Analysis of the model of magnetoconvection with nonlinearity due to modified Taylor's constraint

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Abstract. A problem of magnetoconvection is considered where the nonlinear effect of geostrophic flow determined by Ekman suction is included. Perturbation techniques are adopted in order to construct slowly varying periodic solution branching from the steady state conductive solution. The analysis is also used to determine the relevant bifurcation structure in the vicinity of the critical Rayleigh number.

Key words: Magnetoconvection, modified Taylor's constraint, perturbation techniques, weakly nonlinear analysis

1. Introduction

The fluid motion in Earth-like planet cores can be characterized by magnetostrophic approximation with dominating Lorentz, Coriolis, buoyancy and pressure forces in the equation of motion. The approximation with zero viscous forces has a solution, only if so-called Taylor's constraint is satisfied (see Section 2). A specific problem arises when magnetostrophic approximation holds but small viscous forces in the Ekman boundary layers are present. In this case a non-zero geostrophic flow is induced by the viscous flow in thin Ekman layers and nonlinear dynamics of the whole magnetoconvecting system is affected through the so-called Ekman suction mechanism.

The question is, if such a nonlinear viscous system, which reflects more realistically conditions in the Earth's core, could possibly evolve into the Taylor state. At this particular state, viscous forces have no longer major influence on the dynamics and Taylor's condition is met. The problem of possible achievement of the Taylor state has been studied in simpler planar or cylindrical and also in spherical geometry for both kinematic dynamos and magnetoconvection

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models. It has been shown by Fearn, Proctor and Sellar (1994) that some specific simplifications can be made in the case of magnetoconvection models. Namely, non-axisymmetric instabilities of magnetic field only have to be considered for computation of geostrophic flow, whereas contributions from basic axisymmetric magnetic fields can be neglected (see Section 2).

In this paper we study a problem of finite amplitude rotating magnetoconvection affected by Ekman suction. The investigation has been motivated by the linear stability analysis developed by Soward (1979), (see also Brestenský and Ševčík 1994, 1995, and Šimkanin et al 1997 in this Issue) as well as the nonlinear problem studied in Skinner and Soward (1988, 1991). In contrast to the approach applied in the nonlinear study done by Skinner and Soward (1988, 1991), the purpose of the present paper is to study state of magnetoconvection near the critical Rayleigh number R_c.

The methods and techniques of this paper are based on the regular perturbation theory, linear and nonlinear functional analysis and bifurcation theory. The main idea is to expand a solution into power series in terms of a small unfolding parameter ε corresponding to the small increase in the Rayleigh number beyond its critical value R_c. Let us emphasize that this approach can describe local bifurcation structure near R_c only.

The underlying geometry is a weakly bounded cylinder, i.e. the cylinder with a radius strongly exceeding its height. It can sufficiently approximate the laterally unbounded geometry used in the linear study (Soward 1979). We must emphasize that the finite extension in the radial direction is a crucial assumption of the theory. The reason for dealing with the bounded geometry is twofold. Firstly, it enables us to set up suitable function spaces and operators we will work with. Secondly, as a consequence of the boundedness of the cylinder, the third order approximation of the power series expansion is capable of describing the Hopf bifurcation phenomenon in the amplitude equation (51) in Section 3.3. On the other hand, the main disadvantage of this approach is that we have to set up boundary conditions on vertical boundaries of the cylinder. In this paper we consider the simplest case of Dirichlet boundary conditions which seem to be less physically meaningful. The more realistic boundary conditions will be treated in the forthcoming paper.

The outline of this paper is as follows. In Section 2 we derive a system of nonlinear PDE's governing the motion periodic in both time and the azimuthal variable. Section 3.1 is devoted to the study of the constructed system of nonlinear equations. We present a method on how to obtain a power series expansion of a solution in terms of a small unfolding parameter. Using the so-called solvability condition known from Fredholm's alternative in the functional analysis, we determine leading coefficients of the expansions in Section 3.2. In Section 3.3 we sketch a procedure how to derive an ordinary differential equation for the time dependent amplitude. Numerical results are reported in Section 4. In the Appendix we present formulae for the leading terms in the power series expansions.

2. Formulation of the nonlinear problem

2.1. Basic leading equations

The aim of this paper is a local stability analysis of a nonlinear system of PDE's governing a specified model of magnetoconvection.

The model considered is an infinite horizontal layer of width d rotating rapidly with angular velocity $\Omega_0\hat{\mathbf{z}}$. The layer contains an electrically conducting Boussinesq fluid permeated by an azimuthal magnetic field linearly growing with the distance from the vertical rotation axis. An unstable temperature gradient is maintained by heating the fluid from below and cooling from above. The fluid layer is supposed to have free perfectly electrically and thermally conductive horizontal boundaries.

The convective instability in this rotating system is caused by the vertical temperature gradient and manifests itself by perturbations of the velocity $\mathbf{u}$, the magnetic field $\mathbf{b}$ and the temperature ϑ which refer to the basic state represented by $\mathbf{U}_0$, $\mathbf{B}_0$, T_0.

In this paper, we investigate the existence of periodic solution for these perturbations in the vicinity of the basic state determined by

$$\mathbf{U}_0 = \mathbf{0}\,, \quad \mathbf{B}_0 = B_M \frac{s}{d}\,\hat{\boldsymbol{\varphi}}\,, \quad T_0 = T_1 - \frac{\Delta T}{d}\,(z + \frac{d}{2})\,. \tag{1}$$

We non-dimensionalise the problem with the use of characteristic length d, magnetic diffusion time d^2/η, magnetic field B_M, and temperature difference across the layer ΔT. In the cylindrical polar coordinates $(s,\,\varphi,\,z)$ the equations governing the evolution of perturbations $\mathbf{u}$, $\mathbf{b}$, $\widetilde{\vartheta}$ of the basic state gain the following form

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \Lambda\,[\,(\,\nabla \times s\,\hat{\boldsymbol{\varphi}}) \times \mathbf{b} + (\,\nabla \times \mathbf{b}\,) \times s\,\hat{\boldsymbol{\varphi}}] + R\,\vartheta\,\hat{\mathbf{z}}\,, \tag{2}$$

$$\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \times \mathbf{b}\,) = \nabla \times (\,\mathbf{u} \times s\,\hat{\boldsymbol{\varphi}}\,) + \nabla^2 \mathbf{b}\,, \tag{3}$$

$$\frac{1}{q_R}\left(\frac{\partial \widetilde{\vartheta}}{\partial t} + (\,s\,\Omega(s)\,\hat{\boldsymbol{\varphi}} \cdot \nabla\,)\,\widetilde{\vartheta}\right) = -\,\mathbf{u} \cdot \nabla T_0 + \nabla^2 \widetilde{\vartheta}\,, \tag{4}$$

$$\nabla \cdot \mathbf{b} = 0\,, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

where $\hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{z}}$ are the unit azimuthal and axial vectors, respectively. The dimensionless parameters, the modified Rayleigh number R, the Elsasser number Λ, the Ekman number E and the Roberts number q_R, are defined by

$$R = \frac{gd\Delta T\alpha}{2\Omega_0\kappa}\,, \qquad \Lambda = \frac{B_M^2}{2\Omega_0\rho_0\eta\mu}\,, \qquad E = \frac{\nu}{2d^2\Omega_0}\,, \qquad q_R = \frac{\kappa}{\eta}$$

where κ and η are the thermal and magnetic diffusivities, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, μ is the permeability and ρ_0 is the density.

