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Viscously controlled nonlinear magnetoconvection in a non-uniformly stratified horizontal fluid layer

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Abstract

Weakly nonlinear analysis is adopted in order to study a model of magnetoconvection in a rotating horizontal fluid layer. The layer is supposed to be non-uniformly stratified and is permeated by an azimuthal magnetic field. The only nonlinearity brought in this convecting system is due to presence of Ekman layers along the horizontal mechanical boundaries. The governing equations for this model together with the expression for geostrophic flow, i.e., modified Taylor's constraint are analysed with help of perturbation methods. As a result, the bifurcation structure in the vicinity of the critical Rayleigh number is revealed.

Keywords: Magnetoconvection; Modified Taylor's constraint; Non-uniform stratification; Perturbation techniques; Weakly nonlinear analysis

1. Introduction

In a large class of MHD models, the assumption is often made that the primary force balance in the Earth's core is entered by Lorentz, Coriolis, buoyancy and pressure forces in the equation of motion. Such a force balance, familiar also as magnetostrophic approximation, has a solution, if and only if so-called Taylor's constraint is satisfied (Taylor, 1963). In the case of small but non-zero viscosity, the magnetostrophic approximation still holds as a primary force balance, but Taylor's constraint has to be modified to include viscous effects. In this case, modified Taylor's constraint can be taken as a predictive formula for evaluation of geostrophic flow which is thus expressed explicitly (Fearn, 1994).

In this paper, we focus our attention on a finite amplitude rotating magneto-convection in a horizontal layer permeated by azimuthal magnetic field. The linearized version of this problem for the model of uniformly stratified rotating layer with infinite horizontal extension has been studied by Soward (1979) and Brestenský and Ševčík (1994). In the model of rotating annulus (Skinner and Soward, 1988, 1991), the effect of geostrophic flow was incorporated making the whole problem nonlinear. Here, conditions

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for the onset of instability in the regime of so-called Taylor state [1] were found.

Unlike the above mentioned references, in our recent studies made in Brestenský et al. (1997) and Revallo et al. (1997), [2] we restricted our attention on the early evolution of instability in the vicinity of the basic static state, i.e., in the vicinity of critical Rayleigh number R_c. This leads to a specific weakly nonlinear problem where the ensuing magnetoconvection is affected by the presence of geostrophic flow. The interested reader is referred to BRS for some mathematical aspects as well as for the method of solution. In RSB we considered a simple model of radially bounded horizontal rotating layer with free infinitely electrically and thermally conducting boundaries. We found that the oscillatory convection in this system sets in via Hopf bifurcation which is typically supercritical for $q = 0.005$. Furthermore the convective instability has a form of travelling waves whose frequency has a decreasing tendency as the Rayleigh number is increased beyond its critical value.

In this paper, we pursue the weakly nonlinear analysis made in BRS and RSB for a more complicated model of magnetoconvection where non-uniform stratification of the layer is incorporated. Our modification of the model is based upon an idea originally proposed by Bod'a (1988) and later developed by Ševčík (1989). [3] In these references a linear problem of magnetoconvection in a horizontally unbounded geometry was set up in which the density gradient changes its sign across the layer (non-uniform stratification). The concept of a non-uniformly stratified layer appears to be reasonable as it incorporates more realistic conditions in the Earth's interior which is known to be non-uniformly stratified.

Note that several interesting features were isolated in the model introduced by Bod'a (1988), e.g., existence of the magnetic mode in the layer gravitationally stable in the top half and unstable in the bottom half. Moreover, under the assumption of non-uniform stratification, the excitation of thermal mode was observed in S89 even for the layer cooled from below and heated from above (the case of negative Rayleigh number).

The paper is organised as follows. In Section 2 we formulate the nonlinear problem and we state the expression for the geostrophic flow. In Section 3 we present a system of nonlinear PDE's governing motion. Such a motion is periodic in both time and in the azimuthal variable. Section 4 refers back to the original linear problem for a non-uniformly stratified layer solved in S89. In Section 5 we briefly outline the solution of PDE's by a perturbation technique and we quote resulting amplitude equation. The results of bifurcation analysis are presented in Section 6. The corresponding bifurcation diagrams are also shown in this section. Section 7 is devoted to conclusions.

2. Description of the nonlinear model

The model under consideration is a weakly bounded cylinder [4] of width d, $z \in \langle 0, d \rangle$ and radius s_n, $s \in \langle 0, s_n \rangle$, rotating rapidly with angular velocity $\Omega_0 \hat{\boldsymbol{z}}$ about the vertical rotation axis. The cylinder contains an electrically conducting Boussinesq fluid permeated by an azimuthal magnetic field B_0 linearly growing with the distance from the rotation axis. The instability of this system can be caused by the vertical temperature gradient. Therefore, we consider the temperature difference ΔT maintained between the lower, T_1, and the upper boundaries, $T_1 - \Delta T$. Non-uniform stratification can be modelled by negative heat sources, $H < 0$, distributed within the layer. This has a consequence of non-linear (quadratic) dependence of basic temperature profile, T_0.

