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# FREE NON-DISTRIBUTIVE MORGAN-STONE ALGEBRAS

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Abstract. In this paper we investigate free non-distributive Morgan-Stone algebras. We construct the free non-distributive Morgan-Stone algebra as a free lattice generated by a suitable partially ordered set endowed by a unary operation of involution. A positive answer to the word problem is also proven.

## 1. Introduction

In [**2**] Blyth and Varlet have studied a new variety of so-called Morgan-Stone algebras as a common abstraction of the well known classes of De Morgan and Stone algebras. Such algebras are bounded distributive lattices with a unary operation of involution fulfilling certain identities.

The aim of this note is to investigate a larger variety of algebras containing, in particular, Morgan-Stone algebras. In such algebras the distributive identity need not be necessarily satisfied. We are mainly concerned with the construction of free non-distributive Morgan-Stone algebras. The idea of construction is based on the concept of a free lattice generated by a partially ordered set $P$ and preserving bounds prescribed by chosen subsets of $P$ due to Dean [**3**]. We then analyze the word problem for the varieties under consideration. We show that there is an algorithm for deciding when two words in a free algebra are equal.

The approach to the construction of free algebras was significantly influenced by the work of Katriňák. In [**5**] he has treated a similar task for the class of p-algebras. Based on the characterization of a free p-algebra Katriňák and the author were able to characterize projective p-algebras [**6**] as well as bounded endomorphisms of free p-algebras [**7**]. It is hoped that an analogous technique can be also applied in the study of projective non-distributive Morgan-Stone algebras.

The outline of the paper is as follows. In Section 2 we recall definitions of De Morgan and Morgan-Stone algebras. New varieties of non-distributive De Morgan and Morgan-Stone algebras are introduced. We also present some of results due to Dean [**3**] regarding free lattices generated by partially ordered sets. Section 3 is focused on the construction of a free non-distributive De Morgan algebra. In Section 4 we construct a free non-distributive Morgan-Stone algebra and give the positive answer to the word problem in this variety. Finally, some examples showing free non-distributive De Morgan and Morgan-Stone algebras with a simple generator are also presented.

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## 2. Preliminaries

We start by recalling definitions of De Morgan algebras, Stone algebras and Morgan-Stone algebras. By *De Morgan algebra* we understand a universal algebra $(M, \vee, \wedge, {}^-, 0, 1)$ where $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation of involution satisfies the identities: $x = x^{--}$, $(x \wedge y)^- = x^- \vee y^-$, $1^- = 0$. *Stone algebra* is a universal algebra $(S, \vee, \wedge, *, 0, 1)$ where $(S, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation of complementation satisfies: $x \wedge x^* = 0$, $(x \wedge y)^* = x^* \vee y^*$, $0^* = 1$. Finally, *Morgan-Stone algebra* (or MS algebra) is a universal algebra $(M, \vee, \wedge, {}^\circ, 0, 1)$ where $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation of involution satisfies: $x \le x^{\circ\circ}$, $(x \wedge y)^\circ = x^\circ \vee y^\circ$, $1^\circ = 0$. We refer to a book by Balbes and Dwinger [**1**] for a broader discussion regarding De Morgan and Stone Algebras.

Now we introduce two new varieties of so called generalized De Morgan and Morgan-Stone algebras in such a way that all the identities for the unary operation of complementation (involution) are preserved. We will consider a larger equational class of algebras satisfying all the above identities of Morgan-Stone algebras lattice skeletons of which are not assumed to be distributive lattices.

**Definition 1.** *A generalized De Morgan algebra* (or GM – algebra) is a universal algebra $(M, \vee, \wedge, {}^-, 0, 1)$ where $(M, \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation of involution satisfies the identities:

$$\mathrm{GM}_1 : x = x^{--}, \quad \mathrm{GM}_2 : (x \wedge y)^- = x^- \vee y^-, \quad \mathrm{GM}_3 : 1^- = 0.$$

**Definition 2.** *A generalized Morgan-Stone algebra* (or GMS – algebra) is a universal algebra $(M, \vee, \wedge, {}^\circ, 0, 1)$ where $(M, \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation of involution satisfies the identities:

$$\mathrm{GMS}_1 : x \le x^{\circ\circ}, \quad \mathrm{GMS}_2 : (x \wedge y)^\circ = x^\circ \vee y^\circ, \quad \mathrm{GMS}_3 : 1^\circ = 0.$$

