BOUNDED ENDOMORPHISMS OF FREE P-ALGEBRAS

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1. Introduction. The present note deals with bounded endomorphisms of free p-algebras (pseudocomplemented lattices). The idea of bounded homomorphisms was introduced by R. McKenzie in [**8**]. T. Katriňák [**5**] subsequently studied the properties of bounded homomorphisms for the varieties of p-algebras. This concept is also an efficient tool for the characterization of, so-called, splitting as well as projective algebras in the varieties of all lattices or p-algebras. For details the reader is referred to [**2**], [**5**], [**6**], [**7**] and other references therein. Let us emphasize that the main results that are contained in the above mentioned references strongly depend on the boundedness of each endomorphism of any finitely generated free algebra in a given variety.

In [**8**], R. McKenzie showed that each endomorphism of a finitely generated free lattice $\mathrm{FL}(X)$ is bounded. For the variety of all p-algebras, the same statement was proved by T. Katriňák in [**5**]. More precisely, he has considered the countable chain of equational classes of p-algebras

$$P_1 \subset P_2 \subset \ldots \subset P_n \subset \ldots \subset P_\omega,$$

where P_ω is the variety of all p-algebras and the nth variety P_n is determined by Lee's identity $\mathscr{L}_n$ (the definition is recalled in the next section). In [**5**, Lemmas 11, 12], it is shown that each endomorphism of $\mathrm{FP}_\omega(X)$ (a free p-algebra in P_ω freely generated by a finite set X) is bounded. Using this result, it is possible to characterize both the splitting p-algebras [**5**, Theorem 3] and projective p-algebras [**6**, Proposition 5] in the variety P_ω.

Concerning the boundedness of endomorphisms of $\mathrm{FP}_n(X)$, $n \geq 1$, Katriňák posed the question whether the above mentioned result for $\mathrm{FP}_\omega(X)$ can be extended to the remaining varieties P_n, $n \geq 1$. Hence the aim of this note is to investigate bounded endomorphisms of free p-algebras $\mathrm{FP}_n(X)$, where $n \geq 1$ and X is finite.

The paper is organized as follows. In Section 4, we recall the basic notions and some of the known results of the theory of free p-algebras. The main results of this paper are contained in Section 3. We give necessary and sufficient conditions for an endomorphism of $\mathrm{FP}_n(X)$ to be bounded. We make use of the constructive method of limit tables for endomorphisms of $\mathrm{FP}_n(X)$. Of course, the concept of limit tables is well known from lattice theory (see, for example [**8**]). However, in the case of p-algebra limit tables we have to take into account the principal inner antisymmetry of p-algebras as well as Lee's identity $\mathscr{L}_n$. An effective algorithm for determining the boundedness of a given endomorphism is also presented. In Section 4, we investigate the variety P_1, i.e. the equational class of all p-algebras satisfying the Stone identity $x^* \vee x^{**} = 1$. In this class, we shall construct explicit examples of endomorphisms of $\mathrm{FP}_1(X)$ that are not bounded.

2. Preliminaries. A *p-algebra* (*pseudocomplemented lattice*) is a universal algebra $(L; \vee, \wedge, {}^*, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation * is defined by $a \wedge b = 0$ if and only if $a \leq b^*$.

DEFINITION (R. McKenzie [**5**], T. Katriňák [**4**]). Suppose A, B are p-algebras and f is a homomorphism of A into B. We say f is *upper bounded* if and only if, for each $b \in B$,

$\{a \in A; f(a) \leq b\}$ has a greatest element $f^{-}(b)$. We say f is *lower bounded* if and only if, for each $b \in B$, $\{a \in A; f(a) \geq b\}$ has a least element $f_{-}(b)$. We say f is *bounded* if and only if it is both upper and lower bounded.

