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Limiting Behaviour of Invariant Manifolds for a System of Singularly Perturbed Evolution Equations

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In this paper we consider a class of specific singularly perturbed abstract evolution equations. It is shown that, for small values of the singular parameter, the invariant manifold for the perturbed equation is C^1 close to that of the unperturbed equation. The results obtained are applied to the second-order evolution equations with strong damping arising in the mathematical théory of viscoelasticity.

1. Introduction

In this paper we will treat the qualitative properties of semiflows generated by the following system of abstract evolution equations:

$$\begin{aligned} &u' + A_\alpha u = g(u, w), \\ &\alpha w' + B_\alpha w = f(u), \end{aligned} \tag{1.1$_\alpha$}$$

where $\alpha \in [0, \alpha_1]$, $\{A_\alpha\}_{\alpha \geqslant 0}$ and $\{B_\alpha\}_{\alpha \geqslant 0}$ are continuously depending families of sectorial operators in the Banach spaces X and Y, respectively, $g: X^\gamma \times Y^\beta \to X$ and $f: X^\gamma \to Y$ are C^1 bounded functions for some $\gamma, \beta \in [0, 1)$. Hereafter, X^γ and Y^β will denote the fractional power spaces with respect to the sectorial operators A_0 and B_0, respectively (cf. [8, chapter 1]).

The goal of this paper is to establish the existence of a finite dimensional invariant C^1 manifold $\mathscr{M}_\alpha$ for the semiflow $\mathscr{S}_\alpha(t)$, $t \geqslant 0$, generated by system $(1.1)_\alpha$. We furthermore prove that both $\mathscr{M}_\alpha$ and the vector field on $\mathscr{M}_\alpha$ converge in the C^1 topology towards the ones corresponding to $\alpha = 0$ (Theorem 3.11). By combining this result with the well-known theory of Morse–Smale vector fields (cf. [12]) one can prove topological equivalence of vector fields on $\mathscr{M}_\alpha$ and $\mathscr{M}_0$ whenever the vector field on $\mathscr{M}_0$ is Morse–Smale.

The techniques used in the proof of Theorem 3.11 are similar in spirit to those developed by Mora and Solà-Morales [11]. The construction of an invariant manifold for (1.1) is based upon the well-known method of integral equations due to Lyapunov and Perron. In this method the major role is played by the choice of functional spaces we work with. For the proof of the existence of $\mathscr{M}_\alpha$, we notice that

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the usual choice would be the Banach space consisting of all continuous functions on $(-\infty, 0]$ with values in $X^{\gamma} \times Y^{\beta}$ equipped with some exponentially weighted sup or integral norm. Then one can look for $\mathcal{M}_\alpha$ as the union of all solutions of (1.1) belonging to this functional space. We refer to [5, 9–11] for details. However, it turns out that such a setting does not capture the singular limit behaviour of the derivative of the vector field on $\mathcal{M}_\alpha$ as $\alpha \to 0^+$. In order to overcome this difficulty, in contrast to the approach of [11], we will operate with Banach spaces consisting of all Hölder continuous functions growing exponentially at $-\infty$. In the proof of Theorem 3.11 an important tool is a slightly modified version of the two-parameter contraction theorem due to Mora and Solà-Morales covering differentiability and continuity of a family of non-linear contraction mappings operating between a pair of Banach spaces.

The paper is organized as follows. Section 2 is devoted to preliminaries. Perturbations of sectorial operators are investigated in section 2.1. The existence of solutions of (1.1) is established in section 2.2. In section 2.3 we introduce functional spaces we will work with. The core of the paper is contained in section 3. First, we prove the existence of a family of invariant manifolds $\{M_\alpha,\}_{\alpha \geqslant 0}$ for system (1.1). The singular limit dynamics of $\mathcal{M}_\alpha$, $\alpha \to 0^+$, is investigated in section 3.2. The main results are summarized in Theorem 3.11. Section 4 illustrates an application of the results obtained to the second-order evolution equations with strong damping arising in the mathematical theory of elastic systems with dissipation:

$$\alpha u_{tt} + A^{\kappa} u_t + Au = f(u), \tag{1.2$_\alpha$}$$

where $\alpha > 0$ is a small parameter, $\kappa \in [1/2, 1)$, A is a self-adjoint elastic operator in a real Hilbert space X and $f: X^{\varrho} \to X$ is a non-linear C^1 function for some $\varrho \in [\kappa, 1)$. We prove that, for any α small enough, the invariant manifold $\mathcal{M}_\alpha$ for system $(1.2)_\alpha$ is C^1 close to that of the limiting equation

$$u_t + A^{1-\kappa} u = A^{-\kappa} f(u). \tag{1.2$_0$}$$

2. Preliminaries

2.1. *Properties of a family of sectorial operators*

The goal of this section is to establish perturbation results for a family of closed densely defined operators. Let $L: D(L) \subset \mathscr{X} \to \mathscr{X}$ be a sectorial operator in the Banach space $\mathscr{X}$ (cf. [8, Definition 1.3.1]). It is well known (see [8, Theorem 1.3.4]) that if L is sectorial then the operator $-L$ generates an analytic semigroup $\exp(-Lt)$, $t \geqslant 0$, and $\exp(-Lt) = (1/2\pi i)\int_\Gamma e^{\lambda t}(\lambda + L)^{-1}\,d\lambda$, where Γ is a contour in $\varrho(-L)$ such that $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\pi/2, \pi)$.

Consider a family $\{L_\alpha\}$ of closed densely defined operators in a Banach space $\mathscr{X}$ satisfying the following hypothesis:

$$(\mathrm{H1}) \begin{cases} (1)\ D(L_0) = D(L_\alpha), \\ (2)\ 0 \in \varrho(L_\alpha) \text{ and } L_0 L_\alpha^{-1} \to I \text{ as } \alpha \to 0^+ \text{ in } L(\mathscr{X}, \mathscr{X}), \\ (3)\ L_0^{-1} L_\alpha^{-1} = L_\alpha^{-1} L_0^{-1}, \\ (4)\ L_0 \text{ is a sectorial operator in } \mathscr{X} \text{ such that } \operatorname{Re}\sigma(L_0) > \omega > 0 \end{cases}$$

for any $\alpha \in [0, \alpha_0]$. Applying [8, Theorem 1.3.2] we obtain the following lemma.

Lemma 2.1. *Assume that hypothesis* (H1) *is satisfied. Then*

(a) $(\lambda - L_0)^{-1}$ *commutes with* $(\mu - L_\alpha)^{-1}$ *for any* $\alpha \in [0, \alpha_0]$, $\lambda \in \varrho(L_0)$ *and* $\mu \in \varrho(L_\alpha)$,
(b) *there exists* $\alpha_1 \leqslant \alpha_0$ *such that* $L_\alpha L_0^{-1} \in L(\mathscr{X}, \mathscr{X})$ *for any* $\alpha \in [0, \alpha_1]$ *and* $L_\alpha L_0^{-1} \to I$ *as* $\alpha \to 0^+$ *and*
(c) L_α *is a sectorial operator in* $\mathscr{X}$ *for any* $\alpha \in [0, \alpha_1]$.

Assume that L is a sectorial operator in the Banach space $\mathscr{X}$. Suppose that $\sigma(L) = \sigma_1 \cup \sigma_2$, where σ_1, σ_2 are disjoint spectral sets and σ_1 is bounded in C. Let $P: \mathscr{X} \to \mathscr{X}$ denote the projector associated with the operator L and the spectral set σ_1 (cf. [8, chapter 1.5]).

Besides (H1) we also make the following hypothesis:

$$
\text{(H2)} \begin{cases} (1) & L_0^{-1} \text{ is a compact linear operator on } \mathscr{X} \text{ and} \\ (2) & \text{there are } 0 < \lambda_- < \lambda_+ < \infty \text{ such that } \sigma(L_0) = \sigma_1^0 \cup \sigma_2^0, \text{ where} \\ & \sigma_1^0 = \{\lambda \in \sigma(L_0);\ \operatorname{Re}\lambda < \lambda_-\} \text{ and } \sigma_2^0 = \{\lambda \in \sigma(L_0);\ \operatorname{Re}\lambda > \lambda_+\}. \end{cases}
$$

Notice that condition $(\mathrm{H2})_1$ implies that σ_1^0 is finite and $\dim \mathscr{X}_{1,0} < \infty$, where $\mathscr{X}_{1,0} := P_0\mathscr{X}$, P_0 is the projector in $\mathscr{X}$ associated with L_0 and σ_1^0. If $L_0 L_\alpha^{-1} \to I$ as $\alpha \to 0^+$ then, by Lemma 2.1, $L_\alpha L_0^{-1} \to I$ and $L_\alpha^{-1} \to L_0^{-1}$ in $L(\mathscr{X}, \mathscr{X})$ as well. Hence $\sigma(L_\alpha^{-1}) \to \sigma(L_0^{-1})$ when $\alpha \to 0^+$ in the Hausdorff set distance metric of the complex plane. With this one can easily verify the following lemma.

Lemma 2.2. *Assume that hypotheses* (H1) *and* (H2) *are satisfied. Then there is* $\alpha_1 > 0$ *sufficiently small and such that, for any* $\alpha \in [0, \alpha_1]$:

(a) $\sigma(L_\alpha) = \sigma_1^\alpha \cup \sigma_2^\alpha$, *where*

$$\sigma_1^\alpha = \{\lambda \in \sigma(L_\alpha);\ \operatorname{Re}\lambda < \lambda_-\} \quad \textit{and} \quad \sigma_2^\alpha = \{\lambda \in \sigma(L_\alpha);\ \operatorname{Re}\lambda > \lambda_+\}.$$

(b) $P_\alpha \to P_0$ *in* $L(\mathscr{X}, \mathscr{X})$ *as* $\alpha \to 0^+$, *where* P_α *is the projector associated with* L_α *and* σ_1^α. *Furthermore,* $P_0 P_\alpha = P_\alpha P_0$.
(c) $P_\alpha|_{\mathscr{X}_{1,0}}: \mathscr{X}_{1,0} \to \mathscr{X}_{1,\alpha}$ *is a linear isomorphism, where* $\mathscr{X}_{1,\alpha} := P_\alpha \mathscr{X}$. *Moreover,* $\dim \mathscr{X}_{1,0} = \dim \mathscr{X}_{1,\alpha} < \infty$.

Remark 2.3. For any $\alpha \geqslant 0$ small enough, we denote by

$$P_\alpha^{(-1)} := (P_\alpha|_{\mathscr{X}_{1,0}})^{-1} : \mathscr{X}_{1,\alpha} \to \mathscr{X}_{1,0} \tag{2.1}$$

the inverse operator of $P_\alpha|_{\mathscr{X}_{1,0}}: \mathscr{X}_{1,0} \to \mathscr{X}_{1,\alpha}$. Since $P_\alpha \to P_0$ as $\alpha \to 0^+$ the linear operator $P_\alpha^{(-1)} P_\alpha$ converges to P_0 in the space $L(\mathscr{X}, \mathscr{X})$.

If L_0 is a sectorial operator in $\mathscr{X}$ with $\operatorname{Re}\sigma(L_0) > \omega > 0$ then the fractional powers L_0^γ, $\gamma \in R$, can be defined (see [8, Definition 1.4.7]). Under hypothesis (H1) we have shown that $-L_\alpha$ generates an analytic semigroup $\exp(-L_\alpha t)$, $t \geqslant 0$ for $\alpha > 0$ small. In the following lemma we give some estimates on the decay of $\exp(-L_\alpha t)$.

Lemma 2.4. *Assume that hypothesis* (H1) *is satisfied. Then there is a* $C > 0$ *such that, for any* $\alpha \in [0, \alpha_1]$ *and* $\gamma \geqslant 0$, *the following estimates hold*:

(a) $\|\exp(-L_\alpha t)\| \leqslant C e^{-\omega t}$, $t \geqslant 0$,
(b) $\|L_0^\gamma(\exp(-L_\alpha t) - \exp(-L_0 t))\| \leqslant C\|L_0 L_\alpha^{-1} - I\| t^{-\gamma} e^{-\omega t}$, $t > 0$,
(c) $\|L_0^\gamma \exp(-L_\alpha t)\| \leqslant C t^{-\gamma} e^{-\omega t}$, $t > 0$.