Let $L$ be a GMS – algebra. We define the skeleton (the set of closed elements) $S(L)$ of $L$ as follows: $S(L) := \{x \in L, x = x^{\circ\circ}\}$. One can easily verify that the set $S(L)$ endowed with induced operations from $L$ is a GM – algebra. More precisely,

**Lemma 1.** *Let* $(L, \vee, \wedge, {}^\circ, 0, 1)$ *be a GMS – algebra. Then the skeleton* $\big(S(L), \vee, \wedge, {}^\circ, 0, 1\big)$ *is a GM – algebra.*

Throughout the paper the following simple rules for computation in GMS – algebras will be frequently used:

$$\text{if} \quad x \le y \quad \text{then} \quad y^\circ \le x^\circ, \;\; x^{\circ\circ\circ} = x^\circ, \;\; (x \vee y)^\circ = x^\circ \wedge y^\circ, \;\; 0^\circ = 1.$$

The construction of free GM – as well as GMS – algebras is based on the well known characterization of free lattices generated by partially ordered sets and preserving bounds due to Dean [**3**]. Let us therefore summarize his results.

Let $P$ be a partially ordered set (poset) with an order relation $\prec$. Let $\mathcal{U}, \mathcal{L}$ be families of subsets of $P$ such that

if $p \prec q$, $p, q \in P$ then $\{p, q\} \in \mathcal{U}$ and $\{p, q\} \in \mathcal{L}$

if $S \in \mathcal{U}$ ($S \in \mathcal{L}$) then there is $\sup_P S$ ($\inf_P S$) in the poset $P$.

According to [**3**, Theorem 6] there exists a free lattice $FL(P, \mathcal{U}, \mathcal{L})$ generated by $P$ and preserving bounds prescribed by the sets from $\mathcal{U}$ and $\mathcal{L}$. We also recall that, by [**3**, Theorem 10], the word problem in $FL(P, \mathcal{U}, \mathcal{L})$ has an affirmative solution if there is an affirmative solution to the problem of determining whether two ideals of $P$ of the form $M(a) = \{p \in P, p \geq a\}$, $J(b) = \{p \in P, p \leq b\}$ have a common element. More precisely, in the free lattice $FL(P, \mathcal{U}, \mathcal{L})$ $a \leq b$ if and only if one or more of the following hold:

$$(W) \qquad \begin{array}{lllll} a \equiv a_1 \vee a_2 & \text{and} & a_i \leq b & \text{for } i = 1 & \text{and } i = 2, \\ a \equiv a_1 \wedge a_2 & \text{and} & a_i \leq b & \text{for } i = 1 & \text{or } i = 2, \\ b \equiv b_1 \vee b_2 & \text{and} & a \leq b_i & \text{for } i = 1 & \text{or } i = 2, \\ b \equiv b_1 \wedge b_2 & \text{and} & a \leq b_i & \text{for } i = 1 & \text{and } i = 2, \\ \multicolumn{5}{l}{\text{there is a } p \in P \text{ such that } a \leq p \leq b,} \end{array}$$

(c.f. [**3**, Theorem 7]). With regard to [**3**, Definition 2] the order relation $p \leq b$ for $p \in P$ means $p \leq b(k)$ for some integer $k \geq 0$ where

$$(J) \qquad \begin{array}{ll} p \leq b(0) & \text{iff } b \equiv q \in P \text{ and } p \prec q \text{ in } P; \\ p \leq b(k) & \text{iff either } b \equiv b_1 \vee b_2 \text{ and } p \leq b_i(k-1) \text{ for } i = 1 \text{ or } i = 2, \\ & \text{or } b \equiv b_1 \wedge b_2 \text{ and } p \leq b_i(k-1) \text{ for } i = 1 \text{ and } i = 2, \\ & \text{or there is a } S \in \mathcal{U} \text{ such that } p \prec \sup\limits_P S \text{ and} \\ & \qquad s \leq b(k-1) \text{ for all } s \in S. \end{array}$$

Analogously, $a \leq p$ for $p \in P$ means $a \leq p(k)$ for some integer $k \geq 0$ where

$$(M) \qquad \begin{array}{ll} a \leq p(0) & \text{iff } a \equiv q \in P \text{ and } q \prec p \text{ in } P; \\ a \leq p(k) & \text{iff either } a \equiv a_1 \vee a_2 \text{ and } a_i \leq p(k-1) \text{ for } i = 1 \text{ and } i = 2, \\ & \text{or } a \equiv a_1 \wedge a_2 \text{ and } a_i \leq p(k-1) \text{ for } i = 1 \text{ or } i = 2, \\ & \text{or there is a } S \in \mathcal{L} \text{ such that } \inf\limits_P S \prec p \text{ and} \\ & \qquad a \leq s(k-1) \text{ for all } s \in S. \end{array}$$

Suppose that $p \prec q$ in $P$. With regard to the above definition of the ordering in the free lattice $FL$ we observe that if $q \leq b(k)$ then $p \leq b(k)$ also. Similarly, if $a \leq p(k)$ then $a \leq q(k)$. The proof utilizes an induction argument with respect to $k \geq 0$.