In this paper we deal with the p-algebra varieties

$$P_1 \subset P_2 \subset \ldots \subset P_n \subset \ldots \subset P_\omega,$$

where $P_\omega = P$ is the equational class of all p-algebras and for a p-algebra L, $L \in P_n$ for $1 \leq n < \omega$ if and only if L satisfies the identity

$$(\mathcal{L}_n): (x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge x_2 \wedge \ldots \wedge x_n^*)^* = 1.$$

In what follows, the symbol $\mathrm{FP}_n(X)$, $1 \leq n \leq \omega$, denotes a free p-algebra freely generated by a set X. We shall frequently use the following rules of computation in p-algebras:

(a) $a \leq b$ implies $b^* \leq a^*$, (d) $(a \vee b)^* = a^* \wedge b^*$,
(b) $a \leq a^{**}$, (e) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,
(c) $a^* = a^{***}$, (f) $0^* = 1$ and $1^* = 0$.

If, in any p-algebra L, we write $B(L) = \{a \in L; a = a^{**}\}$ then $(B(L); +, \wedge, {}^*, 0, 1)$ is a Boolean algebra when $a + b$ is defined by $a + b = (a \vee b)^{**}$.

Later we will need some of Katriňák's results concerning free p-algebras (see [**4**, Lemmas 2, 3 and Theorem 3]).

Let a p-algebra L be generated by a subset X, i.e. $[X] = L$. Then the set $X^{**} = \{x^{**}; x \in X\}$ generates $B(L)$ in the class of Boolean algebras, i.e. $B(L) = [X^{**}]_{\mathrm{bool}}$. The set $X \cup B(L)$ generates L in the class of lattices, i.e. $L = [X \cup B(L)]_{\mathrm{lat}}$.

Suppose that K is a nontrivial equational class of p-algebras. Let $L = FK(X)$ be a free p-algebra freely generated by X in K. Then $B(L) = \mathrm{FB}(X^{**})$ (the free Boolean algebra freely generated by the set X^{**}).

Put

$$\mathcal{P}(X) = X \cup B(\mathrm{FP}_n(X)).$$

From the above results, it follows that $\mathcal{P}(X) = X \cup \mathrm{FB}(X^{**})$ and $\mathrm{FP}_n(X) = [\mathcal{P}(X)]_{\mathrm{lat}}$.

For $1 \leq n < \omega$, a family $\mathcal{U}_n$ of subsets of $B(\mathrm{FP}_n(X))$ is defined as follows:

$S \in \mathcal{U}_n$ if and only if

$$S = \{(a_1 \wedge a_2 \wedge \ldots \wedge a_n)^*, (a_1^* \wedge a_2 \wedge \ldots \wedge a_n)^*, \ldots, (a_1 \wedge a_2 \wedge \ldots \wedge a_n^*)^*\}$$

for some $a_1, a_2, \ldots, a_n \in B(\mathrm{FP}_n(X))$.

For $n = \omega$, we simply set $\mathcal{U}_\omega = \varnothing$.

We see that $\bigvee S = 1$ for each $S \in \mathcal{U}_n$, $1 \leq n < \omega$, because $\mathrm{FP}_n(X) \in P_n$.

The following lemmas give an algorithm enabling us to decide whether $a \leq b$ in $\mathrm{FP}_n(X)$ for given words $a, b \in \mathrm{FP}_n(X) = [\mathcal{P}(X)]_{\mathrm{lat}}$.

LEMMA 1 [**3**, Lemma 10]. *Let $a, b \in \mathrm{FP}_n(X)$ and $p \in \mathcal{P}(X)$. Then $p \leq a \vee b$ if and only if $p \leq a$ or $p \leq b$ or there exists $S \in \mathcal{U}_n$ such that $s \leq a$ or $s \leq b$ for every $s \in S$.*

LEMMA 2 [**3**, Lemma 8]. *$a \wedge b \leq c \vee d$ in $\mathrm{FP}_n(X)$, $1 \leq n \leq \omega$, if and only if*
(W) $a \leq c \vee d$ or $b \leq c \vee d$ or $a \wedge b \leq c$ or $a \wedge b \leq d$.