Proof. Using the translation operator $L_\alpha - \omega I$ it is sufficient to prove the lemma with $\omega = 0$. The proof of (a) immediately follows from Lemma 2.1 and [8, Theorem 1.3.4]. To show part (b), we make use of the integral representation of $\exp(-L_\alpha t)$. We obtain, for $t > 0$,

$$\begin{aligned} &L_0^\gamma(\exp(-L_\alpha t) - \exp(-L_0 t)) \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} L_0^\gamma(\lambda + L_0)^{-1}(L_0 L_\alpha^{-1} - I)L_\alpha(\lambda + L_\alpha)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\mu} L_0^\gamma(\mu/t + L_0)^{-1}(L_0 L_\alpha^{-1} - I)L_\alpha(\mu/t + L_\alpha)^{-1}\frac{d\mu}{t}. \end{aligned}$$

Since $\operatorname{Re}\sigma(L_\alpha) > \omega > 0$ one can choose a contour Γ with the property $\operatorname{Re}\lambda < 0$ for any $\lambda \in \Gamma$. By [8, Theorem 1.4.4] and Lemma 2.1, there is an $M > 0$ such that $\| L_0^\gamma(\lambda + L_0)^{-1}\| \leqslant M|\lambda|^{\gamma-1}$ and $\| L_\alpha(\lambda + L_\alpha)^{-1}\| \leqslant M$ for any $\lambda \in \Gamma$ and $\alpha \in [0, \alpha_1]$, α_1 small. Hence,

$$\| L_0^\gamma(\exp(-L_\alpha t) - \exp(-L_0 t))\| \leqslant C\| L_0 L_\alpha^{-1} - I\| t^{-\gamma}, \quad t > 0.$$

Because of the well-known estimate $\| L_0^\gamma \exp(-L_0 t)\| \leqslant C t^{-\gamma}$, $t > 0$ [8, Theorem 1.4.3], it is clear that part (c) follows from (b). □

Assume that a family $\{L_\alpha, \alpha \in [0, \alpha_1]\}$ satisfies (H1) and (H2). For any $\alpha \in [0, \alpha_1]$, we denote $Q_\alpha := I - P_\alpha$ and let

$$L_{1,\alpha} := P_\alpha L_\alpha = L_\alpha P_\alpha, \quad L_{2,\alpha} := Q_\alpha L_\alpha = L_\alpha Q_\alpha, \qquad \mathscr{X}_{1,\alpha} := P_\alpha \mathscr{X}, \quad \mathscr{X}_{2,\alpha} := Q_\alpha \mathscr{X}.$$

Then $L_{1,\alpha}$ is a bounded linear operator, $\operatorname{Re}\sigma(L_{1,\alpha}) < \lambda_-$ in $\mathscr{X}$, and $L_{2,\alpha}$ is a sectorial operator in $\mathscr{X}$, $\operatorname{Re}\sigma(L_{2,\alpha}) > \lambda_+$. Moreover, $\| L_{1,0}^\gamma\| \leqslant C\lambda_-^\gamma$, $\gamma \in [0, 1)$. Applying Lemma 2.4 to the operators $\lambda_- - L_{1,\alpha}$ and $L_{2,\alpha} - \lambda_+$, respectively, one obtains the following lemma.

Lemma 2.5. *Assume that hypotheses* (H1) *and* (H2) *are satisfied. Then there is a* $C > 0$ *such that, for any* $\alpha \in [0, \alpha_1]$ *and* $\gamma \geqslant 0$, *the following estimates are true*:

(a) $\| L_0^\gamma \exp(-L_{1,\alpha}t)P_\alpha\| \leqslant C\lambda_-^\gamma e^{-\lambda_- t}$, $t \leqslant 0$,

(b) $\| L_0^\gamma(\exp(-L_{1,\alpha}t)P_\alpha - \exp(-L_{1,0}t)P_0)\| \leqslant C\lambda_-^\gamma \| L_0 L_\alpha^{-1} - I\| e^{-\lambda_- t}$, $t \leqslant 0$,

(c) $\| L_0^\gamma \exp(-L_\alpha t)Q_\alpha\| \leqslant C t^{-\gamma} e^{-\lambda_+ t}$, $t > 0$,

(d) $\| L_0^\gamma(\exp(-L_\alpha t)Q_\alpha - \exp(-L_0 t)Q_0)\| \leqslant C t^{-\gamma}\| L_0 L_\alpha^{-1} - I\| e^{-\lambda_+ t}$, $t > 0$.

In what follows, by C we will always denote the positive constant, the existence of which is ensured by Lemmas 2.4 and 2.5.

We end this section by a useful lemma referring to the Hölder continuity of the exponential mapping $t \mapsto \exp(-Lt)$.

Lemma 2.6. *Assume that hypotheses* (H1) *and* (H2) *are satisfied. Suppose that* $\mu \in (\lambda_-, \lambda_+)$. *Then, for any* $\gamma \in [0, 1)$, $\varrho \in (0, 1 - \gamma)$ *and* $\alpha \in [0, \alpha_1]$, *the following estimates are true*:

(a) $\| L_0^\gamma[\exp((\mu - L_{1,\alpha})r) - \exp((\mu - L_{1,\alpha})(r - h))]P_\alpha x\|$

$$\leqslant hC\lambda_-^{1+\gamma} e^{(\mu - \lambda_-)r}\| (\mu - L_\alpha)L_0^{-1}\| \, \| x\| \quad \textit{for any } h > 0,\ r \leqslant 0,$$

(b) $\| L_0^\gamma [\exp(-(L_\alpha - \mu)(r+h)) - \exp(-(L_\alpha - \mu)r)] Q_\alpha x \|$

$$\leqslant h^{(1-\gamma+\varrho)/2} C^2 r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+ - \mu)r} \| (\mu - L_\alpha) L_0^{-1} \| \, \| x \|$$

for any $h > 0$, $r \geqslant 0$.

Proof. (a) Clearly, for any $r \leqslant 0$ and $h > 0$,

$$I_1 := L_0^\gamma [\exp((\mu - L_{1,\alpha})r) - \exp((\mu - L_{1,\alpha})(r-h))] P_\alpha x$$

$$= (\mu - L_{1,\alpha}) L_0^{-1} \int_{r-h}^{r} L_0^{1+\gamma} \exp((\mu - L_{1,\alpha})\xi) \, d\xi \, P_\alpha x.$$

By Lemma 2.5(a),

$$\| I_1 \| \leqslant \| (\mu - L_\alpha) L_0^{-1} \| C \lambda_-^{1+\gamma} \int_{r-h}^{r} e^{(\mu - \lambda_-)\xi} \, d\xi \, \| x \|$$

$$\leqslant h C \lambda_-^{1+\gamma} e^{(\mu - \lambda_-)r} \| (\mu - L_\alpha) L_0^{-1} \| \, \| x \|.$$

To show (b) we will argue similarly as above. We have, for any $r \geqslant 0$ and $h > 0$,

$$I_2 := L_0^\gamma [\exp(-(L_\alpha - \mu)(r+h)) - \exp(-(L_\alpha - \mu)r)] Q_\alpha x$$

$$= (\mu - L_\alpha) L_0^{-1} \int_0^h L_0^{(1+\gamma-\varrho)/2} \exp(-(L_\alpha - \mu)\xi) Q_\alpha \, d\xi \, L_0^{(1+\gamma+\varrho)/2} \times \exp(-(L_\alpha - \mu)r) Q_\alpha x.$$

Hence,

$$\| I_2 \| \leqslant \| (\mu - L_\alpha) L_0^{-1} \| C \int_0^h \xi^{-(1+\gamma-\varrho)/2} e^{-(\lambda_+ - \mu)\xi} \, d\xi \, C r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+ - \mu)r} \| x \|$$

$$\leqslant h^{(1-\gamma+\varrho)/2} C^2 \| (\mu - L_\alpha) L_0^{-1} \| r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+ - \mu)r} \| x \|$$

for any $r \geqslant 0$, $h > 0$. □

2.2. *Existence of solutions of the system of abstract equations*

In this section, the aim is to show local and global solvability of a family of abstract equations $(1.1)_\alpha$, $\alpha \in (0, \alpha_1]$, and

$$u' + A_0 u = g(u, B_0^{-1} f(u)). \tag{$(1.1)_0$}$$

Assume that the families $\{A_\alpha, \alpha \in [0, \alpha_1]\}$ and $\{B_\alpha, \alpha \in [0, \alpha_1]\}$, $\alpha_1 > 0$, small enough, fulfil hypotheses (H1)–(H2) and (H1) on the Banach spaces X and Y, respectively.

Henceforth, we denote by

$$X^\gamma := [D(A_0^\gamma)], \qquad Y^\beta := [D(B_0^\beta)], \quad \gamma, \beta \geqslant 0,$$

the fractional power spaces with graph norms of A_0^γ and B_0^β, respectively, i.e. $\| u \|_\gamma := \| A_0^\gamma u \|$ and $\| w \|_\beta := \| B_0^\beta w \|$ (cf. [8, chapter 1]).

By a globally defined solution of $(1.1)_\alpha$ with initial data $(u_0, w_0) \in X^\gamma \times Y^\beta$, we understand a function

$$t \mapsto (u(t), w(t)) \in C([0, T]; X^\gamma \times Y^\beta) \cap C^1((0, T); X \times Y) \quad \text{for any } T > 0$$

such that $(u(0), w(0)) = (u_0, w_0)$; $(u(t), w(t)) \in D(A) \times D(B)$ for $t > 0$ and $(u(.), w(.))$ satisfies $(1.1)_\alpha$ for any $t > 0$.

By a globally defined solution of $(1.1)_0$ with initial data $u_0 \in X^\gamma$, we understand a function

$$t \mapsto u(t) \in C([0, T]; X^\gamma) \cap C^1((0, T); X) \quad \text{for any } T > 0$$

such that $u(0) = u_0$; $u(t) \in D(A)$ for $t > 0$ and $u(.)$ satisfies $(1.1)_0$ for any $t > 0$.

As usual, for Banach spaces E_1, E_2 and $\eta \in (0, 1]$ we denote by $C^1_{\text{bdd}}(E_1, E_2)$ the Banach space consisting of the mappings $F: E_1 \to E_2$ which are Fréchet differentiable and such that F and DF are bounded and uniformly continuous, the norm being given by $\|F\|_1 := \sup|F| + \sup|DF|$. $C^{1+\eta}_{\text{bdd}}(E_1, E_2)$ will denote the Banach space consisting of the mappings $F \in C^1_{\text{bdd}}(E_1, E_2)$ such that DF is η-Hölder continuous, the norm being given by

$$\|F\|_{1,\eta} := \|F\|_1 + \sup_{\substack{x \neq y \\ x, y \in E_1}} \frac{\|DF(x) - DF(y)\|}{\|x - y\|^\eta}.$$

Concerning functions g and f we will assume

$$\text{(H3)} \quad \begin{cases} g \in C^1_{\text{bdd}}(X^\gamma \times Y^\beta; X),\ f \in C^{1+\eta}_{\text{bdd}}(X^\gamma; Y^\xi) \\ \text{for some } \gamma, \beta \in [0, 1),\ \beta > \xi > \beta - 1 \text{ and } \eta \in (0, 1]. \end{cases}$$

First, we will consider the case $\alpha > 0$. According to Lemmas 2.1 and 2.4 the operator A_α (B_α) is sectorial in X (Y). In Lemma 2.5 we have shown the estimates

$$\begin{aligned} &\|A_0^\gamma \exp(-A_\alpha t)x\| \leqslant Ct^{-\gamma} e^{-\omega t} \|x\|, \\ &\|B_0^\beta \exp(-B_\alpha t)y\| \leqslant Ct^{-(\beta-\xi)} e^{-\omega t} \|B_0^\xi y\|, \end{aligned} \qquad x \in X,\ y \in Y^\xi,\ t > 0.$$

With the help of these inequalities one can easily adapt the proofs of [8, Theorems 3.3.3 and 3.3.4] to establish local and global existence of solutions of $(1.1)_\alpha$, $\alpha \in (0, \alpha_1]$, for initial data belonging to the phase space $X^\gamma \times Y^\beta$. Local and global existence of solutions of $(1.1)_0$ with initial data from X^γ follows from [8, Theorems 3.3.3 and 3.3.4].

In this way we have shown that system $(1.1)_\alpha$, $\alpha \in (0, \alpha_1]$, generates a semiflow $\mathscr{S}_\alpha(t)$, $t \geqslant 0$, on $X^\gamma \times Y^\beta$ defined by $\mathscr{S}_\alpha(t)(u(0), w(0)) := (u(t), w(t))$. Similarly, system $(1.1)_0$ generates a semiflow $\tilde{\mathscr{S}}_0(t)$, $t \geqslant 0$, on X^γ.

2.3. *Banach spaces with exponentially weighted norms*

Let $\mathscr{X}$ be a Banach space and $\mu \in R$. Following the notation of [5, 11] we denote

$$C^-_\mu(\mathscr{X}) := \left\{u: (-\infty, 0] \to \mathscr{X},\ u \text{ is continuous and } \sup_{t \leqslant 0} e^{\mu t} \|u(t)\|_{\mathscr{X}} < \infty\right\}$$

and

$$\|u\|_{C^-_\mu(\mathscr{X})} := \sup_{t \leqslant 0} e^{\mu t} \|u(t)\|_{\mathscr{X}}.$$

The linear space $C^-_\mu(\mathscr{X})$ endowed with the norm $\|\cdot\|_{C^-_\mu(\mathscr{X})}$ is a Banach space. If $\mu \leqslant \nu$ then embedding $C^-_\mu(\mathscr{X}) \hookrightarrow C^-_\nu(\mathscr{X})$ is continuous with an embedding constant equal to 1.

