Finite amplitude magnetoconvection determined by modified Taylor's constraint

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Abstract. The complexity of the full MHD problem has prompted the study of simpler problems focusing on part of the whole problem. The two main classes are kinematic mean-field dynamos and convection in the presence of a prescribed magnetic field (magnetoconvection). In their simplest form both are linear. In this paper we consider the problem of magnetoconvection where the nonlinear effect of a geostrophic flow determined by Ekman suction is included. A weakly nonlinear theory, based on perturbation techniques and functional analysis, is used to determine the relevant bifurcation structure in the vicinity of the critical Rayleigh number.

Key words: magnetoconvection, modified Taylor's constraint, perturbation techniques, weakly nonlinear analysis

1. Introduction

The fluid motion in Earth-like planet cores can be characterized by magnetostrophic approximation with dominating Lorentz, Coriolis, buoyancy and pressure forces in the equation of motion. The approximation with zero viscous forces has a solution, if and only if the so called Taylor's constraint is satisfied (see Section 2).

The detailed study of the fine dynamics in planetary interiors, however, gives rise to a question if such an approximation can reflect real conditions with extremely small but non-zero viscous forces. Therefore, the case of the so called modified Taylor's constraint arises when the geostrophic flow is encountered into the governing equations. This makes the whole problem nonlinear.

In this paper we study a model of finite amplitude rotating magnetoconvection. The research has been motivated by linear study of (Soward 1979),

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(Brestenský, Ševčík 1994)[1] and is similar to nonlinear model studied by Skinner and Soward. Unlike the approach of (Skinner, Soward 1988, 1991) where solutions satisfying Taylor's constraint have been found for sufficiently high Rayleigh number we focus our attention on an early state of magnetoconvection in the presence of geostrophic flow. This means that local structure of solutions near the critical Rayleigh number is studied. Thus the methods and techniques employed in the present paper are different from those of the previously mentioned references.

The outline of the paper has been designed in order to meet the criteria on both physical understanding as well as mathematical clarity. In Sections 2 and 3 we derive a system of nonlinear PDE's governing motion which is periodic in both time and in the azimuthal variable. In Section 3 we present a method on how to obtain a power series expansion of a solution in terms of a small unfolding parameter. We make use of the so-called solvability condition in order to determine leading coefficients of the expansions. Mathematical and physical conclusions of the results are presented in Section 5. Appendix provides mathematical support for Sections 3, 4.

2. Description of the nonlinear model

The model considered is an infinite horizontal layer of width d rotating rapidly with angular velocity $\Omega_0\hat{\mathbf{z}}$. The layer contains an electrically conducting Boussinesq fluid permeated by an azimuthal magnetic field $\mathbf{B}_0$ linearly growing with the distance from the vertical rotation axis. An unstable temperature gradient is maintained by heating the fluid from below and cooling from above. The fluid layer is supposed to have free perfectly electrically and thermally conductive horizontal boundaries.

The convective instability in this rotating system is caused by the vertical temperature gradient and manifests itself by perturbations of the velocity $\mathbf{u}$, the magnetic field $\mathbf{b}$ and the temperature ϑ which refer to the basic state represented by $\mathbf{U}_0$, $\mathbf{B}_0$, T_0.

Considering very small perturbations the whole problem can be solved as linear (see e.g. BS). When viscosity and inertia in the layer are neglected, an arbitrary geostrophic flow, which is aligned with the applied azimuthal magnetic field and independent of the axial coordinate, can be superimposed on the basic axisymmetric state. In this inviscid limit, the angular velocity $\Omega(s)$ of geostrophic flow $s\Omega(s)\hat{\boldsymbol{\varphi}}$ which occurs at the onset of convection is such that the net torque on geostrophic cylinders vanishes (Taylor's condition).

If viscous boundary-layer effects (i.e. Ekman suction) are included, the dynamics of the model results into differential rotation of the fluid layer which acts to inhibit convection. The interaction between boundary layer and rest of the

[1] Henceforth referred to as BS.

fluid is non-linear and is expressed in terms of modified (or relaxed) Taylor's constraint.

As a driving force (heating) is increased, the non-linear system determined by the Ekman suction usually evolves to the special equilibrium state, i.e. Taylor state, when viscous effects have no longer a major influence on the solution. In duct model (Soward 1986) and the cylindrical annulus (Skinner, Soward 1988) Taylor solutions were found for a sufficiently high Rayleigh number.

Here, we focus our attention on the early state of magnetoconvection, affected by Ekman suction, with the Rayleigh number slightly beyond the critical value R_c which is known from linear stability analysis BS[2].

We investigate the stability of the basic state

$$\mathbf{U}_0 = \mathbf{0}\,, \quad \mathbf{B}_0 = B_M \frac{s}{d}\,\hat{\boldsymbol{\varphi}}\,, \quad T_0 = T_1 - \frac{\Delta T}{d}\,(z + \frac{d}{2})\,. \tag{1}$$

We non-dimensionalise the problem with the use of characteristic length d, magnetic diffusion time d^2/η, magnetic field B_M, and temperature difference across the layer ΔT. The equations in the cylindrical polar coordinates $(s,\ \varphi,\ z)$ governing the evolution of perturbations $\mathbf{u}$, $\mathbf{b}$, ϑ thus gain the following form

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \Lambda\,[\,(\,\nabla \times s\,\hat{\boldsymbol{\varphi}}) \times \mathbf{b} + (\,\nabla \times \mathbf{b}\,) \times s\,\hat{\boldsymbol{\varphi}}] + R\,\vartheta\,\hat{\mathbf{z}}\,, \tag{2}$$

$$\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \times \mathbf{b}\,) = \nabla \times (\,\mathbf{u} \times s\,\hat{\boldsymbol{\varphi}}\,) + \nabla^2 \mathbf{b}\,, \tag{3}$$

$$\frac{1}{q_R}\left(\frac{\partial \vartheta}{\partial t} + (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \cdot \nabla\,)\,\vartheta\right) = -\mathbf{u} \cdot \nabla T_0 + \nabla^2 \vartheta\,, \tag{4}$$

$$\nabla \cdot \mathbf{b} = 0\,, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

where $\hat{\mathbf{z}}$ is the non-dimensionalised axial vector.

In (2 - 6) the dimensionless parameters, the modified Rayleigh number R, the Elsasser number Λ, the Ekman number E and the Roberts number q_R, are defined by

$$R = \frac{g d \Delta T \alpha}{2\Omega_0 \kappa}\,, \qquad \Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \eta \mu}\,, \qquad E = \frac{\nu}{2d^2\Omega_0}\,, \qquad q_R = \frac{\kappa}{\eta}$$

where κ, η are thermal and magnetic diffusivities, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, μ is permeability and ρ_0 is density.

Notice that the presence of geostrophic flow[3] , $\Omega(s)$ involved in nonlinear convective terms in induction equation (3) and heat equation (4), is the only difference from the linearized model in BS.