With regard to the previous lemmas, the word problem for $\mathrm{FP}_n(X)$, $1 \leq n \leq \omega$, has an affirmative solution. In addition, an algorithm is given which can be used to decide whether $a = b$ in $\mathrm{FP}_n(X)$ for words $a, b \in \mathrm{FP}_n(X)$.

THEOREM 3 [4, Lemmas 11 and 12]. *Let* $f : \mathrm{FP}_n(X) \to \mathrm{FP}_n(X)$ *be an endomorphism. Then*

(i) *f is upper bounded,*

(ii) *f is lower bounded for $n = \omega$,*

(iii) *f is lower bounded, $1 \leq n < \omega$, whenever the set $\{a \in \mathrm{FP}_n(X); f(a) = 1\}$ has a smallest element $f_-(1)$.*

Finally, we recall that $\mathrm{FP}_n(X)$ is infinite, whenever $|X| \geq 2$ and $1 \leq n \leq \omega$ (see [4, Theorem 2]).

3. Limit tables for endomorphisms of $\mathrm{FP}_n(X)$. In this section, we give necessary and sufficient conditions for an endomorphism of $\mathrm{FP}_n(X)$ to be bounded. The characterization is based on the properties of limit tables for a given endomorphism of $\mathrm{FP}_n(X)$. In the class of lattices the idea of limit tables was introduced by B. Jónsson and widely exploited by R. McKenzie and A. Kostinsky. In what follows, we shall introduce a p-algebra type of limit table similar to that in [8]. However, there are principal difficulties arising in the direct application of the known lattice theoretical type of limit table. More precisely, we must carefully take into account inner antisymmetries of $\mathrm{FP}_n(X)$ (the lattice theoretical dual of a given p-algebra need not be a p-algebra) as well as the identity $\mathscr{L}_n$.

From now on we shall suppose that $f : \mathrm{FP}_n(X) \to \mathrm{FP}_n(X)$ is an endomorphism, $1 \leq n < \omega$, and X is a finite freely generating set.

Let the maps $\beta_m : B(\mathrm{FP}_n(X)) \to \mathrm{FP}_n(X)$, $m \geq 0$, be defined inductively as follows:

$$\beta_0(a) = \bigwedge \{p \in \mathscr{P}(X); f(p) \geq a\}$$

and, for $m \geq 0$,

$$\beta_{m+1}(a) = \beta_0(a) \wedge \bigwedge \bigvee \{\beta_m(S); S \in \mathscr{U}_n\} \tag{1}$$

for any $a \in B(\mathrm{FP}_n(X))$.

We call the family $\{\beta_m\}_{m \geq 0}$ a *limit table* for the endomorphism f. From the definition (1), one can easily verify the following rules:

$$\begin{aligned}
&\text{(a) } \beta_{m+1}(a) \leq \beta_m(a);\\
&\text{(b) } a \leq b \text{ implies } \beta_m(a) \leq \beta_m(b);\\
&\text{(c) } a \leq f(\beta_m(a));\\
&\text{(d) } f(\beta_m(a)) = f(\beta_0(a));\\
&\text{(e) } \beta_m(a) \leq \beta_m(b) \text{ if and only if } \beta_0(a) \leq \beta_0(b);\\
&\text{(f) } \beta_m(a) = \beta_0(a) \wedge \beta_m(1);\\
&\text{(g) } \beta_{m+1}(1) = \bigwedge \bigvee \{\beta_m(S); S \in \mathscr{U}_n\}.
\end{aligned} \tag{2}$$

We say that a limit table $\{\beta_m\}_{m \geq 0}$ is *closed* if and only if there is $k \geq 0$ such that $\beta_{k+1}(a) = \beta_k(a)$ for each $a \in \mathrm{FB}(X^{**})$. It is easy to see that $\{\beta_m\}_{m \geq 0}$ is closed if and only if there is $k \geq 0$ such that $\beta_{k+1}(1) = \beta_k(1)$.

