

On a Numerical Approximation Scheme for Construction of the Early Exercise Boundary for a Class of Nonlinear Black–Scholes Equations

Daniel Ševčovič

Abstract The purpose of this paper is to construct the early exercise boundary for a class of nonlinear Black–Scholes equations with a volatility function depending on the option price. We review and revisit a method how to transform the problem into a solution of a time depending nonlinear parabolic equation defined on a fixed domain. An example of numerical computation of the early exercise boundary for a nonlinear Black–Scholes equation is also presented.

1 Black–Scholes Equations with a Nonlinear Volatility Function

The main purpose of this paper is to review and revisit the fixed domain transformation method adopted for solving a class of nonlinear Black–Scholes equations having the form:

$$\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S^2 \partial_S^2 V, S, T - t)S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad S > 0, t \in (0, T). \quad (1)$$

A solution $V = V(S, t)$ can be identified with a price V of the option contract in the future maturity time $T > 0$ (e.g. call or put) where $S > 0$ is the underlying asset value at the present time $t \in [0, T)$. Here, $r > 0$ is the riskless interest rate, $q \geq 0$ is the dividend yield rate of the underlying asset. For American style of a call option, the free boundary problem consists in construction of the early exercise position $S_f = S_f(t)$ and the solution $V = V(S, t)$ to (1) defined on the time dependent

D. Ševčovič (✉)

Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics & Informatics, Comenius University, 842 48 Bratislava, Slovak Republic
e-mail: sevcovic@fmph.uniba.sk

domain $0 < S < S_f(t)$, $0 < t < T$ (cf. Kwok [13]). V is subjected to the boundary conditions yielding C^1 smooth pasting of $V(S, t)$ and $V(S, T)$ at $S = S_f(t)$:

$$V(0, t) = 0, \quad V(S_f(t), t) = S_f(t) - E, \quad \partial_S V(S_f(t), t) = 1, \quad (2)$$

and the terminal pay-off condition at expiry $t = T$,

$$V(S, T) = (S - E)^+, \quad (3)$$

where $E > 0$ is the exercise price.

We briefly mention a motivation for studying the nonlinear Black–Scholes equation having the form of (1). Such equations with a volatility function $\sigma(S^2 \partial_S^2 V, S, T - t)$ arise from modeling the option prices by taking into account nontrivial transaction costs (cf. Leland [14], Hoggard et al. [10], Avellaneda and Paras [3]), market feedbacks and effects due to large traders choosing given stock-trading strategies (Frey [7], Frey and Patie [8], Frey and Stremme [9], During et al. [6], Schönbucher and Wilmott [15]), the risk adjusted pricing methodology model due to Kratka [12] and its modification developed by Jandačka and Ševčovič [11, 17]). As an example for application of the numerical method, we consider a nonlinear model taking into account imperfect replication and investor's preferences which has been proposed by Barles and Soner in [4]. If investor's preferences are characterized by an exponential utility function they derived a nonlinear Black–Scholes equation with the volatility function σ given by

$$\sigma^2(S^2 \partial_S^2 V, S, \tau) = \hat{\sigma}^2 (1 + \Psi(a^2 e^{r\tau} S^2 \partial_S^2 V)). \quad (4)$$

Here $\hat{\sigma}^2 > 0$ is a constant historical volatility of the asset price returns, Ψ is the unique solution to the ODE: $\Psi'(x) = (\Psi(x) + 1)/(2\sqrt{x\Psi(x)} - x)$, $\Psi(0) = 0$ and $a \geq 0$ is a constant depending transaction costs and investor's risk aversion parameter (see [4] for details). The function Ψ satisfies: $\Psi(x) = O(x^{\frac{1}{3}})$ for $x \rightarrow 0$ and $\Psi(x) = O(x)$ for $x \rightarrow \infty$. For practical purposes, the solution $\Psi(x)$ can be constructed from an implicit equation obtained in [5].

