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On the Ginzburg–Landau system of complex modulation equations for a rotating annulus with radial magnetic field

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Abstract

The problem of convection in a rotating annulus in the presence of a radial magnetic field is considered in a local Cartesian approximation. Linear stability analysis known from earlier studies shows the formation of two minima of the dispersion relation. In this paper, the problem is extended to the weakly nonlinear regime and the system of complex Ginzburg–Landau (G–L) equations is derived. The asymptotic behaviour and stability properties of solutions are studied in terms of the physical parameters. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is known that thermal convection in the Earth-like planet cores and in stars is subjected to the effects of magnetic field and rotation. Due to the fast rotation of the Earth the dominant motion of the liquid in the Earth core appears in the azimuthal direction. Upon this idea, the *cylindrical annulus model* (Fig. 1 has been introduced by Busse and Or [4], to study the convective instabilities. Assuming the radius of the annulus to be large and the convective zone to be thin, a local Cartesian approximation of the annulus can be made, see e.g. [4]. The underlying model is thus an infinite horizontal fluid channel bounded by vertical sidewalls rotating about a vertical axis and is commonly referred to as a *duct model* (Fig. 2).

Adding the magnetic field makes the problem more complex. A linear stability problem of rotating magnetoconvection has been studied by Busse and Finocchi [3]. The basic magnetic field was chosen to have various directions with respect to the equatorial plane, varying from the radial to the azimuthal direction. They have identified the most unstable solution having the form of traveling wave propagating in the azimuthal direction. The conditions

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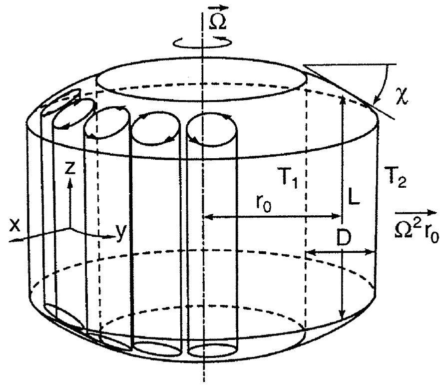

Fig. 1. Convection in a rotating annulus [2].

for the onset of convection in terms of the critical Rayleigh number and the critical frequency were found. The remarkable feature has been observed, that the dispersion curve, i.e. the dependence of the Rayleigh number on the wave number, possesses two minima which can be identified as the most unstable modes. A similar linear problem has been considered by Busse et al. [2] with more realistic sidewall boundary conditions.

In this paper, we extend the investigation by Busse and Finocchi [3] to the weakly nonlinear regime. We restrict ourselves to the case with radial magnetic field. We focus on the case of the two modes emerging simultaneously at the same Rayleigh number. Assuming the spatial and temporal modulation of the solutions, the interaction of the two modes can be described by the two coupled Ginzburg–Landau (G–L) equations. Properties and stability of the modes can be studied in terms of the equations' coefficients. The asymptotic case of high rotation rate allows for analytical tractability of the weakly nonlinear problem.

It is remarkable for the underlying model that coefficients of the G–L system are complex. This makes the mathematical analysis a bit more complicated. Note that a single G–L equation with complex coefficients has been studied before by e.g. Kapitula and Maier-Paape [8] and Mielke [10]. Systems of G–L equations with complex coefficients were investigated by e.g. van den Berg and van der Vorst [1] and Riecke and Kramer [11].

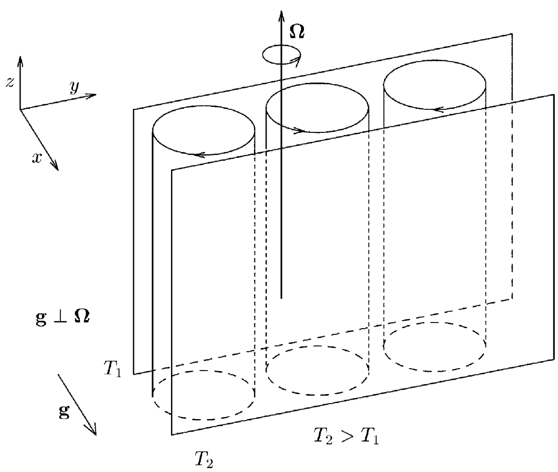

Fig. 2. Convection in a rotating duct.

We also focus on mathematical properties of the G–L system of equations (referred to as *the G–L system* in the following). We derive a priori estimates for various norms of solutions to the G–L system. We furthermore analyze stability of the so-called phase winding solutions to the G–L system. Within this class of spatially nonhomogeneous solutions, it is possible to investigate the stability of both convective modes with respect to each other and with respect to zero solution.

The structure of the paper is as follows. The description of the model and mathematical formulation are outlined in Section 2. In Section 3, the linear stability analysis is performed and asymptotic results are found as well. The derivation of the the G–L system is described in Section 4. Section 5 is dedicated to the qualitative analysis, focusing on the asymptotic behaviour of the solutions. In Section 6, the stability analysis is performed for phase winding solutions. Finally in Section 7, main results are summarized. The coefficients of the G–L system and their asymptotics are given in Appendix A.

2. Description of the model

Upon the local Cartesian aproximation, the model considered is an infinite horizontal duct (Fig. 2), containing an electrically conducting Boussinesq fluid. The duct rotates about the vertical axis and is permeated by a homogeneous horizontal magnetic field perpendicular to the sidewalls. The buoyancy is provided by the centrifugal force. The duct is exposed to the unstable temperature gradient which is directed opposite to the centrifugal force.

The fluid is subjected to a convective instability occurring when heating measured by the Rayleigh number is strong enough. Convection in the underlying model can be described in terms of two scalar functions, the velocity potential ψ and temperature θ. We do not derive the mathematical formulation in this paper, for reference see [3].

The governing equations (those of [3], Eqs. (6a and b)) are as follows:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}-\frac{\partial}{\partial x}\psi\frac{\partial}{\partial y}-\Delta_2\right)\Delta_2\psi-\eta\frac{\partial}{\partial y}\psi+Ra\,\frac{\partial}{\partial y}\theta+Q\frac{\partial^2}{\partial x^2}\psi=0,$$

$$Pr\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}-\frac{\partial}{\partial x}\psi\frac{\partial}{\partial y}\right)\theta-\Delta_2\theta+\frac{\partial}{\partial y}\psi=0, \tag{1}$$

where Δ_2 is the two dimensional Laplacian, $\Delta_2=\partial_x^2+\partial_y^2$. The dimensionless parameters in the above equations are the Rayleigh number *Ra*, the Prandtl number *Pr*, the Chandrasekhar number Q and the rotation parameter η.

