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Dissipative Feedback Synthesis for a Singularly Perturbed Model of a Piston Driven Flow of a Non-Newtonian Fluid

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The limiting behaviour of solutions of a system of singularly perturbed equations is studied. The goal is to construct a dissipative feedback control synthesis that stabilizes the prescribed output functional along trajectories of solutions. The results are applied to a singularly perturbed Johnson–Sagelman–Oldroyd model of shearing motions of a piston driven flow of a non-Newtonian fluid.

1. Introduction

The aim of this paper is to construct a dissipative feedback control synthesis that stabilizes a given output functional along solutions of the following system of singularly perturbed evolution equations

$$\begin{aligned} x_t &= G_\varepsilon(x, y, z), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, z), \end{aligned} \tag{1.1}$$

where $0 \leqslant \varepsilon \ll 1$ is a small parameter, $x \in X$, $y \in Y$, X and Y are Banach spaces, B is a sectorial operator in Y. In this paper we consider a specific feedback control mechanism of the form

$$z = \Xi(x),$$

where Ξ is a smooth function from X into another Banach space Z. In other words, a synthesis $z = \Xi(x)$ should only depend on the slow variable x. It is well-known that the Cauchy problem for the full system of equations, $\varepsilon > 0$,

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \Xi(x)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \Xi(x)) \end{aligned} \tag{1.2}$$

generates a globally defined semi-flow $\mathscr{S}_\varepsilon(t)$, $t \geqslant 0$, on a phase-space $\mathscr{X} = X \times Y^\beta$, provided that the nonlinearities G_ε, F_ε and Ξ satisfy certain regularity and growth conditions (cf. [6]). Furthermore, under a suitable assumption on a function F_0,

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system (1.2) generates a semi-flow $\mathscr{S}_0(t)$, $t \geqslant 0$, on a phase-space $\mathscr{M}_0$ which is a Banach submanifold of $\mathscr{X}$.

Typically, the structure of the reduced system of equations (1.1), $\varepsilon = 0$, allows us to construct a feedback law $z = \Xi_0(x)$ with the property that a prescribed output functional Q_0 asymptotically vanishes along all solutions of (1.2), i.e. $Q_0(\mathscr{S}_0(t)(x_0, y_0)) \to 0$ as $t \to \infty$. We discuss an example of such a reduced dynamics in section 6. Under assumptions made in sections 2 and 3, our goal in this work is to find a feedback synthesis $\Xi = \Xi_\varepsilon$ stabilizing the given output functional Q_ε along trajectories of the full system of equations (1.2) whenever $\varepsilon > 0$ is sufficiently small. It should be noted that an explicit construction of such a synthesis is not obvious, in many cases, and this is why we have to turn to functional analytic methods in order to prove the existence of a stabilizing feedback law and to examine the limiting behavior of Ξ_ε when $\varepsilon \to 0^+$.

Before stating our main result we need several definitions.

Definition 1.1. Let $\mathscr{S}(t)$, $t \geqslant 0$, be a semi-flow on a metric space $(\mathscr{X}, d)$. Let $\mathscr{M}$ be an attracting invariant set for $\mathscr{S}$, i.e. $\mathscr{S}(t)\mathscr{M} = \mathscr{M}$ for any $t \geqslant 0$ and dist $(\mathscr{S}(t)u, \mathscr{M}) \to 0$ as $t \to \infty$ for any $u \in \mathscr{X}$. Let $Q : \mathscr{X} \to E$ be a prescribed output functional, E is a metric space. We say that the semi-flow $\mathscr{S}(t)$ is *asymptotically Q-constrained on $\mathscr{M}$* if $Q(u) = 0$ for any $u \in \mathscr{M}$.

Remark 1.1. Notice that, if $Q : \mathscr{X} \to E$ is continuous then any Q-asymptotically constrained semi-flow $\mathscr{S}(t)$ on the attracting invariant set $\mathscr{M}$, has the property $Q(\mathscr{S}(t)u) \to 0$ as $t \to \infty$ for any $u \in \mathscr{X}$. Clearly, if a functional Q vanishes on $\mathscr{X}$ then any semi-flow on $\mathscr{X}$ is Q-asymptotically constrained on the whole phase-space $\mathscr{X}$.

Definition 1.2. Let $\varepsilon \in [0, \varepsilon_0]$ be fixed. Let $Q_\varepsilon : \mathscr{X} \to E$ be a continuous mapping, $\mathscr{X}$ is the phase-space for (1.1). We say that system of equations (1.1) admits a *dissipative feedback synthesis* $\Xi : X \to Z$ if the semi-flow $\mathscr{S}_\varepsilon(t)$ generated by solutions of (1.2) possesses an attracting invariant manifold $\mathscr{M}_\varepsilon$ and the semi-flow $\mathscr{S}(t)$ is Q_ε-asymptotically constrained on $\mathscr{M}_\varepsilon$.

We also recall the notion of an inertial manifold.

Definition 1.3. Let $\mathscr{S}(t)$, $t \geqslant 0$, be a semi-flow in the Banach space $\mathscr{X}$. We say that a Banach submanifold $\mathscr{M} \subset \mathscr{X}$ is an inertial manifold. for semi-flow $\mathscr{S}$ if:

(a) it is an invariant, i.e. $\mathscr{S}(t)\mathscr{M} = \mathscr{M}$ for any $t \geqslant 0$; and

(b) $\mathscr{M}$ attracts exponentially all solutions, i.e. there is $\mu > 0$ such that dist $(\mathscr{S}(t)u_0, \mathscr{M}) = O(\mathrm{e}^{-\mu t})$ as $t \to \infty$ for any $u_0 \in \mathscr{X}$.

In contrast to the classical definition of an inertial manifold due to Foias *et al.* [4], we allow the exponentially attractive invariant manifold to be an infinite-dimensional Banach submanifold of the phase-space $\mathscr{X}$. (see e.g. [8]).

Given a family of output functionals Q_ε, $\varepsilon \geqslant 0$, the main result can be stated as follows:

Theorem 1.1. *Assume hypotheses* (H1)–(H4) *and the structural condition* (5.1) *below. Then, for any $\varepsilon > 0$ small enough,*

(a) *system* (1.1) *admits a dissipative feedback synthesis* $\Xi_\varepsilon \in C^1_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(X, Z)$ *and, moreover,*

(b) $\lim_{\varepsilon \to 0^+} \Xi_\varepsilon = \Xi_0$ *in* $C^1_{bdd}(\mathscr{B}, Z)$ *for any* $\mathscr{B}$ *bounded and open subset of* X.

(c) *The feedback law* $z = \Xi_\varepsilon(x)$ *stabilizes the prescribed output functional* Q_ε. *This means that* $\lim_{t \to \infty} Q_\varepsilon\,(x(t), y(t)) = 0$ *for any solution* $(x(.), y(.))$ *of* (1.2).

(d) *The semi-flow* $\mathscr{S}_\varepsilon$ *generated by solutions of system* (1.2) *is* Q_ε*-asymptotically constrained on a* C^1 *smooth inertial manifold* $\mathscr{M}_\varepsilon$. *The manifold* $\mathscr{M}_\varepsilon$ *is* C^1 *close to* $\mathscr{M}_0$ *for* $\varepsilon > 0$ *sufficiently small.*

The idea of the proof and the organization of the paper is as follows. In section 3 we find a synthesis $z = \theta_\varepsilon(x, y)$ depending on the both slow and fast variables. Under suitable assumptions (see (H3)) such a function θ_ε can be uniquely determined from the governing equations and the condition that $\varepsilon \mathrm{d}/\mathrm{d}t Q_\varepsilon(x(t), y(t)) + Q_\varepsilon(x(t), y(t)) = 0$, i.e. $\| Q_\varepsilon(x(t), y(t)) \| = O(\mathrm{e}^{-t/\varepsilon})$ as $t \to +\infty$ for any solution of system (1.1) with $z = \theta_\varepsilon(x, y)$. Incorporating the feedback law $z = \theta_\varepsilon(x, y)$ into system (1.1) we then construct an inertial manifold $\mathscr{M}_\varepsilon$ for (1.1) as a smooth graph $\mathscr{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\}$. To this end we make use of the abstract singular perturbation theorem proved in [14]. We recall this result in section 4. Roughly speaking, the existence of such an inertial manifold $\mathscr{M}_\varepsilon$ means that the fast variable y is governed by the slow variable x when restricted on the manifold $\mathscr{M}_\varepsilon$. This enables us to construct Ξ as a composite function $\Xi_\varepsilon(x) = \theta_\varepsilon(x, \Phi_\varepsilon(x))$.