Let X, Y be Banach spaces and $F: X \to Y$ a bounded and Lipschitz continuous mapping. Denote by

$$\tilde{F}: C_\mu^-(X) \to C_\mu^-(Y) \tag{2.2}$$

a mapping defined as $\tilde{F}(u)(t) := F(u(t))$ for any $t \leqslant 0$. By [11, Lemma 5.1], for every $\mu \geqslant 0$, the mapping $\tilde{F}$ is bounded and Lipschitzian with $\sup|\tilde{F}| \leqslant \sup|F|$ and $\mathrm{Lip}|\tilde{F}| \leqslant \mathrm{Lip}|F|$. If $F: X \to Y$ is Fréchet differentiable then $\tilde{F}: C_\mu^-(X) \to C_\mu^-(Y)$ need not be necessarily differentiable. Nevertheless, the following result holds.

Lemma 2.7 (Vanderbauwhede and Van Gils [15, Lemma 5]). *If $F: X \to Y$ is Fréchet differentiable with $DF: X \to L(X, Y)$ bounded and uniformly continuous, then, for every $\nu > \mu$, $\nu > 0$, the mapping $\tilde{F}: C_\mu^-(X) \to C_\nu^-(Y)$ is Fréchet differentiable, its derivative being given by $D\tilde{F}(u)h = DF(u(\cdot))h(\cdot)$ and $D\tilde{F}: C_\mu^-(X) \to L(C_\mu^-(X), C_\nu^-(Y))$ is bounded and uniformly continuous.*

We now recall a notion of uniform equicontinuity of a subset of $C_\mu^-(X)$ (see [11]). By definition, a subset $\mathscr{F} \subset C_\mu^-(X)$ is called C_μ^--uniformly equicontinuous if and only if the set of functions $\{f_\mu, f \in \mathscr{F}\}$, where $f_\mu(t) := \mathrm{e}^{\mu t} f(t)$, is equicontinuous, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{f \in \mathscr{F}} \sup_{\substack{t, s \leqslant 0 \\ |t-s| < \delta}} \|\mathrm{e}^{\mu t} f(t) - \mathrm{e}^{\mu s} f(s)\| < \varepsilon.$$

For any $\varrho \in (0, 1]$, $a \in (0, 1]$ and $\mu \geqslant 0$, we futhermore denote

$$C_{\mu,\varrho,a}^-(\mathscr{X}) := \{u \in C_\mu^-(\mathscr{X});\ [u]_{\mu,\varrho,a} < \infty\},$$

where

$$[u]_{\mu,\varrho,a} := \sup_{\substack{t \leqslant 0 \\ h \in (0, a]}} \frac{\|\mathrm{e}^{\mu t} u(t) - \mathrm{e}^{\mu(t-h)} u(t-h)\|}{h^\varrho},$$

and let

$$\|u\|_{C_{\mu,\varrho,a}^-(\mathscr{X})} := \|u\|_{C_\mu^-(\mathscr{X})} + [u]_{\mu,\varrho,a} \quad \text{for any } u \in C_{\mu,\varrho,a}^-(\mathscr{X}).$$

The space $C_{\mu,\varrho,a}^-(\mathscr{X})$ endowed with the norm $\|\cdot\|_{C_{\mu,\varrho,a}^-}$ is a Banach space continuously embedded into $C_\mu^-(\mathscr{X})$ with an embedding constant equal to 1. Furthermore, the space $C_{\mu,\varrho,a}^-(\mathscr{X})$ is continuously embedded into $C_{\nu,\varrho,a}^-(\mathscr{X})$ for any $0 \leqslant \mu \leqslant \nu$ and $\varrho \in (0, 1]$. Indeed, for any $u \in C_{\mu,\varrho,a}^-(\mathscr{X})$, $t \leqslant 0$ and $h \in (0, a]$, we have

$$\begin{aligned}
&\|\mathrm{e}^{\nu t} u(t) - \mathrm{e}^{\nu(t-h)} u(t-h)\| \\
&\quad \leqslant \|(\mathrm{e}^{(\nu-\mu)t} - \mathrm{e}^{(\nu-\mu)(t-h)}) \mathrm{e}^{\mu t} u(t)\| + \|\mathrm{e}^{(\nu-\mu)(t-h)}(\mathrm{e}^{\mu t} u(t) - \mathrm{e}^{\mu(t-h)} u(t-h))\| \\
&\quad \leqslant \|u\|_{C_\mu^-(\mathscr{X})} (\nu - \mu) h + [u]_{\mu,\varrho,a} h^\varrho.
\end{aligned}$$

Thus, $u \in C_{\nu,\varrho,a}^-(\mathscr{X})$ and the embedding $C_{\mu,\varrho,a}^-(\mathscr{X}) \hookrightarrow C_{\nu,\varrho,a}^-(\mathscr{X})$ is continuous, its embedding constant being less or equal to $\max\{1, (\nu - \mu) a^{1-\varrho}\}$.

For any $K > 0$, the set

$$\mathscr{F}_C := \{u \in C_{\mu,\varrho,a}^-(\mathscr{X});\ \|u\|_{C_{\mu,\varrho,a}^-} \leqslant K\} \tag{2.3}$$

is a C_μ^--uniformly equicontinuous and bounded subset of $C_\mu^-(\mathscr{X})$.

Since $C^-_{\mu,\varrho,a}(\mathscr{X})$ is continuously embedded into $C^-_\mu(\mathscr{X})$ we obtain the following consequence of Lemma 2.7.

Lemma 2.8. *Let $F: X \to Y$ be as in Lemma 2.7. Suppose that $v > \mu$, $v > 0$ and $\varrho \in (0, 1]$. Then the mapping $\tilde{F}: C^-_{\mu,\varrho,a}(X) \to C^-_v(Y)$ is Fréchet differentiable, its derivative $D\tilde{F}: C^-_{\mu,\varrho,a}(X) \to L(C^-_{\mu,\varrho,a}(X), C^-_v(Y))$ being bounded and uniformly continuous.*

3. Invariant manifolds

3.1. *Construction of a family of invariant manifolds*

In this section, we establish the existence of a one-parameter family of invariant manifolds for semiflows generated by abstract singularly perturbed equations $(1.1)_\alpha$, $\alpha \geqslant 0$ small enough.

First, we will deal with solutions of the linear equation

$$\alpha w' + B_\alpha w = f \tag{3.1$_\alpha$}$$

existing on R and satisfying a growth condition of an exponential type when $t \to -\infty$. We will also consider the 'limiting equation'

$$B_0 w = f. \tag{3.1$_0$}$$

Assume that a family $\{B_\alpha, \alpha \in [0, \alpha_1]\}$ satisfies hypothesis (H1). From Lemma 2.1 we know that B_α is sectorial and $\operatorname{Re}\sigma(B_\alpha) > \omega > 0$ for any $\alpha \in [0, \alpha_1]$, α_1 small. Moreover, we choose $\alpha_1 > 0$ such that

$$\omega > v\alpha_1 > 0,$$

where $v > 0$ is given. Now, it is routine to verify that $(3.1)_\alpha$, $\alpha \in (0, \alpha_1]$, has the unique solution $w \in C^-_v(Y^\beta)$, $\beta \in [0, 1)$, for any $f \in C^-_v(Y^\xi)$, $\xi > \beta - 1$. This solution is given by

$$w(t) := \frac{1}{\alpha} \int_{-\infty}^{t} \exp(-B_\alpha(t-s)/\alpha) f(s)\,\mathrm{d}s =: C_\alpha f(t), \quad t \leqslant 0.$$

The unique solution of $(3.1)_0$ is determined by

$$w := B_0^{-1} f =: C_0 f.$$

Concerning the boundedness and limiting behaviour of the linear operators

$$C_\alpha : C^-_v(Y^\xi) \to C^-_v(Y^\beta), \quad \alpha \in [0, \alpha_1], \tag{3.2}$$

we claim the following lemma.

Lemma 3.1. *Assume that the family $\{B_\alpha;\ \alpha \in [0, \alpha_1]\}$ fulfils hypothesis* (H1). *Let $\beta \in [0, 1)$, $\beta > \xi > \beta - 1$ and $0 < v\alpha_1 < \omega$. Then*

(a) *there is a $C > 0$ such that*

$$\| C_\alpha \|_{L(C^-_v(Y^\xi), C^-_v(Y^\beta))} \leqslant C\Gamma(1 - \beta + \xi)(\omega - v\alpha_1)^{\beta - \xi - 1} \quad \textit{for any } \alpha \in [0, \alpha_1],$$

where Γ is the gamma function $\Gamma(\theta) := \int_0^\infty r^{\theta - 1} \mathrm{e}^{-r}\,\mathrm{d}r$ for $\theta > 0$,

(b) *$C_\alpha f \to C_0 f$ as $\alpha \to 0^+$ uniformly with respect to $f \in \mathscr{F}$, where $\mathscr{F}$ is a C^-_v-uniformly equicontinuous and bounded subset of $C^-_v(Y^\xi)$, and*

(c) *$C_\alpha \to C_0$ as $\alpha \to 0^+$ in the norm topology of the space $L(C^-_{v,\varrho,a}(Y^\xi), C^-_v(Y^\beta))$ for any $\varrho \in (0, 1]$, $a > 0$.*

Proof. Denote $w := C_\alpha f$ for $f \in C_\nu^-(Y^\xi)$. With regard to Lemma 2.4 we obtain, for any $t \leqslant 0$ and $\alpha \in (0, \alpha_1]$,

$$\begin{aligned}
e^{\nu t}\|w(t)\|_\beta &\leqslant \frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}\exp(-(B_\alpha - \nu\alpha)(t-s)/\alpha)\|\, e^{\nu s}\|B_0^\xi f(s)\|\,ds \\
&\leqslant \frac{C}{\alpha}\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-(\omega-\nu\alpha)(t-s)/\alpha}\,ds\,\|f\|_{C_\nu^-(Y^\xi)} \\
&\leqslant C\Gamma(1-\beta+\xi)(\omega-\nu\alpha_1)^{\beta-\xi-1}\|f\|_{C_\nu^-(Y^\xi)}.
\end{aligned}$$

For $\alpha = 0$ we have

$$e^{\nu t}\|w(t)\|_\beta \leqslant \|B_0^{\beta-\xi-1}\|\,\|f\|_{C_\nu^-(Y^\xi)} \leqslant C\omega^{\beta-\xi-1}\|f\|_{C_\nu^-(Y^\xi)}.$$

(b) Because $\operatorname{Re}\sigma(B_0) > \omega > 0$, we have the following integral representation of B_0^{-1}:

$$B_0^{-1} = \frac{1}{\alpha}\int_{-\infty}^{t} \exp(-B_0(t-s)/\alpha)\,ds \quad \text{for any } t \leqslant 0,\ \alpha > 0.$$

Let $t \leqslant 0$ and $f \in \mathscr{F}$ be arbitrary. Using Lemma 2.4 we obtain

$$\begin{aligned}
&e^{\nu t}\|C_\alpha f(t) - C_0 f(t)\|_\beta \\
&\quad\leqslant \frac{e^{\nu t}}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}(\exp(-B_\alpha(t-s)/\alpha)B_0^\xi f(s) \\
&\qquad - \exp(-B_0(t-s)/\alpha)B_0^\xi f(t))\|\,ds \\
&\quad\leqslant \frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}\exp(-B_\alpha(t-s)/\alpha)B_0^\xi(f(s)-f(t))e^{\nu t}\|\,ds \\
&\qquad + \frac{1}{\alpha}\int_{-\infty}^{t} \|B_0^{\beta-\xi}(\exp(-B_\alpha(t-s)/\alpha) - \exp(-B_0(t-s)/\alpha))\|\,ds\,\|f\|_{C_\nu^-(Y^\xi)} \\
&\quad\leqslant \frac{C}{\alpha}\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|_\xi\,ds \\
&\qquad + \frac{C}{\alpha}\|B_0 B_\alpha^{-1} - I\|\int_{-\infty}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha}\,ds\,\|f\|_{C_\nu^-(Y^\xi)} \\
&\quad =: I_1 + I_2.
\end{aligned}$$

As is usual in integrals with singular kernels (see e.g. [11]) we decompose the first integral into two parts $I_1 = \int_{-\infty}^{t-\tau} + \int_{t-\tau}^{t} =: I_{1,1} + I_{1,2}$, where $\tau > 0$ will be determined later. Clearly,

$$e^{\nu t}\|f(s) - f(t)\|_\xi \leqslant 2e^{\nu(t-s)}\|f\|_{C_\nu^-(Y^\xi)} \quad \text{for any } -\infty < s \leqslant t \leqslant 0.$$

Then

$$\begin{aligned}
I_{1,1} &:= \frac{C}{\alpha}\int_{-\infty}^{t-\tau} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|_\xi\,ds \\
&\leqslant \frac{2C\tau^{-(\beta-\xi)}\alpha^{\beta-\xi}}{\omega - \nu\alpha_1}\|f\|_{C_\nu^-(Y^\xi)}.
\end{aligned}$$

On the other hand, for any $s \in [t-\tau, t]$, we have

$$
\begin{aligned}
e^{\nu t}\|f(s)-f(t)\|_\xi &\leqslant \|e^{\nu s}f(s)-e^{\nu t}f(t)\|_\xi + (e^{\nu(t-s)}-1)\|f\|_{C_\nu^-(Y^\xi)} \\
&\leqslant e^{\nu(t-s)}(\operatorname{osc}(f_\nu,\tau)+(1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)}),
\end{aligned}
$$

where

$$\operatorname{osc}(f_\nu,\tau) := \sup_{\substack{t,s\leqslant 0\\ |t-s|<\tau}} \|e^{\nu t}f(t)-e^{\nu s}f(s)\|_\xi .$$

Hence,

$$
\begin{aligned}
I_{1,2} &:= \frac{C}{\alpha}\int_{t-\tau}^{t} ((t-s)/\alpha)^{-(\beta-\xi)} e^{-\omega(t-s)/\alpha} e^{\nu t}\|f(s)-f(t)\|\, ds \\
&\leqslant C\Gamma(1-\beta+\xi)(\omega-\nu\alpha_1)^{\beta-\xi-1}(\operatorname{osc}(f_\nu,\tau)+(1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)}).
\end{aligned}
$$

Finally, we have

$$I_2 \leqslant C\Gamma(1-\beta+\xi)\omega^{\beta-\xi-1}\|B_0B_\alpha^{-1}-I\|\,\|f\|_{C_\nu^-(Y^\xi)}.$$

Since the set $\mathscr{F} \subset C_\nu^-(Y^\xi)$ is assumed to be C_ν^--uniformly equicontinuous and bounded, we have

$$\operatorname{osc}(f_\nu,\tau)+(1-e^{-\nu\tau})\|f\|_{C_\nu^-(Y^\xi)} \to 0^+ \quad \text{as } \tau \to 0^+$$

uniformly with respect to $f \in \mathscr{F}$. Now, it is easy to see that $C_\alpha f \to C_0 f$ in $C_\nu^-(Y^\beta)$ when $\alpha \to 0^+$ uniformly for $f \in \mathscr{F}$.