Let $L$ be a $K$ – algebra in the variety $K$. By $[X]_K$ we denote the $K$ – subalgebra of $L$ generated by a subset $X \subseteq L$. As usual, by $[a, b]$ we denote the interval $[a, b] := \{c, a \leq c \leq b\}$.

### 3. Free Generalized De Morgan Algebras

In this section we study free algebras in the variety of all generalized De Morgan algebras. We begin with a general result due to Katriňák regarding free algebras in a variety of $K$ algebras.

**Lemma 2.** ([**5**, Lemma 1]) *Let $K$ be a class of algebras, $X$ any set, and $F_K(X)$ the free algebra in $K$ freely generated by the set $X$. Suppose $A \in K$ is also generated by $X$ and there exists a $K$-homomorphism $h : A \to F_K(X)$ which is the identity function on $X$. Then $h$ is an isomorphism.*

Let $L$ be a GM – algebra and $X \subset L$. We denote $X^-$ the set $X^- := \{x^-, x \in X\}$. By a straightforward induction on the rank of a GM – term $p$ one can easily prove that for any $a_1, a_2, \dots, a_n \in X$ there exist $b_1, b_2, \dots, b_m \in X \cup X^-$ and a lattice term $q$ such that $p(a_1, a_2, \dots, a_n) = q(b_1, b_2, \dots, b_m)$. In other words, we have –

**Lemma 3.** *If a GM – algebra $L$ is generated by the set $X$, i.e. $L = [X]_{GM}$, then the set $X \cup X^-$ generates $L$ in the variety $BL$ of bounded lattices, i.e. $L = [X \cup X^-]_{BL}$.*

Now we are in a position to define a poset $P = P_{GM}(X)$ and two families $\mathcal{U}_{GM}, \mathcal{L}_{GM}$ of subsets of $P_{GM}(X)$ in such a way that the free lattice generated by the poset $P_{GM}(X)$ and preserving bounds from $\mathcal{U}_{GM}, \mathcal{L}_{GM}$ will admit a unary operation of involution with the property that the resulting algebra is free in the category of GM – algebras.

Let $X$ be a set. Let $\widetilde{X}$ be a disjoint copy of $X$, i.e. $\widetilde{X} = \{\widetilde{x}, x \in X\}$ and $X \cap \widetilde{X} \cap \{0,1\} = \varnothing$. Define the set $P_{GM}(X) = X \cup \widetilde{X} \cup \{0,1\}$ and the ordering $\prec$ on $P_{GM}(X)$ as follows: $0 \prec x \prec 1$, $0 \prec \widetilde{x} \prec 1$ for any $x, \widetilde{x} \in P_{GM}(X)$. The families $\mathcal{U}_{GM}, \mathcal{L}_{GM}$ are defined as $\mathcal{U}_{GM} = \mathcal{L}_{GM} = \{\{p,q\} \subset P_{GM}(X),$ $p \prec q$ in $P_{GM}(X)\}$. Then there is a free lattice $FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ generated by the poset $P_{GM}(X)$. In what follows, we will show that there is an operation of involution $^-$ with the property that the free lattice $FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ endowed with such a unary operation is a free GM – algebra. To this end, we first introduce the mapping $\theta : FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM}) \to FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ defined on the set of generators as follows:

$$\theta(x) = \widetilde{x}, \quad \theta(\widetilde{x}) = x, \quad \theta(0) = 1, \quad \theta(1) = 0. \tag{3.1}$$

This mapping extends to a dual endomorphism of $FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ preserving bounds prescribed by sets from $\mathcal{U}_{GM}, \mathcal{L}_{GM}$ ([**3**, Theorem 6]).

Let $p$ be any lattice term. By $\overline{p}$ we denote a lattice term which is obtained from $p$ by replacing all the symbols $\wedge$ by $\vee$, $\vee$ by $\wedge$, 0 by 1 and 1 by 0.