THEOREM 4. *An endomorphism $f:\mathrm{FP}_n(X)\to\mathrm{FP}_n(X)$ is bounded if and only if the limit table for f is closed.*

Proof. The crucial step in the proof consists of proving the following statement:

(P) for any $a\in B(\mathrm{FP}_n(X))$ and $b\in\mathrm{FP}_n(X)$, $f(b)\geq a$ implies $b\geq\beta_k(a)$ for some $k\geq 0$.

As is usual in such circumstances, we shall proceed by induction on the length of a lattice term $b\in\mathrm{FP}_n(X)=[\mathscr{P}(X)]_{\mathrm{lat}}$. For $b\in\mathscr{P}(X)$, it is clear that (P) holds true with $k=0$. We now suppose that (P) holds for b_1, b_2 with corresponding indices $k_1, k_2\geq 0$. Then, using Lemmas 1, 2 and (2), one can easily show that (P) holds for $b=b_1\wedge b_2$ with $k=\max\{k_1,k_2\}$ and for $b=b_1\vee b_2$ with $k=1+\max\{k_1,k_2\}$, respectively. Now, thanks to the property (P), the rest of the proof can be carried out as in [**8**, Section 6] and therefore is omitted.

LEMMA 5. *Let m be an integer such that $\log_2\log_2\log_2 m=|X|+1$. Then an endomorphism $f:\mathrm{FP}_n(X)\to\mathrm{FP}_n(X)$ is bounded if and only if the limit table for f is closed before the m-th column.*

Proof. Only the necessity needs a proof. If $\beta_1(1)=1$ then $\beta_1(1)=\beta_0(1)$. Hence, the limit table is closed in the first column.

We now consider the case $\beta_1(1)<1$. Let an equivalence relation θ on ω be defined as follows:

$\theta(k,m)$ if and only if, for any $a,b\in\mathrm{FB}(X^{**})$,

$$\beta_{k+1}(a)\geq\beta_k(b)\Leftrightarrow\beta_{m+1}(a)\geq\beta_m(b).$$

We shall prove, in a manner similar to that in [**8**, Lemma 6.1], the following statement:

(H) $\theta(k,m)$ implies $\theta(k+1,m+1)$.

In order to prove (H), we assume $\theta(k,m)$ holds and $\beta_{k+2}(a)\geq\beta_{k+1}(b)$.

Let $R\in\mathscr{U}_n$. Then

$$1>\beta_1(1)\geq\bigvee\beta_{k+1}(R)\geq\beta_{k+2}(1)\geq\beta_{k+2}(a)\geq\beta_{k+1}(b)=\beta_0(b)\wedge\bigwedge\bigvee(\beta_k(S);S\in\mathscr{U}_n).$$

From Lemma 1 and 2, we obtain either the existence of $r\in R$ such that $\beta_{k+1}(r)\geq\beta_{k+1}(b)$ or $\bigvee\beta_{k+1}(R)\geq\bigvee\beta_k(S_1)$ for some $S_1\in\mathscr{U}_n$. In the first event, by (2), we have $\bigvee\beta_{m+1}(R)\geq\beta_{m+1}(r)\geq\beta_{m+1}(b)$. In the second event two cases can arise:

(i)$_1$ for every $s\in S_1$, there is $r_s\in R$, $\beta_k(s)\leq\beta_{k+1}(r_s)$,

(ii)$_1$ there exists $s_1\in S_1$, such that $\beta_k(s_1)\not\leq\beta_{k+1}(r)$ for any $r\in R$.