Assuming small but non-zero viscosity leads towards formation of viscous boundary layers (the Ekman layers) along the horizontal boundaries. As a result, non-zero geostrophic flow $\Omega(s)$ is induced by Ekman suction, which couples the interaction between boundary layer and the rest of the fluid, making the whole problem nonlinear.

[1] In this asymptotic regime the magnitude of geostrophic flow is such that viscous forces no longer have major influence on the convection and the net torque on geostrophic cylinders vanishes.

[2] Henceforth referred to as BRS and RSB.

[3] Henceforth referred to as S89.

[4] The radial extension of the layer is much greater than its thickness.

The ensuing convective instability manifests itself by perturbations of the velocity $\boldsymbol{u}$, the magnetic field $\boldsymbol{b}$ and the temperature $\tilde{\vartheta}$ which relate to the basic state represented by

$$\begin{aligned} &\boldsymbol{U}_0 = \boldsymbol{0}, \\ &\boldsymbol{B}_0 = B_M \frac{s}{d}\hat{\boldsymbol{\varphi}}, \\ &T_0 = T_l - \Delta T \frac{z}{d}\left(1 - \frac{z-d}{2 z_M * - d}\right). \end{aligned} \tag{1}$$

The quantity $z_M * = -\rho_0 c_p \kappa \Delta T/(dH) + d/2$, referred to as stratification parameter, is the z-coordinate of the level dividing the layer into the stably and unstably stratified sublayers; $\rho_0 c_p \kappa$ is the thermal conductivity. The temperature $T_0(z)$ reaches minimum and its gradient changes direction at the level $z = z_M *$. Note that the cases of uniform stratification can be obtained as the limiting cases $z_M * \to \pm\infty$.

We non-dimensionalise the basic equations with the use of characteristic length d, magnetic diffusion time d^2/η, magnetic field B_M, and temperature difference across the layer ΔT. The equations in the cylindrical polar coordinates (s, φ, z) governing the evolution of perturbations $\boldsymbol{u}$, $\boldsymbol{b}$, $\tilde{\vartheta}$ are cast as follows

$$\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + \Lambda[(\nabla \times s\hat{\boldsymbol{\varphi}}) \times \boldsymbol{b} + (\nabla \times \boldsymbol{b}) \times s\hat{\boldsymbol{\varphi}}] + R\tilde{\vartheta}\hat{\boldsymbol{z}}, \tag{2}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \nabla \times (s\Omega(s)\hat{\boldsymbol{\varphi}} \times \boldsymbol{b}) = \nabla \times (\boldsymbol{u} \times s\hat{\boldsymbol{\varphi}}) + \nabla^2 b, \tag{3}$$

$$q\left(\frac{\partial \tilde{\vartheta}}{\partial t} + (s\Omega(s)\hat{\boldsymbol{\varphi}} \cdot \nabla)\tilde{\vartheta}\right) = -\boldsymbol{u} \cdot \nabla T_0 + \nabla^2 \tilde{\vartheta}, \tag{4}$$

$$\nabla \cdot \boldsymbol{b} = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \tag{5}$$

where $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\varphi}}$ are the axial and azimuthal unit vectors, respectively.

In Eqs. (2)–(5) the dimensionless parameters, the modified Rayleigh number R, the Elsasser number Λ, the Ekman number E and the Roberts number q, are defined by

$$R = \frac{g d \Delta T \alpha}{2\Omega_0 \kappa}, \quad \Lambda = \frac{B_M^2}{2\Omega_0 \rho_0 \eta \mu}, \quad E = \frac{\nu}{2 d^2 \Omega_0}, \quad q = \frac{\kappa}{\eta}$$

where κ and η are the thermal and magnetic diffusivities, ν is the kinematic viscosity, α is the coefficient of thermal expansion, g is the acceleration due to gravity, μ is the permeability and ρ_0 is the density.

In the case of non-uniform stratification, the temperature gradient entering Eq. (4) can be expressed in terms of the dimensionless parameters in the following form (S89)

$$\begin{aligned} \frac{\mathrm{d}T_0}{\mathrm{d}z} &= -(2az - a + 1) \quad \text{for} \quad \Delta T > 0, \\ \frac{\mathrm{d}T_0}{\mathrm{d}z} &= +(2az - a + 1) \quad \text{for} \quad \Delta T < 0 \end{aligned} \tag{6}$$

where the coefficient a relates to the dimensionless stratification parameter z_M via

$$a = \frac{1}{1 - 2z_M}. \tag{7}$$

Here $z_M = z_M */d$. Note that taking the coefficient $a = 0$ corresponds to uniform stratification.

