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Early Exercise Boundary for American Type of Floating Strike Asian Option and Its Numerical Approximation

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ABSTRACT *In this article, we generalize and analyse the model for pricing American-style Asian options proposed by Hansen and Jørgensen (2000) by including a continuous dividend rate q and a general method of averaging the floating strike. We focus on the qualitative and quantitative analysis of the early exercise boundary. The first-order expansion in terms of $\sqrt{\tau}$ of the early exercise boundary close to expiry is constructed. We furthermore propose an efficient numerical algorithm for determining the early exercise boundary position based on the front-fixing method. Construction of the algorithm is based on a solution to a non-local parabolic partial differential equation for the transformed variable representing the synthesized portfolio. Various numerical results and comparisons of our numerical method and the method developed by Dai and Kwok (2006) are presented.*

KEY WORDS: Option pricing, American-style Asian options, early exercise boundary, limiting behaviour close to expiry

1. Introduction

Evolution of trading systems influences the development of financial derivatives market. First, simple derivatives like forwards and plain vanilla options were used to hedge the risk of a portfolio. Progress in pricing these simple financial instruments pushed traders into inventing less predictable and more complex derivatives. Using financial derivatives with more complicated pay-offs brings also into attention new mathematical problems. Asian options belong to a group of the so-called path-dependent options. Their pay-off diagrams depend on the spot value of the underlying during the whole or some part(s) of the lifespan of the option. Usually Asian options depend on the (arithmetic or geometric) average of the spot price of the underlying. They can also be used as a useful tool for hedging highly volatile assets or goods. Because the price of an underlying varies during the lifespan of the option, the holder of the Asian

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option can be secured for the risk of a sudden price jumping to an undesirable region (too high for the call option holder or too low for the put option holder). Among the path-dependent options Asian options play an important role as they are quite common in currency and commodity markets like oil industry (cf. Dai and Kwok, 2006; Detemple, 2006; Hansen and Jørgensen, 2000; Hull, 1997; Kim and Oh, 2004; Kwok, 2008; Linetsky, 2004; Wilmott *et al.*, 1995; Wu and Fu, 2003; Wu *et al.*, 1999; Wystup, 2006).

In this article, we focus on the so-called floating strike Asian call or put options whose strike price depends on the averaged path history of the underlying asset. More precisely, we are interested in pricing American-style Asian call and put options having the pay-off functions $V_T(S, A) = (S - A)^+$ and $V_T(S, A) = (A - S)^+$, respectively. The strike price A is given as an average of the underlying over the time history $[0, T]$. We are analysing the early exercise boundary for American-style Asian option (cf. Dai and Kwok, 2006; Hansen and Jørgensen, 2000; Kwok, 2008). Recall that American-style options can be exercised at any time until the maturity T. The holder of such an option has the right to exercise it or to keep it depending on the spot price of the underlying S_t at time t and its history $\{S_u, 0 \le u \le t\}$ prior to the time t. The boundary between 'continuation' and 'stopping' regions plays an important role in pricing American-style options. It can be described by the mapping $S_t^* : t \mapsto S_t^*$, where S_t^* is the so-called early exercise boundary (cf. Chadam, 2008; Geske and Johnson, 1984; Geske and Roll, 1984; Hull, 1997; Karatzas, 1988; Kuske and Keller, 1998; Kwok, 2008; Mallier, 2002; Pascucci, 2008).

The article is organized as follows. In Section 2, we discuss the probabilistic model for valuation of American-type Asian options with a floating strike given in the form of an average of the underlying asset price. The model is based on conditioned expected values and theory of martingales. In Section 3, we present a general result enabling us to calculate the value of the limit of the early exercise boundary at expiry. We also calculate the analytical integral formula for an option with continuous geometric average and an approximation for the value of an option with a continuous arithmetic average. The main result of this section is the approximation formula for the first-order Taylor series approximation of the early exercise boundary near to expiry. Similarly, as in the case of plain vanilla call options, we show that the leading order of the expansion is the square root $\sqrt{T-t}$ of the time $T - t$ remaining to maturity. In Section 4, we present the fixed domain transformation method yielding a non-local non-linear parabolic partial differential equation (PDE) for pricing the synthesized portfolio for an Asian option. We also present an efficient and robust numerical scheme for construction of an approximation of the solution to the governing system consisting of a PDE with an algebraic constraint. In Section 5, we also present the results of the presented method on some examples.

2. A Probabilistic Approach for Pricing of American-Style Asian Options

The main purpose of this section is to derive an integral equation for valuation of the early exercise boundary of an American-style Asian option paying continuous dividends. We follow the ideas of derivation proposed by Hansen and Jørgensen (2000). Their formula for a floating strike option was derived using the theory of martingales and conditioned expected values. We extend their formula to Asian options on underlying paying non-zero dividend yield and having a general form of floating strike averaging. In more detail, we discuss geometric, arithmetic and weighted arithmetic averaging operator.

The pricing model is based on the assumption on the stochastic behaviour of the underlying asset in time. Throughout the article we shall assume that the underlying asset price S_t is driven by a stochastic process satisfying the following stochastic differential equation:

$$dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t^P, \quad 0 \le t \le T. \tag{1}$$

It starts almost surely from the initial price $S_0 > 0$. Here the constant parameter $r > 0$ denotes the risk-free interest rate whereas $q \ge 0$ is a continuous dividend rate. The constant parameter σ stands for the volatility of the underlying asset returns and W_t^P is a standard Brownian motion with respect to the standard risk-neutral probability measure P. A solution to Equation (1) corresponds to the geometric Brownian motion

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma W_t^P}, \quad 0 \le t \le T.$$

We shall derive an integral equation determining the value of an American-style Asian option with a floating strike. If we define the optimal stopping time as T^*, the pay-off of the option is set by

$$V_{T^*} = (\rho(S_{T^*} - A_{T^*}))^+\,,$$

where V_T is the value of the option at time t, A_t is a continuous average of the underlying asset value during the interval [0,t] and $\rho = 1$ for a call option and $\rho = -1$ for a put option. We may consider several different types of continuous averages presented in Table 1.

In the case of a weighted arithmetically averaged floating strike Asian option, the kernel function $a(.) \ge 0$ with the property $\int_0^\infty a(s)\,ds < \infty$ is usually defined as $a(s) = e^{-\lambda s}$ where $\lambda > 0$ is a constant.

According to Hansen and Jørgensen (2000), the American-style contingent claims can be priced by the conditioned expected approach. The option price can be calculated by considering all possible stopping times in the interval [t, T]

$$V(t,S,A) = \text{ess}\sup_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}_t^P\left[e^{-r(s-t)}\Big(\rho(S_s - A_s)\Big)^+\Big|S_t = S, A_t = A\right],$$

Table 1. Classification of averaging methods.

Arithmetic average	Geometric average	Weighted arithmetic average
$A_t = \frac{1}{t}\int_0^t S_u\,du$	$\ln A_t = \frac{1}{t}\int_0^t \ln S_u\,du$	$A_t = \frac{1}{\int_0^t a(s)\,ds}\int_0^t a(t-u)S_u\,du$

where $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times in the interval $[t,T]$ and $\mathbb{E}^P_t[X] = \mathbb{E}^P[X|\mathcal{F}_t]$ is the conditioned expectation with information available at time t (the information set is represented by the filtration $\mathcal{F}_t$ of the σ-algebra $\mathcal{F}$ where the Brownian motion is supported). To simplify the formula, we change the probability measure by the martingale

$$\eta_t = e^{-(r-q)t}\frac{S_t}{S_0} = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t^P},$$

the new probability measure $\mathcal{Q}$ being defined as $d\mathcal{Q} = \eta_T\, dP$. According to Girsanov's theorem (Revuz and Yor, 2005), the process

$$W_t^{\mathcal{Q}} = W_t^P - \sigma t$$

is a standard Brownian motion with respect to the measure $\mathcal{Q}$. The value of the underlying asset price under this measure is defined by

$$S_t = S_0 e^{(r-q+\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathcal{Q}}}. \tag{2}$$

All assets priced under this measure are $\mathcal{Q}$-martingales when discounted by the underlying price. According to this fact, we can reduce the dimension of the stochastic variables. We introduce a new variable $x_t = \frac{A_t}{S_t}$. We obtain

$$\begin{aligned} V(t,S,A) &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}^P_t\left[e^{-r(s-t)}\Big(\rho(S_s - A_s)\Big)^+\Big| S_t = S, A_t = A\right], \\ &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}^{\mathcal{Q}}_t\left[\frac{\eta_t}{\eta_T}e^{-r(s-t)}\Big(\rho(S_s - A_s)\Big)^+\Big| S_t = S, A_t = A\right], \\ &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} \mathbb{E}^{\mathcal{Q}}_t\left[e^{-q(s-t)}S_t\left(\rho\left(1 - \frac{A_s}{S_s}\right)\right)^+\Big| S_t = S, A_t = A\right], \\ &= \operatorname*{ess\,sup}_{s\in\mathcal{T}_{[t,T]}} e^{-q(s-t)} S\, \mathbb{E}^{\mathcal{Q}}_t\left[\Big(\rho(1 - x_s)\Big)^+\Big| S_t = S, A_t = A\right]. \end{aligned}$$

The last expression can be rewritten in terms of the new variable $x = \frac{A}{S}$ as follows:

$$\widetilde{V}(t,x) = e^{-qt}\frac{V(t,S,A)}{S} = e^{-qT_t^*}\,\mathbb{E}^{\mathcal{Q}}_t\left[\big(\rho(1 - x_{T_t^*})\big)^+\right], \tag{3}$$

where $T_t^* = \inf\{s \in [t,T] | x_s = x_s^*\}$ and the function $[0,T] \ni t \mapsto x_t^* \in \mathbb{R}$ describes the early exercise boundary.

Definition 2.1. The stopping region $\mathcal{S}$ and continuation region $\mathcal{C}$ for American-style Asian call and put options (3) are defined by

$$\mathcal{S}_{call} = \mathcal{C}_{put} = \{(t,x) | t \in [0,T], 0 \le x < x_t^*\}, \mathcal{C}_{call} = \mathcal{S}_{put} = \{(t,x) | t \in [0,T], x_t^* < x < \infty\},$$

where $[0, T] \in t \mapsto x_t^* \in \mathbb{R}$ is a continuous function determining the early exercise boundary. By $1_{\mathcal{S}}(\cdot)$ we shall denote the indicator function of the set $\mathcal{S}$, that is, $1_{\mathcal{S}}(t, y) = 1$ for $(t, y) \in \mathcal{S}$ and $1_{\mathcal{S}}(t, y) = 0$ otherwise.

In the following theorem, we present a solution to the pricing problem with one stochastic variable x_t formulated in Equation (3). It is a generalization of the result by Hansen and Jørgensen (2000) and Wu *et al.* (1999) for the case of a non-trivial dividend rate $q \geq 0$ and a general form of the averaging of the floating strike price.

Theorem 2.1. The value $\widetilde{V}(t, x_t)$ of the American-style floating strike Asian call ($\rho = 1$) or put option ($\rho = -1$) on the underlying asset x_t paying continuous dividends with a rate $q \geq 0$ is given by $\widetilde{V}(t, x_t) = \widetilde{v}(t, x_t) + \widetilde{e}(t, x_t)$, where

$$\widetilde{v}(t, x_t) \equiv \mathbb{E}_t^{\mathcal{Q}} \left[e^{-qT} \left(\rho(1 - x_T)\right)^+ \right], \tag{4}$$

$$\widetilde{e}(t, x_t) \equiv \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T \rho e^{-qu} x_u 1_{\mathcal{S}}(u, x_u) \left(\frac{dA_u}{A_u} - (r - q x_u^{-1}) du \right) \right], \tag{5}$$

with the average given by the function A_t and stopping region $\mathcal{S}$.