[2]In BS the viscous forces are considered though with no Ekman suction effects.
[3]Hereafter, we will often use for the angular velocity $\Omega(s)$ of geostrophic flow $s\Omega(s)\hat{\boldsymbol{\varphi}}$ simpler shorter terms, i.e. geostrophic flow $\Omega(s)$ or geostrophic velocity $\Omega(s)$.

Denoting averaging over φ by $\langle \dots \rangle^\varphi \equiv 1/(2\pi) \int_0^{2\pi} \dots d\varphi$ and azimuthal component of Lorentz force by $F_{M\varphi} \equiv [(\nabla \times \mathbf{B}) \times \mathbf{B}]_\varphi$ and using the fact of splitting magnetic field $\mathbf{B}$ on basic field $\mathbf{B}_0$ and perturbation $\mathbf{b}$, $\mathbf{B} \equiv \mathbf{B}_0 + \mathbf{b}$ ($\langle \mathbf{B} \rangle^\varphi = \mathbf{B}_0$, $\langle \mathbf{b} \rangle^\varphi = \mathbf{0}$), the angular velocity $\Omega(s)$ of geostrophic flow in our magnetoconvection model can be expressed only in terms of the magnetic field perturbation $\mathbf{b}$, i.e. (see e.g. Skinner, Soward 1988)

$$\Omega(s) = \frac{\Lambda}{(2E)^{1/2} s} \int_{z_B}^{z_T} \langle F_{M\varphi} \rangle^\varphi \, dz \quad \text{with} \quad \langle F_{M\varphi} \rangle^\varphi = \langle [(\nabla \times \mathbf{b}) \times \mathbf{b}]_\varphi \rangle^\varphi \,, \tag{7}$$

because the contribution $\langle [(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0]_\varphi \rangle^\varphi$ from basic field to azimuthally averaged Lorentz force $\langle F_{M\varphi} \rangle^\varphi$ is zero [see also (Fearn, Proctor, Sellar 1994)]. We note that the expression (7) is well-known as modified Taylor's constraint.

The above structure described by equations (2 - 6) and (7) reveals an important feedback in the convecting system affected by Ekman suction, i.e. the nonlinear interaction between perturbations $\mathbf{u}$, $\mathbf{b}$, ϑ and geostrophic flow $\Omega(s)$. This nonlinear problem will be dealt with and solved with the use of perturbation methods in the following sections.

3. From the physical nonlinear problem towards an abstract nonlinear problem

3.1. Nonlinear terms and their representing functions

In this section we rearrange the nonlinear system (2 - 6) into such a form which can be easily solved to obtain bifurcated solutions with the use of perturbation techniques.

Due to the splitting of velocity perturbation $\mathbf{u}$ and magnetic field perturbation $\mathbf{b}$ into their poloidal and toroidal parts

$$\mathbf{u} \;=\; k^{-2}[\nabla \times (\nabla \times \widetilde{w}\,\hat{\mathbf{z}}) + \nabla \times \widetilde{\omega}\,\hat{\mathbf{z}}]\,, \tag{8}$$

$$\mathbf{b} \;=\; k^{-2}[\nabla \times (\nabla \times \widetilde{b}\,\hat{\mathbf{z}}) + \nabla \times \widetilde{j}\,\hat{\mathbf{z}}] \tag{9}$$

the perturbations (or fluctuations) $\mathbf{u}$, $\mathbf{b}$ can be expressed in terms of representing functions $\widetilde{w}$, $\widetilde{\omega}$, $\widetilde{b}$, $\widetilde{j}$ depending on coordinates z, s, φ and time t and being symbolized by $\widetilde{f}(z, s, \varphi, t)$, or shortly $\widetilde{f}$, as in BS. With regard to the above notations, the temperature perturbation ϑ can be also referred to as a representing function and thus denoted by $\widetilde{\vartheta}$.

Let furthermore the representing functions $\widetilde{b}$, $\widetilde{j}$ as well as $\widetilde{w}$, $\widetilde{\omega}$ and $\widetilde{\vartheta}$ be of the form

$$\tilde{f}(z, s, \varphi, t) = \Re e\{f_m(z, s) \exp(im\varphi + \lambda t)\} \tag{10}$$

where the functions of $f_m(z, s)$ describe a preliminarily non-separated dependence on vertical coordinate z and radial coordinate s .

The assumption (10) is analogous to that in (Soward 1979) or BS where the case $\Omega(s) \equiv 0$ has been considered with separable functions $f_m(z,s) = f(z)J_m(ks)$. Therefore the meaning of all parameters entering (10) will be left unchanged. Let us summarize that

m is an azimuthal wavenumber (integer),

k is a radial wavenumber (real),

λ is a complex frequency (related to a real frequency via $\lambda = i\sigma$)

and $J_m(ks)$, Bessel function of the first kind, will be also used below.

The vector nonlinearity in the induction equation (3) being solenoidal, can be split as well as $\mathbf{u}$, $\mathbf{b}$ in (8, 9), i.e.

$$-\nabla \times [\, s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \times \mathbf{b}\,] \;=\; k^{-2}[\nabla \times (\nabla \times \widetilde{P}\,\hat{\mathbf{z}}) + \nabla \times \widetilde{T}\,\hat{\mathbf{z}}]\,, \tag{11}$$

where the functions $\widetilde{P}$, $\widetilde{T}$ are also expressible via (10). As the nonlinear term in the induction equation describes the interaction between magnetic field perturbation $\mathbf{b}$ and geostrophic flow $\Omega(s)$, it is reasonable to expect the functions $\widetilde{P}$ and $\widetilde{T}$ be dependent on functions $\widetilde{b}$, $\widetilde{j}$ and $\Omega(s)$. To make this relation more evident, we simplify the nonlinear term into

$$-\nabla \times [\, s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \times \mathbf{b}\,] \;=\; \Omega(s)\,\frac{\partial_1}{\partial\varphi}\,\mathbf{b} - s\,b_s\frac{\partial\Omega(s)}{\partial s}\,\hat{\boldsymbol{\varphi}} \tag{12}$$

where $\partial_1/\partial\varphi$ is a differential operator (commonly used in (Skinner, Soward 1988)) which keeps the unit vectors $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\mathbf{z}}$ fixed in direction and b_s is a radial component of magnetic field perturbation $\mathbf{b}$. The b_s together with other cylindrical components b_φ, b_z of $\mathbf{b}$ can be expressed using $\widetilde{b}$ and $\widetilde{j}$ as follows

$$\begin{aligned}
b_s \;&=\; \frac{1}{k^2}\left[\frac{\partial^2}{\partial s\,\partial z}\,\widetilde{b} + \frac{1}{s}\,\frac{\partial}{\partial\varphi}\,\widetilde{j}\right], \\
b_\varphi \;&=\; \frac{1}{k^2}\left[\frac{1}{s}\,\frac{\partial^2}{\partial\varphi\,\partial z}\,\widetilde{b} - \frac{\partial}{\partial s}\,\widetilde{j}\right], \\
b_z \;&=\; \mathcal{J}_m\,\widetilde{b}
\end{aligned} \tag{13}$$

where the operator

$$\mathcal{J}_m \equiv \;-\frac{1}{k^2}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\,\frac{\partial}{\partial s} - \frac{m^2}{s^2}\right), \tag{14}$$

will be referred to as the Bessel differential operator, due to the property

$$\mathcal{J}_m\;\{J_m(ks)\} = J_m(ks)\,. \tag{15}$$

To emphasize fact of $\Omega(s)$ being a functional (see the following paragraph, in particular (22)) we use a partial derivative $\partial/\partial s$ in the above formula. The calculations to follow are very tedious and therefore will not be presented in this paper.