The angular velocity $\Omega(s)$ of geostrophic flow entering the convective non-linearities in the above set of equations is determined by modified Taylor's constraint

$$\Omega(s) = \frac{\Lambda}{(2E)^{1/2} s}\int_{z_B}^{z_T} \langle F_{M\varphi}\rangle^{\varphi} dz \quad \text{with} \quad \langle F_{M\varphi}\rangle^{\varphi} = \langle [(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}]_{\varphi}\rangle^{\varphi} \tag{8}$$

where z_B and z_T delimit the horizontal boundaries of the layer, the $\langle \dots \rangle^{\varphi} \equiv 1/(2\pi)\int_0^{2\pi} \dots \mathrm{d}\varphi$ stands for averaging over φ and $F_{M\varphi} \equiv [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]_{\varphi}$ denotes an azimuthal component of Lorentz force.

Eqs. (2)–(5) have to be solved subject to boundary conditions corresponding to rigid [5] perfectly

[5] Only due to the effect of Ekman secondary flow.

electrically and thermally conducting horizontal boundaries, i.e.,

$$u_z = \tilde{\vartheta} = b_z = 0, \quad \hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{b}}{\partial z} = \boldsymbol{0} \quad \text{at} \quad z = 0, d. \quad (9)$$

In addition, we will assume that on the sidewall boundaries, delimited by $s = s_n$ the perturbations are vanishing (see following section).

3. Formulation of the nonlinear problem

It is convenient to rearrange the system of nonlinear equations (Eqs. (2)–(5)) with help of the poloidal–toroidal decomposition of vector fields (for details see BRS). For the velocity perturbation $\boldsymbol{u}$ and magnetic field perturbation $\boldsymbol{b}$ we assume

$$\begin{aligned} \boldsymbol{u} &= k^{-2}\left[\nabla \times (\nabla \times \tilde{w}\hat{\boldsymbol{z}}) + \nabla \times \tilde{\omega}\hat{\boldsymbol{z}}\right], \\ \boldsymbol{b} &= k^{-2}\left[\nabla \times (\nabla \times \tilde{b}\hat{\boldsymbol{z}}) + \nabla \times \tilde{j}\hat{\boldsymbol{z}}\right]. \end{aligned} \quad (10)$$

Here, k is a radial wave number and the representing functions $\tilde{w}$, $\tilde{\omega}$, $\tilde{b}$, $\tilde{j}$ depend on coordinates z, s, φ and time t and will be symbolized by $\tilde{f}(z, s, \varphi, t)$, or shortly $\tilde{f}$, as in Brestenský and Ševčík (1994). The same notation applies for the temperature perturbation, i.e., $\tilde{\vartheta}$.

The representing functions of $\tilde{f}$ can be sought in the form

$$\tilde{f}(z,s,\varphi,t) = \Re e\left\{A(\varepsilon^p t) f_m(z,s) \exp(im\varphi + \lambda t)\right\}. \quad (11)$$

Hereafter the symbol $f_m(z,s)$ stands for one of the complex functions $w_m(z,s)$, $\omega_m(z,s)$, $b_m(z,s)$, $j_m(z,s)$ and $\vartheta_m(z,s)$ dependent on coordinates z and s. Unlike the assumption often made in the linear case, e.g, (Soward, 1979; Šimkanin et al., 1997), each of the functions $f_m(z,s)$ above is modulated by a complex amplitude $A(\varepsilon^p t)$ varying in the so-called slow time scale $\varepsilon^p t$ where ε is a small unfolding parameter and p is a natural number to be specified later. Furthermore, m is an azimuthal wave number (integer) and λ is a complex frequency (related to a real frequency via $\lambda = i\sigma$).

Upon the above assumption, the reduced system of nonlinear equations can be derived from Eqs. (2)–(5). Hereafter, the notation $\tau = \varepsilon^p t$ and $\dot{A}(\tau) = \mathrm{d}A(\tau)/\mathrm{d}\tau$ will be used. The partial differential equations for representing functions take the following forms

$$\begin{aligned}
0 &= -Dw_m(z,s) + 2\Lambda Db_m(z,s) - im\Lambda j_m(z,s), \\
0 &= -D\omega_m(z,s) + 2\Lambda Dj_m(z,s) \\
&\quad + im\Lambda\left(D^2 - k^2\mathscr{I}_m\right)b_m(z,s) - Rk^2\vartheta_m(z,s), \\
&\lambda A(\tau) b_m(z,s) + \varepsilon^p \dot{A}(\tau) b_m(z,s) \\
&\quad + A(\tau)|A(\tau)|^2 P_m(z,s) = imA(\tau) w_m(z,s) \\
&\quad + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)b_m(z,s), \\
&\lambda A(\tau) j_m(z,s) + \varepsilon^p \dot{A}(\tau) j_m(z,s) \\
&\quad + A(\tau)|A(\tau)|^2 T_m(z,s) = imA(\tau)\omega_m(z,s) \\
&\quad + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)j_m(z,s), \\
&(1/q)\left(\lambda A(\tau)\vartheta_m(z,s) + \varepsilon^p \dot{A}(\tau)\vartheta_m(z,s)\right. \\
&\quad \left. + A(\tau)|A(\tau)|^2 S_m(z,s)\right) = A(\tau)\zeta(z) \\
&\quad \times \mathscr{I}_m w_m(z,s) + A(\tau)\left(D^2 - k^2\mathscr{I}_m\right)\vartheta_m(z,s)
\end{aligned} \quad (12)$$

where $\zeta(z) = -\mathrm{d}T_0/\mathrm{d}z$.