Using the properties $(W)$, $(J)$ and $(M)$ of $FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ and recalling that $S = \{p,q\}$, $p \prec q$ in $P_{GM}(X)$ for any $S \in \mathcal{U}_{GM} = \mathcal{L}_{GM}$, we obtain by a straightforward induction on the rank of lattice terms $p, q$ : $p(a_1, a_2, \dots, a_n) \le q(a_1, a_2, \dots, a_n)$ in $FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$, where $a_1, a_2, ..., a_n \in P_{GM}(X)$, implies $\overline{p}(\theta a_1, \theta a_2, \dots, \theta a_n) \ge \overline{q}(\theta a_1, \theta a_2, \dots, \theta a_n)$.

Now, it should be obvious that the mapping $\theta : FL \to FL$ satisfies the identities $\theta(a \wedge b) = \theta(a) \vee \theta(b)$, $\theta(a \vee b) = \theta(a) \wedge \theta(b)$ and $\theta(\theta(a)) = a$ for any $a, b \in FL$.

Let us denote

$$a^- := \theta(a). \tag{3.2}$$

The lattice $FL \equiv FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ endowed with such a unary operation is a GM – algebra.

**Theorem 4.** *Let $X$ be any set. Then the free lattice $FL \equiv FL(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM})$ endowed with the unary operation $^-$ defined in (3.2) is a free GM – algebra, i.e. $F_{GM}(X) \cong (FL, ^-)$.*

**Proof.** Note that $FL = [X]_{GM} = [X \cup \widetilde{X}]_{BL}$. Let us define the mapping $h : P_{GM}(X) \to F_{GM}(X)$ as follows: $h(x) := x$, $h(\widetilde{x}) := x^-$, $h(0) = 0$, $h(1) = 1$. According to [**3**, Theorem 6] the mapping $h$ can be extended to a homomorphism $h : FL \to F_{GM}(X)$. By an induction on the rank of a lattice term $a \in FL$ we will show that $h(a^-) = h(a)^-$. If $a \in P_{GM}(X)$ the statement is obvious. If $a \equiv a_1 \wedge a_2$ or $a \equiv a_1 \vee a_2$ then the statement follows from the induction hypothesis made on terms $a_1, a_2$ and the properties of the mapping $\theta$. Hence $h : FL \to F_{GM}(X)$ is a GM – homomorphism which is an identity function on $X$. According to Lemma 2 $h$ is an isomorphism and the proof of theorem follows. ∎

We end this section by proving that the word problem in $F_{GM}(X)$ has an affirmative solution.

**Lemma 5.** *Let $F_{GM}(X)$ be a free GM – algebra. Then any element $p \in X \cup X^- \cup \{0, 1\}$ is join and meet irreducible.*

**Proof.** With regard to Theorem 4 it is sufficient to show that any $p \in P_{GM}(X)$ is join and meet irreducible in the lattice $FL$. We will proceed by an induction on the rank. According to $(J)$ $p \le a_1 \vee a_2$, $p \in P_{GM}(X)$, $a_1, a_2 \in FL$ iff $p \le a_1 \vee a_2(k)$ for some integer $k \ge 0$. This means that either $p \le a_i(k-1)$ $i = 1$ or $i = 2$, or there is $S \in \mathcal{U}_{GM}$, $S = \{q_1, q_2\}$, $q_1 \prec q_2$ such that $p \prec \sup_P S = q_2$ and $q_2 \le a_1 \vee a_2(k-1)$. The latter implies $p \le a_1 \vee a_2(k-1)$ and so one can again decrease the rank $(k-1)$ by one. Since $p \le a_1 \vee a_2(0)$ is impossible we may conclude that either $p \le a_1$ or $p \le a_2$. Hence $p$ is join irreducible. The proof of meet irreducibility of a $p \in P_{GM}(X)$ utilizes a dual argument because $\mathcal{U}_{GM} = \mathcal{L}_{GM}$. ∎

The next lemma is an immediate consequence of the property $(W)$ and Lemma 5.

**Lemma 6.** *$a_1 \wedge a_2 \le b_1 \vee b_2$ in $F_{GM}(X)$ if and only if $[a_1 \wedge a_2, b_1 \vee b_2] \cap \{a_1, a_2, b_1, b_2\} \ne \varnothing$.*

Knowing the above characterization of the ordering in $F_{GM}(X)$ we are in a position to state the following theorem.