The model of magnetoconvection includes the effect of Ekman suction which is associated with a nontrivial geostrophic flow. This gives rise to the presence of nonlinear terms encountered in the above differential equations, namely in (3) and (4). It is known that geostrophic flow can be expressed via so-called modified Taylor's constraint (see Fearn 1994).

Let $\langle \dots \rangle^\varphi \equiv 1/(2\pi) \int_0^{2\pi} \dots d\varphi$ be averaging over the azimuthal component φ. Denote by $F_{M\varphi} \equiv [(\nabla \times \mathbf{B}) \times \mathbf{B}]_\varphi$ the azimuthal component of Lorentz force. Then splitting magnetic field $\mathbf{B}$ on basic field $\mathbf{B}_0$ and perturbation $\mathbf{b}$, $\mathbf{B} \equiv \mathbf{B}_0 + \mathbf{b}$ ($\langle \mathbf{B} \rangle^\varphi = \mathbf{B}_0,\ \langle \mathbf{b} \rangle^\varphi = \mathbf{0}$), the angular velocity $\Omega(s)$ of geostrophic flow in our magnetoconvection model can be expressed in terms of the magnetic field perturbation $\mathbf{b}$, i.e. (see e.g. Skinner and Soward 1988)

$$\Omega(s) = \frac{\Lambda}{(2E)^{1/2} s} \int_{z_B}^{z_T} \langle F_{M\varphi} \rangle^\varphi \, dz \quad \text{with} \quad \langle F_{M\varphi} \rangle^\varphi = \langle [(\nabla \times \mathbf{b}) \times \mathbf{b}]_\varphi \rangle^\varphi . \tag{7}$$

It is significant for the model under consideration that the possible contribution $\langle [(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0]_\varphi \rangle^\varphi$ from basic field to azimuthally averaged Lorentz force $\langle F_{M\varphi} \rangle^\varphi$ vanishes (see also Fearn, Proctor and Sellar 1994). We note that the expression (7) is well-known as modified Taylor's constraint.

The vector nonlinear equations (2 - 6) together with the expression for geostrophic flow (7) seem rather complicated to be solved analytically. We therefore restrict solutions to a smaller phase space of functions having special structure. Roughly speaking, the main idea is to express all the vector fields in terms of their scalar representing functions which are supposed to have a form of travelling waves, as it is described below.

We split the velocity perturbation $\mathbf{u}$ as well as the magnetic field perturbation $\mathbf{b}$ into their poloidal and toroidal parts

$$\mathbf{u} \;=\; k^{-2}(\nabla \times (\nabla \times \widetilde{w}\, \hat{\mathbf{z}}) + \nabla \times \widetilde{\omega}\, \hat{\mathbf{z}}) , \tag{8}$$
$$\mathbf{b} \;=\; k^{-2}(\nabla \times (\nabla \times \widetilde{b}\, \hat{\mathbf{z}}) + \nabla \times \widetilde{j}\, \hat{\mathbf{z}}) . \tag{9}$$

Similarly as in the papers Brestenský and Ševčík (1994) and Brestenský, Revallo and Ševčovič (1997)[1] we have adopted the tilde notation for representing poloidal and toroidal functions as well as for thermal function. Each of the representing functions $\widetilde{w}$, $\widetilde{\omega}$, $\widetilde{b}$, $\widetilde{j}$, $\widetilde{\vartheta}$ (all symbolized as $\widetilde{f}$) depends on coordinates z, s, φ and time t.

Suppose that the representing functions $\widetilde{f}$ can be decomposed as

$$\tilde{f}(z, s, \varphi, t) = \Re e\{ f_m(z, s) \exp(im\varphi + \lambda t)\} \tag{10}$$

[1]Henceforth the abbreviations (BS) and (BRS) will be used.

where the functions of $f_m(z,s)$, i.e. $b_m(z,s)$, $j_m(z,s)$, $w_m(z,s)$, $\omega_m(z,s)$ and $\vartheta_m(z,s)$ depend on vertical and radial coordinates z and s . Here m is an integer azimuthal wave number, k is a real radial wave number and λ is a complex frequency related to a real frequency via $\lambda = i\sigma$.

Inserting the above ansatz into the governing equations for perturbations (2-6) and into the expression for modified Taylor's constraint enables us to set up a system of nonlinear equations for representing functions $f_m(z,s)$. The resulting nonlinear system is well posed on a suitable function space as it has been yet shown in (BRS). Hereafter, this system of equations will be referred to as an abstract nonlinear problem.

2.2. Abstract nonlinear problem

The procedure leading towards the abstract nonlinear problem presented below is straightforward but rather technically tedious. It is discussed in a more detail in (BRS).

The equations for the representing functions $f_m(z,s)$ can be finally written as follows

$$
\begin{aligned}
0 \;&=\; -\,Dw_m(z,s) + 2\Lambda\, Db_m(z,s) - im\Lambda\, j_m(z,s)\,,\\
0 \;&=\; -\,D\omega_m(z,s) + 2\Lambda\, Dj_m(z,s) + im\Lambda\,(D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s) - Rk^2\,\vartheta_m(z,s)\,,\\
\lambda\, b_m(z,s) + P_m(z,s) \;&=\; im\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, b_m(z,s)\,,\\
\lambda\, j_m(z,s) + T_m(z,s) \;&=\; im\, \omega_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\, j_m(z,s)\,,\\
(1/q_R)\;(\,\lambda\,\vartheta_m(z,s) + S_m(z,s)\,) \;&=\; \mathcal{J}_m\, w_m(z,s) + (D^2 - k^2\,\mathcal{J}_m)\,\vartheta_m(z,s)
\end{aligned}
\tag{11}
$$

where the nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are expressed in terms of $f_m(z,s)$ and the angular velocity $\Omega(s)$ of geostrophic flow as follows

$$
\begin{aligned}
P_m(z,s) &= im\,\Omega(s)\, b_m(z,s) \;-\; im\,\mathcal{J}_m^{-1}\,\{\mathcal{P}_\Omega\; b_m(z,s)\,\}\;,\\
T_m(z,s) &= im\,\Omega(s)\, j_m(z,s) \;+\; \mathcal{J}_m^{-1}\,\{\mathcal{T}_\Omega\; D\, b_m(z,s)\,\}\;,\\
S_m(z,s) &= im\,\Omega(s)\,\vartheta_m(z,s)\,.
\end{aligned}
\tag{12}
$$