In section 6 we are concerned with the problem of the existence of a feedback control law stabilizing a given output of solutions for a system of singularly perturbed equations arising from the non-Newtonian fluid dynamics. Several authors have considered various constitutive models of a non-Newtonian fluid in order to describe flow instability phenomena like e.g. spurt, hysteresis loop under cyclic load for pressure driven flows of a Johnson–Segalman–Oldroyd (JSO) fluid [9, 11, 5], or KBKZ fluid (see [1, 5]). In this paper we consider the JSO model and research which has been motivated by recent rheological experiments due to Lim and Schowalter [7]. Their experimental data suggests that a nearly periodic regime bifurcates from a steady state when the volumetric flow rate was gradually loaded beyond a critical value. In [10] Malkus *et al.* developed a mathematical theory capable of describing bifurcation phenomena in a piston driven flow of shearing motions of a non-Newtonian fluid. They considered the Johnson–Segalman–Oldroyd model of a shear flow of a non-Newtonian fluid leading to a system of three parabolic–hyperbolic equations.

$$
\begin{aligned}
&\varepsilon v_t - v_{\xi\xi} = \sigma_\xi + f,\\
&\sigma_t + \sigma = (1 + n) v_\xi, \quad (t, \xi) \in [0, \infty) \times [0, 1], \qquad (1.3)\\
&n_t + n = -\sigma v_\xi,
\end{aligned}
$$

where v is directional velocity of a planar shear flow, σ is the extra shear stress and n is the normal stress difference. The dimensionless number $\varepsilon > 0$ is proportional to the ratio of the Reynolds number to Deborah number and, in practice, ε is very small compared to other the terms in (1.3), $\varepsilon = O(10^{-12})$. This gives rise to treating $0 < \varepsilon \ll 1$ as a small parameter and to study a reduced system of equations (1.3) in which $\varepsilon = 0$. The problem to be considered here consists in the construction of a driving pressure gradient f as a function of the flow variables σ, n in such a way that the output of the volumetric flow rate per unit cross-section, $Q(t) = \int_0^1 v(t, \xi)\, \mathrm{d}\xi$ is fixed at the prescribed value Q_{fix}. It turns out that f has the form of a non-local functional of σ, $f = \Xi_0(\sigma) = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\, \mathrm{d}\xi$ (see, [10, (FB)]). Numerical simulations performed in [10] showed that such a quasi-dynamic approximation of the full system (1.3) is capable of capturing an interesting phenomenon of the existence of nearly periodic oscillations in the pressure gradient f observed recently in rheological experiments due to Lim and Showalter [7].

We apply Theorem 1.1 in order to show that, for small values of $\varepsilon > 0$, there exists a real valued dissipative feedback synthesis $f = f_\varepsilon(\sigma, n)$ for the pressure gradient such that $Q(t) \to Q_{\mathrm{fix}}$ as $t \to \infty$ along solutions of the full system of equations (1.3). Moreover, there exists an infinite-dimensional inertial manifold $\mathscr{M}_\varepsilon$ for system (1.3), $0 < \varepsilon \ll 1$, and the volumetric flow rate Q of a solution belonging to $\mathscr{M}_\varepsilon$ is fixed at the prescribed value Q_{fix}. These results are summarized in Theorem 6.3. The vector field governing the motion on the invariant manifold $\mathscr{M}_\varepsilon$ is compared to that of the reduced problem. It is shown that they are locally C^1 close for small values of the singular parameter.

2. Preliminaries

Let E_1, E_2 be Banach spaces and $\eta \in (0, 1]$. By $L(E_1, E_2)$ we denote the Banach space of all linear bounded operators from E_1 to E_2. For an open subset $\mathscr{B} \subset E_1$, $C^k(\mathscr{B}, E_2)$ denotes the vector space of all k-times continuously Frechet differentiable mappings $F: \mathscr{B} \to E_2$. By $C^{k,1}(\mathscr{B}, E_2)$ we denote the vector space consisting of all $F \in C^k(\mathscr{B}, E_2)$ such that all derivatives D^iF, $i = 0, 1, \dots, k$ are globally Lipschitz continuous. $C^1_{bdd}(\mathscr{B}, E_2)$ denotes the Banach space consisting of the mappings $F \in C^1(\mathscr{B}, E_2)$ which are Frechet differentiable and such that F, DF are bounded and uniformly continuous, the norm being given by $\|F\|_1^2 := (\sup|F|)^2 + (\sup|DF|)^2$. Finally, $C^{1+\eta}_{bdd}(\mathscr{B}, E_2)$ will denote the Banach space consisting of the mappings $F \in C^1_{bdd}(\mathscr{B}, E_2)$ such that DF is η-Hölder continuous, the norm being given by $\|F\|_{1,\eta} := \|F\|_1 + \sup_{x \neq y} \|DF(x) - DF(y)\| \, \|x - y\|^{-\eta}$.

Throughout the paper we will assume that

$$\text{(H1)}\qquad \begin{aligned} &X, Y, Z \text{ are real Banach spaces;}\\ &B \text{ is a sectorial operator in } X;\\ &\operatorname{Re}\sigma(B) > \omega > 0 \text{ and } B^{-1}: Y \to Y \text{ is compact.} \end{aligned}$$

It follows from the theory of sectorial operators that $-B$ generates the exponentially decaying analytic semigroup of linear operators $\exp(-Bt)$, $t \geqslant 0$, on Y. Moreover, there is a constant $M \geqslant 1$ such that

$$\|\exp(-Bt)\|_{Y^\beta} \leqslant Mt^{-\beta}\mathrm{e}^{-\omega t} \quad \text{for any } t > 0 \text{ and } \beta \geqslant 0. \tag{2.1}$$

By Y^β, $\beta \in \mathbb{R}$ we have denoted a fractional power space with respect to the sectorial operator B, $Y^\beta = [D(B^\beta)]$, $\|y\|_{Y^\beta} = \|B^\beta y\|_Y$. Furthermore, $\|B^{\beta-1}\| \leqslant M\omega^{\beta-1}$ (cf [6, chapter 1]).

3. Construction of an (x, y)-dependent dissipative feedback synthesis

In this section we give a partial answer to the problem of the existence of a dissipative feedback synthesis that stabilizes a given output functional $Q_\varepsilon(x, y)$. We present a constructive method on how to obtain a feedback law of the form $z = \theta_\varepsilon(x, y)$ from the governing equations. In contrast to the required form of the synthesis $z = \Xi_\varepsilon(x)$ we allow the variable z to be a functional of both the x and y variables. The idea is rather simple and a function $\theta_\varepsilon : X \times Y^\beta \to Z$ is constructed in such a way that the E-valued functional $t \mapsto Q_\varepsilon(x(t), y(t))$ decays exponentially along any solution $(x(t), y(t))$ of system (1.1). Obviously, such an asymptotic behaviour is justified in the case when

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} Q_\varepsilon(x(t), y(t)) + \kappa Q_\varepsilon(x(t), y(t)) = 0, \quad t > 0 \tag{3.1}$$

for any solution $(x(.), y(.))$ of (1.1). Here $\kappa > 0$ is a fixed positive constant. Let us assume that G_ε and F_ε are X and Y valued functions, respectively. Using the chain rule the equation for $z = \theta_\varepsilon(x, y)$ can be deduced from equation (3.1), i.e.

$$\mathscr{H}_\varepsilon(x, y, z) = 0, \tag{3.2}$$

where $x = x(t)$, $y = y(t)$, $t > 0$, $z = \theta_\varepsilon(x, y)$ and

$$\begin{aligned}\mathscr{H}_\varepsilon(x, y, z) = \varepsilon D_x Q_\varepsilon(x, y) G_\varepsilon(x, y, z) + D_y Q_\varepsilon(x, y)[F_\varepsilon(x, y, z) - By] \\ + \kappa Q_\varepsilon(x, y).\end{aligned} \tag{3.3}$$

Suppose that there are constants $\beta \in [0, 1)$, $\eta \in (0, 1]$ such that, for any $\varepsilon \in [0, \varepsilon_0]$,

(H2) $Q_\varepsilon \in C^{2,1}(X \times Y^{\beta-1}, E)$, E is a real Banach space.