Notice that when comparing to the original expression proposed by Hansen and Jørgensen (2000) with a zero dividend rate $q = 0$, the only difference is that the parameter r is replaced by the term $r - q$.

The value of $\frac{dA_t}{A_t}$ depends on the method of averaging the underlying asset used in valuation. The expressions for continuous averages are presented in Table 2. In this table, we present the value of the weighted arithmetic average restricted to the kernel $a(s) = e^{-\lambda s}$.

It is worthwhile noting that the above expression for the value $\widetilde{V}(t, x_t) = \widetilde{V}_{\mathrm{am}}(t, x_t)$ of an American-style Asian option can be restated as follows:

$$\widetilde{V}_{\mathrm{am}}(t, x) = \widetilde{V}_{\mathrm{eu}}(t, x) + \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T 1_{\mathcal{S}}(u, x_u) f_b(u, x_u) du \right],$$

where $\widetilde{V}_{\mathrm{eu}}(t, x) = \mathbb{E}_t^{\mathcal{Q}}[e^{-qT}(\rho(1 - x_T))^+]$ stands for the price of the European-style Asian option, and the term $1_{\mathcal{S}}(u, x_u) f_b(u, x_u), u \in [0, T]$ represents a surplus bonus for the difference between American- and European-style Asian options.

Table 2. The value of the differential $\frac{dA_t}{A_t}$.

Arithmetic average	Geometric average	Weighted arithmetic average
$\frac{dA_t^a}{A_t^a} = \frac{1}{t}\left(\frac{1}{x_t^a} - 1\right) dt$	$\frac{dA_t^g}{A_t^g} = -\frac{1}{t} \ln x_t^g \, dt$	$\frac{dA_t^{wa}}{A_t^{wa}} = \frac{\lambda}{1 - e^{-\lambda t}} \left(\frac{1}{x_t^{wa}} - 1\right) dt$

3. Early Exercise Behaviour Close to Expiry

3.1. Limit of the Early Exercise Boundary at Expiry

In this section, we determine the position of the early exercise boundary x_T^* at expiry T. The result is stated for a wide class of integral equations for pricing American-style options.

Theorem 3.1. Consider an American-style (call or put) option $V_{\rm am}$ on the underlying y with the stopping and continuation regions defined by the sets $\mathcal{S} \neq \emptyset$ and $\mathcal{C} \neq \emptyset$, respectively. Let $y_t^* = \partial\mathcal{S}(t,.) \equiv \partial\mathcal{C}(t,.)$ for $t \in [0,T]$ be the early exercise boundary function. Suppose that the value of $V_{\rm am}$ is given by the equation

$$V_{\rm am}(t,y_t) = V_{\rm eu}(t,y_t) + \mathbb{E}_t\left[\int_t^T 1_{\mathcal{S}}(u,y_u) f_b(u,y_u)\, du\right], \tag{6}$$

where $V_{\rm eu}$ denotes the price of the corresponding European-style option and $f_b(t,y)$ is a continuous function representing the early exercise bonus such that the equation $f_b(T,y) = 0$ has a unique root $y^* \geq 0$. Furthermore, we suppose that

$$V_{\rm am}(t,y) \geq V_{\rm am}(T,y) = V_{\rm eu}(T,y) \geq 0,\ V_{\rm am}(t,y) \geq V_{\rm eu}(t,y) \quad \text{for any } t \in [0,T], y \geq 0,$$

and the function $[0,\infty) \ni y \mapsto \frac{\partial V_{\rm eu}}{\partial t}(T,y) \in \mathbb{R}$ is continuous except for the set $ATM = \partial\{y \geq 0, V_{\rm eu}(T,y) > 0\}$. Then, only one of the following cases can occur:

(1) $y_T^* \in ITM = \{y \geq 0, V_{\rm eu}(T,y) > 0\}$. In this case $f_b(T,y_T^*) = 0$.
(2) $y_T^* \in ATM = \partial\{y \geq 0, V_{\rm eu}(T,y) > 0\}$. In this case $f_b(T,y_T^*) \geq 0$.

Remark 3.1. The abbreviation ITM stands for the so-called in-the-money set whereas ATM (the boundary of ITM) denotes the so-called at-the-money set. We denote by OTM the out-the-money set – the complement of sets ITM and ATM, that is, $OTM = (ITM \cup ATM)^c$.

As a consequence of Theorem 3.1, we obtain the starting position of the early exercise boundary for American-style Asian options with various types of the strike price averaging method.

Corollary 3.1. The value of the limit of early exercise boundary at expiry for the floating strike Asian option is summarized in Table 3. In the case of geometric averaging, it follows from Equation (5) that $\widetilde{f}_b(T,x_T) = e^{-qT}(-\frac{x_T}{T}\ln x_T - r x_T + q)$ such that $x_T^* = \widetilde{x}_T \in ITM$ is a solution of the transcendent equation

$$\ln \widetilde{x}_T = \frac{qT}{\widetilde{x}_T} - rT. \tag{7}$$

The formula for the limit of the early exercise boundary at the expiry in Equation (7) for geometric averaging is the same as presented by Wu *et al*. (1999) and Detemple (2006, p. 69). The isolines of x_T^* are shown in Figure 1. Notice that the same values

Table 3. The limit of the early exercise boundary position x_T^* at expiry $t = T$ ($\widetilde{x}_T$ solves Equation (7)).

x_T^*	Arithmetic average	Geometric average	Weighted arithmetic average
Put	$\max\left(\frac{q+\frac{1}{T}}{r+\frac{1}{T}}, 1\right)$	$\max(\widetilde{x}_T, 1)$	$\max\left(\frac{q(1-e^{-\lambda t})+\lambda}{r(1-e^{-\lambda t})+\lambda}, 1\right)$
Call	$\min\left(\frac{q+\frac{1}{T}}{r+\frac{1}{T}}, 1\right)$	$\min(\widetilde{x}_T, 1)$	$\min\left(\frac{q(1-e^{-\lambda t})+\lambda}{r(1-e^{-\lambda t})+\lambda}, 1\right)$

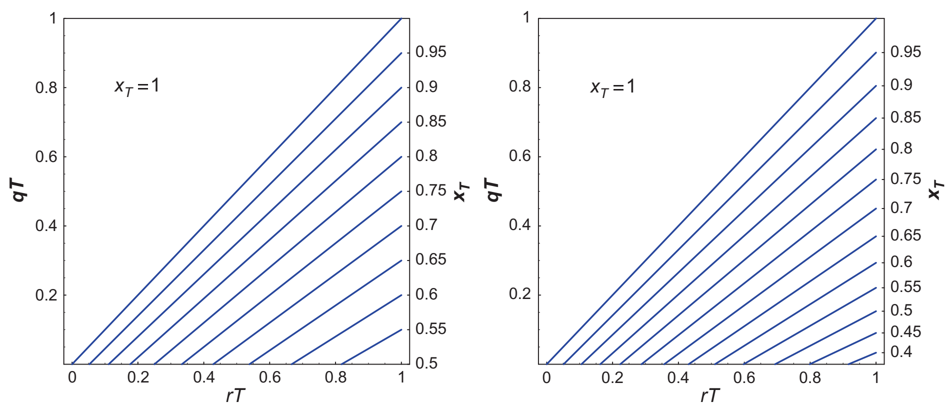

Figure 1. Isolines of the limit x_T^* of the early exercise boundary at expiry T of call option for the continuous arithmetic (left) and the continuous geometric average (right).

of the limit of early exercise boundary at expiry for the continuous arithmetic average type of an Asian option are derived also by Dai and Kwok (2006).

3.2. Integral Equation for Pricing Asian Options

In this section, we calculate the approximate formula for the American-style Asian option with various floating strike averages. The next lemma will be useful in the calculations to follow.

Lemma 3.1. (Hansen and Jørgensen 2000) Let $\omega = \ln\Omega \sim \mathcal{N}(\alpha, \beta^2)$ and define $\gamma \equiv \frac{\alpha+\beta^2-\ln K}{\beta}$, where $K > 0$. We have

$$\mathbb{E}\left[1_{\{\rho\Omega \geq \rho K\}}\right] = \Phi\left(\rho(\gamma-\beta)\right), \mathbb{E}\left[1_{\{\rho\Omega\geq\rho K\}}\Omega\ln\Omega\right] = e^{\alpha+\frac{\beta^2}{2}}\left((\alpha+\beta^2)\Phi(\rho\gamma)+\rho\beta\Phi(\gamma)\right),$$

$$\mathbb{E}\left[1_{\{\rho\Omega \geq \rho K\}}\Omega\right] = e^{\alpha+\frac{\beta^2}{2}}\Phi(\rho\gamma), \mathbb{E}\left[(\rho(\Omega-K))^+\right] = \rho\left(e^{\alpha+\frac{\beta^2}{2}}\Phi(\rho\gamma) - K\Phi(\rho(\gamma-\beta))\right),$$

where $\rho \in \{-1, 1\}$ and $\Phi(\cdot)$ and $\mathit{\Phi}(\cdot)$ are standard normal cumulative distribution and density functions, respectively.

3.2.1 Geometric average. In this section we recall the integral equation for pricing American-style Asian geometrically averaged floating strike options. It was derived for the case $q = 0$ by Hansen and Jørgensen (2000) and for the general case $q \geq 0$ by Wu *et al*. (1999).

Lemma 3.2. (Wu *et al*., 1999) In the case of geometric averaging, the variable $x_t^g = \frac{A_t^g}{S_t}$ has log-normal (conditioned) distribution $\ln x_u^g | \mathcal{F}_t \sim \mathcal{N}(\alpha_{t,u}, \beta_{t,u}^2)$, where $u \geq t$ and the parameters $\alpha_{t,u} = \alpha(t,u,x_t)$ and $\beta_{t,u} = \beta(t,u)$ are defined by

$$\alpha_{t,u}^g = \frac{t}{u} \ln x_t^g - \frac{u^2 - t^2}{2u}(r - q + \frac{\sigma^2}{2}), \quad \beta_{t,u}^g = \frac{\sigma}{u\sqrt{3}}\sqrt{u^3 - t^3}.$$

