and a greatest element for each $a \in L$. (Notation: $f_-(a), f^-(a)$.) Let $\mathcal{B}$ be the class of all p-algebras L being bounded homomorphic image of a free p-algebra.

PROPOSITION 5. *Let L be a finitely generated p-algebra. Then L is projective in the class* $\mathbf{P}$ *if and only if L satisfies* (1), (2) *and*

$$L \in \mathcal{B}. \tag{11}$$

LEMMA 6. *Let L be a Boolean algebra which is a bounded homomorphic image of a free Boolean algebra. Then L is projective in the category of Boolean algebras.*

Proof. Let $f: \mathbf{FB}(X) \to L$ be a bounded epimorphism and let $L = (L; +, \wedge, ', 0, 1)$. We shall prove that there is a homomorphism $g: L \to \mathbf{FB}(X)$ such that $f(g(a)) = a$ for all $a \in L$. (Note, that this assertion is equivalent to the fact that L is a projective Boolean algebra.) Now, choose a prime ideal P of L. We can define $g: L \to \mathbf{FB}(X)$ as follows:

$$g(a) = \begin{cases} f_-(a) & \text{if } a \in P \\ f^-(a) & \text{if } a \notin P. \end{cases}$$

Clearly, $g(0) = 0$, $g(1) = 1$ and $f(g(a)) = a$ for all $a \in L$. Since f is a Boolean homomorphism, we have

$$f_-(a) = f^-(a) \wedge f_-(1) \quad \text{and} \quad f^-(a) = f_-(a) + f^-(0).$$

It is enough to show that g is a lattice homomorphism. Three cases can occur: $a, b \in P$; $a, b \in L \backslash P$ and $a \in P$, $b \notin P$. In the first event $g(a + b) = g(a) + g(b)$, as f_- is join-preserving, and

$$g(a \wedge b) \le g(a) \wedge g(b) = f^-(a) \wedge f^-(b) \wedge f_-(1) = f^-(a \wedge b) \wedge f_-(1) = g(a \wedge b).$$

The second case can be established dually. In the last case we have

$$\begin{aligned} g(a \wedge b) &= f_-(a \wedge b) = f^-(a \wedge b) \wedge f_-(1) = f^-(a) \wedge f^-(b) \wedge f_-(1) \\ &= f_-(a) \wedge f^-(b) = g(a) \wedge g(b), \end{aligned}$$

and by duality, $g(a + b) = g(a) + g(b)$. □

LEMMA 7. *Let $f: \mathbf{FP}(X) \to L$ be a bounded homomorphism onto a p-algebra L. Then*

(i) *the restriction $h = f \mid \mathbf{B}(\mathbf{FP}(X))$ is a bounded Boolean homomorphism onto $\mathbf{B}(L)$ such that*

$$h_-(a) = (f_-(a))^{**} \quad \textit{and} \quad h^-(a) = f^-(a) \qquad \textit{for every } a \in B(L)$$

(ii) $f^-(a^{**}) = (f^-(a))^{**} \qquad$ *for all $a \in L$.*

Proof. (i) is trivial. (ii) Evidently $f^{-}(a^{**}) \geq (f^{-}(a))^{**}$ for any $a \in L$. Since $h(f^{-}(a^{**})) = h((f^{-}(a))^{**})$, we have

$$f^{-}(a^{**}) = (f^{-}(a))^{**} + f^{-}(0) = [(f^{-}(a))^{**} \vee f^{-}(0)]^{**}$$

in $\mathbf{B}(\mathbf{FP}(X))$. But $f^{-}(a) \geq f^{-}(0)$ implies $f^{-}(a^{**}) = (f^{-}(a))^{**}$. $\square$