We revisit an iterative numerical algorithm for solving the free boundary problem (1)–(2) developed by Ševčovič in [17]. The key idea of this method consists in transformation of the free boundary problem into a semilinear parabolic equation defined on a fixed spatial domain coupled with a nonlocal algebraic constraint equation for the free boundary position. This method has been analyzed and utilized in a series of papers [1, 2, 16–19] by Ehrhardt and Ankudinova and the author. The disadvantage of the original method consists in the necessity of solving an algebraic constraint equation. In this approach, highly accurate evaluation of the derivative of a solution at one point entering the algebraic constraint is needed (cf. [17]). In this note, we present a new efficient way how to overcome this difficulty by considering an equivalent integrated form of the algebraic constraint. We also present results

of numerical calculation of the free boundary position for the Barles and Soner nonlinear extension of the Black–Scholes model.

2 Fixed Domain Transformation of the Free Boundary Problem

We recall method how to transform the free boundary problem (1)–(3) into a form of a nonlinear parabolic equation defined on a fixed domain and satisfying a nonlocal algebraic constraint equation developed by the author in [17]. It is based on the following change of independent variables and the transformed function $\Pi = \Pi(x, \tau)$ defined as follows:

$$\tau = T - t, \quad x = \ln(\rho(\tau)/S), \quad \Pi(x, \tau) = V(S, t) - S \partial_S V(S, t), \quad (5)$$

where $\rho(\tau) = S_f(T - \tau)$. Clearly, $\tau \in (0, T)$ and $x \in (0, \infty)$ iff $S \in (0, S_f(t))$. The boundary value $x = 0$ corresponds to the free boundary position $S = S_f(t)$ whereas $x = +\infty$ corresponds to the default value $S = 0$ of the underlying asset. Under the structural assumption

$$0 < q \leq r$$

made on the interest and dividend yield rates and following derivation of the equation for Π , it turns out that the function Π and the free boundary position ρ satisfy the following system of parabolic equation (6) with algebraic constraint (7):

$$\frac{\partial \Pi}{\partial \tau} + \left(b(\tau) - \frac{\sigma^2}{2} \right) \frac{\partial \Pi}{\partial x} - \frac{1}{2} \frac{\partial}{\partial x} \left(\sigma^2 \frac{\partial \Pi}{\partial x} \right) + r \Pi = 0, \quad (6)$$

$$\Pi(0, \tau) = -E, \quad \Pi(+\infty, \tau) = 0, \quad x > 0, \tau \in (0, T),$$

$$\Pi(x, 0) = \begin{cases} -E, & \text{for } x < \ln(r/q), \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho(\tau) = \frac{rE}{q} + \frac{\sigma^2(\partial_x \Pi(0, \tau), \rho(\tau), \tau)}{2q} \frac{\partial \Pi}{\partial x}(0, \tau), \quad \text{with } \rho(0) = \frac{rE}{q}, \quad (7)$$

where $\sigma^2 = \sigma^2(\partial_x \Pi(x, \tau), \rho(\tau)e^{-x}, \tau)$, $b(\tau) = \frac{d}{d\tau} \frac{\rho(\tau)}{\rho(\tau)} + r - q$ (cf. [17]). Notice that (7) is not quite appropriate for construction of a robust numerical approximation scheme since any small inaccuracy in approximation of the value $\partial_x \Pi(0, \tau)$ is immediately transferred in to the entire computational domain $x \in (0, \infty)$ through the free boundary function $\rho(\tau)$ entering (6). Instead of (7), we present a new equivalent integrated equation for the free boundary position $\rho(\tau)$. Indeed, integrating the governing equation (6) for $x \in (0, \infty)$ taking into account the boundary conditions