In case (i)$_1$, the assumption $\theta(k,m)$ implies

$$\beta_m(s)\leq\beta_{m+1}(r_s)\leq\bigvee\beta_{m+1}(R)\qquad\text{for each } s\in S_1.$$

Therefore

$$\beta_{m+1}(b)\leq\beta_{m+1}(1)\leq\bigvee\beta_m(S_1)\leq\bigvee\beta_{m+1}(R).$$

In case (ii)$_1$, by Lemma 1 and 2, we have

$$\bigvee\beta_{k+1}(R)\geq\bigvee\beta_{k-1}(S_2)\qquad\text{for some } S_2\in\mathscr{U}_n.$$

Repeating this procedure, we obtain a sequence $S_j\in\mathscr{U}_n$, $j\geq 1$, such that $\bigvee\beta_{k+1}(R)\geq\bigvee\beta_{k-j+1}(S_j)$. Again two cases can occur:

(i)$_j$ for each $s\in S_j$, there is $r_s\in R$ with $\beta_{k-j+1}(s)\leq\beta_{k+1}(r_s)$, Let $j=k+1$. According to Lemmas 1 and 2, case (ii)$_j$ is not possible. Thus there exist $j\leq k$ and $S_{k-j+1}\in \mathcal{U}_n$ such that, for each $s\in S_{k-j+1}$, there is $r_s\in R$ with $\beta_j(s)\leq \beta_{k+1}(r_s)$. Since $\beta_k\leq \beta_j$ for $j\leq k$, the assumption $\theta(k,m)$ implies $\beta_{m+1}(b)\leq \bigvee \beta_{m+1}(R)$. Thus both (i)$_1$ and (ii)$_1$ also imply $\beta_{m+1}(b)\leq \bigvee \beta_{m+1}(R)$. Therefore $\beta_{m+1}(b)\leq \beta_{m+2}(1)$. Clearly, from the assumption $\beta_{k+2}(a)\geq \beta_{k+1}(b)$, it follows that $\beta_0(a)\geq \beta_0(b)$. Hence

$$\beta_{m+2}(a)=\beta_0(a)\wedge \beta_{m+2}(1)\geq \beta_0(b)\wedge \beta_{m+1}(b)=\beta_{m+1}(b).$$

Thus $\beta_{k+2}(a)\geq \beta_{k+1}(b)$ implies $\beta_{m+2}(a)\geq \beta_{m+1}(b)$ and vice versa. The proof of (H) is complete.

The result of the proof is essentially the same as that of [5, Lemma 6.1]. Indeed, a simple combinatorial argument shows that θ partitions ω into less than $2^{r\cdot r}$ classes of integers, where $r=|\mathrm{FB}(X^{**})|$. This can be visualized by introducing the following one-to-one map

$$[k]\theta\rightarrow \{(a,b)\in B(\mathrm{FP}_n(X)),\, \beta_{k+1}(a)\geq \beta_k(b)\}.$$

Then we infer the existence of l such that $l<m\leq 2^{r\cdot r}$ and $\theta(l,m)$ holds. Clearly, by (H), l will be θ-equivalent to an arbitrarily large integer. Since f is bounded, then, by Theorem 4, there exists k_0 such that $\beta_{k+1}\equiv\beta_k$ for all $k\geq k_0$. Hence $\beta_{m+1}\equiv\beta_m$. The fact that $\log_2\log_2 r=|X^{**}|=|X|$ (see [3, Chapter 2, Section II, Theorem 2]) completes the proof of Lemma 5.

THEOREM 6. *Let f be an endomorphism of* $\mathrm{FP}_n(X)$, *where* $1\leq n<\omega$ *and X is finite. Then there exists an effective algorithm for determining whether f is bounded.*

Proof. In order to decide whether f is bounded, one can construct the first m columns of the limit table for f, where $\log_2\log_2\log_2 m=|X|+1$. By Lemma 5, f is bounded if and only if $\beta_{m+1}(1)=\beta_m(1)$. Since the word problem for $\mathrm{FP}_n(X)$ has a solution, there is an effective algorithm that determines whether $\beta_{m+1}(1)=\beta_m(1)$.

COROLLARY 7. *Let $n\geq 2^{|X|}$. Then each endomorphism of* $\mathrm{FP}_n(X)$ *is bounded.*

Proof. It can readily be shown that $n\geq 2^{|X|}$ implies $1\in S$ for each $S\in\mathcal{U}_n$. Hence $\beta_1(1)=\beta_0(1)$ and f is bounded by Theorem 4.