The sidewalls of the duct are supposed to be stress-free and perfectly thermally conductive, i.e.

$$\psi(x,y,t)=\frac{\partial^2}{\partial x^2}\psi(x,y,t)=\theta(x,y,t)=0\quad\text{at}\quad x=\pm\frac{1}{2}. \tag{2}$$

3. Solution of the linear problem

Considering infinitesimal perturbations the terms which are quadratic in ψ and θ can be neglected. A linearized solution satisfying boundary conditions (2) can be sought in the form

$$\psi(x,y,t)=(Pr\mathrm{i}\omega+m^2\pi^2+\alpha^2)\,\sin[m\pi(x+\tfrac{1}{2})]\exp[\mathrm{i}\alpha y+\mathrm{i}\omega t],$$

$$\theta(x,y,t)=(-\mathrm{i}\alpha)\sin[m\pi(x+\tfrac{1}{2})]\exp[\mathrm{i}\alpha y+\mathrm{i}\omega t], \tag{3}$$

where α is the wave number and ω is the frequency.

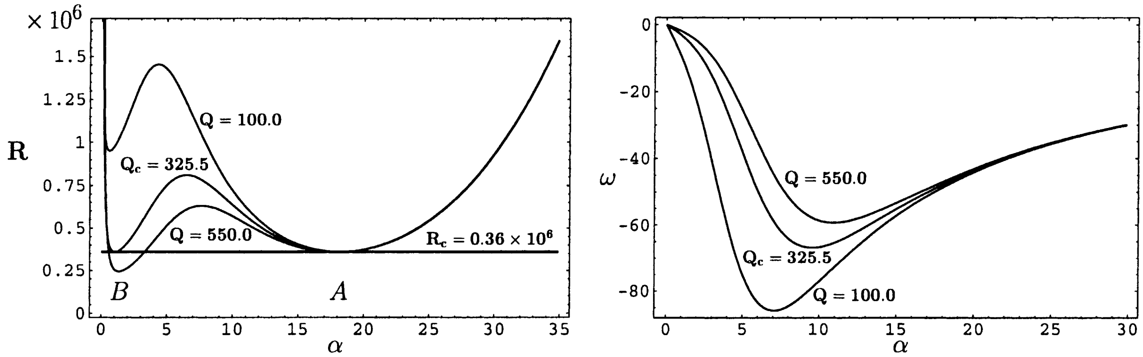

Fig. 3. Dependences of $Ra = Ra(\alpha)$ and $\omega = \omega(\alpha)$ for $Pr = 10$, $\eta = 10^4$, for the critical value Q_c and two other values of Q. Two minima of $Ra = Ra(\alpha)$ correspond to the two most unstable modes A and B.

Linear stability analysis has been performed by Busse and Finocchi [3]. The mode posessing $m = 1$ was found to be the most unstable one and will therefore, be, considered in the following.

Inserting the ansatz (3) into the linearized equations (1), the dispersion equation is obtained:

$$(Pr\mathrm{i}\omega + \pi^2 + \alpha^2)[(\mathrm{i}\omega + \pi^2 + \alpha^2)(\pi^2 + \alpha^2) + Q\pi^2 + \eta\mathrm{i}\alpha] = Ra\alpha^2. \tag{4}$$

Solving the real and imaginary parts of the dispersion equation, yields the analytical formulae for the relations $Ra = Ra(\alpha)$ and $\omega = \omega(\alpha)$.

Note that for $Q \neq 0$ the relation $Ra = Ra(\alpha)$ exhibits two minima which correspond to the most preferred modes (see Fig. 3). It can be observed that varying the rotation rate η, the applied magnetic field measured by Chandrasekhar's number Q can be adjusted in such a way, that both the convective modes emerge simultaneously. Hereafter the modes will be referred to as A and B. The coexistence of the two most critical modes in terms of the critical Rayleigh number can be written $Ra_{cA} = Ra_{cB}$.

The minimization of the dispersion relation $Ra = Ra(\alpha)$ leads to a numerical problem for the critical wave number $\alpha = \alpha_c$. All the parameters evaluated at α_c will be referred to as critical ones. Taking the geophysically interesting limit for large η, analytical progress is possible. We are able to derive the asymptotic results in the limit $\eta \to \infty$ for the modes A and B, namely the critical wave numbers α_{cA} and α_{cB}, the critical frequencies ω_{cA} and ω_{cB}, the critical Rayleigh number Ra_c and the Chandrasekhar number Q_c at which the modes coexist:

$$\alpha_{cA} = \left(\frac{1}{3\hat{Q}}\right)^{1/4}\eta^{1/3}, \qquad \alpha_{cB} = \pi(\pi\hat{Q})^{1/2}\eta^{-1/6}, \qquad \omega_{cA} = -\frac{1}{Pr+1}(3\hat{Q})^{1/4}\eta^{2/3},$$
$$\omega_{cB} = -\frac{\pi}{Pr}\left(\frac{\pi}{\hat{Q}}\right)^{1/2}\eta^{1/6}, \quad Ra_c = \frac{1}{\hat{Q}}\eta^{4/3}, \quad Q_c = \hat{Q}\eta^{2/3}, \quad \text{where } \hat{Q} = \frac{1}{3}\left(\frac{\sqrt{2}(Pr+1)}{Pr}\right)^{4/3}. \tag{5}$$

4. Weakly nonlinear analysis

Considering the nonlinear problem (1) a system of G–L equations for the modes A and B can be derived using perturbation methods. The same bifurcation parameter $Ra - Ra_c = \varepsilon^2 Ra_2$ will be used for both the modes, where $0 < \varepsilon \ll 1$. To resolve the weakly nonlinear problem, two different slow time scales $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ must be introduced and $Y = \varepsilon y$, see [11].

The vector of scalar functions $\Psi = (\psi, \theta)^T$ can be expanded into power series in terms of ε as follows:

$$\Psi = \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \varepsilon^3\Psi_3 + \cdots . \tag{6}$$

The lowest order term is supposed to be the linear combination of the two modes:

$$\Psi_1 = \tfrac{1}{2}(A\Psi_{1A} + B\Psi_{1B} + \text{c.c.}),$$

$$\Psi_{1j} = \begin{pmatrix} (Pr\mathrm{i}\omega_{\mathrm{c}j} + \pi^2 + \alpha_{\mathrm{c}j}^2)\sin[\pi(x + \tfrac{1}{2})]\exp[\mathrm{i}\alpha_{\mathrm{c}j}y + \mathrm{i}\omega_{\mathrm{c}j}t] \\ (-\mathrm{i}\alpha_{\mathrm{c}j})\sin[\pi(x + \tfrac{1}{2})]\exp[\mathrm{i}\alpha_{\mathrm{c}j}y + \mathrm{i}\omega_{\mathrm{c}j}t] \end{pmatrix},$$

where $j = A, B$. Here, the complex modulation amplitudes are the functions of slow time and space coordinates, i.e. $A = A(Y, T_1, T_2)$ and $B = B(Y, T_1, T_2)$.