Proof of Proposition 5. The necessity follows from Theorem 1 [9; Lemmas 11 and 12]. As for the sufficiency we shall show that L satisfies (3)–(7). Since $\mathbf{B}(L)$ is finite, (3) is trivially satisfied (see Lemma B and [6]). By (11) we have a bounded homomorphisms $f : \mathbf{FP}(X) \to L$ onto L. There are two maps $g_i : L \to \mathbf{FP}(X)$, namely, $g_1 = f_{-}$ and $g_2 = f^{-}$. It is easy to verify that g_1 is join-preserving, g_2 is meet-preserving and $f(g_i(a)) = a$ for all $a \in L$. Moreover, $g_2(a^{**}) = (g_2(a))^{**}$ for all $a \in L$ (Lemma 7). Now applying Lemmas 3 and 4 we obtain (4)–(6).

It remains to establish (7). We shall show that there is a map $g : L \to \mathbf{FP}(X)$ satisfying assumptions of Lemma 5. Let P denote a prime ideal of the (finite) Boolean algebra $\mathbf{B}(L)$. With regard to Lemmas 6 and 7 there exists a Boolean homomorphism $k : \mathbf{B}(L) \to \mathbf{B}(\mathbf{FP}(X))$ such that $f(k(a)) = a$ for all $a \in \mathbf{B}(L)$. Now define

$$g(a) = \begin{cases} f_{-}(a) \vee \bigvee (k(b) : b \leq a \quad \text{and} \quad b \in B(L)) & \text{if } a^{**} \in P \\ f^{-}(a) \text{ else.} \end{cases}$$

This map is well defined as $\mathbf{B}(L)$ is finite. One can readily show that g is order-preserving, $f(g(a)) = a$ for all $a \in L$ and the restriction $g \mid \mathbf{B}(L)$ is the Boolean homomorphism k. Since $a \leq a^{**}$, we have $(g(a))^{**} \leq g(a^{**})$ for all $a \in L$. On the other hand, if $a^{**} \in P$, then $g(a) \geq f_{-}(a)$, and consequently,

$$(g(a))^{**} \geq (f_{-}(a))^{**} \geq (f_{-}(a^{**}))^{**} = k(a^{**}) = g(a^{**}).$$

If $a^{**} \notin P$, then

$$(g(a))^{**} = (f^{-}(a))^{**} = f^{-}(a^{**}) = g(a^{**})$$

by Lemma 7. Therefore, $(g(a))^{**} = g(a^{**})$ for all $a \in L$. Applying Lemma 5 we obtain (7). $\square$

PROPOSITION 6. *If there exists a bounded homomorphism of a free p-algebra onto a p-algebra L, then $\mathbf{K}(L) = L = \mathbf{K}'(L)$. Conversely, if $\mathbf{K}(L) = L = \mathbf{K}'(L)$, then every homomorphism of a finitely generated p-algebra into L is bounded.*

Proof. If $f : \mathbf{FP}(X) \to L$ is bounded homomorphism onto L, then we can consider two maps $f_-, f^- : L \to \mathbf{FP}(X)$. Evidently, f_- is join-preserving and f^- is meet-preserving such that $f^-(a^{**}) = (f^-(a))^{**}$ for all $a \in L$ (Lemma 7). According to Lemma 4 we get $\mathbf{K}(L) = L = \mathbf{K}'(L)$.

Now suppose $\mathbf{K}(L) = L$ and consider a homomorphism $f : M \to L$, where M is a p-algebra generated by a finite set X. Let $Y = X \cup \mathbf{B}(M)$. Clearly, Y is finite and $M = [Y]_{\text{lat}}$ (Lemma B). Consider $\mathbf{H}'_n(Y)$ for all $n < \omega$. Applying [7; Theorem 4.2] we see that f is lower bounded. More precisely,

$$f_-(a) = \bigwedge (y \in \mathbf{H}'_r(Y) : f(y) \geq a) \qquad \text{for } a \in \mathbf{K}_r(L).$$