**Theorem 7.** *The word problem in $F_{GM}(X)$ has an affirmative solution.*

**Proof.** Let $a = a(x_1, x_2, \dots, x_n) \in F_{GM}(X)$ be arbitrary GM – term, $x_i \in X$. Taking into account the identities $GM_1, GM_2, GM_3$ one can construct a lattice term $\widetilde{a} = \widetilde{a}(x_1, x_2, \dots, x_n, x_1^-, x_2^-, \dots, x_n^-)$ such that $a = \widetilde{a}$. According to Lemmas 5 and 6 there is an effective algorithm for decision when $\widetilde{a} = \widetilde{b}$ in

$FL\big(P_{GM}(X), \mathcal{U}_{GM}, \mathcal{L}_{GM}\big)$. From this we can conclude that the word problem in $F_{GM}(X)$ has a solution. ∎

**4. Free Generalized Morgan-Stone Algebras**

In this section we focus on the construction of and analyzing the word problem for the free GMS – algebra $F_{GMS}(X)$ freely generated by a set $X$. The next lemma gives us the complete characterization of the skeleton $S\big(F_{GMS}(X)\big)$.

**Lemma 8.** *The skeleton* $S\big(F_{GMS}(X)\big) = \{a \in F_{GMS}(X), a = a^{\circ\circ}\}$ *is isomorphic to the free GM – algebra* $F_{GM}(X^{\circ\circ})$ *generated by the set* $X^{\circ\circ} = \{x^{\circ\circ}, x \in X\}$.

**Proof.** The proof is essentially the same as that of [**5**, Lemma 2 and 3] for the variety of $p$-algebras. We therefore only sketch the main ideas. First, we observe that the skeleton $S\big(F_{GMS}(X)\big)$ is a GM – algebra generated by the set $X^{\circ\circ}$. Indeed, it suffices to show that $a^{\circ\circ} \in [X^{\circ\circ}]_{GM}$ whenever $a \in F_{GMS}(X)$. This can be readily verified by an induction on the rank of a term $a \in F_{GMS}(X)$. Furthermore, let us consider the GMS – homomorphism $h : F_{GMS}(X) \to F_{GM}(X^{\circ\circ})$ defined as $h(x) := x^{\circ\circ}$ for any $x \in X$. The mapping $h$ restricted to the set $S\big(F_{GMS}(X)\big)$ is a GM – homomorphism which is an identity mapping on $X^{\circ\circ}$. With regard to Lemma 2 we obtain that $S\big(F_{GMS}(X)\big)$ is isomorphic to $F_{GM}(X^{\circ\circ})$. ∎

Following the same idea of the construction of $F_{GM}(X)$ as a free lattice $FL(P, \mathcal{U}, \mathcal{L})$ generated by some poset $P$ and preserving bounds prescribed by sets from $\mathcal{U}$ and $\mathcal{L}$ we will define the poset $P_{GMS}(X)$ and two families $\mathcal{U}_{GMS}, \mathcal{L}_{GMS}$ of finite subsets of $P_{GMS}(X)$ and a unary operation of involution $\circ$ in such a way that the resulting free lattice $FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ with the operation $\circ$ is isomorphic to the free GMS – algebra $F_{GMS}(X)$ freely generated by the set $X$.

Let $X$ be a set. Take a disjoint copy $\overline{X} := \{\overline{x}, x \in X\}$ of $X$ with the property $X \cap F_{GM}(\overline{X}) = \varnothing$. Let us define the poset $P_{GMS}(X) = X \cup F_{GM}(\overline{X})$ and the order relation $\prec$ on $P_{GMS}(X)$ as follows:

$0 \prec a \prec 1$ for any $a \in P_{GMS}(X)$;

$x \prec a$, $x \in X$, $a \in F_{GM}(\overline{X})$ iff $\overline{x} \le a$ in $F_{GM}(\overline{X})$;

$a \prec b$, $a, b \in F_{GM}(\overline{X})$ iff $a \le b$ in $F_{GM}(\overline{X})$.

The sets $\mathcal{U}_{GMS}$, $\mathcal{L}_{GMS}$ are defined as follows: a finite subset $S \subseteq P_{GMS}(X)$ belongs to $\mathcal{U}_{GMS} = \mathcal{L}_{GMS}$ iff either $S = \{p, q\}$, $p \prec q$ in $P_{GMS}(X)$ or $S$ is a subset of $F_{GM}(\overline{X})$.

**Lemma 9.** *Let* $FL \equiv FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ *be the free lattice generated by the poset* $P_{GMS}(X)$ *and preserving bounds from* $\mathcal{U}_{GMS}$ *and* $\mathcal{L}_{GMS}$. *Then there exists the unique lattice epimorphism* $\pi : FL \to F_{GM}(\overline{X})$ *such that* $\pi(x) = \overline{x}$ *for any* $x \in X$ *and* $\pi(a) = a$ *for any* $a \in F_{GM}(\overline{X})$. *Moreover,* $a \le \pi(a)$ *for any* $a \in FL$.