$\Pi(0, \tau) = -E$, $\Pi(\infty, \tau) = 0$ (and consequently $\partial_x \Pi(\infty, \tau) = 0$), we obtain the following spatially integrated form of the algebraic constraint:

$$\begin{aligned} & \frac{d}{d\tau} \left(E \ln \rho(\tau) + \int_0^\infty \Pi(x, \tau) dx \right) + q\rho(\tau) - qE \\ & + \int_0^\infty \left(-\frac{1}{2} \sigma^2(\partial_x \Pi(x, \tau), \rho(\tau)e^{-x}, \tau) \frac{\partial \Pi}{\partial x}(x, \tau) + r\Pi(x, \tau) \right) dx = 0. \end{aligned} \quad (8)$$

3 Numerical Scheme Based on Operator Splitting Technique

The idea of the iterative numerical algorithm is based on the original numerical discretization scheme proposed by the author in [17]. We modify this method by considering the alternative integrated form (8) of the constraint between Π and ρ . The spatial domain $x \in (0, \infty)$ is restricted to a finite interval of values $x \in (0, L)$ where $L > 0$ is sufficiently large. For practical purposes one can take $L \approx 3$ (see [17]). Let us denote by $k > 0$ the time step, $k = T/m$ and by $h > 0$ the spatial step, $h = L/n$ where $m, n \in \mathbb{N}$ stand for the number of time and space discretization steps, respectively. We denote by Π_i^j an approximation of $\Pi(x_i, \tau_j)$, $\rho^j \approx \rho(\tau_j)$, $b^j \approx b(\tau_j)$ where $x_i = ih$, $\tau_j = jk$. We furthermore denote by Π^j the vector $\Pi^j = \{\Pi_i^j, i = 1, \dots, n\}$. We approximate the value of the volatility σ at the node (x_i, τ_j) by the finite difference approximation as follows:

$$\sigma_i^j = \sigma((\Pi_{i+1}^j - \Pi_i^j)/h, \rho^j e^{-x_i}, \tau_j).$$

We set $\Pi_i^0(x) = \Pi(x_i, 0)$. Next, following the idea of the operator splitting method discussed in [17], we decompose the above problem into two parts—a convection part and a diffusive part by introducing an auxiliary intermediate step $\Pi^{j-\frac{1}{2}}$. Our discretization of (8) and (6) reads as follows:

(Integrated form of the algebraic part)

$$E \ln \rho^j = E \ln \rho^{j-1} + I_0(\Pi^{j-1}) - I_0(\Pi^j) + k (qE - q\rho^j - I_1(\rho^j, \Pi^j)), \quad (9)$$

where $I_0(\Pi)$ stands for numerical trapezoid quadrature of the integral $\int_0^\infty \Pi(\xi) d\xi$ whereas $I_1(\rho^j, \Pi)$ is a trapezoid quadrature of the second integral in (8), i.e.

$$I_1(\rho^j, \Pi) \approx \int_0^\infty \left(-\frac{1}{2} \sigma^2(\partial_x \Pi(x), \rho^j e^{-x}, \tau_j) \frac{\partial \Pi}{\partial x}(x) + r\Pi(x) \right) dx.$$

(Convective part)

$$\frac{\Pi^{j-\frac{1}{2}} - \Pi^{j-1}}{k} + b^j \frac{\partial}{\partial x} \Pi^{j-\frac{1}{2}} = 0, \quad (10)$$

(Diffusive part)

$$\frac{\Pi^j - \Pi^{j-\frac{1}{2}}}{k} - \frac{(\sigma^j)^2}{2} \frac{\partial}{\partial x} \Pi^j - \frac{1}{2} \frac{\partial}{\partial x} \left((\sigma^j)^2 \frac{\partial}{\partial x} \Pi^j \right) + r \Pi^j = 0. \quad (11)$$

The convective part can be approximated by an explicit solution to the transport equation $\partial_\tau \tilde{\Pi} + b(\tau) \partial_x \tilde{\Pi} = 0$. Thus the spatial approximation $\Pi_i^{j-\frac{1}{2}}$ can be constructed from the formula

$$\Pi_i^{j-\frac{1}{2}} = \begin{cases} \Pi^{j-1}(\xi_i), & \text{if } \xi_i = x_i - \ln \rho^j + \ln \rho^{j-1} - (r - q)k > 0, \\ -E, & \text{otherwise,} \end{cases} \quad (12)$$

where a piecewise linear interpolation between discrete values $\Pi_i^{j-1}, i = 0, 1, \dots, n$, is being used to compute the value $\Pi^{j-1}(x_i - \ln \rho^j + \ln \rho^{j-1} - (r - q)k)$.