We shall consider the case $\mathbf{K}'(L) = L$ in more detail (see [7; Theorem 4.2]). Again suppose that $f : M \to L$ is a homomorphism and $M = [X] = [Y]_{\text{lat}}$, $Y = X \cup \mathbf{B}(M)$ for some finite X. Consider $\mathbf{H}_n(Y)$ for all $n < \omega$. We claim that for all $r < \omega$, $a \in \mathbf{K}'_r(L)$ and $y \in M$

$$f(y) \leq a \text{ implies that } y \leq \bigvee (u \in \mathbf{H}_r(Y) : f(u) \leq a) = \tau_r(a). \tag{12}$$

We shall proceed by induction. Let first $a \in \mathbf{K}'_0(L)$ and $y \in M$. Evidently (12) holds for $y \in \mathbf{H}_0(Y)$. Assume that (12) is true for all $y \in \mathbf{H}_n(Y)$ and some $n \geq 0$. Suppose $y \in \mathbf{H}_{n+1}(Y)$ and $f(y) \leq a$. Then $y = \bigwedge (\bigvee V_i)$ for some $V_i \subseteq \mathbf{H}_n(Y)$. Therefore $f(y) = \bigwedge (\bigvee f(V_i)) \leq a$ implies either $f(y) \leq f(y^{**}) \leq a$ or $\bigvee f(V_i) \leq a$ for some i. Since $y^{**} \in \mathbf{H}_0(Y)$, we get $y \leq y^{**} \leq \tau_0(a)$. In the second case, $f(v) \leq a$ for all $v \in V_i$ implies $v \leq \tau_0(a)$, as $V_i \subseteq \mathbf{H}_n(Y)$. Hence $y \leq \bigvee V_i \leq \tau_0(a)$.

Suppose $a \in \mathbf{K}'_{r+1}(L)$ and (12) holds when $a \in \mathbf{K}'_r(L)$. Moreover, let (12) be true for every $y \in \mathbf{H}_n(Y)$ and some $n \geq r$. Now, assume that $f(y) \leq a$ for some $y \in \mathbf{H}_{n+1}(Y)$, i.e. $y = \bigwedge (\bigvee V_i)$ with $V_i \subseteq \mathbf{H}_n(Y)$ for every $i = 1, \ldots, k$. Clearly, $\{\bigvee f(V_i) : i = 1, \ldots, k\}$ is a dual p-cover of a. If it is a trivial dual p-cover of a, then the proof is straightforward. Suppose that $\{\bigvee f(V_i) : i = 1, \ldots, k\}$ is a nontrivial dual p-cover of a. Then there exists a dual p-cover $U \subseteq \mathbf{K}'_r(L)$ of a with $U \gg^* \{\bigvee f(V_i) : i = 1, \ldots, k\}$. Therefore, for every $u \in U$, $u \geq (f(y))^{**} = f(y^{**})$ or $u \geq \bigvee f(V_i)$ for some i. By inductive hypothesis $y^{**} \leq \tau_r(u)$ in the first case or $\bigvee V_i \leq \tau_r(u)$ in the second one, for every $u \in U$. Therefore $y \leq \bigwedge (\tau_r(u) : u \in U)$. Since $\bigwedge (\tau_r(u) : u \in U) \in \mathbf{H}_{r+1}(Y)$ and $f(\bigwedge(\tau_r(u) : u \in U)) \leq a$, we get $y \leq \bigwedge (\tau_r(u) : u \in U) \leq \tau_{r+1}(a)$. □

THEOREM 3. *Let L be a finitely generated p-algebra. Then the following conditions are equivalent*:

(i) *L is projective in the class* $\mathbf{P}$;

(ii) *L is a subalgebra of a free p-algebra*;

(iii) *L satisfies* (1), (2) *and* (11);
(iv) *L satisfies* (1), (2) *and* (4).