Finally, by (2.3), the set $\mathscr{F}_1 := \{\phi \in C_{\nu,\varrho,a}^-(Y^\xi);\ \|\phi\|_{C_{\nu,\varrho,a}^-} \leqslant 1\}$ is a C_ν^--uniformly equicontinuous and bounded subset of $C_\nu^-(Y^\xi)$. Hence, by (b), $C_\alpha \to C_0$ as $\alpha \to 0^+$ in the topology of the space $L(C_{\nu,\varrho,a}^-(Y^\xi), C_\nu^-(Y^\beta))$. □

We now turn our attention to the construction of an invariant manifold $\mathscr{M}_\alpha$ for the semiflow $\mathscr{S}_\alpha$ generated by system $(1.1)_\alpha$. From now on we will assume that the following hypothesis,

$$
(\mathrm{H})\ \begin{cases}
(1) & \text{the family } \{A_\alpha, \alpha\in[0,\alpha_1]\} \text{ satisfies (H1)–(H2)}\\
 & \text{on a Banach space } X,\\
(2) & \text{the family } \{B_\alpha, \alpha\in[0,\alpha_1]\} \text{ satisfies (H1) on a Banach space } Y,\\
(3) & \text{the functions } g \text{ and } f \text{ satisfy (H3) for some } \gamma,\beta\in[0,1)\\
 & \text{and } \beta>\xi>\beta-1,
\end{cases}
$$

holds.

The idea of the construction of an invariant manifold $\mathscr{M}_\alpha$ for $(1.1)_\alpha$ is fairly standard and is based on the well-known method of integral equations due to Lyapunov and Perron. According to this method, $\mathscr{M}_\alpha$ contains all solutions $(u(\cdot), w(\cdot)) \in X^\gamma \times Y^\beta$ of $(1.1)_\alpha$ existing on R and satisfying an exponential growth condition of the form $\|u(t)\|_\gamma + \|w(t)\|_\beta = O(e^{-\mu t})$ as $t \to -\infty$ where $\mu > 0$ is fixed. In our case, we will take advantage of the particular form of (1.1). With regard to Lemma 3.1, for a given $u \in C_\mu^-(X^\gamma)$ we have $\tilde{f}(u) \in C_\mu^-(Y^\xi)$ ($\tilde{f}$ defined in (2.2)), and hence $w := C_\alpha\tilde{f}(u)$ is the unique solution of $(3.1)_\alpha$ belonging to $C_\mu^-(Y^\beta)$. Roughly speaking, the w-variable of the semiflow $\mathscr{S}_\alpha$ on an invariant manifold $\mathscr{M}_\alpha$ (if it exists) is governed by the u-variable. More precisely, as usual (see e.g. [5, 10, 7]), we will construct $\mathscr{M}_\alpha$ as the union of curves

$(u, C_\alpha \tilde{f}(u))$ where $u \in C_\mu^-(X^\gamma)$ are fixed points of the mapping

$$T_\alpha(x, \cdot): C_\mu^-(X^\gamma) \to C_\mu^-(X^\gamma),$$

$\alpha \in [0, \alpha_1]$, $x \in X_{1,0} := P_0 X$ and, for any $u \in C_\mu^-(X^\gamma)$,

$$T_\alpha(x, u) := \mathscr{K}_\alpha x + \mathscr{T}_\alpha(\mathscr{G}_\alpha(u)). \tag{3.3}$$

The linear operators $\mathscr{K}_\alpha: X_{1,0} \to C_\mu^-(X^\gamma)$, $\mathscr{T}_\alpha: C_\mu^-(X) \to C_\mu^-(X^\gamma)$ are given by

$$\mathscr{K}_\alpha x := \exp(-A_{1,\alpha} t) P_\alpha x \quad \text{for any } x \in X_{1,0},$$

$$\mathscr{T}_\alpha(g)(t) := \int_0^t \exp(-A_{1,\alpha}(t-s)) P_\alpha g(s)\,\mathrm{d}s + \int_{-\infty}^t \exp(-A_\alpha(t-s)) Q_\alpha g(s)\,\mathrm{d}s \quad \text{for any } g \in C_\mu^-(X), \tag{3.4}$$

and the non-linearity $\mathscr{G}_\alpha: C_\mu^-(X^\gamma) \to C_\mu^-(X)$ is given by

$$\mathscr{G}_\alpha(u)(t) := g(u(t), C_\alpha \tilde{f}(u)(t)) \quad \text{for any } u \in C_\mu^-(X^\gamma).$$

By means of the Banach fixed point theorem, we will show that the operator $T_\alpha(x, \cdot)$ has a fixed point $Y_\alpha(x) \in C_\mu^-(X^\gamma)$. To do this, we first establish estimates of norms of $\mathscr{T}_\alpha$ and $\mathscr{K}_\alpha$ and a Lipschitz constant of $\mathscr{G}_\alpha$.

Lemma 3.2. *Suppose $\mu \in (\lambda_-, \lambda_+)$. Then, for any $\alpha \in [0, \alpha_1]$,*

(a) $\mathscr{K}_\alpha \in L(X_{1,0}, C_\mu^-(X^\gamma))$, $\|\mathscr{K}_\alpha\|_{L(X_{1,0}, C_\mu^-(X^\gamma))} \leqslant C\lambda_-^\gamma$,
$\|\mathscr{K}_\alpha - \mathscr{K}_0\|_{L(X_{1,0}, C_\mu^-(X^\gamma))} \leqslant C\lambda_-^\gamma \|A_0 A_\alpha^{-1} - I\|$,

(b) $\mathscr{T}_\alpha \in L(C_\mu^-(X), C_\mu^-(X^\gamma))$, $\|\mathscr{T}_\alpha\|_{L(C_\mu^-(X), C_\mu^-(X^\gamma))} \leqslant CK(\lambda_-, \lambda_+, \mu, \gamma)$,
$\|\mathscr{T}_\alpha - \mathscr{T}_0\|_{L(C_\mu^-(X), C_\mu^-(X^\gamma))} \leqslant C\|A_0 A_\alpha^{-1} - I\| K(\lambda_-, \lambda_+, \mu, \gamma)$,

where

$$K(\lambda_-, \lambda_+, \mu, \gamma) := \frac{\lambda_-^\gamma}{\mu - \lambda_-} + \frac{2-\gamma}{1-\gamma}(\lambda_+ - \mu)^{\gamma - 1}.$$

If, in addition, $\varrho \in (0, 1-\gamma)$ then there is a constant $a = a(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ such that

(c) $\mathscr{K}_\alpha \in L(X_{1,0}, C_{\mu,\varrho,a}^-(X^\gamma))$, $\|\mathscr{K}_\alpha\|_{L(X_{1,0}, C_{\mu,\varrho,a}^-(X^\gamma))} \leqslant 2C\lambda_-^\gamma$,
$\mathscr{T}_\alpha \in L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))$, $\|\mathscr{T}_\alpha\|_{L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))} \leqslant 2CK(\lambda_-, \lambda_+, \mu, \gamma)$,

(d) $\mathscr{K}_\alpha \to \mathscr{K}_0$ *and* $\mathscr{T}_\alpha \to \mathscr{T}_0$ *as* $\alpha \to 0^+$ *in* $L(X_{1,0}, C_{\mu,\varrho,a}^-(X^\gamma))$ *and* $L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))$, *respectively.*

Proof. Using the estimates from Lemma 2.5 the proof of (a) is obvious. Again, with the help of Lemma 2.5, the proof of (b) is an immediate adaptation of that of [5, Lemma 3.1].

In order to prove (c), we make use of Lemma 2.6. Applying Lemma 2.6(a), we obtain

$$[\mathscr{K}_\alpha x]_{\mu,\varrho,a} \leqslant a^{1-\varrho} C\lambda_-^{1+\gamma} \|(\mu - A_\alpha) A_0^{-1}\| \, \|x\| \quad \text{for any } x \in X_{1,0}.$$

Further, by (3.4),

$$P_\alpha \mathscr{T}_\alpha(g)(t) = \int_0^t \exp(-A_{1,\alpha}(t-s))P_\alpha g(s)\,\mathrm{d}s,$$

$$Q_\alpha \mathscr{T}_\alpha(g)(t) = \int_{-\infty}^t \exp(-A_\alpha(t-s))Q_\alpha g(s)\,\mathrm{d}s$$

for any $g \in C_\mu^-(X)$. Hence

$$\begin{aligned} I_1 &:= P_\alpha \mathscr{T}_\alpha(g)(t)\mathrm{e}^{\mu t} - P_\alpha \mathscr{T}_\alpha(g)(t-h)\mathrm{e}^{\mu(t-h)} \\ &= \int_0^t [\exp((\mu - A_{1,\alpha})(t-s)) - \exp((\mu - A_{1,\alpha})(t-h-s))]P_\alpha \mathrm{e}^{\mu s} g(s)\,\mathrm{d}s \\ &\quad + \int_{t-h}^t \exp((\mu - A_{1,\alpha})(t-h-s))P_\alpha \mathrm{e}^{\mu s} g(s)\,\mathrm{d}s. \end{aligned}$$

By taking norms and using Lemmas 2.5 and 2.6(a), we obtain

$$\begin{aligned} \|A_0^\gamma I_1\| &\leqslant \left\{ hC\|(\mu - A_\alpha)A_0^{-1}\|\lambda_-^{1+\gamma} \int_0^t \mathrm{e}^{(\mu-\lambda_-)(t-s)}\,\mathrm{d}s \right. \\ &\quad \left. + C\lambda_-^\gamma \int_{t-h}^t \mathrm{e}^{(\mu-\lambda_-)(t-h-s)}\,\mathrm{d}s \right\} \|g\|_{C_\mu^-(X)} \\ &\leqslant hC\lambda_-^\gamma \left\{ 1 + \frac{\|(\mu - A_\alpha)A_0^{-1}\|\lambda_-}{\mu - \lambda_-} \right\} \|g\|_{C_\mu^-(X)}. \end{aligned}$$

Thus

$$[P_\alpha \mathscr{T}_\alpha(g)]_{\mu,\varrho,a} \leqslant a^{1-\varrho} C\lambda_-^\gamma \left\{ 1 + \frac{\|(\mu - A_\alpha)A_0^{-1}\|\lambda_-}{\mu - \lambda_-} \right\} \|g\|_{C_\mu^-(X)}.$$

Acting similarly as above, we deduce that

$$\begin{aligned} I_2 &:= Q_\alpha \mathscr{T}_\alpha(g)(t)\mathrm{e}^{\mu t} - Q_\alpha \mathscr{T}_\alpha(g)(t-h)\mathrm{e}^{\mu(t-h)} \\ &= \int_{-\infty}^{t-h} [\exp(-(A_\alpha - \mu)(t-s)) - \exp(-(A_\alpha - \mu)(t-h-s))]Q_\alpha \mathrm{e}^{\mu s} g(s)\,\mathrm{d}s \\ &\quad + \int_{t-h}^t \exp(-(A_\alpha - \mu)(t-s))Q_\alpha \mathrm{e}^{\mu s} g(s)\,\mathrm{d}s. \end{aligned}$$