The diffusive part can be solved numerically by means of finite differences. Using a central finite difference approximation of the derivative $\partial_x \Pi^j$ we obtain

$$\begin{aligned} \frac{\Pi_i^j - \Pi_i^{j-\frac{1}{2}}}{k} + r \Pi_i^j - \frac{(\sigma_i^j)^2}{2} \frac{\Pi_{i+1}^j - \Pi_{i-1}^j}{2h} - \frac{1}{2h} \left((\sigma_i^j)^2 \frac{\Pi_{i+1}^j - \Pi_i^j}{h} \right. \\ \left. - (\sigma_{i-1}^j)^2 \frac{\Pi_i^j - \Pi_{i-1}^j}{h} \right) = 0. \end{aligned} \quad (13)$$

Now, (9), (12) and (13) can be rewritten in the operator form:

$$\rho^j = \mathcal{F}(\Pi^j, \rho^j), \quad \Pi^{j-\frac{1}{2}} = \mathcal{T}(\Pi^j, \rho^j), \quad \mathcal{A}(\Pi^j, \rho^j) \Pi^j = \Pi^{j-\frac{1}{2}},$$

where $\mathcal{F}(\Pi^j, \rho^j)$ is the right-hand side of the integrated algebraic equation (9). The operator $\mathcal{T}(\Pi^j, \rho^j)$ is the transport equation solver given by the right-hand side of (12) and $\mathcal{A} = \mathcal{A}(\Pi^j, \rho^j)$ is a tridiagonal matrix with coefficients given corresponding to (13). At each time level $\tau_j, j = 1, \dots, m$, the above system can be solved approximately by means of successive iterations procedure. Given a discrete solution Π^{j-1} , we start up iterations by defining $\Pi^{j,0} = \Pi^{j-1}, \rho^{j,0} = \rho^{j-1}$. Then the $(p + 1)$ -th approximation of Π^j and ρ^j is obtained as a solution to the system:

$$\begin{aligned} \rho^{j,p+1} = \mathcal{F}(\Pi^{j,p}, \rho^{j,p}), \quad \Pi^{j-\frac{1}{2},p+1} = \mathcal{T}(\Pi^{j,p}, \rho^{j,p+1}), \\ \mathcal{A}(\Pi^{j,p}, \rho^{j,p+1}) \Pi^{j,p+1} = \Pi^{j-\frac{1}{2},p+1}, \end{aligned} \quad (14)$$

We repeat the procedure for $p = 0, 1, \dots, p_{max}$, until the prescribed tolerance is achieved.

At the end of this section, we present a numerical example of approximation of the early exercise boundary for the Barles and Soner model by means of a