As a result, we obtain a set of operator equations for the functions $\widetilde{P}$ and $\widetilde{T}$

$$\mathcal{J}_m \widetilde{P} = im \{ \Omega(s) \mathcal{J}_m \} \widetilde{b} , \tag{16}$$

$$\mathcal{J}_m \widetilde{T} = im \{ \Omega(s) \mathcal{J}_m \} \widetilde{j} + im \mathcal{P}_\Omega \widetilde{j} + \mathcal{T}_\Omega D\widetilde{b} \tag{17}$$

where tilded functions depend on variables z, s, φ and t , $D = \partial/\partial z$ and $\mathcal{P}_\Omega$, $\mathcal{T}_\Omega$ are following differential operators

$$\mathcal{P}_\Omega = -\frac{1}{k^2} \left\{ \frac{\partial^2 \Omega(s)}{\partial s^2} + \frac{\partial \Omega(s)}{\partial s} \left[2 \frac{\partial}{\partial s} + \frac{1}{s} \right] \right\} , \tag{18}$$

$$\mathcal{T}_\Omega = -\frac{1}{k^2} \left\{ s \frac{\partial^2 \Omega(s)}{\partial s^2} \frac{\partial}{\partial s} + s \frac{\partial \Omega(s)}{\partial s} \left[\frac{m^2}{s^2} + \frac{2}{s} \frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2} \right] \right\} . \tag{19}$$

Complicated form of the above equations is especially due to geostrophic flow $\Omega(s)$ which enters the nonlinear problem in a very complex implicit manner.

Denoting the nonlinearity in the heat equation (4) by $\widetilde{S}$, we can express it in terms of the representing function $\widetilde{\vartheta}$ in the following way

$$\widetilde{S}(z, s, \varphi, t) = im \, \Omega(s) \, \widetilde{\vartheta}(z, s, \varphi, t) . \tag{20}$$

The above operator equations for the nonlinearity representing functions $\widetilde{P}$, $\widetilde{T}$ and $\widetilde{S}$ will enable us to set up the complete formulation of the abstract nonlinear problem in the end of this section.

3.2. Determination of geostrophic flow $\Omega(s)$ from modified Taylor's constraint and nonlinear terms analysis

In this paragraph, the geostrophic flow $\Omega(s)$ which enters the nonlinearities will be specified in more detail. From Section 2 we already know that the geostrophic flow (i.e. its angular velocity) $\Omega(s)$ is linked with magnetic field perturbation $\mathbf{b}$ through modified Taylor's constraint (7). Our goal is to express $\Omega(s)$ and thereby the nonlinearities $\widetilde{P}$ and $\widetilde{T}$ in terms of the functions $\widetilde{b}$ and $\widetilde{j}$ so that only the representing functions would enter the abstract nonlinear problem. For a horizontal layer in cylindrical coordinates, it follows that

$$\int_{z_B}^{z_T} \langle F_{M\varphi} \rangle^\varphi \, dz = \frac{1}{s^2} \frac{d}{ds} \left(s^2 \int_{z_B}^{z_T} \langle b_\varphi b_s \rangle^\varphi \, dz \right) + \langle b_\varphi b_z \rangle^\varphi \Big|_{z_B}^{z_T} \tag{21}$$

where b_s, b_φ, b_z are given by (9, 13).

In expression (21)[4] z_B and z_T delimit the bottom and top boundary of the layer, respectively, and we recall that $\langle \quad \rangle^\varphi$ denotes averaging over azimuthal coordinate φ . The second term on the right-hand side in (21) is a boundary

[4]Referred to as an alternative form of modified Taylor's constraint in (Fearn 1994).

term vanishing in case of symmetric boundaries with the electric conductivity equal zero or infinity. Plugging the expressions (13) together with the ansatz (10) into (21), modified Taylor's constraint can be rewritten as

$$\Omega(s) = \frac{\Lambda}{2\,(2E)^{1/2}} \cdot \frac{1}{s}\,\Re e\,\left\{ \frac{1}{s^2}\,\frac{\partial}{\partial s}\,[\,s^2\, I(s)\,] - B(s) \right\} \tag{22}$$

where

$$I(s) = \frac{1}{k^4}\int_{z_B}^{z_T} \left(\frac{m^2}{s^2}\, j_m(z,s) D\overline{b_m(z,s)} \;-\; \frac{\partial}{\partial s} j_m(z,s)\,\frac{\partial}{\partial s} D\overline{b_m(z,s)} \right) dz$$

is a functional (integral) and

$$B(s) = \frac{1}{k^2}\frac{\partial}{\partial s} j_m(z,s)\;\mathcal{J}_m\overline{b_m(z,s)}\,\big|_{z_B}^{z_T}$$

is a boundary term while $D = \partial/\partial z$ and an overbar denotes the complex conjugation of $b_m(z,s)$.

Notice that $\Omega(s)$ does not depend on the azimuthal coordinate φ and time t because of the structure of $I(s)$ and $B(s)$ above. Therefore the ansatz (10) will enable us to operate with simpler representing functions $f_m(z,s)$ of only two arguments z and s.

3.3. The abstract nonlinear problem and its analysis

Now we are able to set up the complete formulation of the abstract nonlinear problem for representing functions of $f_m(z,s)$, i.e. for $w_m(z,s)$, $\omega_m(z,s)$ and $b_m(z,s)$, $j_m(z,s)$, $\vartheta_m(z,s)$. This formulation is capable of capturing all the important dynamics that we are interested in. Namely periodic motion will be revealed as a solution of the set of nonlinear equations (23) below which will be solved by means of perturbation techniques in the following section.