The representing functions for nonlinearities $P_m(z,s)$, $T_m(z,s)$, $S_m(z,s)$ are expressed in terms of

$$\begin{aligned}
P_m(z,s) &= im\Omega(s) b_m(z,s) \\
&\quad - im\mathscr{I}_m^{-1}\left\{\mathscr{P}_\Omega\, b_m(z,s)\right\}, \\
T_m(z,s) &= im\Omega(s) j_m(z,s) \\
&\quad + \mathscr{I}_m^{-1}\left\{\mathscr{T}_\Omega\, Db_m(z,s)\right\}, \\
S_m(z,s) &= im\Omega(s)\vartheta_m(z,s).
\end{aligned} \quad (13)$$

Here $D = \partial/\partial z$ and $\mathscr{I}_m^{-1}$ is the inverse operator to the linear Bessel differential operator $\mathscr{I}_m$

$$\mathscr{I}_m \equiv -\frac{1}{k^2}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} - \frac{m^2}{s^2}\right)$$

and P_Ω, T_Ω are differential operators

$$\mathscr{P}_\Omega = -\frac{1}{k^2}\left\{\frac{\partial^2\Omega(s)}{\partial s^2} + \frac{\partial\Omega(s)}{\partial s}\left[2\frac{\partial}{\partial s} + \frac{1}{s}\right]\right\},$$

$$\mathscr{T}_\Omega = -\frac{1}{k^2}\left\{s\frac{\partial^2\Omega(s)}{\partial s^2}\frac{\partial}{\partial s} + s\frac{\partial\Omega(s)}{\partial s} \times\left[\frac{m^2}{s^2} + \frac{2}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s^2}\right]\right\}. \quad (14)$$

The expression for the geostrophic flow $\Omega(s)$ in terms of $b_m(z,s)$ and $j_m(z,s)$ can be derived directly from Eq. (8) (see RSB). Assuming the separability $f_m(z,s) = f(z) J_m(ks)$, it simplifies to

$$\Omega(s) = \mathscr{Z} \frac{1}{s} \frac{d}{ds} J_m^2(ks) \quad \text{where } \mathscr{Z} = \frac{\Lambda}{(8E)^{1/2} k^2} \Re e \int_{z_B}^{z_T} j(z) \overline{Db(z)} dz \tag{15}$$

is a functional depending on the functions $b(z)$ and $j(z)$ known from the linear study.

For the representing functions $f_m(z,s)$, the corresponding boundary conditions can be obtained from Eq. (9)

$$w_m(z,s) = \vartheta_m(z,s) = b_m(z,s) = Dj_m(z,s) = 0, \quad \text{for all } z = 0, d \text{ and } s \in (0, s_n). \tag{16}$$

In a radial direction we impose the following boundary conditions

$$\vartheta_m(z,s) = b_m(z,s) = j_m(z,s) = 0, \quad \text{for all } s = 0, s_n, \text{ and } z \in (0,d). \tag{17}$$

In our model, the parameter s_n, which delimits the layer in a radial direction, has to be chosen to coincide with the n-th root of the Bessel function i.e., $J_m(ks_n) = 0$ for sufficiently large integer n. Since the values of radial wave number k are taken from the linear analysis, the above condition is met for certain values of s_n only.

4. Linearized problem and its solution

Considering small perturbations the whole problem can be linearized (see e.g., Bod'a, 1988; S89). Specifically, in our model the linearization can be fixed by the condition $\Omega(s) = 0$. The linear case allows for the separation of representing functions

$$f_m(z,s) = f(z) J_m(ks) \tag{18}$$

where $J_m(ks)$ is the Bessel function of the first kind, k is a radial wavenumber (real) and $f(z)$ is the complex function of z-coordinate. Respecting the boundary conditions, for each particular $f(z)$ we have

$$w(z) = \sum_{n=1}^{\infty} w_n \sin(\pi nz),$$
$$\omega(z) = \omega_0 + \sum_{n=1}^{\infty} \omega_n \cos(\pi nz),$$
$$b(z) = \sum_{n=1}^{\infty} b_n \sin(\pi nz),$$
$$j(z) = j_0 + \sum_{n=1}^{\infty} j_n \cos(\pi nz),$$
$$\vartheta(z) = \sum_{n=1}^{\infty} \vartheta_n \sin(\pi nz). \tag{19}$$

Inserting the above expansions into the linearized equations, after a series of standard operations we obtain a set of algebraic equations for complex coefficients w_n, ω_n, b_n, j_n, ϑ_n. In S89, the critical Rayleigh number R_c, the critical frequency λ_c the critical radial wave number k_c and the complex coefficients were computed for various sets of parameters (Λ, q, m, a).