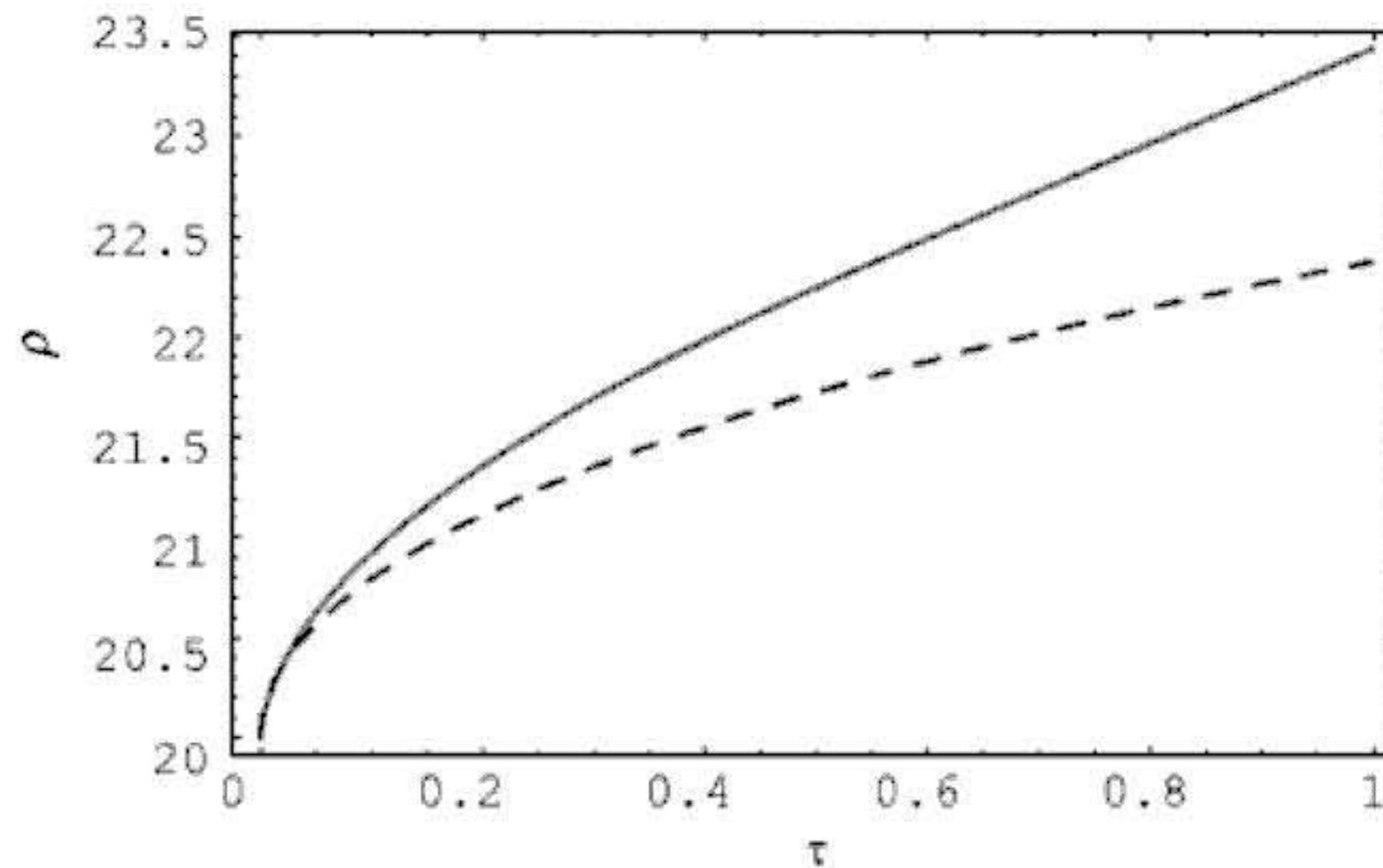


Fig. 1 A comparison of $\rho(\tau) = S_f(T - \tau)$ (solid line) for the Barles and Soner model with $a = 0.15$ and for the Black-Scholes equation, i.e. $a = 0$

solution to the transformed system of equations. In this model the volatility is given by expression (4). A discrete solution pair (Π, ρ) has been computed by our iterative algorithm for the model parameters: $E = 10$, $T = 1$ (one year), $r = 0.1$ (10% p.a.), $q = 0.05$ (5% p.a.) and $\hat{\sigma} = 0.2$. As for the numerical parameters, we chose $n = 750$ spatial points and $m = 225,000$ time discretization steps. The step $k = T/m$ represents 140 s in the real time scale. In order to achieve the precision 10^{-7} we used $p_{max} = 6$ micro-iterates in (14). A graphical plot of the early exercise boundary $\rho(\tau) = S_f(T - \tau)$ is shown in Fig. 1. Taking a positive value of the risk aversion coefficient $a = 0.15$ results in a substantial increase of the free boundary position $\rho(\tau)$ in comparison to the linear Black-Scholes equation with constant volatility $\sigma = \hat{\sigma}$. Notice that the Barles and Soner model for $a = 0$ coincides with the linear Black-Scholes model with constant volatility.