**Proof.** The mapping $\tau : P_{GMS}(X) \to F_{GM}(\overline{X})$ defined by $\tau(x) = \overline{x}$ for any $x \in X$ and $\tau(a) = a$ for any $a \in F_{GM}(\overline{X})$ preserves l.u.b.'s and g.l.b.'s prescribed by sets from $\mathcal{U}_{GMS}$ and $\mathcal{L}_{GMS}$, respectively, i.e. $\tau(\sup_P S) = \bigvee \tau(S)$ and $\tau(\inf_P S) = \bigwedge \tau(S)$ for any $S \in \mathcal{U}_{GMS} = \mathcal{L}_{GMS}$. Hence, by [**3**, Theorem 6] there is a unique lattice homomorphism $\pi : FL \to F_{GM}(\overline{X})$ extending the mapping $\tau$.

Since $F_{GM}(\overline{X}) \subset P_{GMS}(X)$ the homomorphism $\pi$ is also onto. Furthermore, the inequality $a \le \pi(a)$ is satisfied for any $a \in P_{GMS}(X)$. Then one can proceed by an induction on the rank of a lattice term $a \in FL$ to verify that $a \le \pi(a)$ for any $a \in FL$. ∎

For any $a \in FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ we set

$$a^\circ := \big(\pi(a)\big)^- \tag{4.1}$$

where the unary operation $^-$ is taken in the free GM – algebra $F_{GM}(\overline{X})$. Notice that $a^\circ = a^-$ for any $a \in F_{GM}(\overline{X})$.

**Theorem 10.** *Let $X$ be any set. Then the free lattice $FL \equiv FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ endowed with the unary operation $\circ$ defined in (4.1) is a free GMS – algebra freely generated by the set $X$, i.e. $(FL, \circ) \cong F_{GMS}(X)$.*

**Proof.** First we will verify the identities $\text{GMS}_1$, $\text{GMS}_2$ and $\text{GMS}_3$. By Lemma 9, we have $a^{\circ\circ} = \big(\pi\big(\pi(a)^-\big)\big)^- = \pi(a)^{--} = \pi(a) \ge a$. Further $(a \wedge b)^\circ = \big(\pi(a \wedge b)\big)^-$ $= \big(\pi(a) \wedge \pi(b)\big)^- = \pi(a)^- \vee \pi(b)^- = a^\circ \vee b^\circ$. Finally, $1^\circ = \pi(1)^- = 1^- = 0$.

As $x^{\circ\circ} = \big(\pi\big(\pi(x)^-\big)\big)^- = \overline{x}^{--} = \overline{x}$ we have $P_{GMS}(X) = X \cup F_{GM}(\overline{X}) \subset [X]_{GMS}$ and so $(FL, \circ) = [X]_{GMS}$. Let us define the mapping $h : P_{GMS}(X) \to F_{GMS}(X)$ as follows:

$$h(x) := x \quad \text{for} \quad x \in X, \qquad h(\overline{x}) := x^{\circ\circ} \quad \text{for} \quad \overline{x} \in \overline{X}.$$

Since $F_{GM}(\overline{X})$ is a free GM – algebra the mapping $h : \overline{X} \to F_{GMS}(X)$ uniquely extends to a GM – homomorphism $h : F_{GM}(\overline{X}) \to F_{GMS}(X)$. Hence the mapping $h : P_{GMS}(X) \to F_{GMS}(X)$ is well defined. Moreover, it preserves all bounds prescribed by sets from $\mathcal{U}_{GMS} = \mathcal{L}_{GMS}$. Again due to [**3**, Theorem 6] the mapping $h$ can be extended to a lattice homomorphism $h : FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ $\to F_{GMS}(X)$. In what follows, we will show that $h$ is even a GMS – homomorphism, i.e. $h(a^\circ) = h(a)^\circ$. We will argue by an induction on the rank of the lattice term $a \in FL$. For an $a \in P_{GMS}(X)$ the statement is obvious. If $a \equiv a_1 \vee a_2$ then $h(a^\circ) = h\big(\pi(a_1 \vee a_2)^-\big) = h\big(\pi(a_1)^- \wedge \pi(a_2)^-\big) = h\big(\pi(a_1)^-\big) \wedge h\big(\pi(a_2)^-\big)$ $= h(a_1^\circ) \wedge h(a_2^\circ) = h(a_1)^\circ \wedge h(a_2)^\circ = \big(h(a_1) \vee h(a_2)\big)^\circ = h(a)^\circ$. The case $a \equiv a_1 \wedge a_2$ is similar.