Again, by Lemmas 2.5 and 2.6(b),

$$\begin{aligned} \|A_0^\gamma I_2\| &\leqslant \left\{ h^{(1-\gamma+\varrho)/2} C^2 \|(\mu - A_\alpha)A_0^{-1}\| \right. \\ &\quad \times \int_{-\infty}^{t-h} (t-h-s)^{-(1+\gamma+\varrho)/2} \mathrm{e}^{-(\lambda_+ - \mu)(t-h-s)}\,\mathrm{d}s \\ &\quad \left. + C\int_{t-h}^t (t-s)^{-\gamma} \mathrm{e}^{-(\lambda_+ - \mu)(t-s)}\,\mathrm{d}s \right\} \|g\|_{C_\mu^-(X)} \\ &\leqslant h^\varrho \left\{ \frac{a^{(1-\gamma-\varrho)/2} C^2 \|(\mu - A_\alpha)A_0^{-1}\|}{(\lambda_+ - \mu)^{(1-\gamma-\varrho)/2}} \Gamma((1-\gamma-\varrho)/2) \right. \\ &\quad \left. + \frac{Ca^{1-\gamma-\varrho}}{1-\gamma} \right\} \|g\|_{C_\mu^-(X)}. \end{aligned}$$

In this way we have shown that there exists a constant $k = k(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ such that

$$[\mathscr{K}_\alpha x]_{\mu,\varrho,a} \leqslant a^{1-\varrho} k \|x\| \quad \text{and} \quad [\mathscr{T}_\alpha g]_{\mu,\varrho,a} \leqslant a^{(1-\gamma-\varrho)/2} k \|g\|_{C_\mu^-(X)} \quad \text{for any } 0 < a \leqslant 1.$$

Hence, by taking $a = a(\lambda_-, \lambda_+, \mu, \gamma, \varrho, C) > 0$ sufficiently small and using the statements (a) and (b), it follows that

$$\|\mathscr{K}_\alpha\|_{L(X_{1,0} C_{\mu,\varrho,a}^-(X^\gamma))} \leqslant 2C\lambda_-^\gamma \quad \text{and} \quad \|\mathscr{T}_\alpha\|_{L(C_\mu^-(X), C_{\mu,\varrho,a}^-(X^\gamma))} \leqslant 2CK(\lambda_-, \lambda_+, \mu, \gamma).$$

Finally, we will prove that

$$[\mathscr{K}_\alpha x - \mathscr{K}_0 x]_{\mu,\varrho,a} \to 0 \quad \text{and} \quad [\mathscr{T}_\alpha g - \mathscr{T}_0 g]_{\mu,\varrho,a} \to 0 \qquad \text{as } \alpha \to 0^+$$

uniformly with respect to $\|x\| \leqslant 1$ and $\|g\|_{C_\mu^-(X)} \leqslant 1$, respectively.

Denote

$$U_\alpha(t) := \exp((\mu - A_{1,\alpha})t)P_\alpha - \exp((\mu - A_{1,0})t)P_0,$$

$$V_\alpha(t) := \exp(-(A_\alpha - \mu)t)Q_\alpha - \exp(-(A_0 - \mu)t)Q_0.$$

We have the following integral representation of $U_\alpha(t)$: for any $r \leqslant 0$, $h > 0$,

$$\begin{aligned}
U_\alpha(r) - U_\alpha(r-h) &= \int_{r-h}^{r} \frac{\mathrm{d}}{\mathrm{d}\xi} U_\alpha(\xi)\,\mathrm{d}\xi \\
&= \int_{r-h}^{r} [(\mu - A_{1,\alpha})\exp((\mu - A_{1,\alpha})\xi)P_\alpha \\
&\quad - (\mu - A_{1,0})\exp((\mu - A_{1,0})\xi)P_0]\,\mathrm{d}\xi \\
&= (A_{1,0} - A_{1,\alpha}) \int_{r-h}^{r} \exp((\mu - A_{1,\alpha})\xi)P_\alpha\,\mathrm{d}\xi \\
&\quad + (\mu - A_{1,0}) \int_{r-h}^{r} U_\alpha(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Using the above expression and Lemma 2.5 one can proceed similarly as in the proof of Lemma 2.6(a). One obtains

$$\|A_0^\gamma(U_\alpha(r) - U_\alpha(r-h))x\| \leqslant C_1 h e^{(\mu-\lambda_-)r} \|I - A_\alpha A_0^{-1}\| \|x\|, \quad r \leqslant 0,\ h > 0,$$

where $C_1 > 0$ is a constant. Analogously, one also deduces that for any $r \geqslant 0$, $h > 0$

$$\begin{aligned}
&\|A_0^\gamma(V_\alpha(r+h) - V_\alpha(r))x\| \\
&\quad \leqslant C_1 h^{(1-\gamma+\varrho)/2} r^{-(1+\gamma+\varrho)/2} e^{-(\lambda_+ - \mu)r} \|I - A_\alpha A_0^{-1}\| \|x\|.
\end{aligned}$$

With the help of these estimates, statement (d) can be readily proved by repeating the lines of the proof of (c), but now operating with $U_\alpha(t)$ and $V_\alpha(t)$ instead of $\exp((\mu - A_{1,\alpha})t)P_\alpha$ and $\exp(-(A_\alpha - \mu)t)Q_\alpha$, respectively. □

Since $g: X^\gamma \times Y^\beta \to X$ and $f: Y^\beta \to Y^\xi$ are bounded and Lipschitzian we have that the mapping $\mathscr{G}_\alpha: C_\mu^-(X^\gamma) \to C_\mu^-(X)$ is bounded, uniformly with respect to $\alpha \in [0, \alpha_1]$, and, moreover,

$$\|\mathscr{G}_\alpha(u_1) - \mathscr{G}_\alpha(u_2)\|_{C_\mu^-(X)} \leqslant \mathrm{Lip}(g)(1 + \|C_\alpha\| \,\mathrm{Lip}(f)) \|u_1 - u_2\|_{C_\mu^-(X^\gamma)}.$$

By Lemma 3.1,

$$\mathrm{Lip}(\mathscr{G}_\alpha) \leqslant \mathrm{Lip}(g)(1 + C\Gamma(1-\beta+\xi)(\omega-\mu\alpha_1)^{\beta-\xi-1}\,\mathrm{Lip}(f)). \tag{3.5}$$

With this we have established the following lemma.

Lemma 3.3. *Let $\mu \in (\lambda_-, \lambda_+)$. Assume that hypothesis* (H) *holds. Then the operator $T_\alpha(x,\cdot): C_\mu^-(X^\gamma) \to C_\mu^-(X^\gamma)$ is a uniform contraction with respect to $\alpha \in [0, \alpha_1]$ and $x \in X_{1,0}$, provided that the following inequality is satisfied:*

$$\theta := CK(\lambda_-, \lambda_+, \mu, \gamma)\,\mathrm{Lip}(g)(1 + C\Gamma(1-\beta+\xi)(\omega-\mu\alpha_1)^{\beta-\xi-1}\,\mathrm{Lip}(f)) < 1. \tag{3.6}$$

According to the previous lemma, if (3.6) is satisfied then, by the Banach fixed point theorem, there is a family $Y_\alpha(x)$, $\alpha \in [0, \alpha_1]$, $x \in X_{1,0}$, of fixed points of $T_\alpha(x,\cdot)$. Because $\|T_\alpha(x_1, u) - T_\alpha(x_2, u)\| = \|\mathscr{K}_\alpha(x_1 - x_2)\| \leqslant C\lambda_-^\gamma \|x_1 - x_2\|$ we furthermore have

$$\|Y_\alpha(x_1) - Y_\alpha(x_2)\| \leqslant C\lambda_-^\gamma (1-\theta)^{-1} \|x_1 - x_2\|, \tag{3.7}$$

i.e. $Y_\alpha(\cdot)$ are Lipschitz continuous uniformly with respect to $\alpha \in [0, \alpha_1]$.

Now, we can define a set $\mathscr{M}_\alpha$ as follows:

$$\mathscr{M}_\alpha := \{(Y_\alpha(x)(0), C_\alpha \tilde{f}(Y_\alpha(x))(0));\ x \in X_{1,0}\}, \quad \alpha \in (0, \alpha_1].$$

To show invariance of $\mathscr{M}_\alpha$ under the semiflow $\mathscr{S}_\alpha(t)$, $t \geqslant 0$, generated by system $(1.1)_\alpha$, it suffices to prove that

$$\mathscr{M}_\alpha = \{(u(\tau), w(\tau)) \in X^\gamma \times Y^\beta,\ \tau \in R,\ (u, w) \in C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta) \text{ solves } (1.1)_\alpha\}. \tag{3.8}$$

Indeed, let us consider an arbitrary solution $(u(\cdot), w(\cdot))$, belonging to the right-hand side of (3.8). Take a $\tau \in R$ and put $\bar{u}(t) := u(t+\tau)$, $\bar{w}(t) := w(t+\tau)$. Then $(\bar{u}, \bar{w})$ is a solution of $(1.1)_\alpha$ as well, and $(\bar{u}(\cdot), \bar{w}(\cdot)) \in C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta)$. By Lemma 3.1, we have $\bar{w} = C_\alpha \tilde{f}(\bar{u})$ and $\bar{u}$ is therefore a solution of $\bar{u}' + A_\alpha \bar{u} = g(\bar{u}, C_\alpha \tilde{f}(\bar{u})) = \mathscr{G}_\alpha(\bar{u})$. According to [5, Lemma 4.2] $\bar{u}$ is a solution of

$$\bar{u}(t) = \exp(-A_{1,\alpha} t) P_\alpha \bar{u}(0) + \mathscr{T}_\alpha(\mathscr{G}_\alpha(\bar{u}))(t), \quad t \leqslant 0.$$

By Lemmas 2.2 and 2.5, $P_\alpha|_{\mathscr{X}_{1,0}}: \mathscr{X}_{1,0} \to \mathscr{X}_{1,\alpha}$ is a linear isomorphism. Therefore, there exists $x \in X_{1,0}$ such that $P_\alpha x = P_\alpha \bar{u}(0)$. Thus, $\bar{u}$ solves the operator equation $T_\alpha(x, \bar{u}) = \bar{u}$. By uniqueness of a fixed point of $T_\alpha(x,\cdot)$, we have $\bar{u} = Y_\alpha(x)$ and hence

$$(u(\tau), w(\tau)) = (\bar{u}(0), \bar{w}(0)) = (Y_\alpha(x)(0), C_\alpha \tilde{f}(Y_\alpha(x))(0)) \in \mathscr{M}_\alpha.$$

On the other hand, take an arbitrary $x \in X_{1,0}$. Then $(Y_\alpha(x)(\cdot), C_\alpha \tilde{f}(Y_\alpha(x))(\cdot)) \in C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta)$ is a solution of $(1.1)_\alpha$ which can be extended to a solution existing globally on R. Hence $(Y_\alpha(x)(0), C_\alpha \tilde{f}(Y_\alpha(x))(0))$ belongs to the right-hand side of (3.8). In this way we have shown (3.8).

For $\alpha = 0$, we put

$$\tilde{\mathscr{M}}_0 := \{Y_0(x)(0);\ x \in X_{1,0}\}.$$

With regard to [5, Theorem 4.4], $\tilde{\mathscr{M}}_0 \subset X^\gamma$ is an invariant manifold for the semiflow $\tilde{\mathscr{S}}_0$ generated by $(1.1)_0$. This manifold can be naturally embedded into a manifold $\mathscr{M}_0 \subset X^\gamma \times Y^\beta$ defined as

$$\mathscr{M}_0 := \{(u, B_0^{-1} f(u));\ u \in \tilde{\mathscr{M}}_0\}.$$

We note that the manifolds $\mathcal{M}_\alpha$, $\alpha \in [0, \alpha_1]$, are Lipschitz continuous submanifolds of $X^\gamma \times Y^\beta$ (see (3.7)) and $\dim \mathcal{M}_\alpha = \dim X_{1,0} < \infty$ for any $\alpha \in [0, \alpha_1]$.

Denote

$$\Phi_\alpha(x) := Y_\alpha(x)(0) \quad \text{and} \quad \Psi_\alpha(x) := C_\alpha \tilde{f}(Y_\alpha(x))(0).$$

The mapping $X_{1,0} \ni x \mapsto (\Phi_\alpha(x), \Psi_\alpha(x)) \in X^\gamma \times Y^\beta$ is Lipschitz continuous, its Lipschitz constant being independent of $\alpha \in [0, \alpha_1]$.

In terms of Φ_α and Ψ_α, the manifold $\mathcal{M}_\alpha$ is given by

$$\mathcal{M}_\alpha = \{(\Phi_\alpha(x), \Psi_\alpha(x)),\ x \in X_{1,0}\}, \quad \alpha \in [0, \alpha_1],$$

and the semiflow $\mathcal{S}_\alpha$ ($\tilde{\mathcal{S}}_\alpha$) on $\mathcal{M}_\alpha$ ($\tilde{\mathcal{M}}_0$) is determined by solutions of its inertial form. By definition (see [7, chapter 2.1]), an inertial form for (1.1) is an ordinary differential equation in a finite dimensional space $X_{1,0}$ given by

$$p' + P_\alpha^{(-1)} A_{1,\alpha} P_\alpha p = P_\alpha^{(-1)} P_\alpha g(\Phi_\alpha(p), \Psi_\alpha(p)), \tag{3.9$_\alpha$}$$

where the linear operator $P_\alpha^{(-1)}$ was defined in (2.1.4). Indeed, any solution (u, w) of (1.1), $\alpha \in [0, \alpha_1]$, belonging to $\mathcal{M}_\alpha$ can be written as $(u(t), w(t)) = (\Phi_\alpha(p(t)), \Psi_\alpha(p(t)))$, where $p(\cdot)$ is a solution of (3.9) and vice versa.