Acknowledgements The author was supported by VEGA 1/0747/12 grant

References

1. Ankudinova, J., Ehrhardt, M.: Fixed domain transformations and highly accurate compact schemes for nonlinear Black-Scholes equations for American options. In: *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, pp. 243–273. Nova Science Publishers, Inc., Hauppauge (2008)
2. Ankudinova, J., Ehrhardt, M.: On the numerical solution of nonlinear Black-Scholes equations. *Comput. Math. Appl.* **56**(3), 799–812 (2008)
3. Avellaneda, M., Parás, A.: Dynamic hedging portfolios for derivative securities in the presence of large transaction costs. *Appl. Math. Finance* **1**, 165–193 (1994)
4. Barles, G., Soner, H.M.: Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance Stoch.* **2**(4), 369–397 (1998)
5. Company, R., Navarro, E., Pintos, J., Ponsoda, E.: Numerical solution of linear and nonlinear Black-Scholes option pricing equations. *Comput. Math. Appl.* **56**(3), 813–821 (2008)

6. Düring, B., Fournié, M., Jüngel, A.: High order compact finite difference schemes for a nonlinear Black-Scholes equation. *Int. J. Theor. Appl. Finance* **6**(7), 767–789 (2003)
7. Frey, R.: Market illiquidity as a source of model risk in dynamic hedging in model risk. In: *RISK Publications*. R. Gibson Ed., London (2000)
8. Frey, R., Patie, P.: Risk management for derivatives in illiquid markets: a simulation study. In: *Advances in finance and stochastics*, pp. 137–159. Springer, Berlin (2002)
9. Frey, R., Stremme, A.: Market volatility and feedback effects from dynamic hedging. *Math. Finance* **7**(4), 351–374 (1997)
10. Hoggard, T., Whalley, A.E., Wilmott, P.: Hedging option portfolios in the presence of transaction costs. *Adv. Futures Options Res.* **7**, 21–35 (1994)
11. Jandačka, M., Ševčovič, D.: On the risk-adjusted pricing-methodology-based valuation of vanilla options and explanation of the volatility smile. *J. Appl. Math.* **3**, 235–258 (2005)
12. Kratka, M.: No mystery behind the smile. *Risk* **9**, 67–71 (1998)
13. Kwok, Y.K.: *Mathematical models of financial derivatives*. Springer Finance. Springer-Verlag Singapore, Singapore (1998)
14. Leland, H.E.: Option pricing and replication with transaction costs. *J. Finance* **40**, 1283–1301 (1985)
15. Schönbucher, P.J., Wilmott, P.: The feedback effect of hedging in illiquid markets. *SIAM J. Appl. Math.* **61**(1), 232–272 (electronic) (2000)
16. Ševčovič, D.: Analysis of the free boundary for the pricing of an American call option. *European J. Appl. Math.* **12**(1), 25–37 (2001)
17. Ševčovič, D.: An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear Black-Scholes equation. *Can. Appl. Math. Q.* **15**(1), 77–97 (2007)
18. Ševčovič, D.: Transformation methods for evaluating approximations to the optimal exercise boundary for linear and nonlinear Black-Scholes equations. In: *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, pp. 153–198. Nova Science Publishers, Inc., Hauppauge (2008)
19. Stamicar, R., Ševčovič, D., Chadam, J.: The early exercise boundary for the American put near: numerical approximation. *Canad. Appl. Math. Quart.* **7**(4), 427–444 (1999)