5. Solution of the nonlinear problem

A standard way is to represent nonlinear equations (Eq. (12)) in a matrix form (in BRS referred to as an abstract nonlinear problem)

$$A(\tau) \mathscr{L} \boldsymbol{\psi} = N\left(A(\tau), \dot{A}(\tau), \boldsymbol{\psi}\right) \tag{20}$$

where $A(\tau)$ is the complex amplitude, $\mathscr{L}$ is the linear operator and $\boldsymbol{\psi}$ is the vector function

$$\boldsymbol{\psi}^T \equiv \left(w_m(z,s), \omega_m(z,s), b_m(z,s), j_m(z,s), \vartheta_m(z,s)\right) \tag{21}$$

and the right-hand side vector $N(A(\tau), \dot{A}(\tau), \boldsymbol{\psi})$ contains the nonlinearities.

Due to the structure of the geostrophic term, a cubic nonlinearity appears in the system. Taking the symmetry properties into account, the integer parameter p has to be set $p = 2$ and the increment R–R_c of the Rayleigh number will be fixed by the condition

$$R - R_c = \pm \varepsilon^2 \tag{22}$$

which ensures the supercritical or subcritical character of bifurcation. Here ε has the meaning of a small unfolding parameter. The vector of representing functions $\boldsymbol{\psi}$ as well as the complex amplitude $A(\tau)$ have to be expanded in terms of ε, $(\varepsilon \ll 1)$

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \varepsilon \boldsymbol{\psi}_2 + \varepsilon^2 \boldsymbol{\psi}_3 + \cdots,$$
$$A(\tau) = \varepsilon A_1(\tau) + \varepsilon^2 A_2(\tau) + \varepsilon^3 A_3(\tau) + \cdots, \tag{23}$$

where τ is the slow time associated with the physical time through the relation $\tau = \varepsilon^2 t$.

Inserting the above expansions into (20) and collecting terms of the same power of ε, yields a series of non-homogeneous matrix equations. In order to ensure their solvability, the complex amplitudes A_i, $(i = 1, 2, \ldots)$ must be adjusted at each stage of expansion, giving rise to amplitude equations. At the leading order the final form of amplitude equation reads

$$\frac{\mathrm{d}A(\varepsilon^2 t)}{\mathrm{d}t} = (R - R_c)\alpha A(\varepsilon^2 t) - \beta A(\varepsilon^2 t)|A(\tau)|^2, \tag{24}$$

which describes the evolution of the amplitude $A(\tau)$ in the physical time t instead of τ.

The equation quoted above is the normal form for the Hopf bifurcation in $R = R_c$. Stability analysis of this normal form enables us to identify the super- or subcriticality, stability and the frequency response of the convecting system in the vicinity of $R = R_c$. We are able to discuss these properties in terms of the complex coefficients α and β which depend on the parameters m, Λ, E, q as well as on the critical

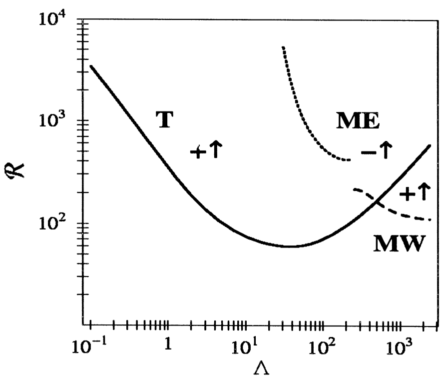

Fig. 1. T, MW and ME modes for $m = 1$, $z_M = 0.6$ and $q = 0.005$.

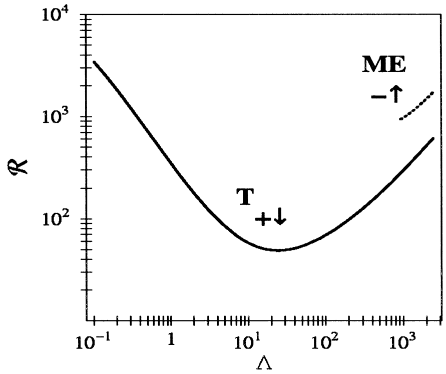

Fig. 2. T and ME modes for $m = 1$, $z_M = 0.6$ and $q = 0.5$.

parameters R_c, k_c and σ_c as shown in Appendix A. Motivated by RSB we introduce the following notation

$$R_2 = \frac{\beta_r}{\alpha_r}, \quad \sigma_2 = \alpha_i \frac{\beta_r}{\alpha_r} - \beta_i. \tag{25}$$

The above expressions are directly associated with the Hopf bifurcation properties, namely if $R_2 > 0$ the Hopf bifurcation at R_c is supercritical, otherwise it is subcritical. The sign of α_r causes the change of stability of the solutions in the vicinity of R_c. In case $\sigma_2 > 0$ the frequency response of the nonlinear system is such that frequency of the oscillations grows; in case $\sigma_2 < 0$ frequency of the oscillations decreases (if $\sigma_c > 0$).