On the set $\mathcal{U}_n$ we define a quasiordering $\ll$ in the following manner:

for $S_1, S_2\in\mathcal{U}_n$, $S_1\ll S_2$ if and only if either $\bigvee\beta_0(S_2)=\beta_0(1)$ or, for each $s_1\in S_1$, there is $s_2\in S_2$ such that $\beta_0(s_1)\leq\beta_0(s_2)$. (3)

Defining $S_1\equiv S_2$ if and only if $S_1\ll S_2$ and $S_2\ll S_1$, one gets an equivalence relation and the resulting classes are made into a partially ordered set $(\mathcal{U}_n,\ll)$ in the standard fashion. In what follows we shall ignore the classes and refer directly to their representatives.

LEMMA 8. *Assume $\bigvee\beta_k(S_0)=\beta_{k+1}(1)$ for some $S_0\in\mathcal{U}_n$. Then the poset $(\mathcal{U}_n,\ll)$ has a least element S_m.*

Proof. We shall proceed by induction on $k\geq 0$. Assume $k=0$. Put $S_m=S_0$. Then $\bigvee\beta_0(S_m)=\beta_1(1)\leq\bigvee\beta_0(S)$ for each $S\in\mathcal{U}_n$. Let $S\in\mathcal{U}_n$. Then either $\bigvee\beta_0(S)=1$ or, by Lemma 1, for each $s\in S_m$, there is $s'\in S$ such that $\beta_0(s)\leq\beta_0(s')$. In both cases we have $S_m\ll S$. Hence S_m is the least element of $(\mathcal{U}_n,\ll)$.

Assume $k>0$. First we consider the case where $S_0 \ll S$ for each $S \in \mathcal{U}_n$. Then $S_m = S_0$ is the least element of $(\mathcal{U}_n, \ll)$. Now we suppose that there exists $S_1 \in \mathcal{U}_n$ with the property S_0 is not $\ll S_1$. Then $\bigvee \beta_0(S_1) \neq 1$ and there is $s_0 \in S_0$ such that $\beta_0(s_0) \nleq \beta_0(s)$ for any $s \in S_1$. Since

$$\beta_0(s_0) \wedge \bigwedge (\bigvee \beta_{k-1}(S); S \in \mathcal{U}_n) = \beta_k(s_0) \leq \bigvee \beta_k(S_0) = \beta_{k+1}(1) \leq \bigvee \beta_k(S_1)$$

and $\bigvee \beta_k(S_1) \leq \bigvee \beta_0(S_1) \neq 1$, then, by Lemmas 3 and 4, we get $\bigvee \beta_{k-1}(S'_0) \leq \bigvee \beta_k(S_1)$ for some $S'_0 \in \mathcal{U}_n$. But this yields

$$\beta_k(1) \leq \bigvee \beta_{k-1}(S'_0) \leq \bigvee \beta_k(S_1) \leq \beta_k(1).$$

From the induction hypothesis, we obtain that $(\mathcal{U}_n, \ll)$ has a least element S_m.

THEOREM 9. *Assume that $f: \mathrm{FP}_n(X) \to \mathrm{FP}_n(X)$ is an endomorphism, $1 \leq n < \omega$ and X is finite. Then*

(i) *if f is bounded then the poset $(\mathcal{U}_n, \ll)$ has a least element,*

(ii) *if $(\mathcal{U}_n, \ll)$ has a least element and $\beta_0(1) < 1$ then f is bounded.*