The equations are presented in a rather schematic way

$$\begin{aligned}
0 \;&=\; -\,Dw_m(z,s) + 2\Lambda\, Db_m(z,s) - im\Lambda\, j_m(z,s)\,,\\
0 \;&=\; -\,D\omega_m(z,s) + 2\Lambda\, Dj_m(z,s) + im\Lambda\,(D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s) - Rk^2\,\vartheta_m(z,s)\,,\\
\lambda\, b_m(z,s) + P_m(z,s) \;&=\; im\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s)\,,\\
\lambda\, j_m(z,s) + T_m(z,s) \;&=\; im\,\omega_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, j_m(z,s)\,,\\
\frac{1}{q_R}\,(\,\lambda\,\vartheta_m(z,s) + S_m(z,s)\,) \;&=\; \mathcal{J}_m\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\,\vartheta_m(z,s)
\end{aligned} \tag{23}$$

where the nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are expressed in terms of the eigenfunctions $f_m(z,s)$ and geostrophic velocity $\Omega(s)$ as follows

$$\begin{aligned}
P_m(z,s) &= im\,\Omega(s)\, b_m(z,s) \;-\; im\,\mathcal{J}_m^{-1}\,\{\mathcal{P}_\Omega\; b_m(z,s)\,\}\;,\\
T_m(z,s) &= im\,\Omega(s)\, j_m(z,s) \;+\; \mathcal{J}_m^{-1}\,\{\mathcal{T}_\Omega\; Db_m(z,s)\,\}\;,\\
S_m(z,s) &= im\,\Omega(s)\,\vartheta_m(z,s)
\end{aligned} \tag{24}$$

while $\mathcal{J}_m^{-1}$ is the inverse Bessel differential operator, $\mathcal{P}_\Omega$ and $\mathcal{T}_\Omega$ are differential operators (18, 19) and $\Omega(s)$ is given by (22). It is shown in Appendix that $\mathcal{J}_m^{-1}$ is a well defined bounded linear operator on a suitable function space. The three above expressions can be obtained from the implicit operator equations (16, 17) and equation (20) after a series of simple formal operations.

With regard to the previous paragraph we already know that the geostrophic velocity is neither arbitrary nor is prescribed, but is nonlinearly related to the functions $b_m(z,s)$ and $j_m(z,s)$. Furthermore from (22) it is obvious that $\Omega(s)$ is a quadratic nonlinearity in $b_m(z,s)$ and $j_m(z,s)$. Taking into account (24), it is obvious that $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are cubic nonlinearities in $b_m(z,s)$, $j_m(z,s)$ and $\vartheta_m(z,s)$.

Due to the above facts, the nonlinear system (23) has the following symmetry: $f_m(z,s)$ solves (23) if and only if $-f_m(z,s)$ does. The magnetohydrodynamic system, the evolution of which is ruled by such a nonlinear system, manifests itself by the Hopf bifurcation when being driven in the vicinity of the critical Rayleigh number R_c. This is an important property of the dynamic system and must be taken into consideration when solving our nonlinear problem.

Note that the linearization of (23) at a trivial solution leads to the linear system of equations presented in (Soward 1979) or e.g. in BS for zero geostrophic flow ($s\Omega(s)\hat{\boldsymbol{\varphi}} = \mathbf{0}$). In this case the perfect separability

$$f_m(z,s) = f(z) J_m(ks) \tag{25}$$

can be assumed and the linear problem for representing functions of the simplest kind $f(z)$ can be obtained.

To conclude this section, we emphasize that our abstract nonlinear problem was set up as a problem arising from the linear one. Therefore the linear stability analysis in BS can be carried over the linear system to the nonlinear one.

4. The solvability condition of nonlinear problem

4.1. The adjoint problem and its solution

The goal of this section is to show the existence of a solution of the system of equations (23) as well as to study the dependence of a solution on the Rayleigh number R. To this end, we will rewrite (23) as an abstract nonlinear problem in a suitable functional space. We first turn our attention to the linear part of the system (23). Similarly as in (Proctor, Weiss 1982) the corresponding linear operator $\mathcal{L}$ admits a matrix representation

$$\mathcal{L} \equiv \begin{pmatrix} -D & 0 & 2\Lambda\, D & -im\Lambda & 0 \\ 0 & -D & im\Lambda\mathcal{D}^2 & 2\Lambda D & -R_c k^2 \\ im & 0 & (\mathcal{D}^2 - \lambda_c) & 0 & 0 \\ 0 & im & 0 & (\mathcal{D}^2 - \lambda_c) & 0 \\ \mathcal{J}_m & 0 & 0 & 0 & (\mathcal{D}^2 - \lambda_c/q_R)) \end{pmatrix} \tag{26}$$

where $\mathcal{D}^2 = D^2 - k^2\mathcal{J}_m$. Thus the linear part of (23) has the form $\mathcal{L}\psi$ where ψ is a vector function

$$\psi^T \equiv\ (w_m(z,s),\, \omega_m(z,s),\, b_m(z,s),\, j_m(z,s),\, \vartheta_m(z,s))\,. \tag{27}$$

Recall that the linear eigenvalue problem has been studied in BS. The aim of this paper was to determine critical values of Rayleigh number R_c and complex frequency $\lambda_c = i\sigma_c$, as well as to construct a solution ψ of the homogeneous matrix equation

$$\mathcal{L}\,\psi = 0\,. \tag{28}$$

Now the nonlinear problem (23) can be written as

$$\mathcal{L}\,\psi = N(\psi) \tag{29}$$

where the term $N(\psi)$ contains all nonlinearities $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$ involved in (23).

To solve the above semilinear problem by means of functional analysis we have to find the kernel of the corresponding adjoint operator $\mathcal{L}^+$. More precisely, one has to find a solution ψ^+ of the adjoint linear equation

$$\mathcal{L}^+\psi^+ = 0\,. \tag{30}$$

A solution of the above problem will be taken as so-called test function in order to determine higher order terms in power series expansion for a solution ψ of (29). It is worthwhile noting that since we are dealing with a nonselfadjoint operator $\mathcal{L}$ the kernel of $\mathcal{L}^+$ need not coincide with that of $\mathcal{L}$. Moreover, the domains of the definition of $\mathcal{L}$ and $\mathcal{L}^+$ are different. Hereafter Z will denote the space of all Lebesgue square integrable functions defined on $G = G_n = (z_B, z_T)\times(0, s_n)$. The domains of definition of $\mathcal{L}$ and $\mathcal{L}^+$ are the Sobolev spaces X and X^+ . In Appendix they will be defined with respect to boundary conditions (34, 38) and (35, 39), respectively. Some of useful properties of the spaces X, X^+ are also discussed there. At this point, we only stress the fact that one has to be careful in a choice of functional spaces. A wrong functional setting may lead to wrong conclusions, especially when using Fredholm alternative techniques.

We proceed by a definition of a bilinear form $\langle\,.\,|\,.\,\rangle$ on $Z\times Z$. Such a bilinear form plays a crucial role in further analysis and is defined as

$$\langle\,\psi\,|\,\chi\,\rangle = \langle\,\psi\,\overline{\chi}\,\rangle^{zs} \equiv \sum \int_G f(z,s)\,\overline{g(z,s)}\;s\,ds\,dz \tag{31}$$

where $\sum$ denotes the summation over all components f and g of vectors ψ and χ , respectively. Then, by the Cauchy-Schwartz inequality, $\langle\,.\,|\,.\,\rangle$ is well-defined and continuous with respect to $(\psi,\chi)\in Z\times Z$.