Remark 3.4. Assume that $\theta \ll 1$ is sufficiently small. Then, following the lines of the proof of [5, Theorem 5.1], one can also prove exponential attractivity of $\mathcal{M}_\alpha$. It means that, for any $(u, w) \in X^\gamma \times Y^\beta$, there is a unique $(u^*, w^*) \in \mathcal{M}_\alpha$ such that $\| \mathcal{S}_\alpha(t)(u, w) - \mathcal{S}_\alpha(t)(u^*, w^*)\| = O(\mathrm{e}^{-\mu t})$ as $t \to \infty$. Hence, $\mathcal{M}_\alpha$ is an inertial manifold for the semiflow $\mathcal{S}_\alpha$ in the sense of [7].

3.2. *The singular limit dynamics of invariant manifolds*

In this section, our objective is to study singular limit dynamics of invariant manifolds $\mathcal{M}_\alpha$ when $\alpha \to 0^+$. The main purpose is to show

$$(\Phi_\alpha, \Psi_\alpha) \to (\Phi_0, \Psi_0) \quad \text{as } \alpha \to 0^+ \tag{3.10}$$

in the topology of the space $C^1_{\mathrm{bdd}}(B, X^\gamma \times Y^\beta)$, where B is an arbitrary bounded and open subset of $X_{1,0}$.

The proof uses abstract results due to Mora and Solà-Morales regarding the limiting behaviour of fixed points of a two-parameter family of non-linear mappings. With regard to Lemma 2.7, we note that the mapping $T_\alpha(x, \cdot): C^-_\mu(X^\gamma) \to C^-_\mu(X^\gamma)$ need not be generally C^1 differentiable. One can, however, expect that T_α is a C^1 mapping when considering $T_\alpha(x, \cdot)$ as a mapping from $C^-_\mu(X^\gamma)$ into $C^-_\nu(X^\gamma)$ for some $\nu > \mu$. Therefore, we need a version of a contraction theorem covering the case in which differentiability involves a pair of Banach spaces.

First, we recall the assumptions of [11, Theorem 5.1]. Let $\mathcal{X}$, U be Banach spaces, $\alpha_1 > 0$. Let T_α, $\alpha \in [0, \alpha_1]$, be a family of mappings from $\mathcal{X} \times U$ into U such that

(T) $\begin{cases} \text{(1) there is } \theta < 1 \text{ such that } \| T_\alpha(x, u_1) - T_\alpha(x, u_2)\|_U \leqslant \theta \| u_1 - u_2 \|_U \\ \quad \text{for any } x \in \mathcal{X},\ u_1, u_2 \in U \text{ and } \alpha \in [0, \alpha_1], \\ \text{(2) there is a } Q < \infty \text{ such that } \| T_\alpha(x_1, u) - T_\alpha(x_2, u)\| \leqslant Q \| x_1 - x_2 \|_{\mathcal{X}} \\ \quad \text{for any } x_1, x_2 \in \mathcal{X},\ u \in U \text{ and } \alpha \in [0, \alpha_1], \text{ and} \\ \text{(3) for any } B \subset \mathcal{X} \text{ bounded and open } \sup_{x \in B} \| T_\alpha(x, Y_0(x)) - \\ \quad T_0(x, Y_0(x))\|_U \to 0 \text{ as } \alpha \to 0^+, \text{ where } Y_\alpha(x),\ x \in \mathcal{X},\ \alpha \in [0, \alpha_1], \\ \quad \text{is the unique fixed point of } T_\alpha(x, Y) = Y. \end{cases}$

Remark 3.5. Note that, by the Banach fixed point theorem, $(\mathrm{T})_1$ and $(\mathrm{T})_2$ ensure the existence of a family of fixed points $Y_\alpha(x)$ of $T_\alpha(x,\cdot)$ such that the mapping $x \mapsto Y_\alpha(x)$ is Lipschitzian, its Lipschitz constant being $Q(1-\theta)^{-1}$. Furthermore, $(\mathrm{T})_3$ implies $Y_\alpha(x) \to Y_0(x)$ as $\alpha \to 0^+$ uniformly with respect to $x \in B$, B is an arbitrary bounded and open subset of $\mathscr{X}$.

We assume that the space U is continuously embedded into a Banach space $\bar{U}$ through a linear embedding operator J. We also denote $\bar{T}_\alpha := JT_\alpha$ and $\bar{Y}_\alpha := JY_\alpha$. We are now in a position to state a slightly modified version of [11, Theorem 5.1].

Theorem 3.6 (Mora and Solà-Morales [11, Theorem 5.1]). *Besides hypothesis* (T) *we also assume that the mappings* $\bar{T}_\alpha : \mathscr{X} \times U \to \bar{U}$, $\alpha \in [0, \alpha_1]$, *satisfy the following conditions:*

(1) *For any* $\alpha \in [0, \alpha_1]$, $\bar{T}_\alpha$ *is Fréchet differentiable, with* $D\bar{T}_\alpha : \mathscr{X} \times U \to L(\mathscr{X} \times U, \bar{U})$ *bounded and uniformly continuous, and there exist mappings*

$$d_u T_\alpha : \mathscr{X} \times U \to L(U, U), \qquad \bar{d}_u T_\alpha : \mathscr{X} \times U \to L(\bar{U}, \bar{U}), \qquad d_x T_\alpha : \mathscr{X} \times U \to L(\mathscr{X}, U)$$

such that

$$D_u \bar{T}_\alpha(x, u) = J d_u T_\alpha(x, u) = \bar{d}_u T_\alpha(x, u) J, \qquad D_x \bar{T}_\alpha(x, u) = J d_x T_\alpha(x, u),$$

$$\| d_u T_\alpha(x, u) \|_{L(U, U)} \leqslant \theta, \qquad \| \bar{d}_u T_\alpha(x, u) \|_{L(\bar{U}, \bar{U})} \leqslant \theta, \qquad \| d_x T_\alpha(x, u) \|_{L(\mathscr{X}, U)} \leqslant Q.$$

(2) *For any* B *bounded and open subset of* $\mathscr{X}$, $D\bar{T}_\alpha(x, u) \to D\bar{T}_0(x, u)$ as $\alpha \to 0^+$ *uniformly with respect to* $(x, u) \in B \times \mathscr{F}_B$, *where*

$$\mathscr{F}_B := \{ Y_\alpha(x) \in U;\ x \in B,\ \alpha \in [0, \alpha_1] \}.$$

Then the mappings $\bar{Y}_\alpha : \mathscr{X} \to \bar{U}$ *have the following properties:*

(a) *For any* $\alpha \in [0, \alpha_1]$, $\bar{Y}_\alpha : \mathscr{X} \to \bar{U}$ *is Fréchet differentiable, with* $D\bar{Y}_\alpha : \mathscr{X} \to L(\mathscr{X}, \bar{U})$ *bounded and uniformly continuous*

(b) *For any* B *bounded and open subset of* $\mathscr{X}$, $D\bar{Y}_\alpha(x) \to D\bar{Y}_0(x)$ *as* $\alpha \to 0^+$ *uniformly with respect to* $x \in B$.

Proof. The only difference between the assumptions of the above theorem and those made in [11, Theorem 5.1] lies in part (2). Hence, the proof of part (1) remains the same as that of [11, Theorem 5.1, part (K1)].

Recall that in [11, Theorem 5.1] Mora and Solà Morales required a uniform convergence of $D\bar{T}_\alpha \to D\bar{T}_0$ instead of (2). Nevertheless, they have shown the estimate

$$\begin{aligned} &\| D\bar{Y}_\alpha(x) - D\bar{Y}_0(x) \|_{L(\bar{U}, \bar{U})} \\ &\quad \leqslant \frac{1}{1-\theta} \left\{ \frac{Q}{1-\theta} \| D_u \bar{T}_\alpha(x, Y_\alpha(x)) - D_u \bar{T}_0(x, Y_0(x)) \|_{L(\bar{U}, \bar{U})} \right. \\ &\qquad \left. + \| D_x \bar{T}_\alpha(x, Y_\alpha(x)) - D_x \bar{T}_0(x, Y_0(x)) \|_{L(\mathscr{X}, \bar{U})} \right\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\| D_i \bar{T}_\alpha(x, Y_\alpha(x)) - D_i \bar{T}_0(x, Y_0(x)) \| \\ &\quad \leqslant \| D_i \bar{T}_\alpha(x, Y_\alpha(x)) - D_i \bar{T}_0(x, Y_\alpha(x)) \| + \omega_i(\| Y_\alpha(x) - Y_0(x) \|), \end{aligned}$$

where i stands either for u or for x and ω_i denotes the modulus of continuity of $D_i \bar{T}_0$.

Hence assumption (2) is sufficient for the proof of the local uniform convergence $D\bar{Y}_\alpha \to D\bar{Y}_0$ as stated in (b). □

Henceforth, we will assume that

$$\varrho \in (0, 1-\gamma) \quad \text{and} \quad \lambda_- < \mu < (1+\eta)\mu \leqslant \kappa < \bar{\mu} < \lambda_+. \tag{3.11}$$

In order to apply Theorem 3.6 to fixed points $Y_\alpha(x)$ of the non-linear operator $T_\alpha(x,\cdot)$ defined in (3.3), we choose the following Banach spaces,

$$\mathcal{U} := C^-_{\mu,\varrho,a}(X^\gamma) \quad \text{and} \quad \bar{\mathcal{U}} := C^-_{\bar{\mu},\varrho,a}(X^\gamma),$$

and denote by

$$J : C^-_{\mu,\varrho,a}(X^\gamma) \to C^-_{\bar{\mu},\varrho,a}(X^\gamma)$$

a linear embedding operator. A constant $0 < a \leqslant 1$ will be determined later. Before proving that the family of mappings T_α, $\alpha \in [0, \alpha_1]$, fulfils the assumptions of Theorem 3.6 we need several auxiliary lemmas, each of which is under hypothesis (H) and (3.11). First, we introduce the notation.

In the following, with regard to Lemma 3.2(c), (d), the mappings $\mathscr{K}_\alpha$ and $\mathscr{T}_\alpha$ will be considered as bounded linear operators acting on

$$\mathscr{K}_\alpha : X_{1,0} \to \mathcal{U}, \qquad \mathscr{T}_\alpha : C^-_\mu(X) \to \mathcal{U}.$$

We also denote by

$$\bar{\mathscr{K}}_\alpha : X_{1,0} \to \bar{\mathcal{U}}, \qquad \bar{\mathscr{T}}_\alpha : C^-_{\bar{\mu}}(X) \to \bar{\mathcal{U}}$$

the bounded linear operators analogous to $\mathscr{K}_\alpha$ and $\mathscr{T}_\alpha$, respectively, but operating on exponentially weighted spaces with weight $e^{\bar{\mu}t}$. We recall that the boundedness of $\mathscr{K}_\alpha$, $\mathscr{T}_\alpha$, $\bar{\mathscr{K}}_\alpha$, $\bar{\mathscr{T}}_\alpha$ follows from Lemma 3.2(c), (d). Because Rank $T_\alpha \subseteq \mathcal{U}$ (see Lemma 3.2), we obtain

$$Y_\alpha(x) = T_\alpha(x, Y_\alpha(x)) \in \mathcal{U} \quad \text{for any } x \in X_{1,0} \text{ and } \alpha \in [0, \alpha_1].$$

Moreover, we have the following lemma.

Lemma 3.7. *Let B be a bounded subset of $X_{1,0}$. Then the set*

$$\mathscr{F}_B := \{ Y_\alpha(x) \in U;\ x \in B,\ \alpha \in [0, \alpha_1] \}$$

is a bounded subset of $\mathcal{U}$. At the same time, $\mathscr{F}_B$ is a C^-_μ-uniformly equicontinuous and bounded subset of $C^-_\mu(X^\gamma)$.

Proof. Since $Y_\alpha(x) = T_\alpha(x, Y_\alpha(x)) = \mathscr{K}_\alpha x + \mathscr{T}_\alpha(\mathscr{G}_\alpha(Y_\alpha(x)))$ and $\mathscr{G}_\alpha$ is bounded, the proof follows from Lemma 3.2(c) and (2.3). □

Because of the assumption $f \in C^{1+\eta}_{\mathrm{bdd}}(X^\gamma, Y^\xi)$ we have that the mapping

$$\tilde{f} : C^-_\mu(X^\gamma) \to C^-_\mu(Y^\xi), \quad \tilde{f}(u)(t) := f(u(t)),$$

is bounded and Lipschitz continuous. Recall that the space $\mathcal{U}$ is continuously embedded into $C^-_\mu(X^\gamma)$. Hence the mapping

$$H_\alpha : \mathcal{U} \to C^-_\mu(Y^\beta), \quad H_\alpha(u) := C_\alpha \tilde{f}(u),$$

is bounded and Lipschitz continuous as well. We also denote by $\bar{C}_\alpha$ the linear operator defined in (3.2) and operating from $C_\kappa^-(Y^\xi) \to C_\kappa^-(Y^\beta)$. Define

$$\bar{H}_\alpha := J'H_\alpha : \mathscr{U} \to C_\kappa^-(Y^\beta),$$

where $J' : C_\mu^-(Y^\beta) \to C_\kappa^-(Y^\beta)$ is a linear embedding operator.