The function $\mathscr{H}_\varepsilon : X \times Y^\beta \times Z \to E$ is well-defined because $F_\varepsilon(x, y, z) - By \in Y^{\beta-1}$ for any $(x, y, z) \in X \times Y^\beta \times Z$ and $D_y Q_\varepsilon \in L(Y^{\beta-1}, E)$.

(H3) For any bounded and open subset $\mathscr{B} \subset X \times Y^\beta$ there is a function $\theta_\varepsilon \in C^{1,1}_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(X \times Y^\beta, Z)$ such that $\mathscr{H}_\varepsilon(x, y, z) = 0$ iff $z = \theta_\varepsilon(x, y)$ for any $(x, y) \in X \times Y^\beta$, and $\theta_\varepsilon \to \theta_0$ as $\varepsilon \to 0^+$ in $C^{1,1}_{bdd}(\mathscr{B}, Z)$

If, in addition to (H2), hypothesis (H3) is fulfilled then by (3.1) we have

$$\begin{aligned} &Q_\varepsilon(x(t), y(t)) = O(\mathrm{e}^{-\kappa t/\varepsilon}) \text{ as } t \to \infty \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_0 \\ &Q_0(x(t), y(t)) = 0 \quad \text{for any } t \geqslant 0. \end{aligned} \tag{3.4}$$

Henceforth, the property

$$\lim_{\varepsilon \to 0^+} \phi_\varepsilon = \phi_0 \text{ in } C^1_{bdd}(\mathscr{B}, E_2) \text{ for any bounded and open subset } \mathscr{B} \subset E_1$$

will be referred to as local C^1 closeness of ϕ_ε and ϕ_0.

Up to this point we did not make any precise assumptions on smoothness of non-linearities G_ε and F_ε appearing in (1.1) as right-hand sides. Henceforth, we will assume that G_ε and F_ε are such that

(H4) $\mathscr{G}_\varepsilon \in C^1_{bdd}(X \times Y^\beta, X)$, $\mathscr{F}_\varepsilon \in C^{1+\eta}_{bdd}(X \times Y^\beta, Y)$, $\|\mathscr{G}_\varepsilon - \mathscr{G}_0\|_1 + \|\mathscr{F}_\varepsilon - \mathscr{F}_0\|_1 = O(\varepsilon)$ as $\varepsilon \to 0^+$, where $\mathscr{G}_\varepsilon(x, y) := G_\varepsilon(x, y, \theta_\varepsilon(x, y))$, $\mathscr{F}_\varepsilon(x, y) := F_\varepsilon(x, y, \theta_\varepsilon(x, y))$.

We remark that $G_\varepsilon(F_\varepsilon)$ need not be necessarily a function from $X \times Y^\beta \times Z$ into $X(Y)$. We only require that the composite function $\mathscr{G}_\varepsilon(\mathscr{F}_\varepsilon)$ takes $X \times Y^\beta$ into $X(Y)$.

According to the theory of abstract parabolic equations due to Henry [6, Theorems 3.3.3, 3.3.4], the initial value problem for the system of equations

$$\begin{aligned} x_t &= \mathscr{G}_\varepsilon(x, y), \\ \varepsilon y_t + By &= \mathscr{F}_\varepsilon(x, y) \end{aligned} \tag{3.5}$$

possesses global-in-time strong solutions and system (3.5) generates a global C^1 semiflow $\bar{\mathscr{S}}_\varepsilon$, $t \geqslant 0$, on the phase-space

$$\mathscr{X} = X \times Y^\beta.$$

By a global strong solution of (3.5) with an initial condition $(x_0, y_0) \in \mathscr{X}$ we mean a function $(x, y) \in C_{\text{loc}}([0, \infty); \mathscr{X}) \cap C^1_{\text{loc}}((0, \infty); \mathscr{X})$ such that $(x(t), y(t)) \in X \times D(B)$ for any $t > 0$, and $(x(\cdot), y(\cdot))$ solves system (3.5) on $(0, \infty)$.

Let us denote

$$\delta(F_0, \theta_0) = \sup_{(x,y)} \| D_y \mathscr{F}_0(x, y) \|, \quad \text{where } \mathscr{F}_0(x, y) := F_0(x, y, \theta_0(x, y)). \tag{3.6}$$

If $\delta(F_0, \theta_0) < \omega^{1-\beta}/M$ then we have

$$\| B^{-1} D_y \mathscr{F}_0(x, y) \|_{L(Y^\beta, Y^\beta)} \leqslant \| B^{\beta-1} \| \sup \| D_y \mathscr{F}_0 \| \leqslant M\omega^{\beta-1}\delta < 1.$$

By the implicit function theorem there exists a C^1_{bdd} function $\Phi_0 : X \to Y^\beta$ such that $By = \mathscr{F}_0(x, y)$ iff $y = \Phi_0(x)$. By a global strong solution of (3.5), $\varepsilon = 0$, with an initial condition $x_0 \in X$ we mean a function $x \in C_{\text{loc}}([0, \infty); X) \cap C^1_{\text{loc}}(0, \infty); X)$ such that $x(\cdot)$ solves the equation $x_t = \mathscr{G}_0(x, \Phi_0(x))$ on $\mathbb{R}^+$. Again due to the above references to Henry's lecture notes this equation generates a global semi-flow $\hat{\mathscr{S}}_0(t)$, $t \geqslant 0$, on X. The semi-flow $\hat{\mathscr{S}}_0$ can be naturally extended to a semi-flow $\bar{\mathscr{S}}_0$ acting on the Banach submanifold

$$\mathscr{M}_0 = \{(x, \Phi_0(x)), x \in X\} \subset \mathscr{X} \tag{3.7}$$

by $\bar{\mathscr{S}}_0(t)(x, \Phi_0(x)) := \hat{\mathscr{S}}_0(t)x$ for any $x \in X$. In what follows, we will identify the semi-flow $\hat{\mathscr{S}}_0$ with $\bar{\mathscr{S}}_0$.

4. Abstract singular perturbation theorem

This section is focused on the C^1 singular limiting behaviour of inertial manifolds $\mathscr{M}_\varepsilon$ for semiflows $\bar{\mathscr{S}}_\varepsilon$ generated by solutions of the ε-parameterized system of equations (3.5). We recall an abstract result on limiting behaviour of inertial manifolds for a singularly perturbed system of evolution equations (3.5). The theorem below ensures both the existence of $\mathscr{M}_\varepsilon$ as well as C^1 closeness of $\mathscr{M}_\varepsilon$ and $\mathscr{M}_0$ for $\varepsilon > 0$ small enough.

Theorem 4.1. ([14, Theorem 3.9]). *Assume that hypotheses* (H1) *and* (H4) *hold. Then there are constants* $\delta_0 > 0$ *and* $0 < \varepsilon_1 \leqslant \varepsilon_0$ *such that if* $\sup_{(x,y)} \| D_y \mathscr{F}_\varepsilon(x, y) \|_{L(Y^\beta, Y)} \leqslant \delta_0$ *then, for any* $\varepsilon \in [0, \varepsilon_1]$, *there exists an inertial manifold* $\mathscr{M}_\varepsilon$ *for the semi-flow* $\bar{\mathscr{S}}_\varepsilon$ *generated by the system of evolution equations* (3.5) *and, moreover,*
(a) $\mathscr{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\}$, *where* $\Phi_\varepsilon \in C^1_{bdd}(X, Y^\beta)$;
(b) $\Phi_\varepsilon \to \Phi_0$ *as* $\varepsilon \to 0^+$ *in* $C^1_{bdd}(\mathscr{B}, Y^\beta)$ *for any bounded open subset* $\mathscr{B} \subset X$.

If $\dim(X) = \infty$ *then* $\mathscr{M}_\varepsilon$ *is an infinite-dimensional Banach submanifold of the phase-space* $\mathscr{X} = X \times Y^\beta$. *If* $\dim(Y) = \infty$ *then* $\text{codim}(\mathscr{M}_\varepsilon) = \infty$.