Here $D = \partial/\partial z$ and $\mathcal{J}_m^{-1}$ is the inverse operator to the linear Bessel differential operator $\mathcal{J}_m$. The operator $\mathcal{J}_m$ is defined as

$$
\mathcal{J}_m \equiv\; -\frac{1}{k^2}\,\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\,\frac{\partial}{\partial s} - \frac{m^2}{s^2}\right)
\tag{13}
$$

and for the Bessel function $J_m(ks)$ it has a useful property $\mathcal{J}_m\;\{J_m(ks)\} = J_m(ks)$. Furthermore, $\mathcal{P}_\Omega$, $\mathcal{T}_\Omega$ are differential operators

$$
\mathcal{P}_\Omega = -\frac{1}{k^2}\,\left\{\frac{\partial^2\Omega(s)}{\partial s^2} + \frac{\partial\Omega(s)}{\partial s}\,\left[2\,\frac{\partial}{\partial s} + \frac{1}{s}\right]\right\}\,,
\tag{14}
$$

$$
\mathcal{T}_\Omega = -\frac{1}{k^2}\,\left\{s\,\frac{\partial^2\Omega(s)}{\partial s^2}\,\frac{\partial}{\partial s} + s\,\frac{\partial\Omega(s)}{\partial s}\,\left[\frac{m^2}{s^2} + \frac{2}{s}\,\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2}\right]\right\}
\tag{15}
$$

where the partial derivative $\partial/\partial s$ reflects the fact of $\Omega(s)$ being a functional (see below). The interested reader is referred to (BRS) for the complete derivation of the above system of nonlinear PDE's. Furthermore, it has been shown in Appendix of (BRS) that $\mathcal{J}_m^{-1}$ is a well defined bounded linear operator on a suitable function space. We notice that the above expressions for P_m and T_m in (12) emerge after decomposition of the vector nonlinearity in induction equation (3) into poloidal and toroidal fields. The expression for S_m represents the scalar nonlinearity in the heat equation (4).

The geostrophic flow $\Omega(s)$ entering the set of equations (12) is given by formula which can be directly obtained by inserting (9) together with the ansatz (10) into (7). A straightforward series of calculations yields

$$\Omega(s) = \frac{\Lambda}{2\,(2E)^{1/2}}\,\frac{1}{s}\cdot \Re e\left\{\frac{1}{s^2}\frac{\partial}{\partial s}\left[\,s^2\, I(s)\,\right] - B(s)\right\} \tag{16}$$

where

$$I(s) = \frac{1}{k^4}\int_{z_B}^{z_T}\left(\frac{m^2}{s^2}\, j_m(z,s) D\overline{b_m(z,s)} \;-\; \frac{\partial}{\partial s} j_m(z,s)\,\frac{\partial}{\partial s} D\overline{b_m(z,s)}\right) dz$$

is the integral part and

$$B(s) = \frac{1}{k^2}\,\frac{\partial}{\partial s} j_m(z,s)\;\mathcal{J}_m \overline{b_m(z,s)}\,\Big|_{z_B}^{z_T}$$

is the boundary term. Here $D = \partial/\partial z$ and an overbar denotes the complex conjugation of $b_m(z,s)$.

It is remarkable that the complex conjugation in the expression for geostrophic flow $\Omega(s)$ eliminates exponentials of the tilded representing functions $\tilde{b}(z,s,\varphi,t)$ and $\tilde{j}(z,s,\varphi,t)$. Therefore upon assumption (10), the expression for $\Omega(s)$ does not involve the variables φ, t and is entered by simpler functions $b_m(z,s)$ and $j_m(z,s)$ only. This is the important fact which approves the choice of $f_m(z,s)$ as representing functions for our nonlinear problem. At this stage it is yet easy to see that $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ are cubic nonlinearities in $f_m(z,s)$.

For the special case of infinitely electrically and thermally conducting horizontal boundaries and vanishing viscosity[2] the following boundary conditions have to be satisfied

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = Dj_m(z,s) = 0\,, \quad \text{for all } z = z_B,\ z_T,\ \text{and } s \in (0, s_n)\,. \tag{17}$$

Notice that the above choice of boundary conditions makes the boundary term in expression (16) vanish.

[2]Recall that viscosity in our model is to be taken non-zero only within the Ekman layers along the horizontal boundaries. It is actually the viscous flow in the Ekman layers which is responsible for Ekman suction and geostrophic flow given by (16).

In a radial direction we impose the following boundary conditions

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = j_m(z,s) = 0\,, \quad \text{for all } s = 0,\ s_n,\ \text{and } z \in (z_B, z_T)\,. \tag{18}$$

Here and after s_n , which delimites the layer in a radial direction, will always stand for the n-th root of the scaled Bessel function $J_m(ks)$, i.e.

$$J_m(ks_n) = 0 \quad \text{for all } n = 1, 2, \ldots\,. \tag{19}$$

Notice that the Dirichlet-like boundary conditions (18) for the representing functions have been set up especially due to mathematical purposes. It should be emphasized again that in our approach the bounded geometry is needed in order to apply some functional analytical results. Roughly speaking, the choice of boundary conditions (18) enables us to guarantee the existence of the inverse operator $\mathcal{J}_m^{-1}$ and, as a consequence, to justify the definitions of the cubic nonlinearities $P_m(z,s)$, $T_m(z,s)$ introduced in (12).

Given a parameter $k > 0$, in our case from the linear stability study for the unbounded geometry, we are forced to restrict ourselves to a certain set of possible radii of the underlying cylinder. Namely, these radii must meet the condition (19).

The relation (19) represents itself a kind of a duality for the choice of the pair $(k,\ s)$; 1) either we firstly fix k and subsequently restrict the radial extension to s_n , or 2) we prescribe the radius, say S , first and then we find a set of possible values of k 's satisfying the relation $J_m(k_n S) = 0$. Although both approaches are beneficial, in this paper we discuss the first approach only.

We also notice that in the approach 1) the minimisation of $R(k)$ leading to the critical R_c and k_c is performed over a continuum of values of k whereas in the approach 2) minimisation is performed over a discrete set of k 's. Finally, we remark that the discrete set of k 's is asymptotically dense in $(0, \infty)$ as $S \to \infty$. Therefore, for large values of the radius S , both approaches appear to be the same from numerical point of view.