Lemma 3.8.
(a) $\bar{H}_\alpha \in C^1_{\mathrm{bdd}}(\mathscr{U}, C_\kappa^-(Y^\beta))$,
(b) *there is an operator* $dH_\alpha : \mathscr{U} \to L(\mathscr{U}, C_\mu^-(Y^\beta))$ *such that* $D\bar{H}_\alpha = J'dH_\alpha$.

Proof. (a) From Lemma 2.8 we have $\tilde{f} \in C^1_{\mathrm{bdd}}(\mathscr{U}, C_\kappa^-(Y^\xi))$ and, by Lemma 3.1, $\bar{C}_\alpha \in L(C_\kappa^-(Y^\xi), C_\kappa^-(Y^\beta))$. Hence $\bar{H}_\alpha \in C^1_{\mathrm{bdd}}(\mathscr{U}, C_\kappa^-(Y^\beta))$ and $D\bar{H}_\alpha = \bar{C}_\alpha D\tilde{f}$.
(b) Since $Df : X^\gamma \to L(X^\gamma, Y^\xi)$ is bounded we obtain that the operator

$$df : \mathscr{U} \to L(\mathscr{U}, C_\mu^-(Y^\xi)), \quad df(u) := Df(u(\cdot)),$$

is well defined and bounded. By Lemmas 2.7 and 2.8, the derivative $D\tilde{f}$ is given by $D\tilde{f} = J'df$. Denote

$$dH_\alpha := C_\alpha df.$$

Then $D\bar{H}_\alpha = \bar{C}_\alpha D\tilde{f} = J'C_\alpha df = J'dH_\alpha$. □

Lemma 3.9. *Let $\mathscr{F}$ be a bounded subset of $\mathscr{U}$. Then $H_\alpha(u) \to H_0(u)$ as $\alpha \to 0^+$ in $C_\mu^-(Y^\beta)$ and $D\bar{H}_\alpha(u) \to D\bar{H}_0(u)$ as $\alpha \to 0^+$ in $L(\mathscr{U}, C_\kappa^-(Y^\beta))$ uniformly with respect to $u \in \mathscr{F}$.*

Proof. (a) Because both f and Df are assumed to be bounded, one can show that the set

$$\mathscr{F}_0 := \{\tilde{f}(u);\ u \in \mathscr{F}\}$$

is a C_μ^--uniformly equicontinuous and bounded subset of $C_\mu^-(Y^\xi)$. By Lemma 3.1, we obtain

$$H_\alpha(u) = C_\alpha \tilde{f}(u) \to C_0 \tilde{f}(u) = H_0(u) \quad \text{as } \alpha \to 0^+$$

in the space $C_\mu^-(Y^\beta)$ uniformly for $u \in \mathscr{F}$.
(b) From (3.11) we have $(\kappa - \mu)/\eta \geqslant \mu$. Since we have assumed that the mapping $X^\gamma \ni u \mapsto Df(u) \in L(X^\gamma, Y^\xi)$ is η-Hölder continuous one can easily verify that the set

$$\mathscr{F}_1 := \{df(u);\ u \in \mathscr{F}\}$$

is a $C_{\kappa-\mu}^-$-uniformly equicontinuous and bounded subset of $C_{\kappa-\mu}^-(L(X^\gamma, Y^\xi))$. Then, by [11, Lemma 5.4(d)] and (2.3), the set

$$\mathscr{F}_2 := \{df(u)h;\ \|h\|_{C^-_{\mu,\varrho,a}} \leqslant 1,\ u \in \mathscr{F}\}$$

is a C_κ^--uniformly equicontinuous and bounded subset of $C_\kappa^-(Y^\xi)$. Again, by Lemma 3.1(b), we obtain

$$D\bar{H}_\alpha(u)h = \bar{C}_\alpha df(u)h \to \bar{C}_0 df(u)h = D\bar{H}_0(u)h \quad \text{as } \alpha \to 0^+$$

uniformly for $\|h\|_{C^-_{\mu,\varrho,a}} \leqslant 1$ and $u \in \mathscr{F}$. □

It follows from Lemma 2.7 and (H3) that

$$\tilde{g} \in C^1_{\mathrm{bdd}}(C_i^-(X^\gamma) \times C_i^-(Y^\beta), C_{\bar{\mu}}^-(X)), \tag{3.12}$$

where i stands either for μ or for κ. Define the operators

$$d_u g : C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta) \to L(C_\mu^-(X^\gamma), C_\mu^-(X))$$

and

$$d_w g : C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta) \to L(C_\mu^-(Y^\beta), C_\mu^-(X))$$

as follows:

$$d_u g(u, w) := D_u g(u(\cdot), w(\cdot)) \quad \text{and} \quad d_w g(u, w) := D_w g(u(\cdot), w(\cdot)).$$

As Dg is bounded, the mappings $d_u g$ and $d_w g$ are bounded as well. Further, the derivative $D\tilde{g}$ when restricted to $C_\mu^-(X^\gamma) \times C_\mu^-(Y^\beta)$ can be expressed as

$$D_u \tilde{g} = J'' d_u g, \qquad D_w \tilde{g} = J'' d_w g, \tag{3.13}$$

where $J'': C_\mu^-(X) \to C_{\bar{\mu}}^-(X)$ is a linear embedding operator.

Lemma 3.10. *Let $\mathscr{F}$ be a bounded subset of $\mathscr{U}$. Then*

(a) $\bar{\mathscr{G}}_\alpha \in C^1_{\mathrm{bdd}}(\mathscr{U}, C_{\bar{\mu}}^-(X))$, *where* $\bar{\mathscr{G}}_\alpha := J''\mathscr{G}_\alpha$,

(b) *there is a mapping* $d\mathscr{G}_\alpha : \mathscr{U} \to L(\mathscr{U}, C_\mu^-(X))$ *such that* $D\bar{\mathscr{G}}_\alpha = J'' d\mathscr{G}_\alpha$,

(c) $\mathscr{G}_\alpha(u) \to \mathscr{G}_0(u)$ *in* $C_\mu^-(X)$ *and* $D\bar{\mathscr{G}}_\alpha(u) \to D\bar{\mathscr{G}}_0(u)$ *as* $\alpha \to 0^+$ *in* $L(\mathscr{U}, C_{\bar{\mu}}^-(X))$ *uniformly with respect to* $u \in \mathscr{F}$.

Proof. The proof of statement (a) follows from Lemma 3.8 and (3.12). Let us define $d\mathscr{G}_\alpha$ as follows:

$$d\mathscr{G}_\alpha := d_u g + d_w g \, dH_\alpha.$$

By Lemmas 2.7 and 3.8 and (3.13),

$$D\bar{\mathscr{G}}_\alpha = D_u \tilde{g} + D_w \tilde{g} D\bar{H}_\alpha = J'' d_u g + J'' d_w g \, dH_\alpha = J'' d\mathscr{G}_\alpha.$$

Since $\mathscr{G}_\alpha(u) = g(u(\cdot), C_\alpha \tilde{f}(u)(\cdot)) = g(u(\cdot), H_\alpha(u)(\cdot))$, the first part of statement (c) follows from Lemma 3.9. As $D\tilde{g}_w$ is bounded, the second part is a consequence of Lemmas 2.7, 2.8 and 3.9. □

Now we can apply Theorem 3.6 to the family of non-linear operators $\{T_\alpha\}$, introduced in section 3.1.

Since the mapping $\mathscr{G}_\alpha : \mathscr{U} \to C_\mu^-(X)$ is Lipschitz continuous, its Lipschitz constant being estimated by the right-hand side of (3.5), using Lemma 3.2(c), (d), we obtain that the family $T_\alpha(x, \cdot)$ satisfies hypotheses T_1 and T_2 in the Banach space $\mathscr{U}$ with the constants

$$\bar{\theta} := 2\theta \quad \text{and} \quad \bar{Q} := 2C\lambda^{\gamma}_{-}, \tag{3.14}$$

where the constant $\theta > 0$ was defined in (3.6). Furthermore, according to Lemmas 3.2 and 3.10, assumption $(\mathrm{T})_3$ is also fulfilled. Let us define the operators

$$d_u T_\alpha : X_{1,0} \times \mathscr{U} \to L(\mathscr{U}, \mathscr{U}), \quad d_x T_\alpha : X_{1,0} \times \mathscr{U} \to L(X_{1,0}, \mathscr{U}),$$

$$\bar{d}_u T_\alpha : X_{1,0} \times \mathscr{U} \to L(\bar{\mathscr{U}}, \bar{\mathscr{U}})$$

as follows:

$$d_u T_\alpha(x, u) := \mathscr{T}_\alpha d\mathscr{G}_\alpha(u), \quad d_x T_\alpha(x, u) := \mathscr{K}_\alpha, \quad \bar{d}_u T_\alpha(x, u) := \bar{\mathscr{T}}_\alpha \bar{d}\mathscr{G}_\alpha(u),$$

where $\bar{d}\mathscr{G}_\alpha$ is defined in the same way as the operator $d\mathscr{G}_\alpha$ but operating from $C_{\mu,\varrho,a}^-(X^\gamma)$ to $L(C_{\bar{\mu},\varrho,a}^-(X^\gamma), C_{\bar{\mu}}^-(X))$. Denote

$$\bar{T}_\alpha := J T_\alpha : X_{1,0} \times \mathscr{U} \to \bar{\mathscr{U}} \quad \text{and} \quad \bar{Y}_\alpha := J Y_\alpha.$$

By Lemma 3.10, $\bar{T}_\alpha \in C^1_{\mathrm{bdd}}(X_{1,0} \times \mathscr{U}, \bar{\mathscr{U}})$. Moreover, from Lemmas 3.2 and 3.7 we obtain $D\bar{T}_\alpha(x, u) \to D\bar{T}_0(x, u)$ as $\alpha \to 0^+$ uniformly with respect to $(x, u) \in X_{1,0} \times \mathscr{F}_B$ for every B bounded and open subset of $X_{1,0}$.

In this way we have shown that the family $T_\alpha(x, \cdot)$ and the mappings $d_u T_\alpha$, $d_x T_\alpha$, $\bar{d}_u T_\alpha$ satisfy the assumptions of Theorem 3.6, provided that the constant $\bar{\theta}$ defined by (3.14) is less than 1. In the case when $\bar{\theta} < 1$, by Theorem 3.6 we obtain

$$\bar{Y}_\alpha \to \bar{Y}_0 \quad \text{as} \quad \alpha \to 0^+ \quad \text{in} \quad C^1_{\mathrm{bdd}}(B, C^-_{\bar{\mu}, \varrho, a}(X^\gamma)) \tag{3.15}$$

for any $B \subset X_{1,0}$ bounded and open.

Recall that

$$(\Phi_\alpha(x), \Psi_\alpha(x)) := (\bar{Y}_\alpha(x)(0), \tilde{H}_\alpha(\bar{Y}_\alpha(x))(0)),$$

where $\tilde{H}_\alpha$ is now considered as a C^1 mapping from $C^-_{\bar{\mu}, \varrho, a}(X^\gamma)$ into $C^-_v(Y^\beta)$ for some $v > \bar{\mu}$. In view of Lemma 3.9, statement (3.15) readily implies a C^1-local uniform convergence of $(\Phi_\alpha, \Psi_\alpha)$ towards (Φ_0, Ψ_0) as stated in (3.10).

In accordance with Lemma 3.3, we remind ourselves that the assumption $\theta < 1$ is sufficient for the existence of a family of Lipschitz continuous invariant manifolds $\mathscr{M}_\alpha$ for the semiflows $\mathscr{S}_\alpha$, $\alpha \in [0, \alpha_1]$. On the other hand, the assumption $\bar{\theta} = 2\theta < 1$ guarantees a 'C^1-closeness' of $\mathscr{M}_\alpha$ and $\mathscr{M}_0$ which can be precisely expressed by (3.10). Clearly, one way of ensuring the condition $\bar{\theta} < 1$ is to require smallness of the constant $K > 0$.

Having developed the previous background we can state the main result of this paper.