5. Construction of an x-dependent dissipative feedback synthesis. Proof of Theorem 1.1

Now we are in a position to prove the existence of a dissipative feedback synthesis of the required form $z = \Xi_\varepsilon(x)$. We assume that hypotheses (H1)–(H4) hold and, moreover,

$$\delta(F_0, \theta_0) < \delta_0, \tag{5.1}$$

where $\delta_0 > 0$ is the constant of Theorem 4.1. Then $\sup_{(x,y)} \|D_y \mathscr{F}_\varepsilon(x, y)\| < \delta_0$ for any $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough. As an immediate consequence of Theorem 4.1 we obtain the existence of an inertial manifold

$$\mathscr{M}_\varepsilon = \{(x, \Phi_\varepsilon(x)), x \in X\} \subset \mathscr{X} \tag{5.2}$$

for the semi-flow $\bar{\mathscr{S}}_\varepsilon$ generated by system (3.5). Moreover, $\Phi_\varepsilon \in C^1_{bdd}(X, Y^\beta)$ and

$$\Phi_\varepsilon \to \Phi_0 \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in } C^1_{bdd}(\mathscr{B}, Y^\beta) \tag{5.3}$$

for any bounded and open subset $\mathscr{B} \subset X$. Let us define the feedback law $\Xi_\varepsilon : X \to Z$ as follows:

$$\Xi_\varepsilon(x) := \theta_\varepsilon(x, \Phi_\varepsilon(x)) \quad x \in X. \tag{5.4}$$

Since we have assumed $\theta_\varepsilon \in C^{1,1}_{bdd}(\mathscr{B}, Z) \cap C^{0,1}(\mathscr{X}, Z)$ and $\theta_\varepsilon \to \theta_0$ in $C^1_{bdd}(\mathscr{B}, Z)$ as $\varepsilon \to 0^+$ for any bounded and open subset $\mathscr{B} \subset \mathscr{X} = X \times Y^\beta$ we infer from Theorem 4.1 that

$$\Xi_\varepsilon \in C^{0,1}(X, Z) \cap C^1_{bdd}(\mathscr{B}, Z), \qquad \Xi_\varepsilon \to \Xi_0 \text{ in } C^1_{bdd}(\mathscr{B}, Z) \text{ as } \varepsilon \to 0^+, \tag{5.5}$$

where $\mathscr{B}$ is an arbitrary bounded and open subset of X. Again due to Henry's theory the system

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \Xi_\varepsilon(x)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \Xi_\varepsilon(x)) \end{aligned} \tag{5.6}$$

generates a global semiflow $\mathscr{S}_\varepsilon$ on $\mathscr{X}$ for $0 < \varepsilon \leqslant \varepsilon_1$ and $\mathscr{S}_0$ on $\mathscr{M}_0$, respectively. Furthermore, we observe that the right-hand side of system (5.6) and that of system (3.5), i.e.

$$\begin{aligned} x_t &= G_\varepsilon(x, y, \theta_\varepsilon(x, y)), \\ \varepsilon y_t + By &= F_\varepsilon(x, y, \theta_\varepsilon(x, y)) \end{aligned} \tag{5.7}$$

coincide on the set $\mathscr{M}_\varepsilon, \varepsilon \in [0, \varepsilon_1]$. Thus $\mathscr{S}_\varepsilon(t)(x_0, y_0) = \bar{\mathscr{S}}_\varepsilon(t)(x_0, y_0)$ for any $(x_0, y_0) \in \mathscr{M}_\varepsilon$ and $t \geqslant 0$. Since $\mathscr{M}_\varepsilon$ is invariant for the semi-flow $\bar{\mathscr{S}}_\varepsilon$ we conclude that the set $\mathscr{M}_\varepsilon$ is an invariant manifold for the semi-flow $\mathscr{S}_\varepsilon$ as well. Notice that $\mathscr{S}_0$ and $\bar{\mathscr{S}}_0$ are defined on $\mathscr{M}_0$ and they are equal. Although the set $\mathscr{M}_\varepsilon$ is an attractive invariant manifold (inertial manifold) for $\bar{\mathscr{S}}_\varepsilon$ it should be emphasized that it is not obvious that $\mathscr{M}_\varepsilon$ is an attractive set for $\mathscr{S}_\varepsilon$. The reason is that governing systems (5.6) and (5.7) may differ outside the set $\mathscr{M}_\varepsilon$. Nevertheless, we will show that the semi-flows $\mathscr{S}_\varepsilon$ and $\bar{\mathscr{S}}_\varepsilon$ are exponentially asymptotically equivalent.

Lemma 5.1. *There exists a constant* $\mu > 0$ *such that for any* $(x_0, y_0) \in \mathscr{X}$ *there is* $(x_0^*, y_0^*) \in \mathscr{M}_\varepsilon$ *with the property*

$$\| \mathscr{S}_\varepsilon(t)(x_0, y_0) - \bar{\mathscr{S}}_\varepsilon(t)(x_0^*, y_0^*) \|_{\mathscr{X}} = O(\mathrm{e}^{-\mu t}) \quad \text{as } t \to \infty. \tag{5.8}$$

Proof. This is just the proof of [3, Theorem 5.1] and it follows the lines of the proof of the existence of exponential tracking to a centre-unstable manifold. A slightly modified version of this proof is also contained in [14, Lemma 3.5]. This version utilizes compactness of the operator B^{-1}.

The idea is as follows. Let us fix $0 < \varepsilon \leqslant \varepsilon_1$. Given a solution $(x(\cdot), y(\cdot)) = \mathscr{S}_\varepsilon(\cdot)(x_0, y_0)$ of (5.6) we will prove the existence of an initial condition $(x_0^*, y_0^*) \in \mathscr{M}_\varepsilon$ with the property $(u(\cdot), v(\cdot)) \in C_\mu^+(\mathscr{X})$, where $(u(t), v(t)) = \bar{\mathscr{S}}_\varepsilon(t)(x_0^*, y_0^*) - \mathscr{S}_\varepsilon(t)(x_0, y_0)$ and C_μ^+ is the Banach space

$$C_\mu^+(\mathscr{X}) := \{ f \in C([0, \infty), \mathscr{X}),\ \|f\|_{C_\mu^+} = \sup_{t \geqslant 0} \mathrm{e}^{\mu t} \|f(t)\|_{\mathscr{X}} < \infty \}.$$

Obviously, the existence of such an initial condition (x_0^*, y_0^*) implies statement (5.8).

Let us choose $\mu > 0$. Taking into account the decay estimate (2.1) for the semigroup $\exp(-Bt)$ we have that (u, v) belongs to C_μ^+, if and only if it is a solution of the following pair of integral equations:

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} g(s, u(s), v(s))\,\mathrm{d}s \\ v(t) &= \exp(-Bt/\varepsilon)\xi + \frac{1}{\varepsilon}\int_0^t \exp(-B(t-s)/\varepsilon) f(s, u(s), v(s))\,\mathrm{d}s, \quad t \geqslant 0, \end{aligned} \tag{5.9}$$

for some $\xi \in Y^\beta$, where

$$\begin{aligned} g(s, u, v) &= G_\varepsilon(x^*(s), y^*(s), \theta_\varepsilon(x^*(s), y^*(s))) - G_\varepsilon(x^*(s) - u, y^*(s) \\ &\quad - v, \Xi_\varepsilon(x^*(s) - u)), \\ f(s, u, v) &= F_\varepsilon(x^*(s), y^*(s), \theta_\varepsilon(x^*(s), y^*(s))) - F_\varepsilon(x^*(s) - u, y^*(s) \\ &\quad - v, \Xi_\varepsilon(x^*(s) - u)). \end{aligned}$$

Since $\mathscr{M}_\varepsilon$ is invariant for $\bar{\mathscr{S}}_\varepsilon$ we have $y^*(s) = \Phi_\varepsilon(x^*(s))$ and hence $\theta_\varepsilon(x^*(s), y^*(s)) = \Xi_\varepsilon(x^*(s))$ for any $s \geqslant 0$. Thus, $\|\zeta(s, u, v)\|_X \leqslant C(\|u\|_X + \|v\|_{Y^\beta})$ where ζ stands either for g or f and $C > 0$ is a positive constant depending only on the Lipschitz constants of the mappings $G_\varepsilon, F_\varepsilon, \theta_\varepsilon, \Phi_\varepsilon$. Notice that the constant $C > 0$ can be chosen to be independent of $\varepsilon \in (0, \varepsilon_1]$. The rest of the proof is essentially the same as that of [3, Theorem 5.1] or [14, Lemma 3.5] and therefore is omitted. We only remind ourselves that, using the integral equations (5.9), the main idea is to set-up a suitable fixed point equation for $\xi \in Y^\beta$ by requiring that $(x_0^*, y_0^*) = (x_0 - u(0), y_0 - \xi)$ must be an element of the manifold $\mathscr{M}_\varepsilon = \mathrm{Graph}(\Phi_\varepsilon)$. To solve such a fixed point equation $\mu > 0$ must be chosen large enough. □

Lemma 5.2. *The output functional* Q_ε *vanishes on* $\mathscr{M}_\varepsilon$, *i.e.* $Q_\varepsilon(x_0, y_0) = 0$ *for any* $(x_0, y_0) \in \mathscr{M}_\varepsilon$.

