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A TRANSFORMATION METHOD FOR SOLVING THE HAMILTON–JACOBI–BELLMAN EQUATION FOR A CONSTRAINED DYNAMIC STOCHASTIC OPTIMAL ALLOCATION PROBLEM

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Abstract

We propose and analyse a method based on the Riccati transformation for solving the evolutionary Hamilton–Jacobi–Bellman equation arising from the dynamic stochastic optimal allocation problem. We show how the fully nonlinear Hamilton–Jacobi–Bellman equation can be transformed into a quasilinear parabolic equation whose diffusion function is obtained as the value function of a certain parametric convex optimization problem. Although the diffusion function need not be sufficiently smooth, we are able to prove existence and uniqueness and derive useful bounds of classical Hölder smooth solutions. Furthermore, we construct a fully implicit iterative numerical scheme based on finite volume approximation of the governing equation. A numerical solution is compared to a semi-explicit travelling wave solution by means of the convergence ratio of the method. We compute optimal strategies for a portfolio investment problem motivated by the German DAX 30 index as an example of the application of the method.

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1. Introduction

The purpose of this paper is to propose and analyse a method based on the Riccati transformation for solving a time dependent Hamilton–Jacobi–Bellman (HJB) equation arising from a dynamic stochastic optimal allocation problem on a finite time

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horizon, in which our aim is to maximize the expected value of the terminal utility subject to constraints on portfolio composition.

Investment problems with state constraints were considered and analysed by Zariphopoulou [41], where the purpose was to maximize the total expected discounted utility of consumption for the optimal portfolio investment consisting of a risky and a risk-free asset, over an infinite and finite time horizon. It was shown that the value function of the underlying stochastic control problem is the unique smooth solution to the corresponding HJB equation, and the optimal consumption and portfolio were presented in feedback form. Furthermore, it was shown that the value function is a constrained viscosity solution of the associated HJB equation. Classical methods for solving HJB equations are discussed by Benton [5]. Musiela and Zariphopoulou [30] applied a power-like transformation in order to linearize the nonlinear partial differential equation (PDE) for the value function in the case of an exponential utility function. In a seminal paper, Karatzas et al. [19] investigated a similar problem of consumption–investment optimization where the problem is to maximize total expected discounted utility of consumption over time horizon $[0, T]$. For a class of utility functions, they derived explicit solutions to the HJB equation. However, in our case the aim is to maximize the expected value of the terminal utility from the portfolio for a general utility function under constraints imposed on the control function and for the case of nontrivial defined contributions to the portfolio. As a consequence, we have to solve the dynamic HJB equation and, in general, explicit solutions to such nonlinear PDEs are no longer available.

Regarding numerical approaches for solving HJB equations associated with portfolio optimization, we can refer to finite difference methods for approximating the viscosity solution developed and analysed by Crandall et al. [9], Nayak and Papanicolaou [32] and Tourin and Zariphopoulou [39]. Muthamaran and Sunil [31] solved a multidimensional portfolio optimization problem with transaction costs. They used the finite element method and an iterative procedure that converts a free boundary problem into a sequence of fixed boundary problems. Peyrl et al. [33] applied a successive approximation algorithm for solving the corresponding HJB equation. The fixed point policy iteration scheme for solving discretized HJB equations is discussed by Huang et al. [13]. Witte and Reisinger [36] presented a penalty approach for the numerical solution of discrete continuously controlled HJB equations.

We follow a different approach. Rather than solving the fully nonlinear HJB equation directly, we first transform it into a quasilinear parabolic equation by means of the Riccati transformation. We prove existence and uniqueness of a solution to the transformed quasilinear parabolic equation. Moreover, we derive useful bounds on the solution. These bounds can be interpreted as estimates for the coefficient of risk aversion. Special attention is given to a solution of an auxiliary parametric quadratic programming problem. It is shown that the derivative of the value function of such a convex problem plays the role of a diffusion coefficient of the quasilinear equation. Although the diffusion function need not be sufficiently smooth, we are able to prove existence and uniqueness and derive useful bounds of classical Hölder smooth solutions.

The resulting equation can be solved numerically by an iterative method based on finite volume approximation. There is an analogy between a solution of fully nonlinear generalizations of the Black–Scholes equation for pricing derivative securities [37] and the fully nonlinear HJB equation investigated in this paper. Jandačka and Ševčovič [18] suggested a numerical method for solving a fully nonlinear generalization of the Black–Scholes equation by means of its transformation to the so-called gamma equation stated for the second derivative of the option price. In fact, the Riccati transformation is the logarithmic derivative of the derivative of the value function. Here we apply the Riccati transformation proposed and analysed in a series of papers by Ishimura et al. [1, 16]. In the context of a class of HJB equations with range constraints, such a transformation has been analysed recently by Ishimura and Ševčovič [17], who also constructed a travelling wave solution to the HJB equation. Concerning numerical methods for solving the transformed quasilinear parabolic PDE, there are recent papers by Ishimura, Koleva and Vulkov [14, 15, 21, 22] in which the authors consider a simplified problem without inequality constraints on the optimal control function.