3. Solution of abstract nonlinear problem by perturbation methods

3.1. Properties of the adjoint operator

In this section we recall derivation of the so called solvability condition made in (BRS). The computations to follow are based on methods of the functional analysis, namely on the Fredholm alternative argument which is applicable to linear operators on Hilbert spaces. In this paper we will not report all the relevant mathematics except of some remarks on the choice of function spaces setting.

Following the idea of a matrix representation (see e.g. Proctor and Weiss 1982) we rewrite the linear part of equations (11) in the matrix form

$$\mathcal{L} \equiv \begin{pmatrix} -D & 0 & 2\Lambda\, D & -im\Lambda & 0 \\ 0 & -D & im\Lambda\mathcal{D}^2 & 2\Lambda D & -R_c k^2 \\ im & 0 & (\mathcal{D}^2-\lambda_c) & 0 & 0 \\ 0 & im & 0 & (\mathcal{D}^2-\lambda_c) & 0 \\ \mathcal{J}_m & 0 & 0 & 0 & (\mathcal{D}^2-\lambda_c/q_R) \end{pmatrix} \tag{20}$$

where $\mathcal{D}^2 = D^2 - k^2\mathcal{J}_m$. Thus the linear part of (11) has the form $\mathcal{L}\psi$ where ψ is a vector function

$$\psi(z,s) \equiv\, (w_m(z,s),\, \omega_m(z,s),\, b_m(z,s),\, j_m(z,s),\, \vartheta_m(z,s))^T\,.$$

The linear kernel problem, i.e. the homogeneous matrix equation

$$\mathcal{L}\,\psi = 0 \tag{21}$$

has been studied in Soward (1979) where the critical values of Rayleigh number R_c , the complex frequency $\lambda_c = i\sigma_c$ as well as the solution ψ have been found.

The full nonlinear problem (11) can be rewritten as

$$\mathcal{L}\,\psi = N(\psi) \tag{22}$$

where the term $N(\psi)$ contains all the nonlinearities $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$ involved in (11).

At this stage it is worthwhile noting that the nonlinear problem (11) has an important symmetry, i.e. the vector function $\psi = (w_m, \omega_m, b_m, j_m, \vartheta_m)^T$ solves (11) if and only if $-\psi$ does. This is based upon the useful property of the nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ being cubic in representing functions $f_m(z,s)$.

To solve the above semilinear problem by means of the functional analysis we have to find the kernel of the corresponding adjoint operator $\mathcal{L}^+$, i.e. a solution ψ^+ of the adjoint linear equation

$$\mathcal{L}^+\psi^+ = 0\,. \tag{23}$$

A solution of the above problem will be taken for as so-called test function in order to determine higher order terms in power series expansion for a solution ψ of (22).

We define a bilinear form $\langle\,.\,|\,.\,\rangle$ as follows

$$\langle\, \psi \,|\, \chi \,\rangle = \langle\, \psi\, \overline{\chi}\, \rangle^{zs} \equiv \sum \int_G f(z,s)\, \overline{g(z,s)}\; s\, ds\, dz \tag{24}$$

where $\sum$ denotes the summation over all components f and g of vectors ψ and χ , respectively. Here G_n is a bounded domain of the vertical and radial variable, $G_n = (z_B, z_T) \times (0, s_n)$.

Now we are in a position to define an adjoint operator to $\mathcal{L}$ with respect to the inner product $\langle\,.\,|\,.\,\rangle$. The adjoint linear operator $\mathcal{L}^+$ is completely determined by the relation

$$\langle\, \mathcal{L}\,\psi \,|\, \psi^+ \,\rangle = \langle\, \psi \,|\, \mathcal{L}^+\,\psi^+ \,\rangle \qquad \text{for all } \psi \in X\,,\ \psi^+ \in X^+ \tag{25}$$

where X and X^+ are domains of definitions of the linear operators $\mathcal{L}$ and $\mathcal{L}^+$, respectively. Applying Green's formula on $\langle\, \mathcal{L}\,\psi \,|\, \psi^+ \,\rangle$ yields

$$\langle\, \mathcal{L}\,\psi \,|\, \psi^+ \,\rangle = \langle\, \psi \,|\, \mathcal{L}^+\,\psi^+ \,\rangle + \mathcal{B} \tag{26}$$

where $\mathcal{B}$ is a boundary term. With the use of (26) it can be shown that the matrix linear operator

$$\mathcal{L}^+ = \begin{pmatrix} D & 0 & -im & 0 & \mathcal{J}_m \\ 0 & D & 0 & -im & 0 \\ -2\Lambda\, D & -im\Lambda\mathcal{D}^2 & (\mathcal{D}^2+\lambda_c) & 0 & 0 \\ im\Lambda & -2\Lambda\, D & 0 & (\mathcal{D}^2+\lambda_c) & 0 \\ 0 & -R_c k^2 & 0 & 0 & (\mathcal{D}^2+\lambda_c/q_R) \end{pmatrix} \tag{27}$$

obeys the definition (25) (i.e. the boundary term $\mathcal{B}$ vanishes), provided that $\psi(z,s)$ satisfies the boundary conditions (17, 18) and $\psi^+(z,s) = (w_m^+(z,s),\ \omega_m^+(z,s),\ b_m^+(z,s),\ j_m^+(z,s),\ \vartheta_m^+(z,s))^T$ satisfies dual boundary conditions at $z = z_B,\ z_T$

$$\omega_m^+(z,s) = \vartheta_m^+(z,s) = b_m^+(z,s) = Dj_m^+(z,s) = 0\,, \qquad \text{for all } z = z_B, z_T \text{ and } s \in (0, s_n) \tag{28}$$

and radial boundary conditions at $s = 0,\ s_n$

$$\psi^+(z,0) = \psi^+(z,s_n) = 0\,, \qquad \text{for all } s = 0, s_n \text{ and } z \in (z_B, z_T)\,. \tag{29}$$

We proceed by construction of a kernel function ψ^+ satisfying the adjoint equation $\mathcal{L}^+\psi^+ = 0$. The components of a vector $\psi^+ = (w_m^+,\ \omega_m^+,\ b_m^+,\ j_m^+,\ \vartheta_m^+)^T$ are assumed to be separated as follows

$$f_m^+(z,s) = f^+(z)\, J_m(ks) \tag{30}$$

where the adjoint functions $f^+(z)$ depend only on a vertical coordinate while the radial dependence is expressed here by the Bessel function $J_m(ks)$. Plugging the above ansatz into the matrix equation $\mathcal{L}^+\psi^+ = 0$, we obtain a system of linear differential equations in z variable (see BRS). The existence of a nontrivial solution of this adjoint system satisfying the dual boundary conditions (28) in the z variable is a consequence of the spectral theorem for the adjoint operator and the fact that the equation $\mathcal{L}\psi = 0$ has a solution decomposable in each vector component to the form $f_m(z,s) = f(z)J_m(ks)$ (see BS).