Proof. The proof utilizes a simple invariance argument. Let $(x_0, y_0) \in \mathscr{M}_\varepsilon$ be fixed. Since $\mathscr{M}_\varepsilon$ is invariant for the semi-flow $\bar{\mathscr{S}}_\varepsilon$, for any $t \geqslant 0$, there is $(x_{-t}, y_{-t}) \in \mathscr{M}_\varepsilon$ such that $\bar{\mathscr{S}}_\varepsilon(t)(x_{-t}, y_{-t}) = (x_0, y_0)$. Clearly, $x_0 = x_{-t} + \int_{-t}^{0} \mathscr{G}_\varepsilon(\bar{\mathscr{S}}_\varepsilon(s)(x_{-t}, y_{-t}))\,\mathrm{d}s$. Hence, $\|x_0 - x_{-t}\| \leqslant \|\mathscr{G}_\varepsilon\|_0 t$. Furthermore, as $(x_{-t}, y_{-t}) \in \mathscr{M}_\varepsilon$ we have $y_{-t} = \Phi_\varepsilon(x_{-t})$ and so $\|y_{-t}\| \leqslant \|\Phi_\varepsilon\|_0$. Solving the linear homogeneous equation (3.1) we obtain $Q_\varepsilon(x_0, y_0) = Q_\varepsilon(\bar{\mathscr{S}}_\varepsilon(t)(x_{-t}, y_{-t})) = Q_\varepsilon(x_{-t}, y_{-t})\mathrm{e}^{-\kappa t/\varepsilon}$, $t \geqslant 0$. We remind ourselves that the output functional is assumed to be globally Lipschitz continuous and this is why

$$\|Q_\varepsilon(x_0, y_0)\|(\mathrm{e}^{\kappa t/\varepsilon} - 1) = \|Q_\varepsilon(x_{-t}, y_{-t}) - Q_\varepsilon(x_0, y_0)\|$$
$$\leqslant \operatorname{lip}(Q_\varepsilon)(\|x_{-t} - x_0\|_X + \|y_{-t} - y_0\|_{Y^\beta}) \leqslant \operatorname{lip}(Q_\varepsilon)(2\|\Phi_\varepsilon\|_0 + \|\mathscr{G}_\varepsilon\|_0 t).$$

Comparing the growth in $t \geqslant 0$ of the left- and right-hand sides of the above inequality we conclude $Q_\varepsilon(x_0, y_0) = 0$. Since $(x_0, y_0) \in \mathscr{M}_\varepsilon$ was arbitrary the proof of the lemma follows. □

Proof of Theorem 1.1. Under hypotheses (H1)–(H4) and assumption (5.1) we have established the existence of a dissipative feedback synthesis Ξ_ε (see (5.4) and Lemma 5.2). The regularity and convergence properties of Ξ_ε were shown in (5.5). Since, Q_ε is globally Lipschitz continuous the statement c) of Theorem 1.1 follows from Lemmas 5.1 and 5.2. Again with regard to Lemma 5.1, the manifold $\mathscr{M}_\varepsilon$ is an inertial manifold for the semi-flow $\mathscr{S}_\varepsilon$ generated by system (5.6). By (5.2) $\mathscr{M}_\varepsilon$ is a C^1 graph over the space X and the convergence property $\Phi_\varepsilon \to \Phi_0$ as $\varepsilon \to 0^+$ follows from (5.3). Hence, the statement (d) also holds. □

6. An application to the Johnson–Segalman–Oldroyd model of shearing motions of a piston driven non-Newtonian fluid

6.1. Governing equations

In order to examine the behaviour of a piston driven flow of a non-Newtonian fluid we consider the Johnson–Segalman–Oldroyd constitutive model of shearing motions of a planar Poiseuille flow within a thin channel. The channel is aligned along the y-axis and extends between $x \in [-1, 1]$. The flow is assumed to be symmetric with respect to $x = 0$ and the fluid undergoes simple shearing. Therefore, we can restrict ourselves to the interval $x \in [0, 1]$. Moreover, the flow variables (velocity and stresses) are independent of y so $\vec{v} = (0, v(t, x))$. To determine the extra stress tensor as a functional of the rate of a deformation tensor we consider the Johnson–Segalman–Oldroyd constitutive law (see [9] for details). In non-dimensional units the system of partial differential equations governing the motion of such a fluid is a system of parabolic–hyperbolic equations:

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)v_x, \\ n_t &= -n - \sigma v_x, \\ \varepsilon v_t &= \eta v_{xx} + \sigma_x + f, \end{aligned} \tag{6.1}$$

$(t, x) \in [0, \infty) \times [0, 1]$, subject to boundary and initial conditions

$$v_x(t, 0) = v(t, 1) = \sigma(t, 0) = 0 \quad \text{for any } t \geqslant 0$$
$$v(0, x) = v_0(x),\ \sigma(0, x) = \sigma_0(x),\ n(0, x) = n_0(x) \quad \text{for } x \in [0, 1]. \tag{6.2}$$

Here σ is the extra shear stress, n is the normal stress difference. It should be noted that in the case of a pressure driven flow studied in [9, 11, 15] the pressure gradient $f \in R$ is fixed. On the other hand, in the case of a piston driven flow (see [10] or [5, chapter 3]) the pressure gradient f is assumed to vary with respect to time. The parameters $\varepsilon > 0$ and $\eta > 0$ are proportional to the ratio of the Reynolds number to the Deborah number and the Newtonian viscosity to shear viscosity, respectively. In rheological experiments the number ε is very small compared to other terms in (6.1), $\varepsilon = O(10^{-12})$ (see [9]). This gives rise to treating $0 < \varepsilon \ll 1$ as a small parameter and investigate the singular limiting behavior of system (6.1)–(6.2) when $\varepsilon \to 0^+$. We refer to [9] for the complete derivation of a system of governing equations.

For the purpose of this analysis, let us introduce the following change of variables:

$$(\sigma, n, v) \leftrightarrow (\Sigma, n, u), \quad \Sigma(x) := -\int_x^1 \sigma(\xi)\,\mathrm{d}\xi, \; u := \eta v + \Sigma. \tag{6.3}$$

In terms of the new variables (Σ, n, u) system (6.1) has the form

$$\begin{aligned} \Sigma_t &= G^{(\Sigma)}, \\ n_t &= G^{(n)}, \\ \varepsilon u_t - \eta u_{xx} &= \eta f + \varepsilon G^{(\Sigma)}, \end{aligned} \tag{6.4}$$

where the non-linear functions $G^{(\Sigma)}$, $G^{(n)}$ are defined as

$$\begin{aligned} G^{(\Sigma)} &= G^{(\Sigma)}(\Sigma, n, u) = -\Sigma - \frac{1}{\eta}\int_x^1 (1 + \eta(\xi))[u_x(\xi) - \Sigma_x(\xi)]\,\mathrm{d}\xi, \\ G^{(n)} &= G^{(n)}(\Sigma, n, u) = -n - \frac{1}{\eta}\Sigma_x[u_x - \Sigma_x]. \end{aligned} \tag{6.5}$$

The corresponding boundary conditions are

$$u_x(t, 0) = u(t, 1) = \Sigma_x(t, 0) = \Sigma(t, 1) = 0 \quad \text{for any } t \geqslant 0. \tag{6.6}$$

Let $Q_{\text{fix}} \in R$ be a prescribed value of the volumetric flow rate. If Q denotes the variation in the volumetric flow rate of a planar flow per unit cross-section, i.e. $Q = \int_0^1 v(\xi)\,\mathrm{d}\xi - Q_{\text{fix}}$ then Q can be rewritten in terms of Σ and u as

$$Q(\Sigma, u) = \frac{1}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi - Q_{\text{fix}}. \tag{6.7}$$