Now one can apply Lemma 3.1 to calculate the formula for option with the geometric averaging. Recall that for the floating strike Asian call or put option, the stopping region $\mathcal{S} = \{(t,x), x \geq 0, \rho x_t^* > \rho x\}$, where x_t^* is the exercise boundary and $\rho = 1$ for the case of a call option, whereas $\rho = -1$ for a put option. If we insert the expression $\frac{dA_t^g}{A_t^g}$ for the geometric average (see Table 1) into Equations (4) and (5), we obtain the formula for the European-style option

$$\begin{aligned}
\widetilde{v}^g(t,x_t) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\rho\left(1 - x_T^g\right)^+\right)\right] = e^{-qT}\mathbb{E}_t^{\mathcal{Q}}\left[\left(\rho\left(1 - x_T^g\right)^+\right)\right], \\
&= \rho e^{-qT}\left(\Phi\left(-\rho\left(\frac{\alpha_{t,T}^g}{\beta_{t,T}^g}\right)\right) - e^{\alpha_{t,T}^g + \frac{(\beta_{t,T}^g)^2}{2}}\Phi\left(-\rho\left(\frac{\alpha_{t,T}^g}{\beta_{t,T}^g} + \beta_{t,T}^g\right)\right)\right),
\end{aligned}$$

and the value of the American-style early exercise bonus premium

$$\begin{aligned}
\widetilde{e}^g(t,x_t) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \rho e^{-qu} x_u^g 1_{\mathcal{S}}(u, x_u^g)\left(\frac{dA_u^g}{A_u^g} - \left(r - q\left(x_u^g\right)^{-1}\right)\right) du\right], \\
&= \int_t^T \rho e^{-qu}\mathbb{E}_t^{\mathcal{Q}}\left[1_{\{\rho x_u^* > \rho x_u^g\}}\left(-\frac{1}{u}x_u^g \ln x_u^g - r x_u^g + q\right)\right] du, \\
&= \int_t^T \rho e^{-qu}\left(q\ \Phi(\rho(\beta_{t,u}^g - \gamma_{t,u}^g))\right. \\
&\quad \left. + e^{\alpha_{t,u}^g + \frac{(\beta_{t,u}^g)^2}{2}}\left(\rho\frac{\beta_{t,u}^g}{u}\Phi'(\gamma_{t,u}^g)\right) - \left(r + \frac{\alpha_{t,u}^g + (\beta_{t,u}^g)^2}{u}\right)\Phi\left(-\rho\gamma_{t,u}^g\right)\right) du,
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and

$$\gamma_{t,u}^g = \frac{\alpha_{t,u}^g - \ln x_u^*}{\beta_{t,u}^g} + \beta_{t,u}^g.$$

Returning to the original variables, we obtain the formula of American-style floating strike Asian option with geometrically averaged floating strike:

$$V^g(t,S,A) = Se^{qt}\widetilde{V}^g(t,\frac{A}{S}) = Se^{qt}\left(\widetilde{v}^g\left(t,\frac{A}{S}\right)+\widetilde{e}^g\left(t,\frac{A}{S}\right)\right).$$

If we formally set value of the continuous dividend rate to 0, that is, $q=0$, the result is identical to the expression obtained in the paper (Hansen and Jørgensen, 2000).

3.2.2 Approximation for the arithmetic average. Unfortunately, in the case of an arithmetically averaged floating strike Asian option, the probabilistic distribution function of the arithmetic average cannot be expressed in an explicit way. Following Hansen and Jørgensen (2000), we approximate the probabilistic distribution of the variable $x_t^a = \frac{A_t^a}{S_t}$ for the continuous arithmetic average A_t^a by the log-normal conditioned distribution, that is, $\ln x_u^a|\mathcal{F}_t \sim \mathcal{N}(\alpha_{t,u}^a,(\beta_{t,u}^a)^2)$ at time t, where

$$\alpha_{t,u}^a = 2\ln \mathbb{E}_t^{\mathbb{Q}}[x_u^a] - \frac{1}{2}\ln \mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2], \quad \beta_{t,u}^a = \sqrt{\ln \mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2] - 2\ln \mathbb{E}_t^{\mathbb{Q}}[x_u^a]}. \tag{8}$$

Lemma 3.3. Consider the variable $x_u = \frac{A_u}{S_u}$, where A_u and S_u are defined as the arithmetic average (see Table 1) and as in Equation (2), respectively. First, two conditioned moments $\mathbb{E}_t^{\mathbb{Q}}[x_u^a]$ and $\mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2]$ of x_u entering the expressions for the functions $\alpha_{t,u}^a = \alpha^a(t,u,x_t^a)$ and $\beta_{t,u}^a = \beta^a(t,u,x_t^a)$ can be calculated, for t $\leq u$, as follows:

$$\mathbb{E}_t^{\mathbb{Q}}[x_u^a] = x_t^a\frac{t}{u}e^{-(r-q)(u-t)} + \frac{1}{(r-q)u}(1-e^{-(r-q)(u-t)}), \tag{9}$$

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2] = {}&\left(x_t^a\right)^2\frac{t^2}{u^2}e^{-2\left(r-q-\frac{\sigma^2}{2}\right)(u-t)} + x_t^a\frac{2te^{-(r-q)(u-t)}}{u^2(r-q)}(1-e^{-(r-q)(u-t)})\\ &+\frac{(r-q-\sigma^2)-2\left(r-q-\frac{\sigma^2}{2}\right)e^{-(r-q)(u-t)}+(r-q)e^{-2\left(r-q-\frac{\sigma^2}{2}\right)(u-t)}}{u^2(r-q)\left(r-q-\frac{\sigma^2}{2}\right)(r-q-\sigma^2)}.\end{aligned} \tag{10}$$

Remark 3.2. If we set the value of the continuous dividend rate $q=0$ in Lemma 3.3, we obtain almost identical expression to that of Hansen and Jørgensen (2000) except of the second moment $\mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2]$ entering Equation (8). Recall that expression for $\mathbb{E}_t^{\mathbb{Q}}$ by Hansen and Jørgensen (2000)

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[(x_u^a)^2]_{HJ} = {}&\left(x_t^a\right)^2\frac{t^2}{u^2}e^{-2\left(r-\frac{\sigma^2}{2}\right)(u-t)} + x_t^a\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}(1-e^{-(r-\sigma^2)(u-t)})\\ &+\frac{(r-\sigma^2)-2\left(r-\frac{\sigma^2}{2}\right)e^{-r(u-t)}+re^{-2\left(r-\frac{\sigma^2}{2}\right)(u-t)}}{u^2r\left(r-\frac{\sigma^2}{2}\right)(r-\sigma^2)}\end{aligned}$$

differs from ours (see Equation (10)) in the expression for the second summand where the term $r - \sigma^2$ is replaced by r in both the denominator and the exponent. But this seems to be just a typo in the paper (Hansen and Jørgensen, 2000).

Now we can return to the problem of valuation of an Asian option. First, we replace the general form of the average in Equation (5) by the expression for the arithmetic average defined as arithmetic average in Table 1. The stopping region is the same as for the case of geometric averaging.

Using Lemma 3.1, we calculate the value of both Equations (4) and (5). The European part of the option has value

$$\begin{aligned}
\widetilde{v}^a(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\rho\left(1-x_T^a\right)^+\right)\right],\\
&= \rho e^{-qT}\left(\Phi\left(-\rho(\frac{\alpha_{t,T}^a}{\beta_{t,T}^a}\right)\right) - e^{\alpha_{t,T}^a+\frac{\left(\beta_{t,T}^a\right)^2}{2}}\Phi\left(-\rho\left(\frac{\alpha_{t,T}^a}{\beta_{t,T}^a}+\beta_{t,T}^a\right)\right)\right),
\end{aligned}$$

and the American-style early exercise bonus premium

$$\begin{aligned}
\widetilde{e}^a(t,x) &= \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \rho e^{-qu} x_u^a 1_{\mathcal{S}}(u,x_u^a)\left(\frac{dA_u^a}{A_u^a} - \left(r - q\left(x_u^a\right)^{-1}\right)\right)du\right],\\
&= \int_t^T \rho e^{-qu}\mathbb{E}_t^{\mathcal{Q}}\left[1_{\{\rho x_u^* > \rho x_u^a\}}\left(\frac{1}{u}\left(1-x_u^a\right) - r x_u^a + q\right)\right]du,\\
&= \int_t^T \rho e^{-qu}\left(\left(q+\frac{1}{u}\right)\Phi\left(\rho\left(\beta_{t,u}^a - \gamma_{t,u}^a\right)\right) - \left(r+\frac{1}{u}\right)e^{\alpha_{t,u}^a+\frac{\left(\beta_{t,u}^a\right)^2}{2}}\Phi\left(-\rho\gamma_{t,u}^a\right)\right)du,
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and

$$\gamma_{t,u}^a = \frac{\alpha_{t,u}^a - \ln x_u^*}{\beta_{t,u}^a} + \beta_{t,u}^a.$$

Returning to the original variables we have the approximate value of American-style Asian option with a continuous arithmetic averaging

$$V^a(t,S,A) = Se^{qt}\widetilde{V}^a(t,x) = Se^{qt}(\widetilde{v}^a(t,x) + \widetilde{e}^a(t,x)).$$

3.3. *Expansion of the Early Exercise Boundary Close to Expiry*

Throughout this section, we shall assume the structural assumption on the interest and dividend rates:

$$r > q \geq 0. \tag{11}$$

We shall calculate an approximation of the call option early exercise boundary function for a call option. The approximation is obtained by the first-order Taylor series expansion in the $\sqrt{\tau}$ variable, where $\tau = T - t$ is the time to expiry. To approximate the early exercise boundary function by the Taylor expansion, we need to calculate the first derivative of x_t^* at expiry T with respect to $\sqrt{T-t}$ variable. Following Kuske and Keller (1998), Dewynne *et al.*, Wilmott (1993) and Ševčovič (2001), we propose an approximation of the early exercise boundary x_t^* in the form

$$x_t^* = x_T^*(1 + C\sqrt{T-t}) + O(T-t) \quad \text{as } t \to T,$$

where $C \in \mathbb{R}$ is a constant. To calculate C, we use the condition of smoothness of the early exercise boundary of the call option across the early exercise boundary – smooth pasting principle (c.f. Dai and Kwok, 2006; Kwok, 2008). Because $\widetilde{V}(T,x) = e^{-qT}(1-x)^+$, we have

$$-1 = e^{qt}\frac{\partial \widetilde{V}}{\partial x}(t,x_t^*) = e^{qt}\frac{\partial \widetilde{v}_t}{\partial x}(t,x_t^*) + e^{qt}\frac{\partial \widetilde{e}_t}{\partial x}(t,x_t^*) = \widehat{v}_x(t,x_t^*) + \int_t^T \widehat{e}_{Ix}(t,x_t^*,u,x_u^*)\,du. \tag{12}$$

In further derivation, we use the following limits (according to Lemma 3.2, Lemma 3.3 and Table 3). We recall that $\alpha_{t,u} = \alpha(t,u,x)$, $\beta_{t,u}^a = \beta^a(t,u,x^a)$ and $\beta_{t,u}^g = \beta^g(t,u)$.

$$\lim_{\tau\to 0}\alpha_{T-\tau,T} = \lim_{\tau\to 0}\alpha_{T-\tau,T-\tau(1-\theta)} = \alpha_{T,T} = \ln x_T^* < 0,$$

$$\lim_{\tau\to 0}\beta_{T-\tau,T} = \lim_{\tau\to 0}\beta_{T-\tau,T-\tau(1-\theta)} = \beta_{T,T} = 0^+,$$

$$\lim_{\tau\to 0}\Phi\left(-\frac{\alpha_{T-\tau,T}}{\beta_{T-\tau,T}}\right) = \Phi\left(-\frac{\ln x_T^*}{0^+}\right) = 1,$$

$$\lim_{\tau\to 0}\frac{\Phi'\left(-\frac{\alpha_{T-\tau,T}}{\beta_{T-\tau,T}}\right)}{(\beta_{T-\tau,T})^n} = 0,$$

$$\lim_{\tau\to 0}\frac{\ln x_T^*(1+h\sigma\sqrt{\tau(1-\theta)}) - \alpha_{T-\tau,T-\tau(1-\theta)}}{\beta_{T-\tau,T-\tau(1-\theta)}} = -h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}},$$

$$\lim_{\tau\to 0}\partial_x\alpha_{T-\tau,T} = \lim_{\tau\to 0}\partial_x\alpha_{T-\tau,T-\tau(1-\theta)} = \frac{1}{x_T^*},$$

$$\lim_{\tau\to 0}\beta_{T-\tau,T}\,\partial_\tau\beta_{T-\tau,T} = \frac{\sigma^2}{2}. \tag{13}$$

Because we have assumed $r > q \geq 0$, we have $0 < x_T^* < 1$ (see Table 3). Notice that both α and β have polynomial order in τ and the derivative of the normal cumulative distribution function (i.e. the probability density function) has exponential order in τ variable. In both derivations, we have used several properties of the derivative of normal cumulative distribution function $\Phi(x)$, for example, $\Phi'(x) = \Phi'(-x)$, $\Phi''(x) = -x\Phi'(x)$ and $\Phi'(\frac{a}{b}+b) = e^{-a-\frac{b^2}{2}}\Phi'(\frac{a}{b})$.