The linear operators $\mathcal{L}$ and $\mathcal{L}^+$ are defined on suitable Hilbert spaces X and X^+, respectively, with values in a Hilbert space Z. These function spaces can be constructed with respect to boundary conditions for vector functions ψ and ψ^+, respectively. It turns out that these spaces are subclasses of Sobolev spaces $W^{2,2}(G_n)$. The space Z is the weighted Lebesgue space $L^2_\varrho(G_n)$ with the weight $\varrho(s) = s$. The reader is referred to the analysis made in (BRS) for further details of construction and properties of the underlying function spaces.

Let us emphasize that the crucial assumption of the theory is that we operate with function spaces defined on a bounded domain G_n. Then the operator $\mathcal{J}_m$ defined on a subclass of a Sobolev space has a discrete spectrum bounded away from zero. This justifies the usage of the inverse operator $\mathcal{J}_m^{-1}$ in (12). Furthermore, the boundedness of the domain implies that the coefficients β defined in Appendix and consequently R_2 determined in (43) are generically non-zero. Thus the amplitude equation (51) in Section 3.3 is indeed a prototype for the Hopf bifurcation phenomenon.

3.2. Derivation of the solvability condition

At this stage, we are yet able to make use of perturbation techniques and adjointness properties in order to solve the abstract nonlinear problem (11) in its matrix representation (22).

Suppose that the unknown function ψ and the Rayleigh number R (the system parameter) can be expanded into a power series in terms of a small unfolding parameter ε, $(\varepsilon \ll 1)$

$$\psi = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 + \varepsilon^3\,\psi_3 + \dots\,, \tag{31}$$

$$R = R_c + \varepsilon\,R_1 + \varepsilon^2\,R_2 + \dots \tag{32}$$

where the first order term ψ_1 is identical to the solution of the linearized problem (21) and R_c is a critical value of Rayleigh number known from linear stability analysis made in (BS). Higher order coefficients in the expansion are assumed to satisfy $\psi_k \notin \mathrm{Ker}(\mathcal{L})$ for $k \geq 2$.

The nonlinear system (11), however, when being driven through the critical value R_c within its parameter regime, gives rise to the oscillatory instability. Therefore a complex frequency λ has to be expanded into a power series as well

$$\lambda = \lambda_c + \varepsilon\,\lambda_1 + \varepsilon^2\,\lambda_2 + \dots \tag{33}$$

where λ_c is a critical frequency corresponding to R_c. Now we can insert the above expansions (31 - 33) into the system (11). Collecting the terms of the same power of ε and using the well-known matrix representation one obtains a series of linear problems.

In the first order of ε^1 , we obtain a homogeneous linear problem

$$\mathcal{L}\,\psi_1 = 0 \tag{34}$$

where the components of the vector ψ_1 can be sought in the form $f_{m1}(z,s) = f(z)J_m(ks)$. The exact expression for each vector component $f(z)$ can be found e.g. in (BS) or in (Šimkanin et al 1997) in this Issue.

In the second order of ε^2 , we have

$$\mathcal{L}\,\psi_2 = \begin{pmatrix} 0 \\ R_1 k^2\,\vartheta_{m1}(z,s) \\ \lambda_1\, b_{m1}(z,s) \\ \lambda_1\, j_{m1}(z,s) \\ (\lambda_1/q_R)\,\vartheta_{m1}(z,s) \end{pmatrix} \tag{35}$$

where the components $f_{m2}(z,s)$ of a vector ψ_2 are yet unknown. At this order of perturbation expansion the influence of the cubic nonlinearities $P_m(z,s)$, $T_m(z,s)$ and $S_m(z,s)$ is still not present. Taking the inner product $\langle\,.\,|\,.\,\rangle$ of (35) with the dual kernel function ψ^+ yields a simple complex equation

$$-\,\alpha_1\, R_1 + \lambda_1 = 0\,. \tag{36}$$

With regard to the requirement $\lambda_1 = i\sigma_1$, σ_1 is real, the unique solution of this equation is $R_1 = 0$, $\lambda_1 = 0$ and so $\mathcal{L}\psi_2 = 0$. As ψ_2 does not belong to the kernel of $\mathcal{L}$ we finally obtain $\psi_2 = 0$. This property can be also seen from the symmetry of the abstract nonlinear problem.

In the third order of ε^3 , the solvability condition yields a nonhomogeneous problem

$$\mathcal{L}\,\psi_3 = \begin{pmatrix} 0 \\ R_2 k^2 \vartheta_{m1}(z,s) \\ P_{m1}(z,s) \,+\, \lambda_2 b_{m1}(z,s) \\ T_{m1}(z,s) \,+\, \lambda_2 j_{m1}(z,s) \\ (1/q_R) S_{m1}(z,s) \,+\, (\lambda_2/q_R)\vartheta_{m1}(z,s) \end{pmatrix}\,. \tag{37}$$

It is obvious that the nonlinear terms in first order representing functions $f_{m1}(z,s)$, namely $P_{m1}(z,s)$, $T_{m1}(z,s)$ and $S_{m1}(z,s)$, arise at this order of expansion. Now the angular velocity $\Omega(s)$ of geostrophic flow (in its leading term) is a function of $b_{m1}(z,s)$ and $j_{m1}(z,s)$. We therefore adopt the notation $\Omega_1(s)$ for convenience.

We briefly sum up the notation used for this stage of perturbation method. All the nonlinearities are functions of $f_{m1}(z,s)$ which are separable in z and s coordinate. They can be therefore expressed in terms of the simple representing functions $f(z)$, known from the linear stability study, as follows

$$\begin{aligned} P_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,b(z) - im\,\mathcal{J}_m^{-1}\{\mathcal{P}_{\Omega_1}\,J_m(ks)\}\,b(z)\,, \\ T_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,j(z) + \mathcal{J}_m^{-1}\{\mathcal{T}_{\Omega_1}\,J_m(ks)\}\,Db(z)\,, \\ S_{m1}(z,s) &= im\,\Omega_1(s)\,J_m(ks)\,\vartheta(z) \end{aligned} \tag{38}$$

with $\mathcal{P}_{\Omega_1}$, $\mathcal{T}_{\Omega_1}$ corresponding to $\mathcal{P}_{\Omega}$, $\mathcal{T}_{\Omega}$ in (14, 15) where $\Omega(s)$ has been substituted by $\Omega_1(s)$.