Now we are in a position to define an adjoint operator to $\mathcal{L}$ with respect to duality $\langle\, .\, |\, .\, \rangle$. The adjoint linear operator $\mathcal{L}^+$ is completely determined by the relation

$$\langle\, \mathcal{L}\,\psi \,|\, \psi^+ \,\rangle = \langle\, \psi \,|\, \mathcal{L}^+\, \psi^+ \,\rangle \qquad \text{for all } \psi \in X\,,\ \psi^+ \in X^+\,. \tag{32}$$

We will show that the matrix linear operator

$$\mathcal{L}^+ = \begin{pmatrix} D & 0 & -im & 0 & \mathcal{J}_m \\ 0 & D & 0 & -im & 0 \\ -2\Lambda\, D & -im\Lambda\mathcal{D}^2 & (\mathcal{D}^2+\lambda_c) & 0 & 0 \\ im\Lambda & -2\Lambda\, D & 0 & (\mathcal{D}^2+\lambda_c) & 0 \\ 0 & -R_c k^2 & 0 & 0 & (\mathcal{D}^2+\lambda_c/q_R) \end{pmatrix} \tag{33}$$

obeys the definition (32). Suppose that functions ψ and ψ^+ satisfy the boundary conditions

$$\psi(z,0) = \psi(z,s_n) \ = 0 \qquad \text{for all } z \in (z_B, z_T)\,, \tag{34}$$

$$\psi^+(z,0) = \psi^+(z,s_n) \ = 0 \qquad \text{for all } z \in (z_B, z_T)\,. \tag{35}$$

Then using Green's formula we obtain

$$\langle\, \mathcal{L}\,\psi \,|\, \psi^+ \,\rangle = \langle\, \psi \,|\, \mathcal{L}^+\, \psi^+ \,\rangle + \mathcal{B} \tag{36}$$

where $\mathcal{B}$ is a boundary term,

$$\begin{aligned} \mathcal{B} = &- w_m(z,s) w_m^+(z,s) - \omega_m(z,s)\omega_m^+(z,s) \\ &- b_m(z,s)\ \left[\, 2\Lambda w_m^+(z,s) + im\Lambda\, D\omega_m^+(z,s)\,\right] \\ &+ Db_m(z,s)\ \left[\, im\Lambda\,\omega_m^+(z,s) + b_m^+(z,s)\,\right] \\ &+ j_m(z,s)\ \left[\, 2\Lambda\omega_m^+(z,s) - Dj_m^+(z,s)\,\right] + Dj_m(z,s) j_m^+(z,s) \\ &- \vartheta_m(z,s) D\vartheta_m^+(z,s) + D\vartheta_m(z,s)\vartheta_m^+(z,s)\,. \end{aligned} \tag{37}$$

Our next goal is to set up suitable boundary conditions which will guarantee that the term $\mathcal{B}$ vanishes for $\psi \in X$, $\psi^+ \in X^+$.

Let us suppose that the boundaries $z = z_B,\, z_T$ are infinitely thermally and electrically conductive. This means that

$$\begin{aligned} w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = Dj_m(z,s) = 0\,, \\ \text{for all } z = z_B,\, z_T,\ \text{and } s \in (0, s_n)\,. \end{aligned} \tag{38}$$

Now it is clear that $\mathcal{B}$ vanishes provided that ψ^+ satisfies dual boundary conditions at $z = z_B,\, z_T$

$$\begin{aligned} \omega_m^+(z,s) = \vartheta_m^+(z,s) = b_m^+(z,s) = Dj_m^+(z,s) = 0\,, \\ \text{for all } z = z_B, z_T,\ \text{and } s \in (0, s_n)\,. \end{aligned} \tag{39}$$

We proceed by a construction of a function ψ^+ satisfying the adjoint equation $\mathcal{L}^+\psi^+ = 0$. We assume that all the components of a vector $\psi^+ = (w_m^+(z,s),\, \omega_m^+(z,s),\, b_m^+(z,s),\, j_m^+(z,s),\, \vartheta_m^+(z,s$ can be separated as follows

$$f_m(z,s) = f^+(z)\, J_m(ks) \tag{40}$$

where the adjoint functions $f^+(z)$ depend only on a vertical coordinate while the radial dependence is here expressed by the Bessel function $J_m(ks)$. Plugging the above ansatz into the matrix equation (29) we obtain a system of linear differential equations in z variable

$$\begin{aligned} Dw^+(z) \,-\, im\, b^+(z) \,-\, \vartheta^+(z) \,&=\, 0\,, \\ D\omega^+(z) \,-\, im\, j^+(z) \,&=\, 0\,, \\ -\,2\Lambda\, Dw^+(z) \,-\, im\Lambda\,(D^2-k^2)\,\omega^+(z) \,+\, (D^2-k^2+\lambda_c)\, b^+(z) \,&=\, 0\,, \\ im\Lambda\, w^+(z) \,-\, 2\Lambda\, D\omega^+(z) \,+\, (D^2-k^2+\lambda_c)\, j^+(z) \,&=\, 0\,, \\ -\,R_c k^2\, \omega^+(z) \,+\, (D^2-k^2+(\lambda_c/q_R))\, \vartheta^+(z) \,&=\, 0\,. \end{aligned} \tag{41}$$

It is worthwhile noting that the above system of equations has a nontrivial solution $\psi_z^+ = (w^+(z),\, \omega^+(z),\, b^+(z),\, j^+(z),\, \vartheta^+(z))^T$ satisfying the dual boundary conditions (39) in the z variable. This is due to the fact that the equation $\mathcal{L}\psi = 0$ possesses a nontrivial solution ψ having the form $\psi_z J_m(ks)$ where $\psi_z = (w(z),\, \omega(z),\, b(z),\, j(z),\, \vartheta(z))^T$. The resulting linear system of differential equations in z variable is adjoint to the above system (41). Therefore the kernel of (41) is nontrivial and so there is a nontrivial solution of (41).

Notice that the linear problem $\mathcal{L}\psi = 0$ has a solution $\psi \in X$ provided that the Bessel function $J_m(ks)$ vanishes at $s = s_n$, i.e.

$$J_m(ks_n) = 0\,. \tag{42}$$

Clearly, the restriction of any function of the form $f(z)J_m(ks)$ on the domain $G_n = (z_B,\, z_T) \times (0, s_n)$ vanishes on the lateral boundaries $(z_B,\, z_T) \times 0$ and $(z_B,\, z_T) \times s_n$. This is why the restriction $\psi|_{G_n}$ of the solution ψ constructed in BS belongs to the space X. Similarly, the vector function $\psi^+ = \psi_z^+ J_m(ks)$ belongs to the space X^+.

4.2. The derivation of a solvability condition

In this paragraph, we are yet able to take a use of perturbation techniques and adjointness properties to solve the abstract nonlinear problem (29) which has been set up in the previous section.