The following lemma will be useful in derivation of asymptotic behaviour of the early exercise. Its proof is straightforward and follows from monotonicity of the right-hand side of Equation (14) in the h variable.

Lemma 3.4. The implicit equation

$$0 = 1 - \int_0^1 \Phi\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) d\theta + h\int_0^1 \frac{\sqrt{1-\theta}}{\sqrt{\theta}}\Phi'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) d\theta \quad (14)$$

has the unique solution h^* having its approximate value $h^* \doteq -0.638833$.

Notice that the first-order asymptotic expansion as $t \to T$ of the early exercise boundary $S_f(t) \approx S_f(T)(1 + 0.638833\sigma\sqrt{T-t})$ for the American call option derived by Dewynne *et al.* (1993) and Ševčovič (2001) contains the same constant $-h^* \doteq 0.638833$ where h^* is a solution of Equation (14).

Remark 3.3. For the early exercise boundary function $x_t^* = x_t^*(T, r, q, \sigma^2)$ as a function of the model parameters $T, r, \sigma^2 > 0, q \geq 0$, we have the following scaling property:

$$x_t^*(T, r, q, \sigma^2) = x_{\frac{t}{T}}^*(1, rT, qT, \sigma^2 T).$$

3.3.1 Geometric average. We calculate the derivative of the European part of the expression (for the call option). Recall that α depends on the variable x, but β does not depend on this variable.

$$\widehat{v}_x^g(t,x) = e^{qt}\frac{\partial}{\partial x}\widetilde{v}^g(t,x) = -e^{-q(T-t)}e^{\alpha_{t,T}^g + \frac{(\beta_{t,T}^g)^2}{2}}\Phi\left(-\frac{\alpha_{t,T}^g}{\beta_{t,T}^g} - \beta_{t,T}^g\right)\frac{\partial \alpha_{t,T}^g}{\partial x}.$$

Now we calculate the derivative of the integral function of American-style option bonus:

$$\begin{aligned}\widehat{e}_{Ix}^g(t,x,u,x_u^*) = e^{-q(u-t)}\frac{1}{u}\Bigg(\left(\frac{-qu + x_u^* ru + x_u^*\ln x_u^* + x_u^*\left(\beta_{t,u}^g\right)^2}{\beta_{t,u}^g}\right)\Phi'\left(\frac{\ln x_u^* - \alpha_{t,u}^g}{\beta_{t,u}^g}\right) \\ -e^{\alpha_{t,u}^g + \frac{\left(\beta_{t,u}^g\right)^2}{2}}\left(ru + \alpha_{t,u}^g + (\beta_{t,u}^g)^2 + 1\right)\Phi\left(\frac{\ln x_u^* - \alpha_{t,u}^g}{\beta_{t,u}^g} - \beta_{t,u}^g\right)\Bigg)\frac{\partial \alpha_{t,u}^g}{\partial x}.\end{aligned}$$

We want to determine the behaviour of the early exercise boundary near the expiry T. The limit in Equation (12) leads to the trivial identity. By rearranging all the elements on the right-hand side of the equation, we have an expression of order $T - t$. We substitute $\tau = T - t$ and $x_T^* = G$ and divide the equation by τ. We have

$$\lim_{\tau\to 0}\frac{1+\widehat{v}_x(T-\tau, G(1+h\sigma\sqrt{\tau}))}{\tau} = \lim_{\tau\to 0}\frac{\partial \widehat{v}_x}{\partial \tau}(T-\tau, G(1+h\sigma\sqrt{\tau})).$$

According to Lemma 3.2 and Table 3, we have

$$\lim_{\tau\to 0}\partial_\tau \alpha^g_{T-\tau,T}\,\partial_x \alpha^g_{T-\tau,T} + \partial^2_{x,\tau}\alpha^g_{T-\tau,T} = \frac{1}{G}\left(q - \frac{q}{G} - \frac{1}{T} - \frac{\sigma^2}{2}\right).$$

The only non-zero elements of the first partial limit are the elements multiplied by the cumulative distribution function.

$$\begin{aligned}\lim_{\tau\to 0}\frac{1+\widehat{v}_x(T-\tau, G(1+h\sigma\sqrt{\tau}))}{\tau} &= \lim_{\tau\to 0}\frac{\partial}{\partial\tau}\left(1 - e^{-q\tau} e^{\alpha^g_{T-\tau,T}+\frac{(\beta^g_{T-\tau,T})^2}{2}}\frac{\partial \alpha^g_{T-\tau,T}}{\partial x}\right)\\ &= \frac{q}{G}+\frac{1}{T}.\end{aligned}$$

The second term represents the limit of the integral part divided by τ. If we substitute $u = T - \tau(1-\theta)$, then we obtain

$$\begin{aligned}&\lim_{\tau\to 0}\frac{\int_{T-\tau}^{T}\widehat{e}^g_{Ix}(T-\tau, G(1+h\sigma\sqrt{\tau})), u, G(1+h\sigma\sqrt{T-u}))du}{\tau}\\ &\quad = \int_0^1 \lim_{\tau\to 0}\widehat{e}^g_{Ix}(T-\tau, G(1+h\sigma\sqrt{\tau})), T-\tau(1-\theta), G(1+h\sigma\sqrt{\tau}\sqrt{1-\theta}))d\theta.\end{aligned}$$

The last expression can then be simplified, using Equation (7) and limits in Equation (13) for the limit of early exercise boundary at expiry G, that is, $r = \frac{q}{G} - \frac{\ln G}{T}$, and by calculating the limit (using L'Hospital rule) of the expression multiplied by the derivative of the cumulative density function. The final limit of the integral has the following form:

$$\begin{aligned}&\lim_{\tau\to 0}\widehat{e}^g_{Ix}(T-\tau, G(1+h\sigma\sqrt{\tau})), T-\tau(1-\theta), G(1+h\sigma\sqrt{\tau}\sqrt{1-\theta}))\\ &\quad = \left(\frac{q}{G}+\frac{1}{T}\right)\left(-\Phi\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right) + h\frac{\sqrt{1-\theta}}{\sqrt{\theta}}\Phi'\left(-h\frac{1-\sqrt{1-\theta}}{\sqrt{\theta}}\right)\right).\end{aligned} \tag{15}$$

Integrating Equation (15) with respect to $\theta \in [0, 1]$, putting both partial limits together, dividing by the non-zero constant $\frac{q}{G} + \frac{1}{T}$ and by Lemma 3.4 and Table 3, we finally obtain

$$x^*_t = G(1 + h^*\sigma\sqrt{T-t}) + O(T-t) \quad \text{as t} \to \text{T}, \quad \text{where h}^* \doteq -0.638833.$$

3.3.2 Arithmetic average. The derivation for the case of the arithmetic average is very similar to the geometric one. We calculate the derivatives of both parts of the value function. The European part

$$\widehat{v}^a_x(t,x_t) = e^{qt}\frac{\partial}{\partial x}\widetilde{v}^a(t,x_t) = e^{-q(T-t)}\Phi'\left(-\frac{\alpha^a_{t,T}}{\beta^a_{t,T}}\right)\frac{\partial \beta^a_{t,T}}{\partial x}$$

$$- e^{-q(T-t)+\alpha^a_{t,T}+\frac{(\beta^a_{t,T})^2}{2}}\Phi\left(-\frac{\alpha^a_{t,T}}{\beta^a_{t,T}} - \beta^a_{t,T}\right)\left(\frac{\partial \alpha^a_{t,T}}{\partial x} + \beta^a_{t,T}\frac{\partial \beta^a_{t,T}}{\partial x}\right),$$

and the American-style bonus

$$\widehat{e}^a_{Ix}(t,x,u,x^*_u) = e^{-q(u-t)}\left(\left(\left(q+\frac{1}{u}\right) - \left(r+\frac{1}{u}\right)x^*_u\right)\Phi'\left(\frac{\ln x^*_u - \alpha^a_{t,u}}{\beta^a_{t,u}}\right)\frac{\partial}{\partial x}\left(\frac{\ln x^*_u - \alpha^a_{t,u}}{\beta^a_{t,u}}\right)\right.$$

$$+\left(r+\frac{1}{u}\right)x^*_u\Phi'\left(\frac{\ln x^*_u - \alpha^a_{t,u}}{\beta^a_{t,u}}\right)\frac{\partial \beta^a_{t,u}}{\partial x}$$

$$\left.-\left(r+\frac{1}{u}\right)e^{\alpha^a_{t,u}+\frac{(\beta^a_{t,u})^2}{2}}\Phi\left(\frac{\ln x^*_u - \alpha^a_{t,u}}{\beta^a_{t,u}} - \beta^a_{t,u}\right)\left(\frac{\partial \alpha^a_{t,u}}{\partial x} + \beta^a_{t,u}\frac{\partial \beta^a_{t,u}}{\partial x}\right)\right).$$

The rest of the derivation was performed following the same steps used in Section 3.3.1. The main difference in the derivation is that for the arithmetic average also the expression $\beta^a_{t,u} = \beta^a(t,u,x)$ depends on the variable x. Thus, according to the Lemma 3.3 and Table 3, we need to calculate the following limits:

$$\lim_{\tau\to 0}\partial_\tau\alpha^a_{T-\tau,T}\,\partial_x\alpha^a_{T-\tau,T} + \partial^2_{x,\tau}\alpha^a_{T-\tau,T} = \frac{1}{A}\left(q - r - \frac{1}{T} - \frac{\sigma^2}{2}\right),$$

$$\lim_{\tau\to 0}\partial_x\beta^a_{T-\tau,T} = \lim_{\tau\to 0}\partial_x\beta^a_{T-\tau,T-\tau(1-\theta)} = \lim_{\tau\to 0}\partial_x\beta^a_{T-\tau,T}\,\partial_x\beta^a_{T-\tau,T} + \beta^a_{T-\tau,T}\partial^2_{x,\tau}\beta^a_{T-\tau,T} = 0.$$

We recall that for the case of a continuous arithmetic average we have $x^*(T) = A \equiv \frac{q+\frac{1}{T}}{r+\frac{1}{T}}$. Because we have assumed $r > q \geq 0$ we obtain $A < 1$ (see Table 3).

The derivation leads to the same equation as in Lemma 3.4 when multiplied by the constant $r + \frac{1}{T}$. In summary, we obtain the following approximation of the limiting behaviour of the early exercise boundary near expiry also for the arithmetic average.

$$x^*(t) = \frac{1+qT}{1+rT}(1 + h^*\sigma\sqrt{T-t}) + O(T-t), \quad \text{as t} \to \text{T}, \quad \text{where h}^* \doteq -0.638833. \tag{16}$$

4. Transformation Method for Asian Call Options

The purpose of this section is to propose an efficient numerical algorithm for determining the free boundary position x^*_t for American-style Asian options. Construction of the algorithm is based on a solution to a non-local parabolic PDE. The governing PDE is constructed for a transformed variable representing the so-called δ-synthesized portfolio. Furthermore, we employ a front- fixing method (referred also to as Landau's fixed domain transformation) developed by Wu and Kwok (1997) Stamicar *et al.* (1999) and Ševčovič (2001) for plain vanilla options as well as for a class of non-linear Black–Scholes equations (Ševčovič, 2007, 2009). At the end of this section, we present numerical results and comparisons achieved by these methods to the recent method developed by Dai and Kwok (2006).