Inserting the perturbation expansions (6) into the nonlinear equations (1) we obtain a series of nonhomogeneous problems at different orders of ε. Note that an assumption is made throughout the derivation that no spatial resonance is possible, i.e. $n_1\alpha_{\mathrm{c}A} + n_2\alpha_{\mathrm{c}B} \neq 0$ for $n_1, n_2 \in \mathbb{N}$. As the wave number α can be varied continuously this condition is satisfied generically.

The linear balance occurs at the lowest order $\mathrm{O}(\varepsilon^1)$. At the order $\mathrm{O}(\varepsilon^2)$ the following solvability conditions are obtained:

$$A_{T_1} = -v_{\mathrm{g}A}A_Y, \qquad B_{T_1} = -v_{\mathrm{g}B}B_Y, \tag{7}$$

where $v_{\mathrm{g}A} = -\partial_\alpha\omega|_{\alpha=\alpha_{\mathrm{c}A}}$ and $v_{\mathrm{g}B} = -\partial_\alpha\omega|_{\alpha=\alpha_{\mathrm{c}B}}$ are the group velocities. The subscripts Y and T_1 denote differentiating with respect to the slow spatial coordinates and time, respectively. It results from (7) that $A = A(Y_A, T_2)$ and $B = B(Y_B, T_2)$ where $Y_A = Y - v_{\mathrm{g}A}T_1$ and $Y_B = Y - v_{\mathrm{g}B}T_1$ are the shifted coordinates.

The solution at the order ε^2 is

$$\psi_2 = 0, \qquad \theta_2 = \frac{Pr}{8\pi}\sin[2\pi(x + \tfrac{1}{2})]((\pi^2 + \alpha_{\mathrm{c}A}^2)\alpha_{\mathrm{c}A}^2|A|^2 + (\pi^2 + \alpha_{\mathrm{c}B}^2)\alpha_{\mathrm{c}B}^2|B|^2). \tag{8}$$

At $\mathrm{O}(\varepsilon^3)$ nonlinear effects are brought into the problem and the balance between terms yields the following amplitude equations:

$$p_A A_{T_2} = \alpha_{\mathrm{c}A}^2 Ra_2 A + q_A A_{Y_A Y_A} - (a_{AA}|A|^2 + a_{AB}\langle|B|^2\rangle)A,$$

$$p_B B_{T_2} = \alpha_{\mathrm{c}B}^2 Ra_2 B + q_B B_{Y_B Y_B} - (a_{BA}\langle|A|^2\rangle + a_{BB}|B|^2)B. \tag{9}$$

These equations are known as coupled complex *G–L equations*, referred to as *the G–L system*. It is important to realize that there are two O(1) different group velocities, $v_{\mathrm{g}A}$ and $v_{\mathrm{g}B}$, in this problem and therefore two frames of reference are used. We will be interested in spatially periodic solutions to the G–L system. This enables us to resolve the nonlinear coupling by applying the spatial average $\langle\,\rangle$ to (9), which gives rise to nonlocal cross-coupling terms, see [9,11].

Finally, we multiply Eqs. (7) and (9) by the relevant powers of ε and sum them side to side. We return to the original independent variables y, t and introduce the rescaled amplitudes $\tilde{A} = \varepsilon A$ and $\tilde{B} = \varepsilon B$. The resulting G–L system will be considered in the following, which gains the form

$$\dot{\tilde{A}} = -v_{\mathrm{g}A}\tilde{A}' + \lambda_A\tilde{A}'' + \frac{1}{p_A}(r\alpha_{\mathrm{c}A}^2 - a_{AA}|\tilde{A}|^2 - a_{AB}\langle|\tilde{B}|^2\rangle)\tilde{A},$$

$$\dot{\tilde{B}} = -v_{\mathrm{g}B}\tilde{B}' + \lambda_B\tilde{B}'' + \frac{1}{p_B}(r\alpha_{\mathrm{c}B}^2 - a_{BA}\langle|\tilde{A}|^2\rangle - a_{BB}|\tilde{B}|^2)\tilde{B}. \tag{10}$$

Henceforth, $\dot{\phi}$ stands for the time derivative of ϕ whereas ϕ' and ϕ'' denote the derivatives with respect to the spatial variable y. The coefficients in the G–L system (10) together with their asymptotics are given in Appendix A. The diffusion coefficients are $\lambda_A = q_A/p_A$ and $\lambda_B = q_B/p_B$ and $r = Ra - Ra_c$ is the bifurcation parameter. The tilde denoting the rescaled amplitudes will be dropped hereafter.

5. Asymptotic behaviour of solutions

The goal of this section is to discuss long time behaviour of spatially periodic solutions to the G–L system (10). We analyze asymptotic properties of the solutions and derive a priori bounds for various norms of solutions implying, in particular, bounded dissipativity of the corresponding semi-dynamical system. Without loss of generality, we will suppose that solutions to (10) have the unit spatial period. Moreover, we assume

$$\lambda_j, p_j, a_{jk} \in \mathbb{C}, v_{gj} \in \mathbb{R}, \quad Re(\lambda_j) > 0, \quad Re(p_j) > 0, \quad Re(a_{jk} p_j^{-1}) > 0 \tag{11}$$

for $j, k \in \{A, B\}$. The above structural assumptions can be verified for the physical model studied in Section 2. Details can be found in the Appendix A.

In order to prove local existence, uniqueness and continuation of solutions to the G–L system (10) we rewrite it as an abstract parabolic equation

$$\dot{\Phi} + \mathcal{L}\Phi = \mathcal{F}(\Phi), \quad \Phi(0) = \Phi_0, \tag{12}$$

where $\Phi = (A, B)$, $\mathcal{L}$ is a linear operator defined as $\mathcal{L}\Phi = (-\lambda_A A'', -\lambda_B B'')$ and $\mathcal{F}(A, B) = (-v_{gA} A' + \varrho_A A, -v_{gB} B' + \varrho_B B)$ where

$$\varrho_A = \varrho_A(A, B) = \frac{1}{p_A}(r\alpha_{cA}^2 - a_{AA}|A|^2 - a_{AB}\langle |B|^2\rangle),$$

$$\varrho_B = \varrho_B(A, B) = \frac{1}{p_B}(r\alpha_{cB}^2 - a_{BA}\langle |A|^2\rangle - a_{BB}|B|^2). \tag{13}$$