This way we have shown that $h$ is a GMS – homomorphism. As $h(x) = x$ for any $x \in X$ and $F_{GMS}(X) = [X]_{GMS}$ we infer that $h$ is a surjection. According to Lemma 2 $h$ is a GMS – isomorphism and the proof of theorem follows. ∎

In accord to the previous theorem we will henceforth identify $F_{GMS}(X)$ with the free lattice $FL\big(P_{GMS}(X), \mathcal{U}_{GMS}, \mathcal{L}_{GMS}\big)$ endowed with the unary operation $\circ$ defined in (4.1).

In the following lemma, we examine the property of join and meet irreducibility of elements of $F_{GMS}(X)$ belonging to the set $X \cup S\big(F_{GMS}(X)\big)$.

**Lemma 11.** *Let $F_{GMS}(X)$ be a free GMS – algebra. Then any $p \in X \cup S(F_{GMS}(X))$ is both join and meet irreducible.*

**Proof.** Notice that, as $X^{\circ\circ} = \overline{X}$ in $FL$ we obtain $S(F_{GMS}(X)) \cong F_{GM}(\overline{X})$. We will distinguish two cases: $p \equiv x \in X$ and $p \in F_{GM}(\overline{X})$.

Let us consider the case $p \equiv x \in X$. From the definition of the ordering in $FL$ we know that $x \leq a_1 \vee a_2$ means $x \leq a_1 \vee a_2(k)$ for some integer $k > 0$. By $(J)$, either $x \leq a_i(k-1)$, $i = 1$ or $i = 2$ or there exists an $S \in \mathcal{U}_{GMS}$ such that $x \prec \sup_P S$ and $s \leq a_1 \vee a_2(k-1)$ for any $s \in S$. The first event immediately implies $x \leq a_i$, $i = 1$ or $i = 2$. On the other hand, if $S = \{p, q\}$, $p \prec q$ in $P_{GMS}(X)$, then we obtain $x \leq a_1 \vee a_2\ (k-1)$. If $S$ is a finite subset of $F_{GM}(\overline{X})$ then with regard to the definition of the order relation we infer that $x < \overline{x} \leq \sup_P S = \bigvee S$. Since any $\overline{x}$ is join irreducible in $F_{GM}(\overline{X})$ (see Lemma 5) there is an $s \in S$ such that $\overline{x} \leq s$. This however means that $x \leq a_1 \vee a_2\ (k-1)$. Now, decreasing step by step the rank $k$ by one and taking into account that the case $x \leq a_1 \vee a_2(0)$ is impossible we end up with the claim that either $x \leq a_1$ or $x \leq a_2$. Hence any $x \in X$ is join irreducible in $F_{GMS}(X)$.

To prove meet irreducibility of an $x \in X$ we cannot employ a dual argument because the ordering in $P_{GMS}(X)$ is not symmetric ($x < \overline{x}$). Nevertheless, if $a_1 \wedge a_2 \leq x$ then there is a $k > 0$ such that $a_1 \wedge a_2 \leq x(k)$. By $(M)$, either $a_i \leq x$, $i = 1$ or $i = 2$ or there is a finite subset $S \subset F_{GM}(\overline{X})$ with the property $\inf_P S \prec x$ and $a_1 \wedge a_2 \leq s(k-1)$ for any $s \in S$. Since there are no nonzero elements in $F_{GM}(\overline{X})$ less than $x$ we have $\inf_P S = 0$ and so $a_1 \wedge a_2 = 0$. Thus $a_1^{\circ\circ} \wedge a_2^{\circ\circ} = 0$ in $F_{GM}(\overline{X})$. We remind ourselves that 0 is meet irreducible in $F_{GM}(\overline{X})$. Hence either $a_1 \leq a_1^{\circ\circ} = 0 < x$ or $a_2 \leq a_2^{\circ\circ} = 0 < x$. Hence $x \in X$ is also meet irreducible.