Following (16) and the boundary conditions (17), for geostrophic flow $\Omega_1(s)$ in terms of the simple representing functions $f(z)$ we have

$$\Omega_1(s) = \mathcal{Z} \cdot \Omega_s(s)\,. \tag{39}$$

Here

$$\mathcal{Z} = \frac{\Lambda}{2\,(2E)^{1/2}\,k^2} \cdot \Re e \left\{ \int_{z_B}^{z_T} j(z)\overline{Db(z)}\,dz \right\} \tag{40}$$

is the functional involving the functions $b(z)$ and $j(z)$ and

$$\Omega_s(s) = \frac{1}{k^2\,s^3}\,\frac{\partial}{\partial s}\left[m^2 J_m^2(ks) - s^2 \left(\frac{\partial}{\partial s} J_m(ks) \right)^2 \right]$$

describes the radial dependence of geostrophic flow. Using the property of the Bessel differential operator $\mathcal{J}_m$ defined by (13), the above expression can be simplified and written as

$$\Omega_s(s) = \frac{1}{s}\,\frac{d}{ds} J_m^2(ks)\,. \tag{41}$$

The solvability condition for the 3-rd order of the expansion yields an inner product equation

$$\langle\, F_3 \,|\, \psi^+ \,\rangle = 0 \tag{42}$$

where F_3 is a vector of right-hand side terms in (37) and ψ^+ is the previously constructed solution of $\mathcal{L}^+\psi^+ = 0$. By straightforward integrations one finds the solvability condition schematically written as

$$-\,\alpha\,R_2 + \lambda_2 - \beta = 0\,. \tag{43}$$

This condition can be thought of as a complex equation for determining the parameters R_2 and $\lambda_2 = i\sigma_2$ where σ_2 is real, giving us information about bifurcation and frequency response of the dynamical system in the vicinity of the critical Rayleigh number R_c.

The complex coefficients α and β entering (43) depend on the parameters m, Λ, E, q_R as well as on the critical parameters R_c, k_c and λ_c. Their full form is given in terms of analytical expressions (see Appendix).

Now the solution ψ of the nonlinear problem $\mathcal{L}\psi = N(\psi)$ has the power series expansion

$$\psi = \varepsilon\psi_1 + \varepsilon^3\psi_3 + o(\varepsilon^3)\,. \tag{44}$$

Similarly, up to the second order terms, we have

$$R \sim R_c + \varepsilon^2 R_2 \,, \tag{45}$$

$$\lambda \sim \lambda_c + \varepsilon^2 \lambda_2 \,. \tag{46}$$

Finally, if we put

$$\varepsilon = \sqrt{(R - R_c)/R_2} \tag{47}$$

then, in the first order approximation, the representing functions $\tilde{f}(z, s, \varphi, t)$ associated to a solution of the evolution problem (2 - 6) through (8, 9) can be written as

$$\tilde{f}(z, s, \varphi, t) \sim \sqrt{\frac{R - R_c}{R_2}}\, \Re e\{f(z) J_m(ks) \exp(im\varphi + \lambda t)\} \,. \tag{48}$$

The expression $\sqrt{(R - R_c)/R_2}$ therefore relates to the amplitude of representing functions $\tilde{f}(z, s, \varphi, t)$. It can be seen that if $R_2 > 0$, the Hopf bifurcation arising in R_c is supercritical. On the other hand, if $R_2 < 0$, the bifurcation is subcritical. The complex frequency in the neighbourhood of R_c varies according to

$$\lambda \sim \lambda_c + \varepsilon^2 \lambda_2 = \lambda_c + \frac{R - R_c}{R_2}\, \lambda_2 \,. \tag{49}$$

Some useful properties of the constructed solution, i.e. its dependence on the system parameters and its asymptotics, are presented on Figures 1 - 4 below.

3.3. The amplitude modulation and stability properties of the solution

In the previous paragraph it has been shown that the nonlinear problem (2 - 6) has a nontrivial periodic solution when Rayleigh number R is increased beyond its critical value R_c . This periodic solution, branching at R_c from trivial one, can be either supercritical or subcritical, depending on the sign of parameter R_2 . Such a behaviour should indicate the Hopf bifurcation arising at the critical Rayleigh number R_c .

The above analysis, however, does not cover stability properties of the periodic solution constructed above. To analyze stability of the basic state and the bifurcating periodic orbit we have to study a larger phase space than the space of all functions periodic in t and φ variable as it has been proposed by ansatz (10). To this end, one may enlarge this class of functions by assuming that the representing functions $\widetilde{b}$, $\widetilde{j}$, $\widetilde{w}$, $\widetilde{\omega}$ and $\widetilde{\vartheta}$ have the form

$$\tilde{f}(z, s, \varphi, t) = \Re e\{A(\varepsilon^2 t)\, f_m(z, s)\, \exp(im\varphi + \lambda_c t)\} \,. \tag{50}$$

Notice that in (50) each of the functions $f_m(z, s)$ is modulated by complex amplitude $A(\varepsilon^2 t)$ varying in the so-called slow time scale $\varepsilon^2 t$ where ε is a small unfolding parameter as in (31 - 33). As it is indicated by expansions (45, 46) we are forced to choose the scale $\varepsilon^2 t$ in order to capture slowly varying periodic solutions with the complex frequency $\lambda \sim \lambda_c + \varepsilon^2 \lambda_2$. The meaning of all other variables and parameters involved in (50) is left unchanged.

Under the above assumption, straightforward computations based on the same Fredholm alternative argument and on the same function spaces setting can be carried out to derive solvability condition. It can be shown that in this case solvability condition gains a form of an ordinary differential equation for the time dependent complex amplitude $A(\varepsilon^2 t)$. For the modulus $|A(\varepsilon^2 t)|$ the third order approximation of the corresponding ordinary differential equation reads as follows

$$\frac{1}{\alpha_r} \frac{d|A(\varepsilon^2 t)|}{dt} = (R - R_c)\,|A(\varepsilon^2 t)| - R_2\,|A(\varepsilon^2 t)|^3 \qquad (51)$$

where the coefficients α_r (the real part of α) and R_2 are the same as in solvability condition (43).

Notice that the amplitude equation (51) is a prototype for the Hopf bifurcation phenomenon and therefore can be conceived as normal form for the Hopf bifurcation. Both the trivial solution and the bifurcating periodic (nontrivial) solution can be sought as stationary solutions (fixed points) of amplitude equation (51). The only nontrivial steady state solution of the ODE (51) is the constant function

$$|A| = \sqrt{\frac{R - R_c}{R_2}} \qquad (52)$$

which in fact coincides with the unfolding parameter ε . Therefore inserting the steady state amplitude (52) into (50) yields the same periodic solution as the one previously constructed in Section 3.2.