Let $L^p = L^p(S^1)$, $p \ge 1$, denote the Banach space of all complex Lebesgue square integrable functions defined on the domain $S^1 \approx [0, 1]$, the norm is given by $\|f\|_p = (\int_0^1 |f|^p)^{1/p}$. By $W^{k,2}(S^1)$ we denote the Sobolev space of all complex valued functions defined on the one-periodic domain S^1 whose distributional derivatives up to the order k belong to the space L^2. The norm on $W^{k,2}(S^1)$ is defined as $\|f\|_{k,2}^2 = \|f^{(k)}\|_2^2 + \|f\|_2^2$. Next we define the scale of complex Hilbert spaces as $X^k = (W^{2k,2}(S^1))^2$ for $k = 0, 1/2, 1$. It follows from compactness of the Sobolev embedding $W^{1,2}(S^1) \hookrightarrow\hookrightarrow C(S^1)$ that embeddings $X^1 \hookrightarrow\hookrightarrow X^{1/2} \hookrightarrow\hookrightarrow X^0$ are also compact. Furthermore, the nonlinearity $\mathcal{F}$ is well defined as a mapping from the phase space $\mathcal{X} = X^{1/2}$ into the space X^0, $\mathcal{F}$ is C^∞ smooth and locally Lipschitz continuous. Since $Re(\lambda_j) > 0$, $j \in \{A, B\}$, it is easy to verify that the linear operator $-\mathcal{L}$ generates an analytic semigroup of operators $\{e^{-\mathcal{L}t}, t \ge 0\}$ in the space X^0. Recall that the space $X^\gamma, \gamma \ge 0$, is a fractional power space with respect to the sectorial operator $\mathcal{L}$. According to the general theory of parabolic equations due to Henry [6] (Chapter 1) the abstract parabolic Eq. (12) and, consequently, the G–L system (10) has a unique solution $\Phi \in \mathcal{C}([0, T) : \mathcal{X}) \cap \mathcal{C}^1((0, T) : X^0)$, $\Phi(t) \in X^1$ for $t \in (0, T)$, provided that the initial condition $\Phi_0 \in \mathcal{X} = (W^{1,2}(S^1))^2$. If $T_{\max} > 0$ is a maximum time of existence of a solution then either $T_{\max} = \infty$ or $T_{\max} < \infty$ and then $\limsup_{t\to T_{\max}^-} \|\Phi(t)\|_{\mathcal{X}} = \infty$.

In the next two auxiliary lemmas, we will prove a priori estimates for various norms of solutions to the G–L system (10).

Lemma 5.1. *Suppose that* $A = A(y,t)$ *is a solution to* $\dot{A} = \lambda A'' + \tilde{\omega} A' + \varrho A$ *subject to periodic boundary conditions at* $y = 0, 1$ *where* $\varrho = \varrho(y,t)$ *is a complex valued function and* $\lambda \in \mathbb{C}$, $\tilde{\omega} \in \mathbb{R}$, *are constants such that* $Re(\lambda) > 0$. *If* $\kappa \geq 2$ *satisfies the inequality*

$$(\kappa - 2)|\mathrm{Im}(\lambda)| \leq \sqrt{2\kappa - 3}\,Re(\lambda) \tag{14}$$

then

$$\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |A|^\kappa\,\mathrm{d}y + \frac{Re(\lambda)}{2}\int_0^1 |A'|^2|A|^{\kappa-2}\,\mathrm{d}y \leq \int_0^1 Re(\varrho)|A|^\kappa\,\mathrm{d}y.$$

Proof. Let us multiply the equation $\dot{A} = \lambda A'' + \tilde{\omega} A' + \varrho A$ by the term $|A|^{\kappa-2}\bar{A}$. If $\lambda \in \mathbb{C}$ satisfies the condition (14) then it easy to verify that $Re(\lambda(|z|^2 + (\kappa-2)zRe(z))) \geq (1/2)Re(\lambda)|z|^2$, for any $z \in \mathbb{C}$. Clearly, $Re(\int_0^1 A'|A|^{\kappa-2}\bar{A}) = (1/\kappa)\int_0^1 (|A|^\kappa)' = 0$. Since $(1/\kappa)(\mathrm{d}/\mathrm{d}t)\int_0^1 |A|^\kappa = \int_0^1 Re(\dot{A}|A|^{\kappa-2}\bar{A})$ and $-\int_0^1 A''|A|^{\kappa-2}\bar{A} = \int_0^1 |A|^{\kappa-4}(|A'\bar{A}|^2 + (\kappa-2)A'\bar{A}Re(A'\bar{A}))$ the proof of the inequality (14) follows. □

Lemma 5.2. *Suppose that* $\Phi = (A, B)$ *is a solution to the G–L system* (10). *Denote* $G_\kappa(t) = \int_0^1 (|A(y,t)|^\kappa + |B(y,t)|^\kappa)\,\mathrm{d}y$. *If* $\kappa \geq 2$ *is such that the inequality* (14) *is fulfilled for both* $\lambda = \lambda_A$ *and* $\lambda = \lambda_B$ *then*

$$G_\kappa(t) \leq 2^{(\kappa+2)/\kappa}\left(\frac{c_1}{c_2}\right)^{\kappa/2} \quad \text{for any } t \geq T_0 = \frac{\ln(2)}{2c_1},$$

where

$$c_1 = \max_{j=A,B} r\alpha_{cj}^2 Re(p_j^{-1}) > 0 \quad \text{and} \quad c_2 = \min_{j=A,B} Re(a_{jj}p_j^{-1}) > 0.$$

Proof. Applying Lemma 5.1, Hölder's inequality, assumptions (11) and the fact that terms $Re(-a_{AB}/p_A)\langle|B|^2\rangle \times \int_0^1 |A|^\kappa\,\mathrm{d}y \leq 0$ and $Re(-a_{BA}/p_B)\langle|A|^2\rangle\int_0^1 |B|^\kappa\,\mathrm{d}y \leq 0$ are nonpositive we obtain

$$\frac{1}{\kappa}\dot{G}_\kappa \leq \int_0^1 (Re(\varrho_A)|A|^\kappa + Re(\varrho_B)|B|^\kappa) \leq c_1 G_\kappa - c_2\int_0^1 (|A|^{\kappa+2} + |B|^{\kappa+2}) \leq c_1 G_\kappa - 2^{-2/\kappa}c_2 G_\kappa^{(\kappa+2)/\kappa}.$$

Solving the above differential inequality we obtain

$$G_\kappa^{-2/\kappa}(t) \geq G_\kappa^{-2/\kappa}(0)\mathrm{e}^{-2c_1 t} + \frac{c_2 2^{-2/\kappa}}{c_1}(1 - \mathrm{e}^{-2c_1 t}) \geq \frac{c_2 2^{-(2/\kappa)}}{2c_1} \tag{15}$$

for any $t \geq T_0$. The proof of Lemma 5.2 now follows. □

Theorem 1. *Any solution* $\Phi = (A, B)$ *to the G–L system* (10) *with an initial condition* $\Phi(0) = (A(\cdot,0), B(\cdot,0)) \in \mathcal{X}$ *is global in time. Moreover, there exists a constant* $c_\infty > 0$ *independent of initial conditions and such that*

$$\limsup_{t\to\infty}\|A(\cdot,t)\|_{1,2}^2 + \|B(\cdot,t)\|_{1,2}^2 \leq c_\infty^2$$

uniformly for initial conditions belonging to a bounded set in $\mathcal{X}$.