6. Results

In the numerical experiments to follow the values of the critical Rayleigh number R_c, the critical radial wave number k_c and the critical complex frequency $\lambda_c = i\sigma_c$ were obtained from the linear stability analysis made in S89. We have evaluated the coefficients R_2 and σ_2 numerically for various sets of parameters m, q, Λ.

Namely, we studied two particular cases related to the azimuthal wave numbers $m = 1$ and $m = 2$ with the Elsasser number Λ ranging from 10^{-3} to 2500. The values of the Roberts number were chosen $q = 0.005$ and $q = 0.5$. Note that the choice of the Ekman number is irrelevant in this case. It can be scaled out of the problem when the nonlinearity is only due to geostrophic flow.

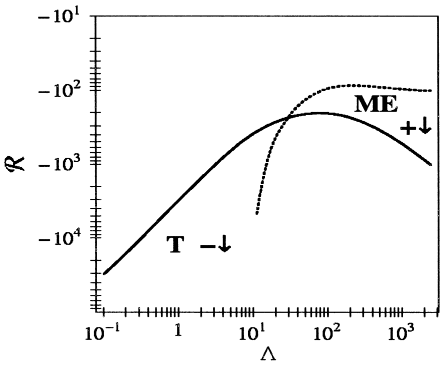

Fig. 3. T and ME modes for $m=1$, $z_M=0.4$ and $q=0.005$.

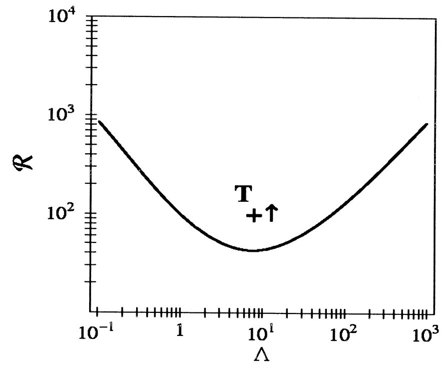

Fig. 5. T modes for $m=2$, $z_M=0.6$ and $q=0.005$.

The figures below are bifurcation diagrams in the space of parameters R (Rayleigh number) and Λ (Elsasser number) where the marked curve $R_c = R_c(\Lambda)$ shows the dependence of the critical Rayleigh number on the Elsasser number for each particular convective mode. Knowing the coefficients α and β of the amplitude equation enables us to classify the domains separated by $R_c = R_c(\Lambda)$. In each of the diagrams, the domains below the curves correspond to trivial conductive solutions whilst the domains above correspond to oscillatory convective solutions.

Here T and MW symbolize thermally and magnetically driven waves propagating westwards, respectively and ME denotes magnetically driven waves propagating eastwards. Other notation adopted here differs from that used in RSB. Hereafter + and − in the diagrams stand for supercritical and subcritical Hopf bifurcation. In both cases the trivial solution loses stability when the parameter R passes its critical value R_c. Recalling properties of the Hopf bifurcation, the arising subcritical and supercritical oscillations are unstable and stable, respectively. It must be emphasized, however, that all of what was said of the stability holds in the case when $\alpha_r > 0$. Analyzing the normal form for $\alpha_r < 0$, we deduce that the stability of the trivial and nontrivial solutions is reversed. Specifically, the case of supercritically bifurcated oscillations which are unstable (only ME modes for $q = 0.5$) will be denoted by $+U$. The arrow symbols ↑ and ↓ in the graphs below denote increase or decrease in frequency of nonlinear convective oscillations.

At this stage, we must realize that also negative values of the Rayleigh number can be considered in the underlying model. This is actually the case when the lower horizontal boundary of the layer is cooled and the upper one is heated. From the physical point

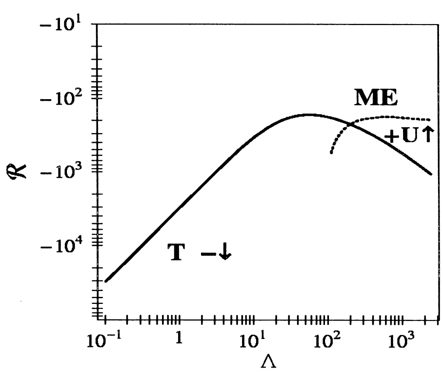

Fig. 4. T and ME modes for $m=1$, $z_M=0.4$ and $q=0.5$.

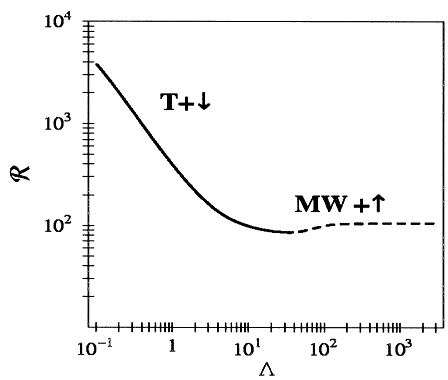

Fig. 6. T and MW modes for $m=1$, $z_M=0.9$ and $q=0.005$.