The feedback law $f = \theta_\varepsilon((\Sigma, n), u)$ can be then readily deduced from equation (3.2). In our application (3.1) and (3.2) become

$$\begin{aligned} &\varepsilon D_\Sigma Q \circ G^{(\Sigma)} + D_u Q \circ [\varepsilon G^{(\Sigma)} + \eta f + \eta u_{xx}] \\ &\quad = -\frac{\varepsilon}{\eta}\int_0^1 G^{(\Sigma)} + \frac{1}{\eta}\int_0^1 [\eta u_{xx}(\xi) + \eta f + \varepsilon G^{(\Sigma)}]\,\mathrm{d}\xi + \frac{\kappa}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi \\ &\qquad - \kappa Q_{\text{fix}} = 0. \end{aligned}$$

Thus, for any $\varepsilon \geqslant 0$, we obtain

$$f = \theta(\Sigma, u) = -u_x(1) - \frac{\kappa}{\eta}\int_0^1 [u(\xi) - \Sigma(\xi)]\,\mathrm{d}\xi + \kappa Q_{\text{fix}}. \tag{6.8}$$

Remark 6.1. It should be noted that in the case of the reduced problem ($\varepsilon = 0$) one can calculate that $u(x) = (1 - x^2)f/2$. Taking into account (6.8) one has $f = 3\eta Q_{\mathrm{fix}} + 3\int_0^1 \Sigma(\xi)\,\mathrm{d}\xi$. In terms of the flow variable σ it means that

$$f = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\,\mathrm{d}\xi$$

which is, up to rescaling, the same formula for the driving pressure gradient as that obtained in [10], formulae (FB).

Incorporating the feedback law $f = \theta(\Sigma, u)$ into system (6.4) we can rewrite the system of governing equations (6.4) in an abstract form

$$\begin{aligned} \Sigma_t &= G^{(\Sigma)}(\Sigma, n, u), \\ n_t &= G^{(n)}(\Sigma, n, u), \\ \varepsilon u_t + Bu &= \mathscr{F}_\varepsilon(\Sigma, n, u), \end{aligned} \tag{6.9}$$

where B is a linear operator, $Bu(x) = -\eta u_{xx}(x) + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$, $x \in [0, 1]$, and

$$\mathscr{F}_\varepsilon(\Sigma, n, u) = \kappa\int_0^1 \Sigma + \kappa\eta Q_{\mathrm{fix}} + \varepsilon G^{(\Sigma)}(\Sigma, n, u) \tag{6.10}$$

and the non-linearities $G^{(\Sigma)}$, $G^{(n)}$ are as defined in (6.5). Notice that the derivative $D_u\mathscr{F}_\varepsilon$ vanishes for $\varepsilon = 0$.

6.2. Function space and operator setting

Let Y denote the real Hilbert space $L^2(0, 1)$ of square integrable functions; $\|u\|_Y^2 = \int_0^1 |u|^2$. For fixed positive real numbers η, $\kappa > 0$, we denote by B the linear operator $Bu = -\eta u_{xx} + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$ its domain being the Sobolev space $D(B) = \{u \in H^2(0, 1),\ u_x(0) = u(1) = 0\}$. B is a non self-adjoint nonlocal operator. In what follows, we will show that B is a sectorial operator in Y, and, moreover, $\operatorname{Re}\sigma(B) > 0$. To this end, we decompose the operator B as $B = \mathscr{B} + \mathscr{L}$ where $\mathscr{L}u = \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi$ and $\mathscr{B}$ is a self-adjoint operator in Y, $\mathscr{B}u = -\eta u_{xx}$ for any $u \in D(\mathscr{B}) = D(B)$. The operator $\mathscr{B}$ is sectorial in Y and $\operatorname{Re}\sigma(\mathscr{B}) \geqslant \eta\pi^2/4 > 0$ (see [6, chapter 1]). Since the embedding $[D(\mathscr{B}^\beta)] \hookrightarrow C^1_{bdd}(0, 1)$ is continuous for any $\beta > 3/4$ we have $\|\mathscr{L}u\|_Y \leqslant C\|\mathscr{B}^\beta u\|$ for any $u \in D(\mathscr{B})$ and $\beta > \frac{3}{4}$. According to [6, Corollary 1.4.5 and Example 11, p. 28] we conclude that the sum $B = \mathscr{B} + \mathscr{L}$ is a sectorial operator in Y as well. Moreover, the norm in the fractional power space $[D(B^\beta)]$ is equivalent to that of $[D(\mathscr{B}^\beta)]$. It remains to estimate the spectrum of B from below. First we notice that the operator $B^{-1}\colon Y \to Y$ exists and is given by $B^{-1}g = \int_0^1 K(.\,, \xi)g(\xi)\,\mathrm{d}\xi$, where K is a Green function.

$$K(x, \xi) = \begin{cases} \dfrac{1 - x}{\eta} + \dfrac{3}{2\kappa}(1 - x^2) - \dfrac{3}{4\eta}(1 - x^2)(1 - \xi^2), & 0 \leqslant \xi \leqslant x \leqslant 1, \\ \dfrac{1 - \xi}{\eta} + \dfrac{3}{2\kappa}(1 - x^2) - \dfrac{3}{4\eta}(1 - x^2)(1 - \xi^2), & 0 \leqslant x < \xi \leqslant 1. \end{cases}$$

Since, the kernel K is bounded the operator B^{-1} is compact and therefore the spectrum $\sigma(B)$ consists of eigenvalues, i.e. $\sigma(B) = \sigma_P(B)$. Let $\lambda \in \sigma(B)$ be an eigenvalue and $u \neq 0$ be the corresponding eigenfunction. Then $-\eta u_{xx}(x) + \eta u_x(1) + \kappa\int_0^1 u(\xi)\,\mathrm{d}\xi = \lambda u(x)$, $x \in [0, 1]$. Integrating this equation over $[0, 1]$ and taking into account the boundary condition $u_x(0) = 0$ we obtain $(\kappa - \lambda) \int_0^1 u = 0$. Then either $\lambda = \kappa > 0$ or $\int_0^1 u = 0$. The latter implies $-\eta u_{xx}(x) + \eta u_x(1) = \lambda u(x)$. By taking the inner product in a complexification of Y with $\bar{u}$ we obtain $\eta \int_0^1 |u_x|^2 = \eta \int_0^1 |u_x|^2 + \eta u_x(1) \int_0^1 \bar{u} = \lambda \int_0^1 |u|^2$. Hence λ is a real number and, moreover, $\lambda \geqslant \inf_{u \neq 0} \eta \| u_x \|^2 / \| u \|^2 = \eta \pi^2/4$. Summarizing we have shown the following proposition.

Lemma 6.2. *Let η, κ be any positive constants. Then the linear operator $Bu = -\eta u_{xx} + \eta u_x(1) + \kappa \int_0^1 u(\xi)\,d\xi$, $D(B) = \{u \in H^2(0,1),\ u_x(0) = u(1) = 0\}$, is sectorial in $Y = L^2(0,1)$. Furthermore, $\sigma(B) \subset [\omega, \infty)$ where $\omega = \min\{\kappa, \eta\pi^2/4\} > 0$. The fractional power space $Y^\beta =: [D(\mathscr{B}^\beta)]$ is imbeded into the Sobolev–Slobodeckii space $H^{2\beta}(0,1)$ for $1 > \beta > 3/4$. The resolvent operator $B^{-1}: Y \to Y$ is compact.*

Let X be the Banach space $X := \{(\Sigma, n) \in C^1_{bdd}(0,1) \times C^0_{bdd}(0,1),\ \Sigma_x(0) = \Sigma(1) = 0\}$. With regard to the continuity of the imbedding $Y^\beta \subsetneq C^1_{bdd}(0,1)$ for $\beta > \frac{3}{4}$, we conclude that the nonlinearities $G := (G^{(\Sigma)}, G^{(n)}): X \times Y^\beta \to X$ and $\mathscr{F}_\varepsilon: X \times Y^\beta \to Y$ are locally Lipschitz continuous. Thus local solvability in $\mathscr{X} = X \times Y^\beta, \frac{3}{4} < \beta < 1$, of system (6.9) follows from [6, Theorem 3.3.3]. To prove global-in-time solvability of solutions we have to find *a priori* estimates of any solution of (6.9).