Proof. Let us multiply the first equation in (10) by $-\bar{A}''$. By taking the real part, integrating over the interval [0, 1], using the Cauchy–Schwartz inequality, the inequalities $\int_0^1 |\phi|^2 \leq 1 + \int_0^1 |\phi|^6$ and $\langle|\phi|^2\rangle = \int_0^1 |\phi|^2 \leq (\int_0^1 |\phi|^6)^{1/3}$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |A'|^2\,\mathrm{d}y + Re(\lambda_A)\int_0^1 |A''|^2\,\mathrm{d}y$$
$$= -Re\int_0^1 \varrho_A A\bar{A}'' \le \frac{Re(\lambda_A)}{2}\int_0^1 |A''|^2\,\mathrm{d}y + 3\frac{r^2\alpha_{cA}^4 + (r^2\alpha_{cA}^4 + |a_{AA}|^2 + |a_{AB}|^2)G_6}{2|p_A|^2 Re(\lambda_A)}$$

Repeating the same argument for the function B yields the inequality

$$\dot{J}_2 + 2c_3 I_2 \le c_4 + c_5 G_6, \qquad (16)$$

where $J_2 = \int_0^1(|A'|^2 + |B'|^2)\,\mathrm{d}y$, $I_2 = \int_0^1(|A''|^2 + |B''|^2)\,\mathrm{d}y$ and

$$c_3 = \frac{1}{2}\min_{j=A,B} Re(\lambda_j), \qquad c_4 = 3r^2\left(\frac{\alpha_{cA}^4}{|p_A|^2 Re(\lambda_A)} + \frac{\alpha_{cB}^4}{|p_B|^2 Re(\lambda_B)}\right),$$

$$c_5 = c_4 + 3\left(\frac{|a_{AA}|^2 + |a_{AB}|^2}{|p_A|^2 Re(\lambda_A)} + \frac{|a_{BA}|^2 + |a_{BB}|^2}{|p_B|^2 Re(\lambda_B)}\right). \qquad (17)$$

According to the Gagliarda–Nirenberg interpolation inequality there exists a constant $c_{gn} > 0$ such that $\|\phi\|_6 \le c_{gn}\|\phi''\|_2^{1/6}\|\phi\|_2^{5/6}$ for any $\phi \in W^{2,2}(S^1)$ [5]. Thus, $G_6 \le c_{gn} I_2^{1/2} G_2^{5/2}$. Now, it follows from (16), Lemma 5.2 and the Young inequality $J^{1/2}G \le \delta J + (1/4)\delta^{-1}G^2$ which is valid for any $J, G \ge 0$, $\delta > 0$, that

$$\dot{J}_2 + 2c_3 I_2 \le c_4 + c_3 I_2 + \frac{1}{4}c_3^{-1}(c_{gn}c_5 G_2^{5/2})^2 \le c_6 + c_3 I_2,$$

where

$$c_6 = c_4 + 2^8 c_{gn}^2 c_3^{-2} c_5^2 c_1^5 c_2^{-5}. \qquad (18)$$

By applying the inequality $\int_0^1 |\phi'|^2 \le \int_0^1 |\phi''|^2$, i.e. $J_2 \le I_2$, we finally obtain $\dot{J}_2 + c_3 J_2 \le c_6$, and hence,

$$J_2(t) \le J_2(0)\mathrm{e}^{-c_3 t} + \frac{c_6}{c_3}(1 - \mathrm{e}^{-c_3 t}) \qquad (19)$$

for any $t \ge 0$. Now, inequalities (15) (with $\kappa = 2$) and (19) enable us to conclude that the norm $\|(A(\cdot,t), B(\cdot,t))\|_{1,2}$ remains bounded within the maximum time interval $t \in [0, T_{\max})$. Thus, $T_{\max} = \infty$ and solutions to the G–L system (10) are global in time. Since $\|A(\cdot,t)\|_{1,2}^2 + \|B(\cdot,t)\|_{1,2}^2 = G_2 + J_2$ the asymptotic estimate $\limsup_{t\to\infty}\|A(\cdot,t)\|_{1,2}^2 + \|B(\cdot,t)\|_{1,2}^2 \le c_\infty^2$ follows from (15) and (19) where

$$c_\infty^2 = 4c_1 c_2^{-1} + c_6 c_3^{-1}. \qquad \square$$

Remark 5.1. By inserting the asymptotic formulae for coefficients $p_j, q_j, \alpha_{cj}, a_{jk}$ (see (5) and (A.2) in Appendix A), it can be shown that for a fixed r the constant $c_\infty = O(\eta^{19/6})$ for $\eta \to \infty$. However, this estimate is not optimal and it can be easily improved by introducing a suitable scaling of amplitudes $A \leftrightarrow \tilde{A}$, $B \leftrightarrow \eta^{7/6}\tilde{B}$. Taking into account the scaling property of the G–L system (10) the corresponding coefficients $\tilde{a}_{jk}$ have the following asymptotics: $|\tilde{a}_{AA}| = O(\eta^{10/3})$, $|\tilde{a}_{AB}| = O(\eta^{13/3})$, $|\tilde{a}_{BA}| = O(\eta^{5/2})$ and $|\tilde{a}_{BB}| = O(\eta^3)$ for $\eta \to \infty$. Hence, for tilded amplitudes $\tilde{A}, \tilde{B}$ we obtain the estimate $\limsup_{t\to\infty}\|\tilde{A}(\cdot,t)\|_{1,2}^2 + \|\tilde{B}(\cdot,t)\|_{1,2}^2 \le \tilde{c}_\infty^2$ where $\tilde{c}_\infty = O(\eta^{-2/3})$ for $\eta \to \infty$. Hence,

$$\limsup_{t\to\infty}\|A(\cdot,t)\|_{1,2} \le O(\eta^{-2/3}), \qquad \limsup_{t\to\infty}\|B(\cdot,t)\|_{1,2} \le O(\eta^{1/2})$$

for $\eta \to \infty$ uniformly with respect to initial conditions belonging to a bounded set in $(W^{1,2}(S^1))^2$. As for the L^2-norm of a solution (A, B), it follows from Lemma 5.2 that

$$\limsup_{t\to\infty}\|A(\cdot,t)\|_2 \le \mathrm{O}(\eta^{-4/3}), \qquad \limsup_{t\to\infty}\|B(\cdot,t)\|_2 \le \mathrm{O}(\eta^{-1/6}).$$