First, we recall the PDE for pricing Asian options (cf. Kwok, 2008). We assume the asset price dynamics follows a geometric Brownian with a drift ρ, continuous dividend yield $q \geq 0$ and volatility σ, that is, $dS = (\rho - q)Sdt + \sigma SdW$ where W is the standard Wiener process. If we apply Itōrsquo;s formula to the function $V = V(t, S, A)$, we obtain

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial A} dA. \tag{17}$$

Recall that for arithmetic, geometric or weighted arithmetic averaging, we have $dA/A = f(A/S, t)dt$ where the function $f = f(x, t)$ is defined as follows (see Table 2):

$$f(x,t) = \begin{cases} \frac{x^{-1}-1}{t} & \text{arithmetic averaging,} \\ \frac{-\ln x}{t} & \text{geometric averaging,} \\ \frac{\lambda(x^{-1}-1)}{1-e^{-\lambda t}} & \text{exponentially weighted arithmetic averaging.} \end{cases} \tag{18}$$

Inserting the expression $dA = Af(A/S, t)dt$ into Equation (17) and following standard arguments from the Black–Scholes theory, we obtain the governing equation for pricing Asian option with averaging given by Equation (18) in the form:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + S(r-q) \frac{\partial V}{\partial S} + Af\left(\frac{A}{S}, t\right) \frac{\partial V}{\partial A} - rV = 0, \tag{19}$$

where $0 < t < T$, $S, A > 0$ (see e.g. Dai and Kwok, 2006; Kwok, 2008). For the Asian call option, the above equation is subject to the terminal pay-off condition $V(T, S, A) = \max(S - A, 0)$, S, $A > 0$. It is well known (see e.g. Dai and Kwok, 2006; Kwok, 2008) that for Asian options with floating strike we can perform dimension reduction by introducing the following similarity variable:

$$x = \frac{A}{S}, \qquad W(x, \tau) = \frac{V(t, S, A)}{A},$$

where $\tau = T - t$. It is straightforward to verify that $V(t, S, A) = A\, W(A/S, T - t)$ is a solution of Equation (19) iff $W = W(x, \tau)$ is a solution to the following parabolic PDE:

$$\frac{\partial W}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial W}{\partial x} \right) + (r-q)x \frac{\partial W}{\partial x} - f(x, T-\tau) \left(W + x \frac{\partial W}{\partial x} \right) + rW = 0, \tag{20}$$

where $x > 0$ and $0 < \tau < T$. The initial condition for W immediately follows from the terminal pay-off diagram for the call option, that is, $W(x, 0) = \max(x^{-1} - 1, 0)$.

4.1. American-Style Asian Call Options

Following Dai and Kwok (2006), the set $\mathcal{E} = \{(t, S, A) \in [0, T] \times [0, \infty) \times [0, \infty), V(t, S, A) = V(T, S, A)\}$ is the exercise region for American-style Asian call options. In the case of a call option, this region can be described by the early exercise boundary function $S_f = S_f(t, A)$ such that $\mathcal{E} = \{(t, S, A) \in [0, T] \times [0, \infty) \times [0, \infty), S \geq S_f(t, A)\}$. For an American-style Asian call option, we have to impose a homogeneous Dirichlet boundary condition $V(t, 0, A) = 0$. According to Dai and Kwok (2006), the C^1 continuity condition at the point $(t, S_f(t, A), A)$ of a contact of a solution V with its pay-off diagram implies the following boundary condition at the free boundary position $S_f(t, A)$:

$$\frac{\partial V}{\partial S}(t, S_f(t, A), A) = 1, \quad V(t, S_f(t, A), A) = S_f(t, A) - A, \tag{21}$$

for any $A > 0$ and $0 < t < T$. It is important to emphasize that the free boundary function S_f can also be reduced to a function of one variable by introducing a new state function x_t^* as follows:

$$S_f(t, A) = A/x_t^*.$$

The function $t \mapsto x_t^*$ is a free boundary function for the transformed state variable $x = A/S$. For American-style Asian call options, the spatial domain for the reduced Equation (20) is given by $1/\rho(\tau) < x < \infty, \ \ \tau \in (0, T)$, where $\rho(\tau) = 1/x_{T-\tau}^*$. Taking into account boundary conditions in Equation (21) for the option price V, we end up with corresponding boundary conditions for the function W:

$$W(\infty, \tau) = 0, \qquad W(x, \tau) = \frac{1}{x} - 1, \ \ \frac{\partial W}{\partial x}(x, \tau) = -\frac{1}{x^2} \ \ \text{at} \ \ \mathrm{x} = \frac{1}{\rho(\tau)}, \tag{22}$$

for any $0 < \tau < T$ and the initial condition

$$W(x, 0) = \max(x^{-1} - 1, 0) \quad \text{for any } \mathrm{x} > 0. \tag{23}$$

4.2. Fixed Domain Transformation

To apply the Landau fixed domain transformation for the free boundary problem in Equations (20), (22) and (23), we introduce a new state variable ξ and an auxiliary function $\Pi = \Pi(\xi, \tau)$ representing a synthetic portfolio. They are defined as follows:

$$\xi = \ln(\rho(\tau)x), \qquad \Pi(\xi, \tau) = W(x, \tau) + x\frac{\partial W}{\partial x}(x, \tau).$$

Clearly, $x \in (\rho(\tau)^{-1}, \infty)$ iff $\xi \in (0, \infty)$ for $\tau \in (0, T)$. The value $\xi = \infty$ of the transformed variable corresponds to the value $x = \infty$, that is, $S = 0$ when expressed in the original variable. On the contrary, the value $\xi = 0$ corresponds to the free boundary position $x = x_t^*$, that is, $S = S_f(t, A)$. After straightforward calculations we conclude that the function $\Pi = \Pi(\xi, \tau)$ is a solution to the following parabolic PDE:

$$\frac{\partial \Pi}{\partial \tau} + a(\xi,\tau)\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi,\tau)\Pi = 0,$$

where the term $a(\xi, \tau)$ depends on the free boundary position ρ. The terms a and b are given as follows:

$$\begin{aligned} a(\xi,\tau) &= \frac{\dot{\varrho}(\tau)}{\varrho(\tau)} + r - q - \frac{\sigma^2}{2} - f\left(\frac{e^{\xi}}{\varrho(\tau)}, T-\tau\right), \\ b(\xi,\tau) &= r - \frac{\partial}{\partial x}\left(x f(x, T-\tau)\right)\big|_{x=\frac{e^{\xi}}{\varrho(\tau)}}. \end{aligned} \tag{24}$$

Notice that $b(\xi,\tau) = r + 1/(T-\tau)$ in the case of arithmetic averaging, that is, $f(x,t) = (x^{-1}-1)/t$.

The initial condition for the solution Π can be determined from Equation (23)

$$\Pi(\xi,0) = \begin{cases} -1 & \xi < \ln\, \varrho(0), \\ 0 & \xi > \ln\, \varrho(0). \end{cases}$$

Because $\partial_x W(x,\tau) = -\frac{1}{x^2}$ and $W(x,\tau) = \frac{1}{x} - 1$ for $x = \frac{1}{\varrho(\tau)}$ and $W(\infty,\tau) = 0$, we conclude the Dirichlet boundary conditions for the transformed function $\Pi(\xi, \tau)$

$$\Pi(0,\tau) = -1, \qquad \Pi(\infty,\tau) = 0.$$

It remains to determine an algebraic constraint between the free boundary function $\rho(\tau)$ and the solution Π. Similarly, as in the case of a linear or non-linear Black–Scholes equation (cf. Ševčovič, 2007) we obtain, by differentiation the condition $W(\frac{1}{\varrho(\tau)},\tau) = \varrho(\tau) - 1$ with respect to τ, the following identity:

$$\frac{d\varrho}{d\tau}(\tau) = \frac{\partial W}{\partial x}(\varrho(\tau)^{-1},\tau)\left(-\varrho(\tau)^{-2}\right)\frac{d\varrho}{d\tau}(\tau) + \frac{\partial W}{\partial \tau}(\varrho(\tau)^{-1},\tau).$$

Because $\partial_x W(\varrho(\tau)^{-1},\tau) = -\varrho(\tau)^2$ we have $\frac{\partial W}{\partial \tau}(x,\tau) = 0$ at $x = \varrho(\tau)$. Assuming continuity of the function $\Pi(\xi, \tau)$ and its derivative $\Pi_\xi(\xi,\tau)$ up to the boundary $\xi = 0$, we obtain

$$x^2\frac{\partial^2 W}{\partial x^2}(x,\tau) \to \frac{\partial \Pi}{\partial \xi}(0,\tau) + 2\varrho(\tau), \quad x\frac{\partial W}{\partial x}(x,\tau) \to -\varrho(\tau) \quad \text{as x} \to \varrho(\tau)^{-1}.$$

Passing to the limit $x \to \varrho(\tau)^{-1}$ in Equation (20), we end up with the algebraic equation

$$q\varrho(\tau) - r + f(\varrho(\tau)^{-1}, T-\tau) = \frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau), \tag{25}$$

for the free boundary position $\varrho(\tau)$ where $\tau \in (0, T]$. Notice that in the case of arithmetic averaging where $f(\varrho(\tau)^{-1}, T-\tau) = (\varrho(\tau)-1)/(T-\tau)$, we can derive an explicit expression for the free boundary position $\varrho(\tau)$

$$\varrho(\tau) = \frac{1 + r(T-\tau) + \frac{\sigma^2}{2}(T-\tau)\frac{\partial \Pi}{\partial \xi}(0,\tau)}{1 + q(T-\tau)}, \quad 0 < \tau < T,$$

as a function of the derivative $\partial_\xi \Pi(0, \tau)$ evaluated at $\xi = 0$. The value $\varrho(0)$ can be deduced from Theorem 3.1. For the arithmetic averaging we have (see also Dai and Kwok, 2006) the following expression:

$$\varrho(0) = \max\left(\frac{1 + rT}{1 + qT}, 1\right).$$

In summary, we derived the following non-local parabolic equation for the synthesized portfolio $\Pi(\xi, \tau)$:

$$\frac{\partial \Pi}{\partial \tau} + a(\xi,\tau)\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} + b(\xi,\tau)\Pi = 0, \quad 0 < \tau < T, \xi > 0,$$

with an algebraic constraint

$$q\varrho(\tau) - r + f(\varrho(\tau)^{-1}, T-\tau) = \frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau), \;\; 0 < \tau < T, \tag{26}$$

subject to the boundary and initial conditions

$$\Pi(0,\tau) = -1, \qquad \Pi(\infty, \tau) = 0,$$

$$\Pi(\xi, 0) = \begin{cases} -1 & \textit{for } \xi < \ln(\varrho(0)), \\ 0 & \textit{for } \xi > \ln(\varrho(0)), \end{cases}$$

where $a(\xi, \tau)$ and $b(\xi, \tau)$ are given by Equation (24), and the starting point $\varrho(0) = 1/x_T^*$ is given by Theorem 3.1.