Suppose that the unknown function ψ and the Rayleigh number R (the system parameter) can be expanded into a power series in terms of a small unfolding parameter ε

$$\psi = \varepsilon\, \psi_1 + \varepsilon^2\, \psi_2 + \varepsilon^3\, \psi_3 + \dots\,, \tag{43}$$

$$R = R_c + \varepsilon\, R_1 + \varepsilon^2\, R_2 + \dots \tag{44}$$

where the first order term ψ_1 is identical to the solution of the linearized problem (28) and R_c is a critical value of Rayleigh number known from linear stability analysis made in BS. Our nonlinear system, however, when being driven through the critical value R_c within its parameter regime, gives rise to the oscillatory instability. It means that a complex frequency λ must be expanded into a series as well

$$\lambda = \lambda_c + \varepsilon\,\lambda_1 + \varepsilon^2\,\lambda_2 + \dots \tag{45}$$

where λ_c is a critical frequency corresponding to R_c. Now we can insert the above expansions (43, 44, 45) into the system (29). Collecting the terms of the same power of ε and using the well-known matrix representation one obtains a series of linear problems. We discuss this procedure in Appendix in a more detail.

In the first order ε^1, we obtain a homogeneous linear problem

$$\mathcal{L}\,\psi_1 = 0\,. \tag{46}$$

We remind ourselves that the components of the vector ψ_1 can be sought in the form $f_{m1}(z,s) = f(z)J_m(ks)$.

In the second order of ε^2, we have

$$\mathcal{L}\,\psi_2 = \begin{pmatrix} 0 \\ R_1 k^2\,\vartheta_{m1}(z,s) \\ \lambda_1\,b_{m1}(z,s) \\ \lambda_1\,j_{m1}(z,s) \\ (\lambda_1/q_R)\,\vartheta_{m1}(z,s) \end{pmatrix} \tag{47}$$

where the components $f_{m2}(z,s)$ of a vector ψ_2 are yet unknown. At this order of perturbation expansion, the cubic nonlinearities, i.e. $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$, are not present. Taking the duality product $\langle\,.\,|\,.\,\rangle$ of (47) with ψ^+ we obtain a simple complex equation

$$-\,\alpha_1\,R_1 + \lambda_1 = 0\,. \tag{48}$$

With regard to the requirement $\lambda_1 = i\sigma_1$, $\sigma_1 \in R$, the only solution of this equation is $R_1 = 0$, $\lambda_1 = 0$ and so $\mathcal{L}\psi_2 = 0$. As ψ_2 does not belong to the kernel of $\mathcal{L}$ we finally obtain $\psi_2 = 0$. It can be also seen from the symmetry of evolution equations.

In the third order of ε^3, the solvability condition yields a nonhomogeneous problem

$$\mathcal{L}\,\psi_3 = \begin{pmatrix} 0 \\ R_2 k^2 \vartheta_{m1}(z,s) \\ P_{m1}(z,s)\;+\;\lambda_2 b_{m1}(z,s) \\ T_{m1}(z,s)\;+\;\lambda_2 j_{m1}(z,s) \\ (1/q_R) S_{m1}(z,s)\;+\;(\lambda_2/q_R)\vartheta_{m1}(z,s) \end{pmatrix}. \tag{49}$$

It is obvious that the nonlinear terms in first order functions $P_{m1}(z,s)$, $T_{m1}(z,s)$ and $S_{m1}(z,s)$, arise at this order of expansion. The angular velocity of geostrophic flow is a function of $b_{m1}(z,s)$ and $j_{m1}(z,s)$ in this case, thus we symbolize it by $\Omega_1(s)$ for convenience.

We briefly sum up the notation used for this stage of perturbation method. All the nonlinearities are functions of $f_{m1}(z,s)$ which are perfectly separable in z and s coordinate. They can be therefore expressed in terms of the simple representing functions $f(z)$ as follows

$$\begin{aligned} P_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,b(z) - im\,\mathcal{J}_m^{-1}\{\mathcal{P}_{\Omega_1}\,J_m(ks)\}\,b(z)\,, \\ T_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,j(z) + \mathcal{J}_m^{-1}\{\mathcal{T}_{\Omega_1}\,J_m(ks)\}\,Db(z)\,, \\ S_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,\vartheta(z) \end{aligned} \tag{50}$$

with $\mathcal{P}_{\Omega_1}$, $\mathcal{T}_{\Omega_1}$ corresponding to $\mathcal{P}_{\Omega}$, $\mathcal{T}_{\Omega}$ in (18, 19) where $\Omega(s)$ has been substituted by $\Omega_1(s)$ which is given by (51, 52) below.

Geostrophic flow $\Omega_1(s)$ entering the above formulae can be expressed in terms of $f(z)$ as well

$$\Omega_1(s) = \Omega_s(s)\cdot\mathcal{Z} \tag{51}$$

where

$$\Omega_s(s) = \frac{1}{s}\,\frac{d}{ds}J_m^2(ks) \tag{52}$$

describes the radial dependence of geostrophic flow. The functional $\mathcal{Z}$ is given by

$$\mathcal{Z} = \frac{\Lambda}{2\,(2E)^{1/2}\,k^2}\,\Re e\left\{\int_{z_B}^{z_T} j(z)\overline{Db(z)}\,dz\right\}. \tag{53}$$

The solvability condition for the 3-rd order of the expansion yields a duality product expansion

$$\langle\, F_3 \,|\, \psi^+ \,\rangle = 0 \tag{54}$$

where F_3 is a vector of right-hand side terms in the order of ε^3 and ψ^+ is the previously constructed solution of $\mathcal{L}^+\psi^+ = 0$. By straightforward integrations one finds the solvability conditions for R_2 and λ_2

$$-\,\alpha\,R_2 + \lambda_2 - \beta = 0\,. \tag{55}$$

This condition can be thought of as a complex equation for determining the parameters R_2 and $\lambda_2 = i\sigma_2$, $\sigma_2 \in R$ which can give us an information about bifurcation and the frequency changes of the dynamic system in the vicinity of R_c. The existence of a solution ψ_3 is assured by Proposition 1 in Appendix.

The complex coefficients α and β entering (55) depend on the parameters m, Λ, E, q_R, k as well as on the critical parameters R_c, λ_c and can be given in terms of analytical expressions.

Now the solution ψ of the nonlinear problem $\mathcal{L}\psi = N(\psi)$ has the power series expansion

$$\psi = \varepsilon\psi_1 + \varepsilon^3\psi_3 + o(\varepsilon^3)\,. \tag{56}$$

Similarly, up to the second order terms, we have

$$R \sim R_c + \varepsilon^2 R_2\,, \tag{57}$$

$$\lambda \sim \lambda_c + \varepsilon^2 \lambda_2\,. \tag{58}$$

Finally, if we put

$$\varepsilon = \sqrt{\frac{R-R_c}{R_2}} \tag{59}$$

then the representing functions $\tilde{f}(z,s,\varphi,t)$ of a solution of the evolution problem (2 - 6) can be written as

$$\tilde{f}(z,s,\varphi,t) \simeq \sqrt{\frac{R-R_c}{R_2}}\ \Re e\{f(z)J_m(ks)\exp(im\varphi+\lambda t)\}\,. \tag{60}$$

The expression $\sqrt{\frac{R-R_c}{R_2}}$ therefore relates to the amplitude of representing functions $\tilde{f}(z,s,\varphi,t)$. It can be seen that if $R_2 > 0$, the Hopf bifurcation arising in R_c is supercritical (see Fig.1). On the other hand, if $R_2 < 0$, the bifurcation is subcritical.