6.3. A priori estimates of solutions, dissipativeness of a semi-flow, modification of governing equations

If (Σ, n, u) is a local solution of (6.9) in the phase space $\mathscr{X}$ then (σ, n, v), $\sigma = \Sigma_x$, $v = (u - \Sigma)/\eta$ is a local solution of (6.1) in $C_{bdd}(0,1) \cap \{\sigma, \sigma(0) = 0\} \times C_{bdd}(0,1) \times Y^\beta$. Let us multiply the first equation in (6.1) by σ and the second one by $(1 + n)$. Their summation leads to the identity $(d/dt)(\sigma^2 + (1+n)^2) + 2(\sigma^2 + (1+n)n) = 0$. As $\sigma^2 + (1+n)^2 \leqslant 2(\sigma^2 + n(1+n)) + 1$ we obtain for Σ and n the estimate

$$\| \Sigma(t,\cdot) \|_1^2 + \| 1 + n(t,\cdot) \|_0^2 \leqslant 2 + 2e^{-t}(\| \Sigma_0 \|_1^2 + \| 1 + n_0 \|_0^2). \tag{6.11}$$

To obtain a bound of a solution u we take the inner product in $Y = L^2(0,1)$ of the equation

$$\varepsilon u_t - \eta u_{xx} + \eta u_x(1) + \kappa \int_0^1 u = \mathscr{F}_\varepsilon \tag{6.12}$$

with $3\kappa u - \eta u_{xx}$. Since $u_x(1) = \int_0^1 u_{xx}$ for any $u \in D(B)$ we have

$$\frac{\varepsilon}{2} \frac{d}{dt} (3\kappa \| u \|^2 + \eta \| u_x \|^2) + \eta (3\kappa \| u_x \|^2 + \eta \| u_{xx} \|^2)$$
$$+ \left(\sqrt{3\kappa} \int_0^1 u + \eta u_x(1)/\sqrt{3} \right)^2 = \frac{4}{3} \eta^2 |u_x(1)|^2 + (\mathscr{F}_\varepsilon, 3\kappa u - \eta u_{xx})_Y.$$

Clearly, $\frac{4}{3}\eta^2 |u_x(1)|^2 = \frac{8}{3}\eta^2 \int_0^1 u_{xx} u_x \leqslant \frac{4}{3}\sqrt{\frac{\eta}{3\kappa}}\,\eta(3\kappa \| u_x \|^2 + \eta \| u_{xx} \|^2)$. Notice that $\frac{4}{3}\sqrt{\frac{\eta}{3\kappa}} < 1$ iff $\kappa > \frac{16}{27}\eta$. Furthermore, as $\| u \|_Y \leqslant \| u_x \|_Y \leqslant \| u_{xx} \|_Y$ for any $u \in D(B)$, we have $\| 3\kappa u - \eta u_{xx} \|_Y^2 \leqslant \max\{6\kappa, 2\eta\}(3\kappa \| u_x \|^2 + \eta \| u_{xx} \|^2)$. Assuming $\kappa > \frac{16}{27}\eta$ and applying Schwartz's inequality to the inner product $(\mathscr{F}_\varepsilon, 3\kappa u - \eta u_{xx})_Y$ one can show the existence of positive constants $\delta, C > 0$ independent of $\varepsilon \geqslant 0$, such that the following Lyapunov-type inequality is satisfied

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}(3\kappa\|u\|^2 + \eta\|u_x\|^2) + \delta(3\kappa\|u_x\|^2 + \eta\|u_{xx}\|^2) \leqslant C\|\mathscr{F}_\varepsilon\|_Y^2. \tag{6.13}$$

Henceforth, C, δ will denote any generic positive constant independent of $\varepsilon \geqslant 0$ and initial conditions. Now, it follows from the definition of $G^{(\Sigma)}$ and $\mathscr{F}_\varepsilon$ that

$$\|\mathscr{F}_\varepsilon\|_Y \leqslant \|\mathscr{F}_\varepsilon\|_0 \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2)(1 + \varepsilon\|u_x\|_Y). \tag{6.14}$$

Then differential inequality (6.13) implies that

$$\varepsilon\frac{\mathrm{d}U}{\mathrm{d}t} + \delta U \leqslant C(1 + \|\Sigma\|_1^4 + \|n\|_0^4)(1 + \varepsilon U), \tag{6.15}$$

where $U(t) := 3\kappa\|u(t,\cdot)\|_Y^2 + \eta\|u_x(t,\cdot)\|_Y^2$. To obtain a bound for $\|u_t\|_Y$ we differentiate equation (6.12) with respect to time. Denoting $w = u_t$, w is a solution of

$$\varepsilon w_t - \eta w_{xx} + \eta w_x(1) + \kappa\int_0^1 w = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon \tag{6.16}$$

subject to the boundary conditions $w_x(t, 0) = w(t, 1) = 0$. Since,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon = \kappa\int_0^1 \Sigma_t + \varepsilon\left(- \Sigma_t - \frac{1}{\eta}\int_x^1 [(1 + n)(w_x - \Sigma_{tx}) + n_t(u_x - \Sigma_x)]\right)$$

and

$$\|\Sigma_t\|_0 \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|u_x\|_Y^2)$$

$$|\Sigma_{tx}(\cdot, x)| \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|1 + n\|_0|u_x(\cdot, x)|)$$

$$|n_t(\cdot, x)| \leqslant C(1 + \|\Sigma\|_1^2 + \|n\|_0^2 + \|\Sigma\|_1|u_x(\cdot, x)|)$$

for a.e. $x \in [0, 1]$, we have

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon\right\|_Y \leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_\varepsilon\right\|_0 \leqslant C(1 + \|\Sigma\|_1^4 + \|n\|_0^4)(1 + \|u_x\|_Y^2 + \varepsilon\|w_x\|_Y). \tag{6.17}$$

Now one can proceed similarly as in the proof of inequality (6.15). By taking the inner product in Y of (6.16) with $3\kappa w - \eta w_{xx}$ we obtain a differential inequality

$$\varepsilon\frac{\mathrm{d}W}{\mathrm{d}t} + \delta W \leqslant C(1 + \|\Sigma\|_1^8 + \|n\|_0^8)(1 + U^2 + \varepsilon W), \tag{6.18}$$

where $W(t) := 3\kappa\|w(t,\cdot)\|_Y^2 + \eta\|w_x(t,\cdot)\|_Y^2$. Now it follows from the evolution equation for u that $\|u\|_{Y^1} = \|Bu\|_Y \leqslant \varepsilon\|u_t\|_Y + \|\mathscr{F}_\varepsilon\|_Y$. Since $\operatorname{Re}\sigma(B) > 0$ the norm $\|u\|_{Y^\beta}$, $3/4 < \beta < 1$ is dominated by $\|Bu\|_Y$. Taking into account estimates (6.11), (6.14), (6.15) and (6.18) and using a simple Gronwall's lemma argument we obtain *a priori* estimate

$$\|\Sigma(t,\cdot)\|_1 + \|n(t,\cdot)\|_0 + \|u(t,\cdot)\|_{Y^\beta} \leqslant \text{const} \quad \text{for any } t \in (0, T_{\max}),$$

where $T_{\max}$ is the maximal time of existence of a solution $(\Sigma(t,\cdot), n(t,\cdot), u(t,\cdot))$. Hence, $T_{\max} = \infty$ and the global-in-time existence of solutions in the phase space $\mathscr{X} = X \times Y^\beta$, $3/4 < \beta < 1$, is established.

In what follows, we will prove the existence of a ball in the phase-space $\mathscr{X}$ that dissipates any solution of (6.9). Let $(\Sigma_0, n_0, u_0) \in \mathscr{X}$ be an initial condition. With regard to (6.11) there exists time $T_1 = T_1(\Sigma_0, n_0) > 0$ such that

$$1 + \|\Sigma\|_1^p + \|n\|_0^p \leqslant 1995 \quad \text{for any } t \geqslant T_1 \;\; p = 4, 8.$$

One can choose $0 < \varepsilon_0 \ll 1$ such that $1995 C\varepsilon_0 < \delta$ where constants $C, \delta > 0$ appear in inequalities (6.15) and (6.18). Then

$$\varepsilon \frac{\mathrm{d}U(t)}{\mathrm{d}t} + \delta U(t) \leqslant C,$$

$$\varepsilon \frac{\mathrm{d}W(t)}{\mathrm{d}t} + \delta W(t) \leqslant C(1 + U^2(t)) \quad \text{for any } t \geqslant T_1,$$

where $C, \delta > 0$ are constants independent of $\varepsilon \in [0, \varepsilon_0]$ and the initial condition (Σ_0, n_0, u_0). It should be noted that the first differential inequality does not involve W. Then solving the above differential inequalities one can show the existence of a time $T = T(\Sigma_0, n_0, u_0) \geqslant T_1$ such that $U(t) + W(t) \leqslant C$ for any $t \geqslant T$. Recall that $\|u_t(t, \cdot)\|_Y^2 \leqslant W(t)$ and $\|\mathscr{F}_\varepsilon\|_Y$ can be estimated in terms of $U(t)$ for $t \geqslant T$ (see (6.15)). Thus, $\|Bu(t, \cdot)\|_Y \leqslant C$ for $t \geqslant T$. In summary, we have shown the existence of a constant $\varrho_0 > 0$ independent of $\varepsilon \in [0, \varepsilon_0]$ and initial data, such that

$$\|u(t, \cdot)\|_{Y^\beta}^2 + \|(\Sigma(t, \cdot), n(t, \cdot))\|_X^2 \leqslant \varrho_0 \quad \text{for any } t \geqslant T(\Sigma_0, n_0, u_0). \tag{6.19}$$

This means that the ball in $X \times Y^\beta$ of radius $\varrho_0^{1/2}$ is a dissipative set for solutions of (6.9), i.e. any solution enters this ball after a certain amount of time. In other words, the long-time behavior of solutions takes place inside this ball.