4.2.1 An equivalent form of the equation for the free boundary. Although Equation (25) provides an algebraic formula for the free boundary position $\varrho(\tau)$ in terms of the derivative $\partial_\xi \Pi(0, \tau)$ such an expression is not quite suitable for construction of a robust numerical approximation scheme. The reason is that any small inaccuracy in approximation of the value $\partial_\xi \Pi(0, \tau)$ is transferred into the entire computational domain $\xi \in (0, \infty)$, making thus a numerical scheme very sensitive to the value of the derivative of a solution evaluated in one point $\xi = 0$. In what follows, we present an equivalent equation for the free boundary position $\varrho(\tau)$ that is more robust from the numerical approximation point of view.

Integrating the governing Equation (26) with respect to $\xi \in (0, \infty)$ yields

$$\frac{d}{d\tau}\int_0^\infty \Pi d\xi + \int_0^\infty a(\xi,\tau)\frac{\partial \Pi}{\partial \xi}d\xi - \frac{\sigma^2}{2}\int_0^\infty \frac{\partial^2 \Pi}{\partial \xi^2}d\xi + \int_0^\infty b(\xi,\tau)\Pi d\xi = 0.$$

Now taking into account the boundary conditions $\Pi(0,\tau) = -1, \Pi(\infty,\tau) = 0$, and consequently $\partial_\xi \Pi(\infty,\tau) = 0$ we obtain, by applying Equation (25), the following differential equation:

$$\frac{d}{d\tau}\left(\ln \varrho(\tau) + \int_0^\infty \Pi(\xi,\tau)d\xi\right) + q\varrho(\tau) - q - \frac{\sigma^2}{2} \\ + \int_0^\infty \left[r - f\left(\frac{e^\xi}{\varrho(\tau)}, T-\tau\right)\right]\Pi(\xi,\tau)d\xi = 0.$$

In the case of arithmetic averaging where $f(x,t) = (x^{-1}-1)/t$, we obtain

$$\frac{d}{d\tau}\left(\ln \varrho(\tau) + \int_0^\infty \Pi(\xi,\tau)d\xi\right) + q\varrho(\tau) - q - \frac{\sigma^2}{2} \\ + \int_0^\infty \left[r - \frac{\varrho(\tau)e^{-\xi}-1}{T-\tau}\right]\Pi(\xi,\tau)d\xi = 0. \tag{27}$$

4.3. *A Numerical Approximation Operator Splitting Scheme*

Our numerical approximation scheme is based on a solution to the transformed system (26). For the sake of simplicity, the scheme will be derived for the case of arithmetically averaged Asian call option. Derivation of the scheme for geometric or weighted arithmetic averaging is similar and therefore omitted.

We restrict the spatial domain $\xi \in (0,\infty)$ to a finite interval of values $\xi \in (0,L)$ where $L > 0$ is sufficiently large. For practical purposes, it is sufficient to take $L \approx 2$. Let $k > 0$ denote by the time step, $k = T/m$, and, by $h = L/n > 0$ the spatial step. Here $m, n \in \mathbb{N}$ denote the number of time and space discretization steps, respectively. We denote by $\Pi^j = \Pi^j(\xi)$ the time discretization of $\Pi(\xi,\tau_j)$ and $\varrho^j \approx \varrho(\tau_j)$ where $\tau_j = jk$. By Π_i^j we shall denote the full space–time approximation for the value $\Pi(\xi_i,\tau_j)$. Then for the Euler backward in time finite difference approximation of Equation (26), we have

$$\frac{\Pi^j - \Pi^{j-1}}{k} + c^j\frac{\partial \Pi^j}{\partial \xi} - \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi}-1}{T-\tau_j}\right)\frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi^j}{\partial^2 \xi} + \left(r + \frac{1}{T-\tau_j}\right)\Pi^j = 0,$$

where c^j is an approximation of the value $c(\tau_j)$ where the $c(\tau) = \frac{\dot\varrho(\tau)}{\varrho(\tau)} + r - q$. The solution $\Pi^j = \Pi^j(x)$ is subject to Dirichlet boundary conditions at $\xi = 0$ and $\xi = L$. We set $\Pi^0(\xi) = \Pi(\xi,0)$ (see Equation (26)). In what follows, we make use of the time step operator splitting method. We split the above problem into a convection part and a diffusive part by introducing an auxiliary intermediate step $\Pi^{j-\frac{1}{2}}$

(*Convective part*)

$$\frac{\Pi^{j-\frac{1}{2}} - \Pi^{j-1}}{k} + c^j \partial_x \Pi^{j-\frac{1}{2}} = 0, \tag{28}$$

(*Diffusive part*)

$$\frac{\Pi^j - \Pi^{j-\frac{1}{2}}}{k} - \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi} - 1}{T - \tau_j}\right)\frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi^j}{\partial^2 \xi} + \left(r + \frac{1}{T - \tau_j}\right)\Pi^j = 0. \tag{29}$$

Similarly as in Ševčovič (2007), we shall approximate the convective part by the explicit solution to the transport equation $\partial_\tau \tilde{\Pi} + c(\tau)\partial_\xi \tilde{\Pi} = 0$ for $\xi > 0$ and $\tau \in (\tau_{j-1}, \tau_j]$ subject to the boundary condition $\tilde{\Pi}(0,\tau) = -1$ and the initial condition $\tilde{\Pi}(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$. It is known that the free boundary function $\rho(\tau)$ need not be monotonically increasing (see e.g. Dai and Kwok, 2006; Ševčovič, 2009 or Hansen and Jørgensen, 2000). Therefore, depending whether the value of $c(\tau)$ is positive or negative the boundary condition $\tilde{\Pi}(0,\tau) = -1$ at $\xi = 0$ is either in-flowing ($c(\tau) > 0$) or out-flowing ($c(\tau) < 0$). Hence the boundary condition $\Pi(0,\tau) = -1$ can be prescribed only if $c(\tau_j) \geq 0$. Let us denote by $C(\tau)$ the primitive function to $c(\tau)$, that is, $C(\tau) = \ln \varrho(\tau) + (r-q)\tau$. Solving the transport equation $\partial_\tau \tilde{\Pi} + c(\tau)\partial_\xi \tilde{\Pi} = 0$ for $\tau \in [\tau_{j-1}, \tau_j]$ subject to the initial condition $\Pi(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$ we obtain: $\tilde{\Pi}(\xi,\tau) = \Pi^{j-1}(\xi - C(\tau) + C(\tau_{j-1}))$ if $\xi - C(\tau) + C(\tau_{j-1}) > 0$ and $\tilde{\Pi}(\xi,\tau) = -1$ otherwise. Hence the full time–space approximation of the half-step solution $\Pi_i^{j-\frac{1}{2}}$ can be obtained from the formula

$$\Pi_i^{j-\frac{1}{2}} = \begin{cases} \Pi^{j-1}(\eta_i), & \text{if } \eta_i = \xi_i - \ln \varrho^j + \ln \varrho^{j-1} - (r-q)k > 0, \\ -1, & \text{otherwise.} \end{cases} \tag{30}$$

To compute the value $\Pi^{j-1}(\eta_i)$, we make use of a linear interpolation between discrete values $\Pi_i^{j-1}, i = 0, 1, \ldots, n$.

Using central finite differences for approximation of the derivative $\partial_x \Pi^j$, we can approximate the diffusive part of a solution of Equation (29) as follows:

$$\frac{\Pi_i^j - \Pi_i^{j-\frac{1}{2}}}{k} + \left(r + \frac{1}{T - \tau_j}\right)\Pi_i^j - \left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T - \tau_j}\right)\frac{\Pi_{i+1}^j - \Pi_{i-1}^j}{2h} - \frac{\sigma^2}{2}\frac{\Pi_{i+1}^j - 2\Pi_i^j + \Pi_{i-1}^j}{h^2} = 0.$$

Therefore, the vector of discrete values $\Pi^j = \{\Pi_i^j, i = 1, 2, \ldots, n\}$ at the time level $j \in \{1, 2, \ldots, m\}$ is a solution of a tridiagonal system of linear equations

$$\alpha_i^j \Pi_{i-1}^j + \beta_i^j \Pi_i^j + \gamma_i^j \Pi_{i+1}^j = \Pi_i^{j-\frac{1}{2}}, \quad \textit{for} \quad i = 1, 2, \ldots, n, \quad \text{where} \tag{31}$$

$$\alpha_i^j(\varrho^j) = -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T - \tau_j}\right),$$

$$\gamma_i^j(\varrho^j) = -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{\sigma^2}{2} + \frac{\varrho^j e^{-\xi_i} - 1}{T - \tau_j}\right), \tag{32}$$

$$\beta_i^j(\varrho^j) = 1 + \left(r + \frac{1}{T - \tau_j}\right)k - (\alpha_i^j + \gamma_i^j).$$

The initial and boundary conditions at $\tau = 0$ and $x = 0$, L, can be approximated as follows:

$$\Pi_i^0 = \begin{cases} -1 & \text{for } \xi_i < \ln\left(\frac{(1+rT)}{(1+qT)}\right), \\ 0 & \text{for } \xi_i \geq \ln\left(\frac{(1+rT)}{(1+qT)}\right), \end{cases}$$

for i =0, 1, . . ., n, and $\Pi_0^j = -1$, $\Pi_n^j = 0$ for $j = 1, \ldots, m$.

Finally, we employ the differential Equation (27) to determine the free boundary position ϱ. Taking the Euler finite difference approximation of $\frac{d}{d\tau}\left(\ln\varrho + \int_0^\infty \Pi d\xi\right)$, we obtain

(*Algebraic part*)

$$\ln\varrho^j = \ln\varrho^{j-1} + I_0(\Pi^{j-1}) - I_0(\Pi^j) + k\left(q + \frac{\sigma^2}{2} - q\varrho^{j-1} - I_1(\varrho^{j-1}, \Pi^j)\right), \tag{33}$$

where I_0 (Π) stands for numerical trapezoid quadrature of the integral $\int_0^\infty \Pi(\xi)d\xi$ whereas $I_1(\varrho^{j-1}, \Pi)$ is a trapezoid quadrature of the integral $\int_0^\infty \left(r - \frac{\varrho^{j-1}e^{-\xi}-1}{T-\tau_j}\right)\Pi(\xi)d\xi$.

We formally rewrite discrete Equations (30), (31) and (33) in the operator form:

$$\varrho^j = \mathcal{F}(\Pi^j), \qquad \Pi^{j-\frac{1}{2}} = \mathcal{T}(\varrho^j), \qquad \mathcal{A}(\varrho^j)\Pi^j = \Pi^{j-\frac{1}{2}}, \tag{34}$$

where $\ln\mathcal{F}(\Pi^j)$ is the right-hand side of Equation (33), $\mathcal{T}(\varrho^j)$ is the transport equation solver given by the right-hand side of Equation (30) and $\mathcal{A} = \mathcal{A}(\varrho^j)$ is a tridiagonal matrix with coefficients given by Equation (32). Equation (34) can be approximately solved by means of successive iterations procedure. We define, for $j \geq 1$, $\Pi^{j,0} = \Pi^{j-1}, \varrho^{j,0} = \varrho^{j-1}$. Then the $(p+1)$-th approximation of Π^j and ρ^j is obtained as a solution to the system:

$$\varrho^{j,p+1} = \mathcal{F}(\Pi^{j,p}),$$

$$\Pi^{j-\frac{1}{2},p+1} = \mathcal{T}(\varrho^{j,p+1}), \tag{35}$$

$$\mathcal{A}(\varrho^{j,p+1})\Pi^{j,p+1} = \Pi^{j-\frac{1}{2},p+1}.$$

Supposing the sequence of approximate discretized solutions $\{(\Pi^{j,p}, \varrho^{j,p})\}_{p=1}^{\infty}$ converges to the limiting value $(\Pi^{j,\infty}, \varrho^{j,\infty})$ as $p \to \infty$ then this limit is a solution to a non-linear system of Equation (34) at the time level j and we can proceed by computing the approximate solution in the next time level $j+1$.