The complex frequency in the neighbourhood of R_c varies according to

$$\lambda \sim \lambda_c + \varepsilon^2\lambda_2 = \lambda_c + \frac{R-R_c}{R_2}\,\lambda_2\,. \tag{61}$$

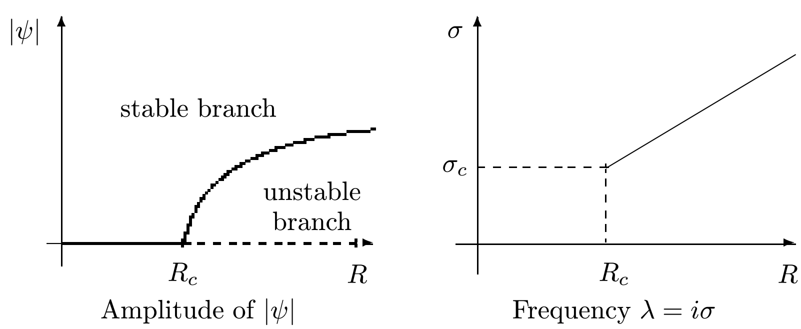

Fig. 1

5. Conclusions

The weakly nonlinear analysis has been here adopted to the magnetoconvecting system with Ekman suction. It has been shown that the effect of nonlinearity, brought into the basic MHD evolution equations via the geostrophic flow $\Omega(s)$ and expressed in terms of modified Taylor's constraint, can be resolved with the use of analytical methods, i.e. perturbation techniques with a support of functional analysis.

The structure of the basic nonlinear equations (namely the cubic nonlinear terms) gives rise to the Hopf bifurcation at critical Rayleigh number R_c. The analytical expressions for second order terms R_2 and λ_2 in power series of Rayleigh number and complex frequency have been found as solutions of the solvability condition. The parameter R_2 determines the supercritical or subcritical character of bifurcation in R_c and λ_2 relates to the frequency response of dynamic system in the vicinity of R_c. The finite amplitude solution for representing functions of $\tilde{f}(z, s, \varphi, t)$ has been also established.

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Appendix

The aim of this section is to establish the existence of a nontrivial solution (ψ, p) of the semilinear equation $\mathcal{L}(p)\psi + N(\psi) = 0$ where $\mathcal{L}(p) = \mathcal{L}_0 + Mp$ is an affine mapping from Y into $L(X,Z)$, i.e. $\mathcal{L}_0 \in L(X,Z)$ and $M \in L(Y, L(X,Z))$, Y is a normed parameter space, X and Z are complex Hilbert spaces such that X is continuously embedded into Z, $X \hookrightarrow Z$. Recall that our aim is to solve equation (29) where the linear operator $\mathcal{L}$ depends on the Rayleigh number and the complex frequency $\lambda = i\sigma$.

Before discussing solvability of equation (29) we have to introduce function spaces we will work with. Let $G_n = (z_B, z_T) \times (0, s_n)$ where s_n is the n-th root of the function $J_m(ks)$. Denote $\mathcal{D}^2 = (D^2 - k^2 \mathcal{J}_m)$. Let U be the weighted Lebesgue space $U = \{f : G_n \to C,\ \|f\|_U^2 = \int_{G_n} |f(z,s)|^2 s\, dz\, ds < \infty\}$. Let $dom(\mathcal{D}^2) = \{f \in C^\infty(G_n),\ \mathcal{J}_m(f) \in U\} \cap \{b.c.\ (34), (38)_a\}$. Here the boundary condition $(38)_a$ where a stands for b, j or ϑ is the corresponding b.c. for a in (38). Then, for any $f,\, g \in dom(\mathcal{D}^2)$ $\langle \mathcal{D}^2 f \,|\, g \rangle_U = \langle f \,|\, \mathcal{D}^2 g \rangle_U$ and, by the Cauchy-Schwartz inequality, for $m \geq 1$, we have the estimate

$$\langle -\mathcal{D}^2 f \,|\, f \rangle_U = \int_{G_n} (|D_z f|^2 + |D_s f|^2) s + m^2 |f|^2 s^{-1}\, dz\, ds \geq (\int_{G_n} |f|^4)^{1/2} .$$

Thus, for any $f \in dom(\mathcal{D}^2)$, we have $\langle -\mathcal{D}^2 f \,|\, f \rangle_U \geq s_n^{-1} meas(G_n)^{-1/2} \|f\|_U^2$. Therefore $-\mathcal{D}^2$ is a positive definite symmetric operator on $dom(\mathcal{D}^2)$. Such an operator can be extended to a self-adjoint in U operator (again denoted by $\mathcal{D}^2$) its domain being $Dom(\mathcal{D}^2) \subset U$. If the space $V_a = Dom(\mathcal{D}^2)$ is equipped by the graph norm $\|f\|_V^2 = \|\mathcal{D}^2 f\|_U^2 + \|f\|_U^2$ then V_a is a Hilbert space continuously embedded into the Hilbert space U. Furthermore, it turns out from the Sobolev embedding theorem that, for any power $p > 0$,

$$\int_{G_n} |f|^p s\, dz\, ds \leq const. \|f\|_V^p .$$

In particular the cubic nonlinearity $f \mapsto f^3$ is well defined and C^∞ smooth mapping from $V_a \to U$. Similarly one can define the function spaces V_w and V_ω for the vector components w and ω, respectively. To define V_w one has to consider the subclass of the Sobolev space $W^{2,2}(G_n)$ of all functions satisfying $(34) - (38)_w$ and such that $\mathcal{J}_m$ maps V_w into U. The space V_ω is the Sobolev space $W^{1,2}(G_n)$.

The spaces V_a^+ of all functions satisfying dual boundary conditions (35)-(39) can be defined in a similar way respecting the structure of the dual operator $\mathcal{L}^+$.