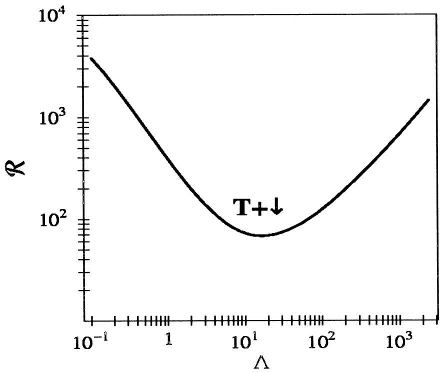

Fig. 7. T modes for $m=1$, $z_M=0.9$ and $q=0.5$.

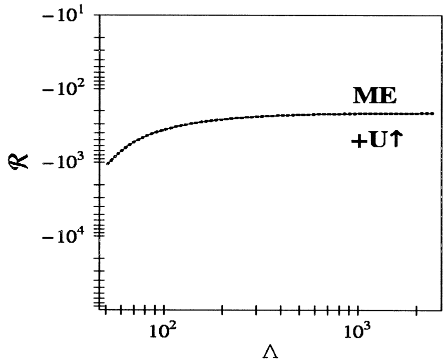

Fig. 9. ME modes for $m=1$, $z_M=0.1$ and $q=0.5$.

of view only the absolute value of the Rayleigh number is of relevance as it is directly related to the energy input into the system. Realizing this fact in what follows, we classify the type of bifurcation (supercritical or subcritical) with respect to the absolute value of the Rayleigh number.

Figs. 1–5 are bifurcation diagrams where the stratification parameter was set $z_M = 0.6$ and $z_M = 0.4$. This choice of z_M relates to the cases of positive and negative Rayleigh number, respectively. The weakly nonlinear behaviour of particular kinds of modes shows some characteristic features. In Figs. 1–5, it can be seen that the value of dimensionless stratification parameter z_M, measuring the thickness of unstably stratified sublayer, is directly related to the sub- or supercriticality of the T, MW and ME modes. Typically, the T modes bifurcate supercritically and the ME modes bifurcate subcritically for $z_M = 0.6$, i.e., when thickness of the unstably stratified sublayer is larger than that of the stably stratified sublayer. On the other hand, for $z_M = 0.4$ the T modes appear to be subcritical and the ME modes are supercritical.

The same applies for different configuration of stratification when z_M was chosen $z_M = 0.9$ or $z_M = 0.1$, as it is presented in Figs. 6–10. This choice of z_M means that the unstably and stably stratified sublayers become more distinct from each other as for their thickness. That is why only one kind of the convective oscillatory mode was isolated for each particular stratification. As for Figs. 8 and 9, only ME modes are depicted. Here, the T modes are off the scale due to the high Rayleigh number R. For $m = 1$, $z_M = 0.9$ and $q = 0.005$ (see Fig. 1) an observation has been made in the linear study that at $\Lambda \sim 50$ the T mode is continuously transformed into MW mode.

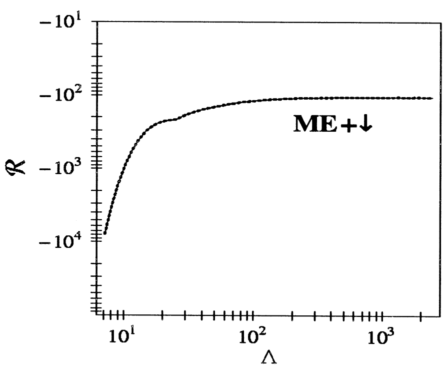

Fig. 8. ME modes for $m=1$, $z_M=0.1$ and $q=0.005$.

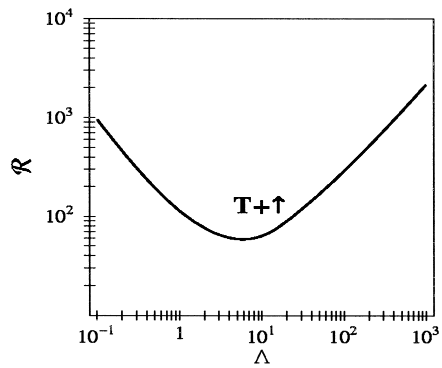

Fig. 10. T modes for $m=2$, $z_M=0.9$ and $q=0.005$.

Table 1
The values of the functional $\mathscr{Z}$ for T modes for $m=1$, $q=0.005$ and various Λ and z_M

	$z_M=0.1$	$z_M=0.4$	$z_M=0.6$	$z_M=0.9$
$\Lambda=100$	0.0245	0.13	0.019	0.0128
$\Lambda=500$	0.0056	0.023	0.0243	0.0016
$\Lambda=1000$	0.0031	0.029	0.0199	0.0032
$\Lambda=2500$	0.0024	0.043	0.032	0.0025

Realizing that frequency of the nonlinear convective oscillations changes at $\Lambda \sim 50$, the weakly nonlinear analysis is capable of identifying this interface as well. It is also remarkable that for $m=1$, $q=0.5$ and for $z_M=0.4$ or $z_M=0.1$ the supercritical oscillations corresponding to ME modes were found to be unstable (the domain below the dotted curve in Figs. 4 and 9). This was the only case when the unstable supercritical convection was observed in our magnetoconvection model.