Finally, according to the Gagliarda–Nirenberg inequality $\|\phi\|_\infty^2 \le c_{gn}\|\phi\|_{1,2}\|\phi\|_2$ where $\|\phi\|_\infty = \sup_{y\in[0,1]}|\phi(y)|$ we obtain L^∞ bounds for solutions to the G–L system (10)

$$\limsup_{t\to\infty}\|A(\cdot,t)\|_\infty \le \mathrm{O}(\eta^{-1}), \qquad \limsup_{t\to\infty}\|B(\cdot,t)\|_\infty \le \mathrm{O}(\eta^{1/6})$$

uniformly for initial conditions belonging to a bounded set in $\mathcal{X}$.

Remark 5.2. From the dynamical system theory point of view, it results from Theorem 5.1 that the semi-dynamical system generated by solutions to (10) is bounded dissipative (cf. [6]). Then, following rather standard arguments, one can easily prove the existence of a compact global attractor having a finite fractal dimension as well as the existence of an inertial manifold for the corresponding semi-dynamical system.

6. Amplitude dynamics of phase winding solutions

Having computed the analytical expressions for coefficients of the G–L system (10) (see (A.2)), we are yet able to study the stability properties. We will analyze the so-called phase winding solutions which posses periodic spatial structure. A phase winding solution to the G–L system (10) is a pair of functions (A, B) having the form

$$A(y,t) = \mathcal{A}(t)\,\mathrm{e}^{\mathrm{i}(k_A y+\Gamma_A(t))}, \qquad B(y,t) = \mathcal{B}(t)\,\mathrm{e}^{\mathrm{i}(k_B y+\Gamma_B(t))} \tag{20}$$

for $(y,t) \in \mathbb{R}\times\mathbb{R}^+$ where $\mathcal{A}, \mathcal{B}, \Gamma_A, \Gamma_B$ are real valued amplitudes and phases, respectively, depending on the time $t \in \mathbb{R}^+$ only, and $k_A,\ k_B \in \mathbb{R}$ are phase winding numbers. Note that under assumption (20) only amplitude instabilities can be analysed. The phase functions do not affect the stability properties in this case. Moreover, amplitude instabilities are insensitive to the averaging of the G–L system (10).

Inserting the ansatz (20) into the G–L system (10) and denoting

$$\beta_{jk} = Re(a_{jk}p_j^{-1}), \qquad d_j = \alpha_{cj}^2 Re(p_j^{-1}), \quad j,k \in \{A,B\},$$

we obtain the planar system of ODEs for real amplitudes $\mathcal{A}, \mathcal{B}$:

$$\dot{\mathcal{A}} = -k_A^2\lambda_A^{\mathrm{R}}\mathcal{A} + (rd_A - \beta_{AA}\mathcal{A}^2 - \beta_{AB}\mathcal{B}^2)\mathcal{A}, \qquad \dot{\mathcal{B}} = -k_B^2\lambda_B^{\mathrm{R}}\mathcal{B} + (rd_B - \beta_{BA}\mathcal{A}^2 - \beta_{BB}\mathcal{B}^2)\mathcal{B}. \tag{21}$$

The phase functions $\Gamma_j,\ j \in \{A,B\}$, are given by

$$\Gamma_j(t) = \Gamma_j^0 - (k_j^2\lambda_j^{\mathrm{I}} + k_j v_{gj})t + \Im m\int_0^t p_j^{-1}(r\alpha_{cj^2} - a_{jA}\mathcal{A}^2(\tau) - a_{jB}\mathcal{B}^2(\tau))\,\mathrm{d}\tau,$$

where $\Gamma_j^0 \in \mathbb{R},\ j \in \{A,B\}$, are initial phases. Hereafter, the superscripts R and I will denote the real and imaginary parts, respectively.

Let $(\mathcal{A}, \mathcal{B})$ be a solution to the planar system of ODEs (21) corresponding to the phase winding numbers $k_A, k_B \in \mathbb{R}$. A straightforward phase–plane analysis enables us to conclude that the first quadrant $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ is invariant with respect to solutions to (21). Furthermore, by introducing the logarithmic transformation of variables $\varrho_A = \log(\mathcal{A})$, $\varrho_B = \log(\mathcal{B})$ and taking into account the Poincaré–Bendixon criterion applied to the transformed planar system of ODEs we are able to conclude that there are neither periodic orbits nor heteroclinic cycles in the planar system (21).

Next we examine stationary, i.e. time independent solutions to (21). Let us introduce the auxiliary function playing a crucial role in the stability analysis of (21):

$$G(r, k_A, k_B; Pr, \eta) = \frac{rd_A - k_A^2 \lambda_A^{\mathrm{R}}}{rd_B - k_B^2 \lambda_B^{\mathrm{R}}}. \tag{22}$$

Here, and after we will assume that the bifurcation parameter $r = Ra - Ra_{\mathrm{c}}$ and the wave numbers k_A, k_B satisfy the so-called *supercritical band conditions*

$$r > \frac{k_j^2 \lambda_j^{\mathrm{R}}}{d_j} \quad \text{for } j = A, B \tag{23}$$

which implies that $G > 0$. Moreover, we will restrict ourselves to the case of

$$d_j > 0, \qquad \beta_{jk} > 0, \qquad D = \beta_{AB}\beta_{BA} - \beta_{AA}\beta_{BB} > 0. \tag{24}$$

This setting is due to physically interesting small Prandtl numbers Pr and high rotation rates η, as it is shown below. In this case the asymptotic results can be directly used. The cases with negative β_{jk} and D will not be considered here.