4.4. Computational Examples of the Free Boundary Approximation

Finally, we present several computational examples of application of the numerical approximation scheme (35) for the solution $\Pi(\xi, \tau)$ and the free boundary position $\varrho(\tau)$ of Equation (26). We consider American-style Asian arithmetically averaged floating strike call options.

In Figure 2, we show the behaviour of the early exercise boundary function $\varrho(\tau)$ and the function $x_t^* = 1/\varrho(T-t)$. In this numerical experiment, we chose $r = 0.06$, $q = 0.04$, $\sigma = 0.2$ and very long expiration time $T = 50$ years. These parameters correspond to the example presented by Dai and Kwok (2006). As far as other numerical parameters are concerned, we chose the mesh of $n = 200$ spatial grid points and we have chosen the number of time steps $m = 10^5$ to achieve very fine time stepping corresponding to 260 min between consecutive time steps when expressed in the original timescale of the problem.

In Figure 3, we can see the behaviour of the transformed function Π in both 3D and contour plot perspectives. We also plot the initial condition $\Pi(\xi, 0)$ and five time steps of the function $\xi \mapsto \Pi(\xi, \tau_j)$ for $\tau_j = 0.1, 1, 5, 25, 50$.

A comparison of early exercise boundary profiles with respect to varying interest rates r and dividend yields q is shown in Figure 4. A comparison of the free boundary position $x_t^* = 1/\varrho(T-t)$ obtained by our method (solid curve) and that of the projected successive over-relaxation algorithm by Dai and Kwok (2006) (dashed curve) for different values of the interest rate r is shown in Figure 4 (right). The algorithm of Dai and Kwok (2006) is based on a numerical solution to the variational inequality for the function $W = W(x, \tau)$, which is a solution to Equation (20) in the continuation region and it is smoothly pasted to its pay-off diagram (22). It is clear that our method and that of Dai and Kwok (2006) give almost the same results. A quantitative comparison of both methods is given in Table 4 for model parameters $T = 50$,

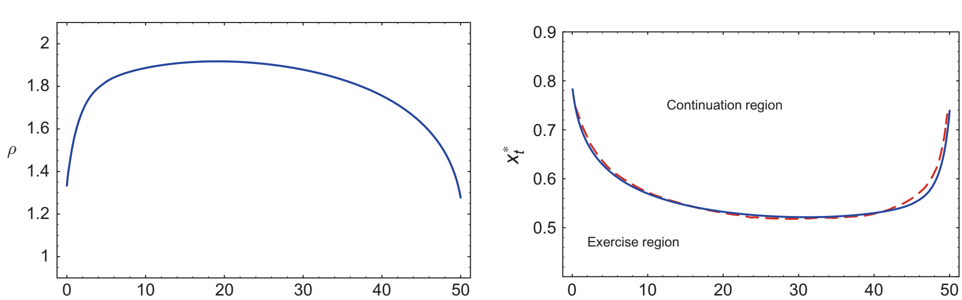

Figure 2. The function $\varrho(\tau)$ (left). A comparison of the free boundary position $x_t^* = 1/\varrho(T-t)$ (right) obtained by our method (solid curve) and that of the projected successive over-relaxation algorithm by Dai and Kwok (dashed curve).

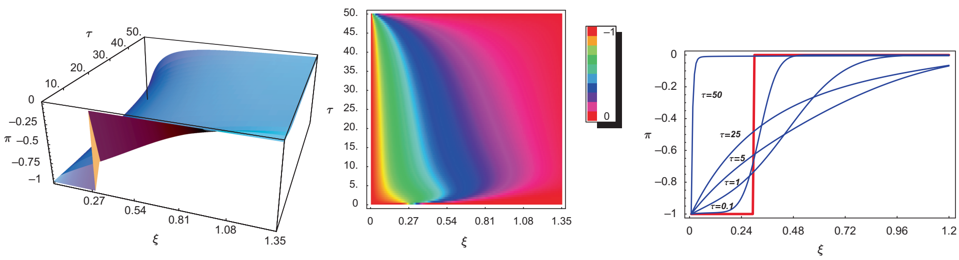

Figure 3. A 3D plot (left) and contour plot (right) of the function $\Pi(\xi, \tau)$. Profiles of the function $\Pi(\xi, \tau)$ for various times $\tau \in [0, T]$.

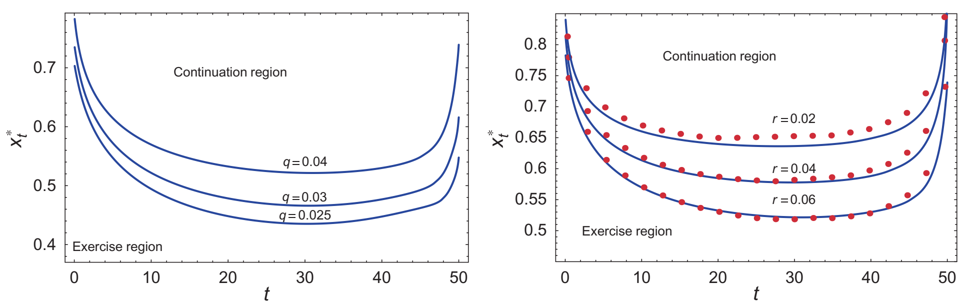

Figure 4. A comparison of the free boundary position x_t^* for various dividend yield rates $q = 0.04, 0.03, 0.025$ and fixed interest rate $r = 0.06$ (left). Comparison of x_t^* for various interest rates $r = 0.06, 0.04, 0.02$ and fixed dividend yield $q = 0.04$. Dots represent the solution obtained by Dai and Kwok (right).

Table 4. Comparison of PSOR and our transformation method for $T = 50$, $\sigma = 0.2$ and $q = 0.04$.

	$r = 0.06$	$r = 0.04$	$r = 0.02$
$\Vert x_t^{*,trans} - x_t^{*,psor} \Vert_\infty$	0.09769	0.03535	0.05359
$\Vert x_t^{*,trans} - x_t^{*,psor} \Vert_1$	0.00503	0.00745	0.01437
$\min x_t^{*,trans}$	0.52150	0.57780	0.63619

$\sigma = 0.2$, $q = 0.04$ and various interest rates $r = 0.02, 0.04, 0.06$. We evaluated discrete $L^\infty(0, T)$ and $L^1(0, T)$ norms of the difference $x_t^{*,trans} - x_t^{*,psor}$ between the numerical solution $x_t^{*,trans}$, $t \in [0, T]$, obtained by our method and that of Dai and Kwok (2006) denoted by $x_t^{*,psor}$. We also show the minimal value $\min_{t\in[0,T]} x_t^{*,trans}$ of the early exercise boundary.

Finally, in Figure 5 we present numerical experiments for shorter expiration times $T = 0.7$ and $T = 1$ (year) with zero dividend rate $q = 0$ and $r = 0.06$ and $\sigma = 0.2$. We also present a comparison of the free boundary position $x_t^* = 1/\varrho(T - t)$ and the analytic approximation in Equation (16) for parameters: $r = 0.06$, $q = 0$, $\sigma = 0.2$ and $T = 1$. It is clear that the analytic approximation in Equation (16) is capable of capturing the behaviour of x_t^* only for times t close to the expiry T. Moreover, the analytic

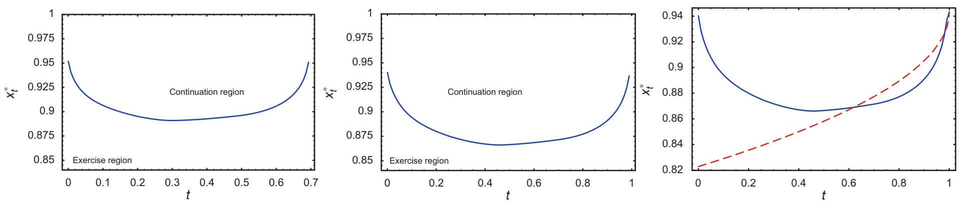

Figure 5. The free boundary position for expiration times $T = 0.7$ (left) and $T = 1$ (centre). A comparison of the free boundary position with its analytic approximation (dashed line).

approximation is a monotone function whereas the true early exercise boundary x_t^* is a decreasing function for small values of t and then it becomes increasing (see e.g. Figure 5).

5. Conclusions

In this article, we analysed American-style Asian options with averaged floating strike. We focused on arithmetic, geometric and weighted arithmetic averaging of the floating strike price. In the first part of this article, we derived an integral representation of the call and put option prices and we provided an integral equation for the free boundary position. We analysed the behaviour of the early exercise boundary close to expiry. We proposed a general methodology how to determine the early exercise position at expiry. We furthermore derived the asymptotic formula for the early exercise boundary close to the expiry. The second part of this article was devoted to the construction of a robust numerical scheme for finding an approximation of the early exercise boundary. Applying the front-fixing method, we derived a non-local parabolic partial differential for the synthesized portfolio and the free boundary position. Using an idea of the operator splitting technique, we moreover constructed a numerical scheme for numerical solution of the problem. The capability of the method has been documented by several computational examples.

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References

Chadam, J. (2008) Free boundary problems in mathematical finance, Progress in industrial mathematics at ECMI 2006, Vol. 12 of Math. Ind., pp. 655–665 (Berlin: Springer).

Dai, M. and Kwok, Y. K. (2006) Characterization of optimal stopping regions of American Asian and lookback options, *Mathematical Finance*, 16(1), pp. 63–82.

Detemple, J. (2006) American-Style Derivatives: Valuation and Computation (Boca Raton, FL: Chapman and Hall/CRC).

Dewynne, J. N., Howison, S. D., Rupf, I. and Wilmott, P. (1993) Some mathematical results in the pricing of American options, *European Journal of Applied Mathematics*, 4(4), pp. 381–398.

Geske, R. and Johnson, H. E. (1984) The American put option valued analytically, *Journal of Finance*, 39, pp. 1511–1524.