Proof. (i) By Theorem 4, there is $k \geq 0$ such that $\beta_{k+1}(1) = \beta_k(1)$. If, for each $S \in \mathcal{U}_n$, $\bigvee \beta_0(S) = \beta_0(1)$ then $S_1 \ll S_2$ for any $S_1, S_2 \in \mathcal{U}_n$. Hence $(\mathcal{U}_n, \ll)$ possesses a least element. Suppose that there exists $S_0 \in \mathcal{U}_n$ with the property $\bigvee \beta_0(S_0) < \beta_0(1)$. Then

$$\bigwedge \bigvee (\beta_{k-1}(S); S \in \mathcal{U}_n) = \beta_k(1) = \beta_{k+1}(1) \leq \bigvee \beta_k(S_0).$$

By Lemma 2, two cases can occur: $\beta_k(1) \leq \beta_k(s_0)$ for some $s_0 \in S_0$ or $\bigvee \beta_{k-1}(S_1) \leq \bigvee \beta_k(S_0)$ for some $S_1 \in \mathcal{U}_n$. In the first case, we have $\beta_0(s_0) = \beta_0(1)$ and therefore $\bigvee \beta_0(S_0) = \beta_0(1)$, a contradiction. Thus only the second case is possible, i.e. $\beta_k(1) \leq \bigvee \beta_{k-1}(S_1) \leq \bigvee \beta_k(S_0) \leq \beta_k(1)$.

Applying Lemma 8, we obtain the existence of a least element of $(\mathcal{U}_n, \ll)$.

(ii) Let S_m be the least element of $(\mathcal{U}_n, \ll)$ and $S \in \mathcal{U}_n$. If $\bigvee \beta_0(S) = \beta_0(1) < 1$ then, from Lemma 1, we obtain $\beta_0(s') = \beta_0(1)$ for some $s' \in S$. By Lemma 1, (2) and (3), we can establish that, for each $s \in S_m$, there is $s' \in S$ such that $\beta_k(s) \leq \beta_k(s')$ for all $k \geq 0$. Thus $\beta_{k+1}(1) = \bigvee \beta_k(S_m)$ for all $k \geq 0$. Therefore $\beta_1(1) = \bigvee \beta_0(S_m)$ and, for each $s \in S_m$, we have $\beta_1(s) = \beta_0(s) \wedge \beta_1(1) = \beta_0(s)$. Then

$$\beta_2(1) = \bigvee \beta_1(S_m) = \bigvee \beta_0(S_m) = \beta_1(1).$$

Hence, by Theorem 4, f is bounded.

REMARK 10. From (1), we see that $\beta_0(1) < 1$ if and only if there exists a in $B(\mathrm{FP}_n(X))$ with $a \neq 1$ and $f(a) = 1$.

4. Examples of nonbounded endomorphisms in the variety P_1. In this section, we shall construct endomorphisms of $\mathrm{FP}_1(X)$ which are not bounded. Recall that P_1 is the equational class of all p-algebras satisfying the Stone identity $x^* \vee x^{**} = 1$.

THEOREM 11. *For every finite set X with $|X| = 2$, there exists a nonbounded endomorphism of $\mathrm{FP}_1(X)$.*

Proof. Consider an endomorphism $f: \mathrm{FP}_1(X) \to \mathrm{FP}_1(X)$ with

$$f(x_1) = (x_1^* \vee x_2^*)^{**}, \quad f(x_2) = (x_1^{**} \vee x_2^*)^{**}, \quad \text{where } X = \{x_1, x_2\} \quad \text{and} \quad x_1 \neq x_2. \quad (4)$$

Let a Boolean endomorphism g of $B(\mathrm{FP}_1(X))$ be defined by $g(a) = f(a)$ for each $a \in B(\mathrm{FP}_1(X))$. Since $B(\mathrm{FP}_1(X))$ is finite, g is a bounded Boolean endomorphism. Clearly, $g_-(a+b) = g_-(a) + g_-(b)$ for every $a, b \in B(\mathrm{FP}_1(X))$. Moreover $g_-(a) = \beta_0^{**}(a)$ for each $a \in B(\mathrm{FP}_1(X))$.

Suppose to the contrary that f is bounded. Then, by Theorem 5, $(\mathscr{U}_1, \ll)$ possesses a least element $S_m = \{a, a^*\}$, where $a \in B(\mathrm{FP}_1(X))$.