In our application we choose the following function spaces
$Z = Z^+ = [U]^5, \quad Y = R^2$,
$X = V_w \times V_\omega \times V_b \times V_j \times V_\vartheta$
$X^+ = V_w^+ \times V_\omega^+ \times V_b^+ \times V_j^+ \times V_\vartheta^+$

We proceed by construction of a nontrivial solution of the abstract equation. The nonlinearity $N : X \to Y$ is assumed to be a C^k Fréchet differentiable function, $k \geq 1$, with the property that there is an $m > 1$ such that for any

$\varepsilon \in R$ and $\psi \in X$, $N(\varepsilon\psi) = \varepsilon^m N(\psi)$. Thus $N(0) = 0$ and this is why the trivial pair $(\psi, p) \equiv (0,0)$ is a solution of $\mathcal{L}(p)\psi + N(\psi) = 0$. Our aim is to find another nontrivial solution $\psi(\varepsilon), p(\varepsilon)$ branching from a trivial one. To this end, one can apply the theorem on a bifurcation from a simple eigenvalue. Since the proof of this theorem is based on the implicit function theorem it turns out that the crucial assumption is that the linear mapping $\mathcal{H} : X^2 \times Y \to Z$, $\mathcal{H}(h,r) = \mathcal{L}_0 h + (Mr)\psi_1$ is surjective. Here $X^2 = Ker(\mathcal{L}_0)^\perp$.

Proposition 1. *Under the above assumptions there exists an $\varepsilon_0 > 0$ and C^k smooth functions $\psi : (-\varepsilon_0, \varepsilon_0) \to X$, $p : (-\varepsilon_0, \varepsilon_0) \to Y$ such that $\mathcal{L}_0(p(\varepsilon))\psi(\varepsilon) + N(\psi(\varepsilon)) = 0$ for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The functions ψ and p can be expanded into power series $\psi(\varepsilon) = \sum_{i=1}^k \varepsilon^i \psi_i + o(\varepsilon^k)$, $p(\varepsilon) = \sum_{i=1}^k \varepsilon^i p_i + o(\varepsilon^k)$.*

Now we are in a position to state so called solvability conditions which enable us to determine the coefficients in the power series expansions for $\psi(\varepsilon)$, $p(\varepsilon)$. These relations are obtained by expanding the term $\mathcal{L}_0(p(\varepsilon))\psi(\varepsilon) + N(\psi(\varepsilon))$ into power series in ε and testing the resulting equations by a dual vector ψ_1^+. More precisely, let X^+ and Z^+ be complex Hilbert spaces, $X^+ \hookrightarrow Z^+$ and $\langle \,.\, | \,.\, \rangle$ is a bilinear mapping from $Z \times Z^+$ into C. Denote by $\mathcal{L}_0^+ : X^+ \to Z^+$ the formally adjoint linear operator to $\mathcal{L}_0$ with respect to $\langle \,.\, | \,.\, \rangle$, i.e. $\mathcal{L}_0^+ \in L(X^+, Z^+)$ and $\langle \mathcal{L}_0\psi \,|\, \psi^+ \rangle = \langle \psi \,|\, \mathcal{L}_0^+\psi^+ \rangle$ for any $\psi \in X$ and $\psi^+ \in X^+$.

Now we plug the power series expansions obtained in Proposition 1 into equation $\mathcal{L}_0(p(\varepsilon))\psi(\varepsilon) + N(\psi(\varepsilon)) = 0$. Counting the coefficients of the j-th power of ε, $j = 1, 2, \ldots, k-1$, we end up with a sequence of determining equations for $p_1, \psi_2, p_2, \psi_3, \ldots$. In particular, we obtain that $\mathcal{L}_0\psi_1 = 0$, $\mathcal{L}_0\psi_2 + Mp_1 = 0$, $\mathcal{L}_0\psi_3 + Mp_3 = 0$ and, in general,

$$\mathcal{L}_0\psi_{j+1} + Mp_j + N_{j-m+1} = 0 \tag{62}$$

where $N_l = 0$ for $l < 0$ and the coefficients N_l, $l \geq 0$ are determined from the series expansion $N(\psi_1 + \varepsilon\psi_2 + \varepsilon^2\psi_3 + \ldots) = N_0 + \varepsilon N_1 + \varepsilon^2 N_3 + \ldots$, i.e.

$N_0 = N(\psi_1)$, $\quad N_1 = DN(\psi_1)\psi_2$, $\quad N_2 = \frac{1}{2}D^2N(\psi_1)[\psi_2, \psi_2] + DN(\psi_1)\psi_3$,
$N_3 = \frac{1}{6}D^3N(\psi_1)[\psi_2, \psi_2, \psi_2] + D^2N(\psi_1)[\psi_2, \psi_3] + DN(\psi_1)\psi_4$.

In general, N_l can be expressed as a linear combination of several multilinear mappings $N_l = \frac{1}{l!}D^lN(\psi_1)[\psi_2, \ldots, \psi_2] + \ldots + DN(\psi_1)\psi_{l+1}$.

Now we can state solvability conditions for the family of equations (29). Let $0 \neq \psi_1^+ \in X^+$ be such that $\mathcal{L}_0^+\psi_1^+ = 0$. As $\langle \mathcal{L}_0\psi_{j+1} \,|\, \psi_1^+ \rangle = \langle \psi_{j+1} \,|\, \mathcal{L}_0^+\psi_1^+ \rangle = 0$ then taking the product map $\langle \,.\, | \,.\, \rangle$ of equation (29) with ψ_1^+ yields

$$\langle Mp_j \,|\, \psi_1^+ \rangle + \langle N_{j-m+1} \,|\, \psi_1^+ \rangle = 0 \qquad \text{for } j = 1, 2, \ldots, k-1 \tag{63}$$

Roughly speaking, the above set of equations enables us to determine the leading coefficients p_2, ψ_2, p_3, $\psi_3 \ldots$ in the power series expansions for $\psi(\varepsilon)$, $p(\varepsilon)$. The above system of equations is sometimes referred to as solvability conditions. Finally, let us turn our attention to the abstract nonlinear system (29). In this case we have $m = 3$. Then conditions (63) for $k = 1$ becomes the set of two determining equations

$$\langle Mp_1 \,|\, \psi_1^+ \rangle = 0\,, \quad \text{and } \langle Mp_2 \,|\, \psi_1^+ \rangle + \langle N(\psi_1) \,|\, \psi_1^+ \rangle = 0\,. \tag{64}$$

Here $p = (R - R_c,\, \sigma - \sigma_c) \in Y$ where R is Rayleigh number and $\lambda = i\sigma$, $Y = R^2$.

We finish this appendix with the following remarks. The operator $\mathcal{L}$ in our application is well defined and bounded linear operator from X to Z. Similarly, $\mathcal{L}^+ : X^+ \to Z^+$ is a bounded linear operator. Since the operator $\mathcal{J}_m$ is defined on function spaces V_a over the bounded domain G_n it turns out that 0 belongs to the resolvent set of $\mathcal{J}_m$ and therefore the inverse operator ${\mathcal{J}_m}^{-1} : U \to V_a$ for each $a \in \{w, \omega, b, j, \vartheta\}$ is well defined and bounded. This justifies the previous formal usage of the inverse operator ${\mathcal{J}_m}^{-1}$ in the definition of nonlinear terms $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$. We have also shown that the cubic nonlinearity is C^∞ smooth from V_a into U. As a consequence one can show that the nonlinearity N in our application is indeed well defined and C^∞ smooth when operating from X to Z.