Proof. (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (iii) follows from Lemma C and [9; Lemmas 11 and 12]. (iii) $\Leftrightarrow$ (iv) Proposition 6 (iii) $\Rightarrow$ (i) Proposition 5. □

Since every finitely generated distributive p-algebra is finite, we have the following

COROLLARY. *Let L be a finite distributive p-algebra. Then L is projective in* **P** *if and only if L satisfies* (1), (2) *and*

$$\mathbf{S}(\mathbf{K}_0(L)) = L = \mathbf{P}(\mathbf{K}_0'(L)). \tag{4'}$$

Proof. In any distributive p-algebra satisfying (2) we have $\mathbf{K}_1(L) = \mathbf{S}(\mathbf{K}_0(L)) = \bigcup_{n \geq 0} \mathbf{K}_n(L)$ and $\mathbf{K}_1'(L) = \mathbf{P}(\mathbf{K}_0'(L)) = \bigcup_{n \geq 0} \mathbf{K}_n'(L)$. Thus (4) is equivalent with (4′). □

5. Concluding remarks

It is not clear whether conditions (1)–(7) of Theorem 1 are logically independent. It is not hard to check that if (7) holds in a p-algebra L, then (7) is true in the Boolean algebra $\mathbf{B}(L)$, as well. The authors of [4] claim (without proof) in the final remark on p. 105 that (7) alone characterizes projective Boolean algebras. (This would mean that (7) implies (3).) We have not succeeded in proving this assertion. Thus, it remains an open question, whether (3) can be dropped in Theorem 1 or not.

As for the remaining conditions (1), (2), (4)–(7), it can be easily demonstrated, adopting the corresponding examples from [4], that they cannot be omitted from Theorem 1. Really, take first the four-element Boolean algebra. This, considered as a p-algebra, satisfies (2)–(7), yet fails (1). For the next example consider first the lattice $\mathbf{3} \times \mathbf{3}$ and add to it a new top element. ($\mathbf{3}$ denotes the three-element chain.) We obtain a 10-element p-algebra, which satisfies (1), (3)–(7), but fails (2). Now, take the lattice L_0 diagrammed in Figure 1 of [4; p. 100]. Add to L_0 a new top element 1 and a new bottom element 0. The new lattice $L = L_0 \cup \{0, 1\}$ is a p-algebra which satisfies (1)–(3) and (5)–(7), but fails (4). Let us start with the lattice L_1, depicted in Figure 2 of [4; p. 101], and add to L_1 a new top element 1. The resulting lattice $L = L_1 \cup \{1\}$ is a p-algebra satisfying (1)–(4) and (6), (7), but L is not projective. By an argument similar to that of [4], one can see that L is a subalgebra of a free p-algebra. Now, take the lattice dual to L_1, depicted in Figure

2 of [4], and add to it a new top element. The resulting lattice is a p-algebra which satisfies (1)–(5) and (7), yet fails (6). For the last example consider the chain $L = \{x : 0 \leq x \leq 1\}$ of real numbers. Evidently, L is a p-algebra satisfying (1)–(6). We sketch the details of the proof that (7) fails. Assume to the contrary that (7) is true for L. Then starting with $b_0 = 1$ one can construct two sequences $1 = b_0 > b_1 > \cdots > b_m > \cdots$ and $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n \leq \cdots$ such that $a_n \leq b_m$ for any n, m; and the following condition is satisfied: $A(y) \cap B(b_n) = \{b_n\}$ for any $a_n \leq y \leq b_n$ and $n \geq 0$. It is well known that there exists a real number a with $a_n \leq a \leq b_m$ for any n, m. Consider $A(a)$. Since $b_n \in A(a)$ for every integer n, $A(a)$ is infinite, a contradiction.

It is an open question whether a countable p-algebra satisfying (1), (2) and (4)–(6) alone is projective or not.

The first author studied in [8] and [9] subvarieties $\mathbf{P}_n$ $(0 \leq n \leq \omega)$ of $\mathbf{P}$. It may be of interest to extend the results of this paper to the classes $\mathbf{P}_n$.

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