Take $S_i = \{x_i^*, x_i^{**}\}$ for $i = 1, 2$. Then $S_m \ll S_i$, $i = 1, 2$. It is routine to check that

$$\beta_0(x_1^*) = x_1, \qquad \beta_0(x_1^{**}) = x_2, \qquad \beta_0(x_2^*) = x_1 \wedge x_2,$$

$$\beta_0(x_2^{**}) = ((x_1^* \wedge x_2^{**}) \vee (x_1^{**} \wedge x_2^*))^{**} \quad \text{and} \quad \beta_0(1) = (x_1^* \wedge x_2^*)^* < 1.$$

Then, by Lemmas 1 and 2, $\bigvee \beta_0(S_i) < \beta_0(1)$ for $i = 1, 2$ and

$$\beta_0(1) \leq g_-(1) = g_-(a + a^*) = g_-(a) + g_-(a^*) = \beta_0^{**}(a) + \beta_0^{**}(a^*).$$

Without loss of generality, we may suppose that

$$\beta_0(a) \leq \beta_0(x_1^*) \quad \text{and} \quad \beta_0(a^*) \leq \beta_0(x_1^{**}).$$

Then either

$$\beta_0(a) \leq \beta_0(x_2^*) \quad \text{and} \quad \beta_0(a^*) \leq \beta_0(x_2^{**})$$

or

$$\beta_0(a) \leq \beta_0(x_2^{**}) \quad \text{and} \quad \beta_0(a^*) \leq \beta_0(x_2^*).$$

In the first case, we see that

$$\beta_0^{**}(a) \leq x_1^{**} \wedge x_2^{**} \quad \text{and} \quad \beta_0^{**}(a^*) \leq x_1^* \wedge x_2^{**}.$$

Then $(x_1^* \wedge x_2^*)^* = \beta_0(1) \leq x_2^{**} < (x_1^* \wedge x_2^*)^*$, a contradiction.

The second case can be handled in the same way. Therefore S_m cannot be the least element of the poset $(\mathscr{U}_n, \ll)$. Hence, by Theorem 5, the endomorphism f is not bounded.

In order to construct nonbounded endomorphisms of $\mathrm{FP}_1(X)$, where $|X| > 2$, let us consider an endomorphism

$$h: \mathrm{FP}_1(X) \to \mathrm{FP}_1(X)$$

defined by $h(a) = f(\tau(a))$, where f is a nonbounded endomorphism of $\mathrm{FP}_1(2)$ and the endomorphism

$$\tau: \mathrm{FP}_1(X) \to \mathrm{FP}_1(2)$$

is defined by $\tau(x_1) = x_1$, $\tau(x_2) = x_2$ and $\tau(x_i) = 1$ for $i > 2$.

Since f is not bounded, h is a nonbounded endomorphism of $\mathrm{FP}_1(X)$.

REFERENCES

1. R. A. Dean, Free lattices generated by partially ordered sets and preserving bounds, *Canad. J. Math.* **16** (1964), 136–148.
2. R. Freese and J. B. Nation, Projective lattices, *Pacific J. Math.* **75** (1978), 93–106.
3. G. Grätzer, *General lattice theory*, Mathematische Reihe 52 (Birkhäuser, 1978.)
4. T. Katriňák, Free p-algebras, *Algebra Universalis* **15** (1982), 176–186.
5. T. Katriňák, Splitting p-algebras, *Algebra Universalis* **18** (1984), 199–224.
6. T. Katriňák and D. Ševčovič, Projective p-algebras, *Algebra Universalis* **28** (1991), 280–300.
7. A. Kostinsky, Projective lattices and bounded homomorphisms, *Pacific J. Math.* **40** (1972), 111–119.
8. R. McKenzie, Equational bases and nonmodular lattice varieties, *Trans. Amer. Math. Soc.* **174** (1972), 1–43.

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