Let us denote the stationary amplitudes as

$$\hat{\mathcal{A}}_{\mathrm{p}} = \left(\frac{rd_A - k_A^2 \lambda_A^{\mathrm{R}}}{\beta_{AA}} \right)^{1/2}, \qquad \hat{\mathcal{B}}_{\mathrm{p}} = \left(\frac{rd_B - k_B^2 \lambda_B^{\mathrm{R}}}{\beta_{BB}} \right)^{1/2},$$
$$\hat{\mathcal{A}}_{\mathrm{m}} = \beta_{AA}^{1/2} \hat{\mathcal{A}}_{\mathrm{p}} \left(\frac{\beta_{AB} G^{-1} - \beta_{BB}}{D} \right)^{1/2}, \hat{\mathcal{B}}_{\mathrm{m}} = \beta_{BB}^{1/2} \hat{\mathcal{B}}_{\mathrm{p}} \left(\frac{\beta_{BA} G - \beta_{AA}}{D} \right)^{1/2}.$$

Depending on the value of function G generically three cases can occur:

(A) $G > \beta_{AB}/\beta_{BB}$: in this case, there are three nonnegative stationary solutions: the zero solution $(0, 0)$ and the pure modes $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$, $(0, \hat{\mathcal{B}}_{\mathrm{p}})$. The solution $(0, 0)$ is an unstable node, $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ is a stable node, $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ is a saddle point and there exists a heteroclinic connection from $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ to $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$.

(AB) $\beta_{AA}/\beta_{BA} < G < \beta_{AB}/\beta_{BB}$: in this case, there are four nonnegative stationary solutions: the zero solution $(0, 0)$, the pure modes $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$, $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ and the mixed mode $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$. The solution $(0, 0)$ is an unstable node, $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ and $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ are stable nodes and $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$ is a saddle point. There exist heteroclinic connections from $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$ to $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ and from $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$ to $(0, \hat{\mathcal{B}}_{\mathrm{p}})$.

(B) $G < \beta_{AA}/\beta_{BA}$: in this case, there are three nonnegative stationary solutions: the zero solution $(0, 0)$ and the pure modes $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$, $(0, \hat{\mathcal{B}}_{\mathrm{p}})$. The solution $(0, 0)$ is an unstable node, $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ is a saddle point, $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ is a stable node and there exists a heteroclinic connection from $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ to $(0, \hat{\mathcal{B}}_{\mathrm{p}})$.

Passing below the critical value $G_{\mathrm{c1}} = \beta_{AB}/\beta_{BB}$ a saddle-node bifurcation occurs, the saddle point $(0, \hat{\mathcal{B}}_{\mathrm{p}})$ becomes a stable node and a new saddle point $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$ bifurcates from $(0, \hat{\mathcal{B}}_{\mathrm{p}})$. Passing through the second critical value $G_{\mathrm{c2}} = \beta_{AA}/\beta_{BA}$ a saddle point $(\hat{\mathcal{A}}_{\mathrm{m}}, \hat{\mathcal{B}}_{\mathrm{m}})$ merges with a stable node $(\hat{\mathcal{A}}_{\mathrm{p}}, 0)$ again via a saddle-node bifurcation (see Fig. 4).

We are yet able to apply the previous stability results to the underlying physical model in the limiting case $\eta \to \infty$. With the asymptotic expressions for the coefficients β_{jk} (see Appendix A) the conditions (24) must be satisfied so that the stability results could be applied. It can be computed that the coefficient $\beta_{AB} > 0$ for $0 < Pr < Pr_{\max} = 0.265$ and $\beta_{AA} > 0$, $\beta_{BA} > 0$, $\beta_{BB} > 0$ for all values of $Pr > 0$.

Moreover, with help of the asymptotics (A.4) available for $\eta \to \infty$ it can be computed that $\beta_{AA}/\beta_{BA} = \mathrm{O}(\eta^{1/3})$ and $\beta_{AB}/\beta_{BB} = \mathrm{O}(\eta)$. Assuming the simplest case of $k_A = 0$ and $k_B = 0$ corresponding to *spatially*

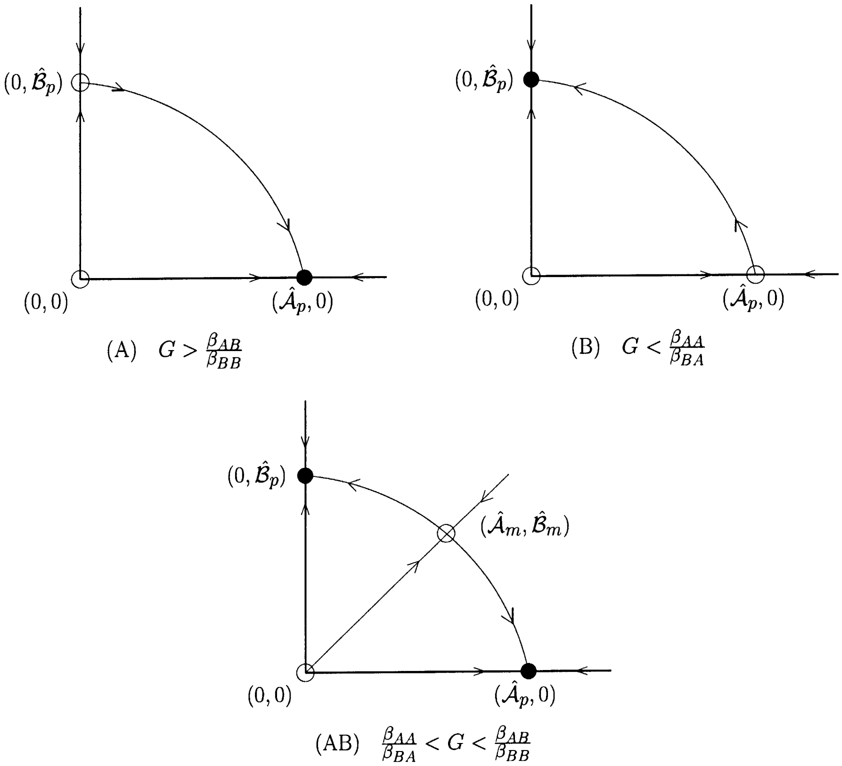

Fig. 4. Phase portraits for the planar system of ODEs (21) for different values of the function G corresponding to the cases (A), (B) and (AB).

homogeneous solutions and using (A.3), it results from (22) that $G(r, k_A, k_B; Pr, \eta) = \mathrm{O}(\eta^{2/3})$ and thus $\beta_{AA}/\beta_{BA} < G(r, k_A, k_B; Pr, \eta) < \beta_{AB}/\beta_{BB}$ for $\eta \to \infty$. The same result applies also for *spatially nonhomogeneous solutions* with the wave numbers $k_A = r^{1/2}\mathrm{O}(\eta^{-1/3})$ and $k_B = r^{1/2}\mathrm{O}(\eta^{-2/3})$ such that supercritical band conditions (23) are satisfied.

As a result, the cases (A) and (B) do not apply for high rotation rates which excludes the single mode instabilities. The case (AB) only occurs corresponding to the two locally asymptotically stable modes A and B (see Fig. 4).