Geske, R. and Roll, R. (1984) On valuing American call options with the Black–Scholes European formula, *Journal of Finance*, 89, pp. 443–455.
Hansen, A. T. and Jørgensen, P. L. (2000) Analytical valuation of American-style Asian options, *Management Science*, 46(8), pp. 1116–1136.
Hull, J. C. (1997) Options, Futures and Other Derivative Securities, 3rd ed. (New Jersey: Prentice Hall).
Karatzas, I. (1988) On the pricing American options, *Applied Mathematics & Optimization*, 17, pp. 37–60.
Kim, B. C. and Oh, S. Y. (2004) Pricing of American-style fixed strike Asian options with continuous arithmetic average. Available at http://ssrn.com/abstract=489103 (accessed 08 February 2004).
Kuske, R. A. and Keller, J. B. (1998) Optimal exercise boundary for an American put option, *Applied Mathematical Finance*, 5, pp. 107–116.
Kwok, Y. K. (2008) Mathematical Models of Financial Derivatives, Springer Finance, 2nd ed. (Berlin: Springer).
Linetsky, V. (2004) Spectral expansions for Asian (average price) options, *Operations Research*, 52(6), pp. 856–867.
Mallier, R. (2002) Evaluating approximations for the American put option, *Journal of Applied Mathematics*, 2, pp. 71–92.
Pascucci, A. (2008) Free boundary and optimal stopping problems for American Asian options, *Finance Stoch*, 12, pp. 21–41.
Revuz, D. and Yor, M. (2005) Continuous Martingales and Brownian Motion, Vol. 293 of Grundlehren der Mathematischen Wissenschaften [A Series of Comprehensive Studies in Mathematics], corrected 3rd ed. (Berlin: Springer-Verlag).
Ševčovič, D. (2001) Analysis of the free boundary for the pricing of an American call option, *European Journal of Applied Mathematics*, 12(1), pp. 25–37.
Ševčovič, D. (2007) An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear Black-Scholes equation, *The Canadian Applied Mathematics Quarterly*, 15(1), pp. 77–97.
Ševčovič, D. (2009) Transformation methods for evaluating approximations to the optimal exercise boundary for a linear and nonlinear Black-Scholes equation. In M. Ehrhardt (Ed.), Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing, pp. 153–198 (New York: Nova Science).
Stamicar, R., Ševčovič, D. and Chadam, J. (1999) The early exercise boundary for the American put near expiry: numerical approximation, *The Canadian Applied Mathematics Quarterly*, 7(4), pp. 427–444.
Wilmott, P., Howison, S. and Dewynne, J. (1995) The Mathematics of Financial Derivatives: A student introduction (Cambridge, UK: Cambridge University Press).
Wu, L. and Kwok, Y. K. (1997) A front-fixing finite difference method for the valuation of American options, *Journal of Financial Engineering*, 6, pp. 83–97.
Wu, L., Kwok, Y. K. and Yu, H. (1999) Asian options with the American early exercise feature, *International Journal of Theoretical and Applied Finance*, 2(1), 101–111.
Wu, R. and Fu, M. C. (2003) Optimal exercise policies and simulation-based valuation for American-Asian options, *Operations Research*, 51(1), pp. 52–66.
Wystup, U. (2006) FX Options and Structured Products (Chichester, UK: John Wiley & Sons).

Appendix: Proofs

In the proof of Theorem 2.1, we shall use the next lemma.

Lemma A.1. The auxiliary variable $x_t = \frac{A_t}{S_t}$ satisfies the following stochastic differential equation:

$$dx_t = x_t \frac{dA_t}{A_t} - (r - q)x_t \, dt - \sigma x_t \, dW_t^Q.$$

Proof of Lemma A.1. We express the differential $dx_t = d\left(\frac{A_t}{S_t}\right)$ as

$$dx_t = \frac{1}{S_t} dA_t - \frac{A_t}{S_t^2} dS_t + \frac{A_t}{S_t^3}(dS_t)^2 = x_t \frac{dA_t}{A_t} - (r-q)x_t\, dt - \sigma x_t\, dW_t^{\mathcal{Q}},$$

and the proof of lemma follows. □

Proof of Theorem 2.1. We follow the proof of the original result by Hansen and Jørgensen (2000) and we include necessary modifications related to the form of averaging and the fact that $q \geq 0$.

First, we suppose that $(t, x) \in \mathcal{S}$. The value of the option is defined by

$$\widetilde{V}(t, x_t) = \varrho e^{-qt}(1 - x_t).$$

Hence the differential $d\widetilde{V} = -\rho q e^{-qt}(1-x)dt - \rho e^{-qt}\, dx$ has the form

$$d\widetilde{V} = -\rho e^{-qt} x \frac{dA}{A} + \rho e^{-qt}(rx - q)dt + \rho e^{-qt} \sigma x dW^{\mathcal{Q}}.$$

In the continuous region $\mathcal{C}$, $\widetilde{V}$ is $\mathcal{Q}$-martingale and thus for both regions we have the following equation:

$$d\widetilde{V}(t, x_t) = -\varrho e^{-qt} 1_{\mathcal{S}}(t, x_t) \left(x_t \frac{dA_t}{A_t} - (rx_t - q)dt \right) + dM_t^{\mathcal{Q}}, \tag{36}$$

where $M_t^{\mathcal{Q}}$ is a $\mathcal{Q}$-martingale. Integrating Equation (36) from t to T and taking expectation we have

$$\begin{aligned} \mathbb{E}_t^{\mathcal{Q}}\left[\widetilde{V}(T, x_T)\right] - \widetilde{V}(t, x_t) = & -\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \varrho e^{-qu} x_u 1_{\mathcal{S}}(u, x_u) \left(\frac{dA_u}{A_u} - \left(r - \frac{q}{x_u} \right) du \right)\right] \\ & + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T dM_u^{\mathcal{Q}}\right]}_{=0}, \end{aligned}$$

$$\begin{aligned} \widetilde{V}(t, x_t) = & \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[e^{-qT}\left(\rho\,(1 - x_T)\right)^+\right]}_{=\widetilde{v}(t, x_t)} \\ & + \underbrace{\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \rho e^{-qu} x_u 1_{\mathcal{S}}(u, x_u) \left(\frac{dA_u}{A_u} - \left(r - \frac{q}{x_u} \right) \right) du\right]}_{=\widetilde{e}(t, x_t)}, \end{aligned}$$

which completes the proof of Theorem 2.1. □

Proof of Theorem 3.1. We have

$$\frac{1}{T-t}\mathbb{E}_t\left[\int_t^T 1_{\mathcal{S}}(u,y_u)f_b(u,y_u)\,du\right] = \frac{1}{T-t}\left(V_{\rm am}(t,y_t) - V_{\rm eu}(t,y_t)\right) \geq 0,$$

for any $t \in [0,T)$. In the limit $t \to T$, we can omit the conditioned expected value operator $\mathbb{E}_t$ and we obtain $1_{\mathcal{S}}(T,y_T)f_b(T,y_T) \geq 0$. Because $(T,y_T^*) \in \partial S$ and the function f_b is continuous we obtain $f_b(T,y_T^*) \geq 0$.

Part 1. Suppose that $y_T^* \in ITM$. We shall prove $f_b(T,y_T^*) = 0$. Notice that in the stopping region we have the identity $V_{\rm am}(t,y) = V_{\rm am}(T,y)$ for any $(t,y) \in \mathcal{S}$ and, consequently, $\frac{\partial V_{\rm am}}{\partial t}(t,y) = 0$. Take any $(T,y_T) \in \mathcal{S}$. Similarly as in above, in the limit $t \to T$ we can omit the conditioned expected value operator $\mathbb{E}_t$ to obtain

$$0 = \frac{\partial V_{\rm am}}{\partial t}(T,y_T) = \frac{\partial V_{\rm eu}}{\partial t}(T,y_T) - f_b(T,y_T).$$

Because the function $y \mapsto \frac{\partial V_{\rm eu}}{\partial t}(T,y)$ is assumed to be continuous in ITM and $y_T^* \in ITM$, we have $\frac{\partial V_{\rm eu}}{\partial t}(T,y_T^*) = f_b(T,y_T^*)$.

On the contrary, take any $(T,y_T) \in \mathcal{C}$. Because $V_{\rm am}(t,y) \geq V_{\rm am}(T,y)$ for any $0 \leq t < T$ we obtain $\frac{\partial V_{\rm am}}{\partial t}(T,y_T) \leq 0$. Therefore, in the limit $t \to T$, we have

$$0 \geq \frac{\partial V_{\rm am}}{\partial t}(T,y_T) = \frac{\partial V_{\rm eu}}{\partial t}(T,y_T) - 1_{\mathcal{S}}(T,y_T)f_b(T,y_T) = \frac{\partial V_{\rm eu}}{\partial t}(T,y_T),$$

because $(T,y_T) \in \mathcal{C}$. Due to continuity of $\frac{\partial V_{eu}}{\partial t}(T,y)$ for $y \in ITM$, we finally obtain $0 \geq \frac{\partial V_{eu}}{\partial t}(T,y_T^*) = f_b(T,y_T^*)$. Hence $f_b(T,y_T^*) = 0$ and the proof of Part 1 follows.

Part 2. It suffices to prove that the case $y_T^* \in OTM$ cannot occur. Suppose to the contrary $y_T^* \in OTM$. Because the set OTM is open we can argue similarly as in Part 1 of the proof to obtain $f_b(T,y_T^*) = 0$. Because we have assumed uniqueness of the root of the equation $f_b(T,y) = 0$ then, in some neighbourhood of y_T^* there exists $\widehat{y}_T \in OTM$ such that $f_b(T,\widehat{y}_T) > 0$ and $(T,\widehat{y}_T) \in \mathcal{S}$. We have

$$0 = \frac{\partial V_{\rm am}}{\partial t}(T,\widehat{y}_T) = \frac{\partial V_{\rm eu}}{\partial t}(T,\widehat{y}_T) - f_b(T,\widehat{y}_T).$$

Thus, $\frac{\partial V_{\rm eu}}{\partial t}(T,\widehat{y}_T) > 0$. For any $y \in OTM$, we have $V_{\rm eu}(T,y) = 0$. Hence $V_{\rm eu}(T,\widehat{y}_T) = 0$ and $V_{\rm eu}(t,\widehat{y}_T) \geq 0$ for all $0 \leq t < T$, a contradiction. Thus, $y_T^* OTM$ and the proof of theorem follows.

Proof. Following the lines of the derivation of $\mathbb{E}_t^{\mathcal{Q}}[x_u^a]$ from Hansen and Jørgensen (2000) adopted for a general dividend yield $q \geq 0$, we obtain the first moment

$$\mathbb{E}_t^{\mathcal{Q}}\left[x_u^a\right] = x_t^a \frac{t}{u} e^{-(r-q)(u-t)} + \frac{1 - e^{-(r-q)(u-t)}}{(r-q)u}.$$

Although we follow the proof by Hansen and Jørgensen (2000), we have to make a slight correction in the derivation of the second conditioned moment. Using the definition of S_t, we have, for all $v \in [t, u]$,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_v}{S_u}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\left(r-q+\frac{\sigma^2}{2}\right)(v-u)+\sigma\left(W_v^{\mathbb{Q}}-W_u^{\mathbb{Q}}\right)}\right] = e^{\left(r-q+\frac{\sigma^2}{2}\right)(v-u)+\frac{\sigma^2}{2}|v-u|} = e^{(r-q)(v-u)}.$$

We need to simplify the expression for the second conditioned moment

$$\mathbb{E}_t^{\mathbb{Q}}\left[\left(x_u^a\right)^2\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\left(\frac{1}{u}\int_0^u \frac{S_v}{S_u} dv\right)^2\right] = \left(x_t^a\right)^2 \frac{t^2}{u^2}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_t}{S_u}\frac{S_t}{S_u}\right]$$
$$+ 2x_t\frac{t}{u^2}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_t}{S_u}\right]\int_t^u \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_v}{S_u}\right] dv + \frac{1}{u^2}\int_t^u\int_t^u \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_z}{S_u}\frac{S_v}{S_u}\right] dv\, dz.$$

Assuming that $u \geq z, v$ and let $m = \min\{z, v\}$ and $M = \max\{z, v\}$, we have

$$e^{-\left(r-q+\frac{\sigma^2}{2}\right)(z+v-2u)}\frac{S_z}{S_u}\frac{S_v}{S_u} = e^{\sigma\left(W_z^{\mathbb{Q}}+W_v^{\mathbb{Q}}-2W_u^{\mathbb{Q}}\right)} = e^{-\sigma\left(2\left(W_u^{\mathbb{Q}}-W_M^{\mathbb{Q}}\right)+\left(W_M^{\mathbb{Q}}-W_m^{\mathbb{Q}}\right)\right)},$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_z}{S_u}\frac{S_v}{S_u}\right] = e^{(r-q)(z+v-2u)+\sigma^2(u-M)}.$$

We have calculated all expressions we need to evaluate the second conditioned moment. If we put all together and perform necessary calculation, we obtain Equation (10) and the proof of lemma follows.