As a result, depending on the sign of R_2 one observes either supercritical or subcritical type of the Hopf bifurcation. The stability of both steady state and periodic solutions depends on the sign of coefficient α_r . More details concerning the amplitude modulation as well as derivation and analysis of the normal form equation (51) will be presented in the forthcoming paper.

4. Bifurcation diagrams and asymptotic properties of the solution

In our numerical experiments the values of the critical Rayleigh number R_c , the critical radial wave number k_c and the critical complex frequency $\lambda_c = i\sigma_c$ were obtained from the linear stability analysis made in (BS). We studied four particular cases related to the azimuthal wave numbers $m = 1$, 2, 3 and 5 , with the Elsasser number Λ ranging from 10^{-3} to 2500 . The Ekman number and the Roberts number were chosen $E = 3 \times 10^{-7}$ and $q_R = 0.005$, respectively. More details concerning the typical values of the critical parameters can be found e.g. in Šimkanin et al (1997) in this Issue.

The Figures 1, 2 are bifurcation diagrams in the space of system parameters R (Rayleigh number) and Λ (Elsasser number). The dependence $R_c = R_c(\Lambda)$ is known from linear stability studies made in Soward (1979) and (BS). The weakly nonlinear analysis from previous sections is capable of describing behaviour of solutions (trivial and nontrivial one) and their stability properties in the underlying space of parameters. This enables us to classify qualitatively the bifurcation diagrams to follow.

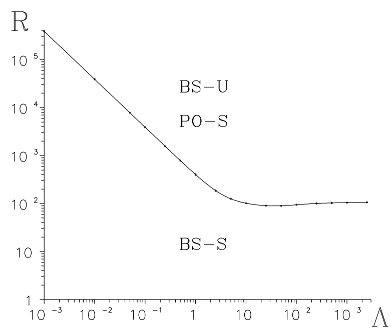

Fig.1. T and MW modes for the azimuthal wave number $m = 1$.

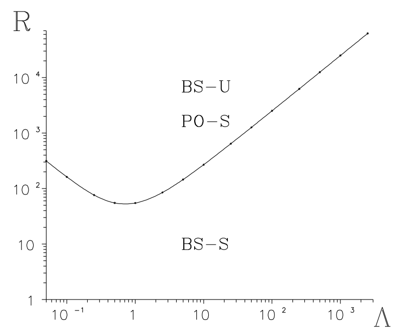

Fig.2. T modes for the azimuthal wave number $m = 5$.

The marked curves in Figures 1, 2 show the dependence of the critical Rayleigh number R_c on the Elsasser number Λ for azimuthal wave numbers $m = 1, 5$. Here T and MW are to symbolize thermally and magnetically driven waves propagating westwards, respectively, as they have been classified in (BS); in Figure 1 the T wave changes into MW wave by increasing Λ at $\Lambda \sim 100$. The parameter space (Λ, R) divided by the curve $R_c = R_c(\Lambda)$, splits into two regions. In the region labeled by $BS - S$ there is no periodic orbit near the locally stable basic state whereas in the region $BS - U$, $PO - S$ the basic state is unstable and there is a stable periodic solution. Here the abbreviation BS stands for "Basic State" and PO for "Periodic Orbit".

The other studied cases of the azimuthal wave number $m = 2, 3$ result into qualitatively same plots and therefore are omitted. We only mention that for large values of the Elsasser number, there is an indication for the Hopf bifurcation to be subcritical for the case $m = 2$. This is due to the change in sign of the coefficient α_r. This special case however needs to be investigated in a more detail. Note that in Skinner and Soward (1990) the subcritical behaviour has been observed for q_R of order unity and for smaller Λ only.

The Figures 3, 4 show asymptotic properties of the finite amplitude solution when the radius of the layer becomes larger. We remind ourselves that the radial extension of the layer measured by s_n has to be finite as it has been proclaimed in previous sections.

Recall that in general the critical Rayleigh number R_c and the critical complex frequency $\lambda_c = i\sigma_c$ are functions of the critical radial wave number k_c . In the linear stability study in (BS) related to the unbounded geometry, for any value of Elsasser number Λ , the wave number $k = k_c$ has been chosen such that the corresponding R_c was minimal. For the particular case of $m = 5$ and for the choice of $\Lambda = 1.0$, it follows from (BS) that $k_c = 5.16$.

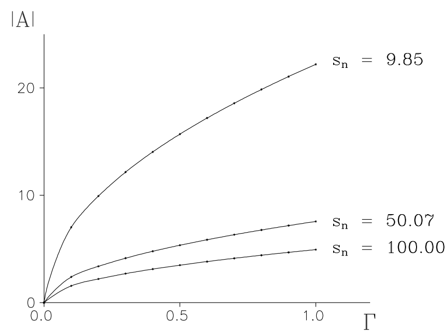

Fig.3. The modulus $|A|$ of the amplitude versus Γ for the azimuthal wave number $m = 5$, the Elsasser number $\Lambda = 1$ and various radii s_n .

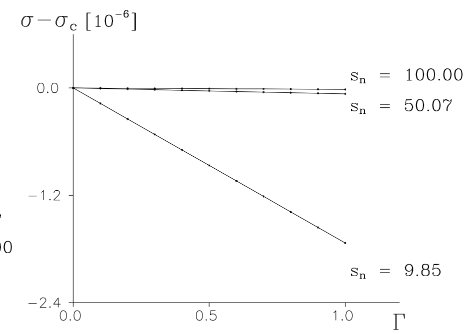

Fig.4. The difference $\sigma - \sigma_c$ of frequencies versus Γ for the azimuthal wave number $m = 5$, the Elsasser number $\Lambda = 1$ and various radii s_n .

Figure 3 above depicts the dependence of the modulus of the amplitude $|A|$, given by (52), on the so-called surplus thermal energy $\Gamma = (R - R_c)s_n^2$. More precisely, the quantity Γ is qualitatively proportional to the thermal energy needed to heat the bottom circular domain of the radius s_n which is, in effect, associated with increase of the Rayleigh number R beyond R_c . This picture can be also viewed as a supercritical bifurcation diagram. Indeed, if $R < R_c$ (i.e. $\Gamma < 0$) there is no periodic solution in the vicinity of the stable basic state. On the other hand, when $R > R_c$ (i.e. $\Gamma > 0$) there is a stable periodic orbit with the modulus of amplitude equal to $|A|$ and the basic state is unstable. The bifurcation curves are plotted for various radial extensions s_n of the cylinder. The reason for introducing the quantity Γ is to compare bifurcation curves for various radii s_n . In terms of the new system parameter Γ , for the amplitude we have $|A| = (\Gamma/R_2)^{1/2} s_n^{-1}$ instead of (52).