The proof of join and meet irreducibility of an element $p \in F_{GM}(\overline{X})$ again follows from Lemma 5. Indeed, if $p \leq a_1 \vee a_2(k)$, $k > 0$ then either $p \leq a_i(k-1)$, $i = 1$ or $i = 2$ or there is a finite subset $S \subset F_{GM}(\overline{X})$ such that $p \prec \sup_P S$ and $s \leq a_1 \vee a_2(k-1)$ for any $s \in S$. Taking into account join irreducibility of any $s \in F_{GM}(\overline{X})$ we infer that either $p \leq a_i(k-1)$, $i = 1$ or $i = 2$ or $p \leq a_1 \vee a_2(k-1)$. Now the standard argument enables us to conclude that $p \leq a_1$ or $p \leq a_2$. On the other hand, if $a_1 \wedge a_2 \leq p$ then $a_1^{\circ\circ} \wedge a_2^{\circ\circ} \leq p^{\circ\circ} = p$. As the element $p \in F_{GM}(\overline{X})$ is meet irreducible in $F_{GM}(\overline{X})$ we obtain $a_i \leq a_i^{\circ\circ} \leq p$ for $i = 1$ or $i = 2$. This means that $p \in F_{GM}(\overline{X})$ is meet irreducible. The proof of lemma is complete. ∎

The next lemma is a direct consequence of Lemma 11 and the property $(W)$.

**Lemma 12.** *$a_1 \wedge a_2 \leq b_1 \vee b_2$ in $F_{GMS}(X)$ if and only if $[a_1 \wedge a_2, b_1 \vee b_2] \cap \{a_1, a_2, b_1, b_2\} \neq \varnothing$.*

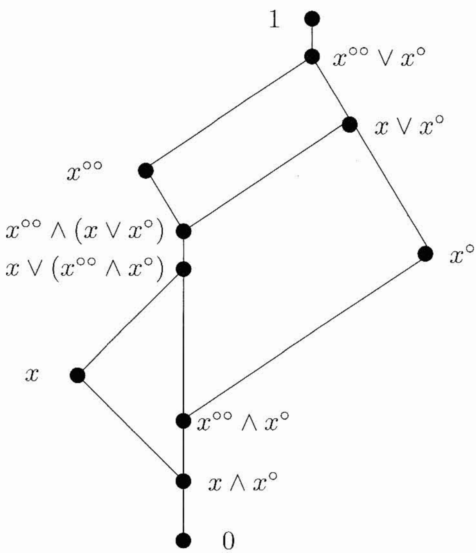
We conclude this paper with the following result proving that the word problem in the free algebra $F_{GMS}(X)$ has a solution, i.e. there is an algorithm for decision when two word in $F_{GMS}(X)$ are equal.

**Theorem 13.** *Let $X$ be any set. Then the word problem in the free generalized Morgan – Stone algebra $F_{GMS}(X)$ has an affirmative solution.*

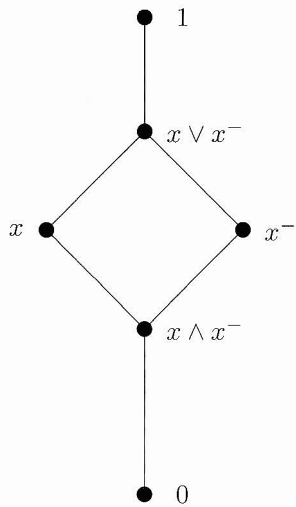
**Proof.** Let $F_{GMS}(X) \cong (FL, ^\circ)$ be a free GMS – algebra and let $a, b \in FL$ be two GMS – terms. Since the identities $x^\circ = x^{\circ\circ\circ}$, $x^\circ \wedge y^\circ = (x \vee y)^\circ$, $x^\circ \vee y^\circ = (x \wedge y)^\circ$ are fulfilled in any GMS – algebra one can construct lattice terms $\widetilde{a}$, $\widetilde{b}$ with letters belonging to the set $X \cup X^\circ \cup X^{\circ\circ}$ and such that $a = \widetilde{a}$, $b = \widetilde{b}$. As $\widetilde{a} = \widetilde{b}$ iff $\widetilde{a} \leq \widetilde{b}$ and $\widetilde{b} \leq \widetilde{a}$ one can recursively apply Lemma 12 in order to decrease the rank of the lattice terms $\widetilde{a}$, $\widetilde{b}$. After performing finite number of steps we end up with the problem to decide whether $p \leq q$ in $F_{GMS}(X)$ for letters from the set $X \cup X^\circ \cup X^{\circ\circ}$. With regard to the definition of the ordering in $P_{GMS}(X)$, this is true iff either $p \equiv q$ or $p \equiv x$ and $q = x^{\circ\circ}$ for some $x \in X$. Hence there is an algorithm for determining whether $a = b$ in $F_{GMS}(X)$. ∎

**Examples.** The free GMS and GM – algebras with $|X| = 1$ have diagrams as shown by the figures below:

$F_{GMS}(1)$




$F_{GM}(1)$

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