7. Conclusions

The nonlinear equations governing the specified model of rotating magnetoconvection by [3] were analyzed and solved in this paper. Special parameter setting corresponding to the case of two mode convection was established in the linearized case. Adopting the limit of high rotation rate $\eta \to \infty$, analytical expressions for the parameters were computed. An interesting feature is that the critical Chandrasekhar number for the two mode convection is scaled as $Q_c \sim \eta^{2/3}$.

For the weakly nonlinear regime the G–L system of complex modulation equations was derived (see Section 4). In the limit of $\eta \to \infty$ analytical expressions for the G–L system coefficients were computed. The qualitative analysis of the G–L system proves bounded dissipativity of the corresponding semi-dynamical system. L_∞ estimates for a compact global attractor were found in terms of the powers of η.

Stability properties of the G–L system were studied for a class of phase winding solutions. It turns out that the existence of convective modes is determined by the amplitude instabilities of phase winding solutions and depends on choice of the physical parameters η and Pr. The modes were found to be supercritical for the Prandtl numbers of $0 < Pr < 0.265$ and rotation rate $\eta \to \infty$. For this parameter setting both the modes A and B are locally asymptotically stable.

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Appendix A

The coefficients of the linear part of the G–L system (9) are

$$\begin{aligned} p_j &= (Pr+1)(\pi^2+\alpha_{cj}^2)^2 + Pr\pi^2 Q_c + i[2Pr(\pi^2+\alpha_{cj}^2)\omega_{cj} + Pr\eta\alpha_{cj}],\\ q_j &= 3(\pi^2+\alpha_{cj}^2)(\pi^2+5\alpha_{cj}^2) - Ra_c + \pi^2 Q_c - Pr[(\pi^2+\alpha_{cj}^2)v_{gj}^2 - 4\alpha_{cj}\omega_{cj}v_{gj} + \omega_{cj}^2 - \eta v_{gj}]\\ &\quad + i[(Pr+1)(-4(\pi^2+\alpha_{cj}^2)\alpha_{cj}v_{gj} + 4\alpha_{cj}^2\omega_{cj} + 2(\pi^2+\alpha_{cj}^2)\omega_{cj}) + 3\eta\alpha_{cj}] \end{aligned} \quad \text{(A.1)}$$

where $j = A, B$. Note that a single mode case with the formal setting $Q_c = 0$ in the expressions for coefficients corresponds to the nonmagnetic case which was studied by [7].

Using the asymptotics (5) for $\eta \to \infty$, the coefficients of the linear part of the G–L system (10) can be expressed as follows (for $Pr \neq 1$):

$$\begin{aligned} p_A &= (Pr+1)\frac{1}{3\hat{Q}}\eta^{4/3} + iPr\frac{Pr-1}{Pr+1}\left(\frac{1}{3\hat{Q}}\right)^{1/4}\eta^{4/3}, \qquad p_B = \pi^2 Pr\hat{Q}\eta^{2/3} + iPr\pi(\pi\hat{Q})^{1/2}\eta^{5/6},\\ q_A &= \left[\frac{4}{\hat{Q}} - Pr\frac{Pr-1}{(Pr+1)^2}(3\hat{Q})^{1/2}\right]\eta^{4/3} + i\left(\frac{1}{3\hat{Q}}\right)^{1/4}\eta^{4/3}, \qquad q_B = \pi\eta + i3\pi(\pi\hat{Q})^{1/2}\eta^{5/6},\\ v_{gA} &= -\frac{1}{Pr+1}(3\hat{Q})^{1/2}\eta^{1/3}, \qquad v_{gB} = \frac{1}{Pr\hat{Q}}\eta^{1/3}, \end{aligned} \quad \text{(A.2)}$$

where $\hat{Q}$ is given by (5).

The asymptotic orders for the diffusion coefficients $\lambda_j = q_j/p_j$ and the coefficients $d_j = \alpha_{cj}^2 Re(p_j^{-1})$ defined in Section 6 are

$$\lambda_A = O(1) + iO(1), \qquad \lambda_B = O(1) + iO(\eta^{1/6}), \qquad d_A = O(\eta^{-2/3}), \qquad d_B = O(\eta^{-4/3}), \quad \text{(A.3)}$$

for $\eta \to \infty$.

The expressions for coefficients entering the nonlinear part of the G–L system (10) are rather complicated. We only refer to their asymptotic behaviour in the limit $\eta \to \infty$ which is sufficient for the stability analysis in this paper:

$$a_{AA} = \frac{Pr^2}{8\hat{Q}}\left(\frac{1}{3\hat{Q}}\right)^{3/2}\eta^{10/3},$$

$$\frac{a_{AB}}{p_A} = \frac{-3Pr^{8/3}\pi^4 Pr_{AB}^{\mathrm{R}}}{2^{1/3}16(1+Pr)^{2/3}(2+5Pr)^2(3-2Pr+3Pr^2)^3}\eta^{2/3} + \mathrm{i}\frac{3Pr^{8/3}\pi^4 Pr_{AB}^{\mathrm{I}}}{(2^{5/6}8(1+Pr)^{2/3}(2+5Pr)^2(3-2Pr+3Pr^2)^3}\eta^{2/3},$$

$$\frac{a_{BA}}{p_B} = \frac{3Pr^{11/3}(4+43Pr-4Pr^2+55Pr^3-14Pr^4+8Pr^5)\pi}{2^{1/3}2(1+Pr)^{5/3}(2+5Pr)^2(3-2Pr+3Pr^2)^2}\eta^{5/3} + \mathrm{i}\frac{3Pr^{13/3}(3\pi)^{1/2}}{2^{2/3}16(1+Pr)^{1/3}(2+5Pr)(3-2Pr+3Pr^2)}\eta^{11/6}, \quad a_{BB} = \pi^6(\pi\hat{Q})^2\frac{Pr^2}{8\hat{Q}}\eta^{2/3}$$

with

$$Pr_{AB}^{\mathrm{R}} = -233 + 489Pr + 1105Pr^2 + 1011Pr^3 + 1201Pr^4 + 455Pr^5 + 7Pr^6 + 125Pr^7,$$
$$Pr_{AB}^{\mathrm{I}} = -133 - 939Pr + 437Pr^2 - 1029Pr^3 + 1133Pr^4 - 109Pr^5 + 355Pr^6 + 29Pr^7.$$

The asymptotic orders for the coefficients $\beta_{jk} = Re(a_{jk}/p_j)$ defined in Section 6 are

$$\beta_{AA} = \mathrm{O}(\eta^2), \quad \beta_{AB} = \mathrm{O}(\eta^{2/3}), \quad \beta_{BA} = \mathrm{O}(\eta^{5/3}), \quad \beta_{BB} = \mathrm{O}(\eta^{-1/3}) \tag{A.4}$$

for $\eta \to \infty$.

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