As is usual, we will modify the governing equation outside the ball of radius $\varrho_0^{1/2}$. Let $\zeta \in C_{bdd}^2(\mathbb{R}^+, \mathbb{R}^+)$ by any smooth cut-off function with the property $\zeta \equiv 1$ on $[0, 2\varrho_0]$, $\zeta \equiv 0$ on $[3\varrho_0, \infty)$. We define the modified functions $\bar{G} = \bar{G}^{(\Sigma)}, \bar{G}^{(n)}: X \times Y^\beta \to X$ and $\bar{\mathscr{F}}_\varepsilon: X \times Y^\beta \to Y$ as follows:

$$\bar{G}^{(i)}(\Sigma, n, u)(x) := \zeta(|\Sigma(x)|^2 + |\Sigma_x(x)|^2 + |n(x)|^2 + \|u\|_{Y^\beta}^2) G^{(i)}(\Sigma, n, u)(x),$$

$$\bar{\mathscr{F}}_\varepsilon(\Sigma, n, u)(x) := \zeta(|\Sigma(x)|^2 + |\Sigma_x(x)|^2 + |n(x)|^2 + \|u\|_{Y^\beta}^2) \mathscr{F}_\varepsilon(\Sigma, n, u)(x)$$

for $x \in [0, 1]$, i stands either for Σ or n. We remind ourselves that the mapping $u \mapsto \|u\|_{Y^\beta}^2$ is a twice continuously Frechet differentiable function from Y^β to R. The modified functions $\bar{\mathscr{G}}$ and $\bar{\mathscr{F}}_\varepsilon$ obey hypothesis (H4). With regard to the definitions of Q and θ (see (6.7), (6.8)) it is easy to verify that hypotheses (H2) and (H3) are also fulfilled. Since $\mathscr{F}_0$ does not depend on u, the structural condition (5.1) is satisfied for any $\delta_0 > 0$. Taking into account Lemma 6.2 and (6.3) we have shown that all the conclusions of Theorem 1.1 hold for system (6.9) except for the statement that $\mathscr{M}_\varepsilon$ is an invariant manifold for the semi-flow generated by solutions of (6.9). This is due to the fact that we have modified the governing equations far from the vicinity of a dissipative ball of the radius $\varrho_0^{1/2}$. Hence, $\mathscr{M}_\varepsilon$ need not be invariant outside this ball. On the other hand, it should be emphasized that the long-time behaviour of solutions of (6.9) takes place inside this ball as it was shown in (6.19). Henceforth, we will therefore refer to $\mathscr{M}_\varepsilon$ as a local invariant manifold for solutions of (6.9).

Now we can rewrite the feedback law in terms of the flow variables σ, n, v as follows: $f = f_\varepsilon(\sigma, n)$ where $f_\varepsilon(\sigma, n) = \Xi_\varepsilon(\Sigma, n)$. For the velocity field on the manifold $\mathscr{M}_\varepsilon$ we obtain the expression $v = \Psi_\varepsilon(\sigma, n) = (u - \Sigma)/\eta = (\Phi_\varepsilon(\Sigma, n) - \Sigma)/\eta$ where $\Sigma(x) = -\int_x^1 \sigma(\xi)\,\mathrm{d}\xi$. We infer from the continuity of the imbedding $Y^\beta \subsetneq C^1_{bdd}(0, 1)$, $\frac{3}{4} < \beta$, (see Lemma 6.2) that

$$\Psi_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to C^1_{bdd}(0, 1)$$

is C^1 smooth and Ψ_ε is locally C^1 close to Ψ_0. Similarly, one has

$$f_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to \mathbb{R}$$

is C^1 smooth and f_ε is locally C^1 close to Ψ_0. Furthermore, with regard to Remark 6.1 we have an explicit formula for f_0 and Ψ_0,

$$f_0 = 3\eta Q_{\mathrm{fix}} + 3\int_0^1 \Sigma(\xi)\,\mathrm{d}\xi$$

$$v(x) = \Psi_0(\sigma, n)(x) = \left((1 - x^2) f_0/2 + \int_x^1 \sigma(\xi)\,\mathrm{d}\xi\right).$$

Summarizing the results of section 6 we can state the following theorem.

Theorem 6.3. *There exists* $0 < \varepsilon_0 \ll 1$ *such that, for any* $\varepsilon \in [0, \varepsilon_0]$, *the system of equations governing the Poiseuille flow of the Johnson–Segalman–Oldroyd fluid* (6.1)–(6.2) *admits a dissipative feedback synthesis of the pressure gradient*

$$f = f_\varepsilon(\sigma, n), \quad \sigma, n \in C^0_{bdd}(0, 1)$$

that stabilizes the volumetric flow rate at the prescribed value Q_{fix}. *The mapping* $f_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to \mathbb{R}$ *is* C^1*-smooth and* f_ε *is locally* C^1 *close to* f_0 *whenever* $\varepsilon > 0$ *is small enough. The feedback law* f_0 *for the reduced system of equations has the form*

$$f_0(\sigma, n) = 3\eta Q_{\mathrm{fix}} - 3\int_0^1 \xi\sigma(\xi)\,\mathrm{d}\xi.$$

The initial-value problem (6.1)–(6.2) *with* $f = f_\varepsilon(\sigma, n)$ *possesses an infinite dimensional locally invariant attractive manifold* $\mathscr{M}_\varepsilon$. *The volumetric flow rate for solutions belonging to* $\mathscr{M}_\varepsilon$ *is fixed at the prescribed value* Q_{fix}. *The manifold* $\mathscr{M}_\varepsilon$ *is a* C^1 *smooth graph,*

$$\mathscr{M}_\varepsilon = \{(\sigma, n, v),\ v = \Psi_\varepsilon(\sigma, n),\ \sigma, n \in C^0_{bdd}(0, 1),\ \|\sigma\|_0^2 + \|n\|_0^2 < \varrho_0\},$$

where $\Psi_\varepsilon : C^0_{bdd}(0, 1) \cap \{\sigma, \sigma(0) = 0\} \times C^0_{bdd}(0, 1) \to C^1_{bdd}(0, 1)$ *is a* C^1 *function which is locally* C^1 *close to* Ψ_0,

$$\Psi_0(\sigma, n)(x) = \frac{1}{\eta}\left((1 - x^2) f_0(\sigma, n)/2 + \int_x^1 \sigma(\xi)\,\mathrm{d}\xi\right), \quad x \in [0, 1].$$

Finally, the flow when restricted to the manifold $\mathscr{M}_\varepsilon$ *is governed by the following system of functional differential equations*:

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)\Psi_\varepsilon(\sigma, n)_x, \\ n_t &= -n - \sigma\Psi_\varepsilon(\sigma, n)_x, \end{aligned} \tag{FDE}$$

$(t, x) \in [0, \infty) \times [0, 1]$, *subject to boundary and initial conditions* (6.2). *For small values of* $\varepsilon > 0$, *the vector field defined by the right-hand side of* (*FDE*) *is locally* C^1 *close to that of the reduced system of equations*

$$\begin{aligned} \sigma_t &= -\sigma + (1 + n)(T - \sigma)/\eta, \\ n_t &= -n - \sigma(T - \sigma)/\eta, \end{aligned} \tag{QFDE}$$

where $T = -f_0(\sigma, n)x$.

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