In Table 1 we show the dependence of the functional $\mathscr{Z}$ on the Elssaser number Λ and the stratification parameter z_M. Recall that $\mathscr{Z}$ enters the expression (15) for the geostrophic flow $\Omega(s)$, i.e., it can be thought as the amplitude of $\Omega(s)$. It turned out that the dependence of $\mathscr{Z}$ on Λ is non-monotonous. More interestingly, for fixed Λ, the factor $\mathscr{Z}$ exhibits local maximum as a function of z_M located near $z_M=0.5$.

7. Concluding remarks

In this paper we studied the weakly nonlinear effect of geostrophic flow on the marginal convection in the non-uniformly stratified horizontal fluid layer. Under the assumption of weak boundedness of the layer, the analysis presented here is based on the data available for the unbounded linear version of the model. We found out that the convective instability in our model sets in via Hopf bifurcation and classified its properties.

For the azimuthal wave number $m=1$, it is apparent from the bifurcation diagrams that the weakly nonlinear behaviour of T and ME modes under the action of geostrophic flow is different while T and MW modes are not distinguished from each other. The global observation says that, for each considered value q, Λ, z_M and for $R>0$, the T modes bifurcate supercritically and the ME modes bifurcate subcritically. On the other hand, for $R<0$ the bifurcations corresponding to T and ME modes are subcritical and supercritical, respectively. Varying the values of the parameters q, Λ, z_M may only cause changes in stability of solutions or change in frequency response.

Two interesting features were isolated for particular modes in certain parametric regimes. Firstly, when $q=0.005$ and stably stratified sublayer is thin enough as it is expressed in terms of stratification parameter equal $z_M=0.9$, the continuous transition between T and MW modes occur. This phenomenon, known from linear study of Soward (1979) and S89, was observed also in our nonlinear problem as a change of frequency of instabilities. Secondly, for $q=0.5$ and the stratification parameter $z_M=0.4$, the ME modes though being preferred to T modes at the linear stage, were identified as unstable supercritically bifurcated ones. Irrespective of the choice of parameters, the convective oscillatory modes with the azimuthal wave number $m=2$, which are always thermally driven, set in via supercritical bifurcation and their frequency grows.

We also found that respect to the choice of the stratification parameter z_M, the maximal amplitude of the geostrophic flow can be expected for the stratification characterised by $z_M=0.5$, i.e., when the stably and unstably stratified sublayers have the same thickness.

In the following, we comment briefly on some mathematical aspects of our analysis. Inserting the perturbation expansions into the modified Taylor constraint, the resulting formula for geostrophic flow $\Omega(s)$ gains quite a simple form (Eq. (15)) which is usable for analytical calculations. Moreover, the structure of the expression for $\Omega(s)$ implies that in this nonlinear problem there is no interaction of oscillatory modes with different azimuthal wave numbers m. Note also that growing the radial extension of the layer, measured in terms of s_n, makes only the amplitude of perturbations vanish, having no impact on the bifurcation properties. This fact emerges from the assumption of weak boundedness of the layer.

Another notable feature is that the Hopf bifurcation is a direct consequence of symmetry of the governing equations which is due to the presence of cubic nonlinearities. Therefore, the same type of bifurcation would appear in spherical geometry where more realistic problem of this kind could be formulated.

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Appendix A

The normal form Eq. (24) coefficients α and β are

$$\alpha = -k_c^2 \frac{\langle \vartheta(z)\,\omega^+(z)\rangle^z}{M},$$

$$\beta = 4\mathscr{Z}\frac{I_2}{I_1}\,\frac{\langle Db(z)\,j^+(z)\rangle^z}{M}$$

where

$$M = \langle b(z)\,b^+(z)\rangle^z + \langle j(z)\,j^+(z)\rangle^z + (1/q)\langle \vartheta(z)\,\vartheta^+(z)\rangle^z$$

and $\mathscr{Z}$ is a functional defined by Eq. (15).

The bracketed terms denote the integrals over the z coordinate

$$\langle f(z)\overline{f^+(z)}\rangle^z = \int_{z_B}^{z_T} f(z)\overline{f^+(z)}\,\mathrm{d}z$$

where the functions $f^+(z)$ solve the corresponding adjoint problem.

The coefficient β involves the integrals over the radial coordinate

$$I_1 = \int_0^{s_n} J_m^2(k_c s)\,s\,\mathrm{d}s,$$

$$I_2 = \int_0^{s_n} J_m^2(k_c s)\left(\frac{d}{\mathrm{d}s}J_m(k_c s)\right)^2 s\,\mathrm{d}s.$$

Being positive for each choice of s_n and irrespective of k_c, these integrals do not affect the properties of the Hopf bifurcation and their ratio $I_2/I_1 \to 0$ as $s_n \to \infty$. The consequence of this asymptotics is that the amplitude of solution decreases as s_n becomes larger, as would be naturally expected from configuration of the model.

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