It follows from (43) and the expressions for α and β in Appendix that $R_2 = O(s_n^{-1})$ as $s_n \to +\infty$. Therefore for fixed values of the parameter Γ we have

$|A| = O(s_n^{-1/2})$ as $s_n \to +\infty$. This is in agreement with an observation that if the input energy proportional to Γ is constant, the amplitude of motion becomes smaller with growth of the radial extension of the layer.

One has to be careful, however, about the asymptotics like this. The proof of existence of finite amplitude periodic solution based on the weakly nonlinear theory is limited to the parameter range $R_c \leq R < \hat{R}(s_n)$ only. Gathering from the expression $\varepsilon = \sqrt{(R - R_c)/R_2}$, where ε has to be chosen small (i.e. $\varepsilon \ll 1$), and from the asymptotics $R_2 = O(s_n^{-1})$ as $s_n \to +\infty$, we can see that $\hat{R}(s_n) \to R_c$ as $s_n \to +\infty$, i.e. the region of parameter space evaporates.

Figure 4 shows the dependence of the complex frequency $\lambda = i\sigma$ on Γ. For $m = 5$ and $\Lambda = 1$ the critical frequency is $\lambda_c = i\sigma_c$ with $\sigma_c = 0.0376392$. Actually, the difference $\sigma - \sigma_c$ has been plotted versus Γ. In terms of Γ we have $\sigma = \sigma_c + (\Gamma\sigma_2)/(R_2 s_n^2)$. Therefore the dependence of σ on Γ is linear.

Notice that the Γ scale in Figures 3, 4 is magnified in order to show the qualitative features of behaviour of amplitude modulus and frequency response of the nonlinear system. The maximal value of the parameter Γ, however, must be chosen small enough as it is interrelated with the small unfolding parameter ε through the relation $\Gamma = \varepsilon^2 R_2 s_n^2$.

5. Conclusions

It has been shown in this paper that the weakly nonlinear analysis is capable of proving the existence of a nontrivial periodic solution in the vicinity of the critical Rayleigh number R_c for a nonlinear model of rotating magnetoconvection affected by Ekman suction. Although the basic governing equations together with modified Taylor's constraint yield a rather complicated structure, they can be solved analytically in the vicinity of R_c. It has been shown that besides the trivial (zero) solution, there is a periodic solution of the nonlinear problem representing wave propagation in the azimuthal direction.

The existence of a non-trivial periodic solution is neither an obvious matter emerging from the corresponding linearized theory nor a direct consequence of the form of nonlinear governing equation. Among the assumptions guaranteeing the existence of such a solution a crucial role is played by boundedness of the underlying geometry. In case of a rotating horizontal layer it naturally means a restriction to the radially bounded cylinder.

The symmetry of governing equations which is due to cubic nonlinearities implies that the transition from a trivial (conductive) solution towards a nontrivial (convective) periodic solution is via Hopf bifurcation. Applying methods and techniques of the functional analysis, namely solvability conditions from Fredholm's alternative, leads towards derivation of the normal form for the Hopf bifurcation and analytical expressions of its coefficients.

The obtained analytical formulae for the normal form coefficients were evaluated numerically. The bifurcation diagrams showing domains of existence and stability of the solutions have been depicted for the parameter space (Λ, R) . Also the asymptotic properties of the amplitude and frequency of periodic solution for different radial extensions of the layer have been portrayed.

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Appendix

The coefficients α and β in the solvability condition (55) are

$$\alpha = -\,k_c^2\,\frac{\langle\,\vartheta(z)\,\omega^+(z)\,\rangle^z}{M}\,, \qquad \beta = 4\mathcal{Z}\,\frac{I_2}{I_1}\,\frac{\langle\,Db(z)\,j^+(z)\,\rangle^z}{M}$$

where $M = \langle\,b(z)\,b^+(z)\,\rangle^z + \langle\,j(z)\,j^+(z)\,\rangle^z + (1/q_R)\,\langle\,\vartheta(z)\,\vartheta^+(z)\,\rangle^z$ and $\mathcal{Z}$ is a functional given by (41).

The above expressions are entered by the integrals over the radial coordinate

$$I_1 = \int_0^{s_n} J_m^2(k_c s)\, s\, ds\,, \qquad I_2 = \int_0^{s_n} J_m^2(k_c s) \left(\frac{d}{ds} J_m(k_c s) \right)^2 s\, ds$$

which are to be computed numerically and by the integrals over the z coordinate

$$\langle\, f(z)\, f^+(z)\,\rangle^z = \int_{z_B}^{z_T} f(z)\, f^+(z)\, dz\,.$$

Particular integrals needed for evaluation of the coefficients are

$$\begin{aligned}
\langle\, \vartheta(z)\, \omega^+(z)\,\rangle^z &= \frac{1}{2R_c k_c^2} \sum_l c_l \gamma_l\,, \\
\langle\, b(z)\, b^+(z)\,\rangle^z &= -\,\frac{1}{2} \sum_l \gamma_l \left(\frac{\pi_l}{m^2\Lambda} s_l^\omega - 1 \right)\,, \\
\langle\, j(z)\, j^+(z)\,\rangle^z &= -\,\frac{1}{2m^2\Lambda} \sum_l s_l^j c_l \gamma_l \pi_l\,, \\
\langle\, \vartheta(z)\, \vartheta^+(z)\,\rangle^z &= -\,\frac{5}{2R_c k_c^2}\,, \\
\langle\, Db(z)\, j^+(z)\,\rangle^z &= -\,\frac{1}{2} \sum_l \gamma_l^2 \pi_l^2 c_l
\end{aligned}$$

where

$$\begin{aligned}
c_l &= \pi_l^2 + k_c^2 + \lambda\,, \\
\gamma_l^{-1} &= \frac{\pi_l^2}{m^2\Lambda} (\pi_l^2 + k_c^2 + \lambda - 2im\Lambda)^2 + m^2\Lambda(\pi_l^2 + k_c^2)\,, \\
s_l^\omega &= \pi_l(\pi_l^2 + k_c^2 + \lambda - 2im\Lambda)\, c_l \gamma_l\,, \\
s_l^j &= \frac{s_l^\omega}{c_l}
\end{aligned}$$

with $\pi_l = (2l-1)\pi$, $\lambda = i\sigma$ and l equals to 5.

Let us emphasize that the integral I_1 diverges to $+\infty$ whereas I_2 converges as $s_n \to +\infty$. Thus the coefficient β vanishes when s_n tends to $+\infty$. We also notice that the integrals over the z coordinate are entered by functions of $f(z)$ which solve the linearized (eigenvalue) problem and by functions of $f^+(z)$ which solve the homogeneous adjoint problem in Section 3.1.