Theorem 3.11. *Assume that hypothesis* (H) *holds. Then there are constants* $\tau > 0$ *and* $\alpha_1 > 0$ *such that, if* $K(\lambda_-, \lambda_+, \mu, \gamma) < \tau$ *then, for every* $\alpha \in (0, \alpha_1]$, *the following hold*:

(a) *There exists an invariant manifold* $\mathscr{M}_\alpha$ $(\tilde{\mathscr{M}}_0)$ *for the semiflow* $\mathscr{S}_\alpha$ $(\tilde{\mathscr{S}}_0)$ *generated by system* $(1.1)_\alpha$. *Moreover,* $\dim \mathscr{M}_\alpha = \dim \mathscr{M}_0 < \infty$ *and* $\mathscr{M}_\alpha$ $(\tilde{\mathscr{M}}_0)$ *is the graph of a* C^1 *continuous mapping* $X_{1,0} \ni x \mapsto (\Phi_\alpha(x), \Psi_\alpha(x)) \in X^\gamma \times Y^\beta$ $(X_{1,0} \ni x \mapsto \Phi_0(x) \in X^\gamma)$.

(b) *For any bounded and open subset* $B \subset X_{1,0}$,

$$(\Phi_\alpha, \Psi_\alpha) \to (\Phi_0, \Psi_0) \text{ as } \alpha \to 0^+ \text{ in } C^1_{\mathrm{bdd}}(B, X^\gamma \times Y^\beta).$$

Remark 3.1.2. In addition to hypothesis (H) we also assume that A_0 is a self-adjoint operator with eigenvalues

$$0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n < \lambda_{n+1} \leqslant \cdots \quad \lambda_n \to \infty \text{ as } n \to \infty.$$

As is usual in such a case, we will let $\lambda_- := \lambda_n$, $\lambda_+ := \lambda_{n+1}$ and $\mu := (\lambda_+ + \lambda_-)/2$. With this setting it should be obvious that the condition 'K is small' reduces to the requirement '$\lambda_n^\gamma(\lambda_{n+1} - \lambda_n)^{-1}$ is small enough'. In the case when $\lambda_m \approx m^2$, $m \in N$, the assumptions of Theorem 3.11 are satisfied, whenever $\gamma \in [0, 1/2)$ and n is large enough.

4. An application

We will consider the second-order abstract evolution equations of the form

$$\begin{aligned} &\alpha u'' + A^\kappa u' + Au = f(u), \\ &u(0) = u_0, \qquad u'(0) = v_0, \end{aligned} \tag{4.1$_\alpha$}$$

where A (the elastic operator) is a self-adjoint positive operator in a real Hilbert space $\mathscr{X}$, $\kappa \in [1/2, 1)$, $\alpha \geqslant 0$ and $f: \mathscr{X}^{\varrho} \to \mathscr{X}$ is a non-linear function for some $\varrho \in [\kappa, 1)$. The operator A^{κ} may represent dissipation in elastic systems (cf. [3]).

In recent years, many authors have studied problems having the general form (4.1) (see e.g. [3, 4] and other references therein). As a motivation for studying systems like (4.1) one can consider some specific beam equations with damping, e.g.

$$\begin{aligned} &u_{tt} - \beta \Delta u_t + \Delta^2 u = m\left(\int_{\Omega} |\nabla u|^2 \right) \Delta u, \\ &u = \Delta u = 0 \quad \text{on} \quad \partial\Omega,\ u(0, x) = u_0(x),\ \beta u_t(0, x) = v_0(x),\ x \in \Omega, \end{aligned} \tag{4.2}$$

where $\Omega \subset R^N$ is a smoothly bounded domain, $\beta > 0$ is a damping coefficient and $m: R^+ \to R$ is a non-decreasing differentiable function measuring non-local character of structural damping of a beam or string (see, for instance [1, 2, 13]). If we let $\mathscr{X} := L_2(\Omega)$, $Au := \Delta^2 u$, $D(A) = \{W^{4,2}(\Omega), u = \Delta u = 0 \text{ on } \partial\Omega\}$ then problem (4.2) can be rewritten abstractly as problem (4.1) with $\varrho = \kappa = 1/2$. After a suitable rescaling time ($\tau := t/\beta$) one obtains $\alpha = 1/\beta^2$ and the singular limit $\alpha \to 0^+$ corresponds to the situation when β tends to infinity (see [14]).

Throughout this section we will assume the following hypothesis:

$$(\mathrm{E}) \begin{cases} A: D_{\mathscr{X}}(A) \subseteq \mathscr{X} \to \mathscr{X} \text{ is a self-adjoint positive unbounded operator in} \\ \text{a real Hilbert space } \mathscr{X}. \text{ The resolvent } A^{-1} \text{ is a compact operator on } \mathscr{X}. \\ \kappa \in [1/2, 1),\ \alpha \geqslant 0. \\ f \in C^{1+\eta}_{\mathrm{bdd}}(\mathscr{X}^{\varrho}, \mathscr{X}) \text{ for some } \varrho \in [\kappa, 1) \text{ and } \eta \in (0, 1]. \end{cases}$$

We recall that an operator A satisfying hypothesis (E) has the spectrum consisting of eigenvalues

$$\sigma(A) = \{\lambda_n; n \in N\}, \quad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_n \to \infty \text{ as } n \to \infty.$$

We denote by ϕ_n the eigenvector of A correponding to λ_n, $n \in N$. According to [8, chapter 1] A is a sectorial operator in $\mathscr{X}$ and the fractional powers of A and $\mathscr{X}$ can be characterized as

$$\begin{aligned} &\mathscr{X}^{\xi} = D_{\mathscr{X}}(A^{\xi}) = \left\{ u \in \mathscr{X};\ \sum_{n=1}^{\infty} \lambda_n^{2\xi} (u, \phi_n)^2 < \infty \right\}, \\ &\|u\|_{\xi}^2 = \|A^{\xi} u\|^2 = \sum_{n=1}^{\infty} \lambda_n^{2\xi} (u, \phi_n)^2. \end{aligned} \tag{4.3}$$

Knowing the above spectral decompositions, one can readily show that, for any $r, s \geqslant 0$, the operator A^r is a self-adjoint positive operator in the Hilbert space $X := \mathscr{X}^s$, its domain being $D_X(A^r) := D_{\mathscr{X}}(A^{r+s})$. The fractional power space X^{γ}, $\gamma \in [0, 1]$, subject to the sectorial operator A^r consists of the domain $D_{\mathscr{X}}(A^{s+\gamma r})$ and the norm on X^{γ} is given by $\|u\|_{X^{\gamma}} = \|A^{s+\gamma r} u\|_{\mathscr{X}}$ for any $u \in X^{\gamma}$. Moreover, $\sigma(A^r) = \{\lambda_n^r; n \in N\}$.

(4.4)

Now we return to system (4.1). We use a change of variables in such a way that the resulting system fits into the abstract setting investigated in section 3. To do so, we let

X, Y denote the real Hilbert spaces

$$X := [D_{\mathscr{X}}(A^{(1-\omega)\kappa})]_{\mathscr{X}} = \mathscr{X}^{(1-\omega)\kappa}, \qquad Y := \mathscr{X},$$

where

$$\omega \in (0, (1-\varrho)/\kappa)$$

is fixed. Let the linear operators A_α, B_α, $\alpha \in [0, \alpha_0]$, in X and Y, respectively, be defined as follows:

$$A_\alpha := \frac{A^\kappa}{2\alpha}(1 - (1 - 4\alpha A^{1-2\kappa})^{1/2}), \qquad B_\alpha := \frac{1}{2}(1 + (1 - 4\alpha A^{1-2\kappa})^{1/2})A^\kappa$$

for $\alpha \in (0, \alpha_0]$, $\alpha_0 > 0$ small, and

$$A_0 := A^{1-\kappa}, \qquad B_0 := A^\kappa,$$

their domains being

$$D_X(A_\alpha) := D_X(A^{1-\kappa}), \qquad D_Y(B_\alpha) := D_Y(A^\kappa) \quad \text{for any } \alpha \in [0, \alpha_0].$$

Since A_0 and B_0 are self-adjoint positive operators in $X$and Y, respectively, with regard to [8, chapter 1], we have that they are sectorial ones. Notice that A_α, $\alpha \in (0, \alpha_0]$, is well defined. Indeed, using (4.3) we obtain

$$\frac{A^{2\kappa-1}}{2\alpha}(1 - (1 - 4\alpha A^{1-2\kappa})^{1/2}) \in L(X, X)$$

for $\alpha \in (0, \alpha_0]$, and hence

$$A_\alpha = \frac{A^{2\kappa-1}}{2\alpha}(1 - (1 - 4\alpha A^{1-2\kappa})^{1/2})A^{1-\kappa}.$$

Furthermore,

$$A_\alpha^{-1} = \frac{A^{\kappa-1}}{2}(1 + (1 - 4\alpha A^{1-2\kappa})^{1/2}).$$

Therefore, $A_0 A_\alpha^{-1} \to I$ in $L(X, X)$. Similarly, $B_\alpha B_0^{-1} \to I$ in $L(Y, Y)$. Hence the families of operators $\{A_\alpha, \alpha \in [0, \alpha_0]\}$ and $\{B_\alpha, \alpha \in [0, \alpha_0]\}$ fulfil hypotheses (H1)–(H2) and (H1) on the Hilbert spaces X and Y, respectively.

In terms of A_α and B_α, system (4.1) can be rewritten as a system of two abstract equations:

$$\begin{aligned} &u' + A_\alpha u = w, \\ &\alpha w' + B_\alpha w = f(u), \quad \alpha \in [0, \alpha_1], \\ &u(0) = u_0, \quad w(0) = w_0 \end{aligned} \tag{4.5$_\alpha$}$$

in the space $X \times Y$. Let us take

$$\gamma := \frac{\varrho - (1-\omega)\kappa}{1-\kappa} \quad \text{and} \quad \beta := 1 - \omega.$$

Then $\gamma, \beta \in (0, 1)$ and the functions

$$g : X^\gamma \times Y^\beta \to X, \quad g(u, w) := w,$$

$$f : X^\gamma \to Y$$

satisfy hypothesis (H3) from section 9 (here X^γ, Y^β denote the fractional power spaces with respect to sectorial operators $A_0 = A^{1-\kappa}$, $B_0 = A^\kappa$, respectively). Indeed, taking (4.4) into account, we obtain

$$X^\gamma = D_{\mathscr{X}}(A^{(1-\omega)\kappa+\gamma(1-\kappa)}) = D_{\mathscr{X}}(A^\varrho) = \mathscr{X}^\varrho,$$

$$Y^\beta = D_{\mathscr{X}}(A^{\beta\kappa}) = D_{\mathscr{X}}(A^{(1-\omega)\kappa}) = X.$$

Hence, $g \in L(X^\gamma \times Y^\beta, X)$ and $f \in C^{1+\eta}_{\mathrm{bdd}}(X^\gamma, Y)$.

Having developed this background we can apply Theorem 3.11 to semiflows generated by systems $(4.5)_\alpha$, $\alpha \in [0, \alpha_0]$, in the phase space $X^\gamma \times Y^\beta$. With regard to Remark 3.12 and (4.4), the assumptions of Theorem 3.11 are fulfilled whenever $\inf_{n\in N} \lambda_n^{(1-\kappa)\gamma}/(\lambda_{n+1}^{1-\kappa} - \lambda_n^{1-\kappa}) = 0$. Because $(1-\kappa)\gamma = \varrho - \kappa + \omega\kappa =: \delta$ and $\omega \in (0, (1-\varrho)/\kappa)$, the last condition becomes

$$\inf_{n\in N} \frac{\lambda_n^\delta}{\lambda_{n+1}^{1-\kappa} - \lambda_n^{1-\kappa}} = 0 \quad \text{for some } \delta \in (\varrho - \kappa, 1-\kappa). \tag{4.6}$$

Theorem 4.1. *Assume that hypothesis* (E) *and* (4.6) *are satisfied. Let* $\gamma := \delta/(1-\kappa)$ *and* $\beta := (\delta - \varrho)/\kappa$. *Then the conclusions of Theorem* 3.11 *hold for semiflows generated by systems* $(4.5)_\alpha$, $\alpha \in [0, \alpha_0]$, $\alpha_0 > 0$ *small enough, in the phase space* $X^\gamma \times Y^\beta$.

Remark 4.2. Let us consider system (4.2). It is known [13] that there exists a compact global attractor for the semiflow generated by (4.2). Then one can smoothly modify the function $f(u) := m(\|\nabla u\|^2)\Delta u$ far from a neighbourhood of an attractor. Hence the assumption $f \in C^{1+\eta}_{\mathrm{bdd}}(X^\varrho, Y)$ is not restrictive when we deal with local invariant manifolds instead of global ones. By classical spectral results (see e.g. [6]), it follows that $\lambda_n \approx n^{4/N^2}$, where λ_n, $n \in N$, are eigenvalues of the self-adjoint operator $A := \Delta^2$ subject to 'hinged ends' boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. In system (4.2) we have $\kappa = \varrho = 1/2$. Hence condition (4.6) is satisfied whenever $N = 1$ and $\delta \in (0, 1/4)$.

Remark 4.3. Theorem 4.1 remains true for the case when the fractional power operator A^κ is replaced by a general self-adjoint linear operator B which commutes with A and is comparable with A^κ (cf. [4]), i.e. there are constants $a, b > 0$ such that

$$a(A^\kappa u, u) \leqslant (Bu, u) \leqslant b(A^\kappa u, u) \quad \text{for any } u \in D(A^\kappa) = D(B).$$

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