The paper is organized as follows. We formulate the problem of interest and the motivation behind it in Section 2. Analysis of the Riccati transformation of the HJB equation into a quasilinear parabolic equation is given in Section 3. The transformed function can be interpreted in terms of the coefficient of relative risk aversion of an investor. In Section 4 we analyse a class of parametric quadratic optimization problems. The goal is to show that the value function is sufficiently smooth and increasing. Lipschitz continuity of the derivative of the value function is a crucial requirement for the proof of existence and uniqueness of a classical solution to the transformed quasilinear parabolic equation presented in Section 5. We also derive useful bounds of a solution to the Cauchy problem for the corresponding quasilinear parabolic equation. Using these bounds and smoothness properties of the value function of the auxiliary parametric quadratic optimization problem, we prove existence of a classical Hölder smooth solution. A special semi-explicit solution having the form of a travelling wave is analysed in Section 6. Such a special solution is then utilized as a benchmark solution in Section 7, where we construct an iterative fully implicit numerical approximation scheme for solving a quasilinear parabolic equation. Section 8 is devoted to application of the method to construction of an optimal response strategy for the German DAX 30 index.

2. Problem statement

Our motivation stems from a dynamic stochastic optimization problem in which the purpose is to maximize the conditional expected value of the terminal utility of a portfolio:

$$\max_{\boldsymbol{\theta}|_{[0,T)}} \mathbb{E}[U(X_T^{\boldsymbol{\theta}}) \mid X_0^{\boldsymbol{\theta}} = x_0], \tag{2.1}$$

where $\{X_t^{\boldsymbol{\theta}}\}$ is Itō's stochastic process on the finite time horizon $[0, T]$, $U : \mathbb{R} \to \mathbb{R}$ is a given terminal utility function and x_0 is a given initial state condition of $\{X_t^{\boldsymbol{\theta}}\}$

at $t = 0$. The function $\boldsymbol{\theta} : \mathbb{R} \times [0, T) \to \mathbb{R}^n$ mapping $(x, t) \mapsto \boldsymbol{\theta}(x, t)$ represents an unknown control function governing the underlying stochastic process $\{X_t^{\boldsymbol{\theta}}\}_{t\geq 0}$. Here $\boldsymbol{\theta}|_{[t,T)}$ for $0 \leq t < T$ denotes the restriction of the control function $\boldsymbol{\theta}$ to the time interval $[t, T)$. We assume that $X_t^{\boldsymbol{\theta}}$ is driven by the stochastic differential equation (SDE)

$$dX_t^{\boldsymbol{\theta}} = (\varepsilon e^{-X_t} + r + \mu(\boldsymbol{\theta}) - \tfrac{1}{2}\sigma(\boldsymbol{\theta})^2)\, dt + \sigma(\boldsymbol{\theta})\, dW_t, \tag{2.2}$$

where W_t denotes the standard Brownian motion and the functions $\mu(\boldsymbol{\theta})$ and $\sigma(\boldsymbol{\theta})$ are the drift and volatility functions depending on the control function $\boldsymbol{\theta}$. The parameter $\varepsilon \in \mathbb{R}$ represents a constant inflow rate of property to the system and $r \geq 0$ is the interest rate. Many European pension systems use $\varepsilon > 0$, representing a defined yearly contribution rate to the saver's pension account as a prescribed percentage of their salary. For example, $\varepsilon = 0.06$–0.09 in Slovakia, $\varepsilon = 0.14$ in Bulgaria and $\varepsilon = 0.02$–0.05 in Sweden [22, 26].

Throughout the paper we assume that the control parameter $\boldsymbol{\theta} \in \mathcal{S}^n$ belongs to the compact simplex

$$\mathcal{S}^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta} \geq \mathbf{0}, \mathbf{1}^T \boldsymbol{\theta} = 1\} \subset \mathbb{R}^n,$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$. It should be noted that the process $\{X_t^{\boldsymbol{\theta}}\}$ is a logarithmic transformation of a stochastic process $\{Y_t^{\tilde{\boldsymbol{\theta}}}\}_{t\geq 0}$ driven by the SDE

$$dY_t^{\tilde{\boldsymbol{\theta}}} = (\varepsilon + (r + \mu(\tilde{\boldsymbol{\theta}}))Y_t^{\tilde{\boldsymbol{\theta}}})\, dt + \sigma(\tilde{\boldsymbol{\theta}})Y_t^{\tilde{\boldsymbol{\theta}}}\, dW_t, \tag{2.3}$$

where $\tilde{\boldsymbol{\theta}}(y, t) = \boldsymbol{\theta}(x, t)$ with $x = \ln y$.

It is known from the theory of stochastic dynamic programming that the so-called value function

$$V(x, t) := \sup_{\boldsymbol{\theta}|_{[t,T)}} \mathbb{E}[U(X_T^{\boldsymbol{\theta}}) \mid X_t^{\boldsymbol{\theta}} = x],$$

subject to the terminal condition $V(x, T) := U(x)$, can be used for solving the stochastic dynamic optimization problem (2.1) [3, 6, 12]. If the process $X_t^{\boldsymbol{\theta}}$ is driven by (2.2) then the value function $V = V(x, t)$ satisfies the HJB equation

$$\partial_t V + \max_{\boldsymbol{\theta}\in\mathcal{S}^n}\{(\varepsilon e^{-x} + r + \mu(\boldsymbol{\theta}) - \tfrac{1}{2}\sigma(\boldsymbol{\theta})^2)\partial_x V + \tfrac{1}{2}\sigma(\boldsymbol{\theta})^2\partial_x^2 V\} = 0 \tag{2.4}$$

for all $x \in \mathbb{R}$, $t \in [0, T)$ subject to the terminal condition $V(x, T) := U(x)$ [17, 26].

As a typical example leading to the stochastic dynamic optimization problem (2.1) in which the underlying stochastic process satisfies the SDE (2.2), one can consider a problem of dynamic portfolio optimization in which the assets are labelled as $i = 1, \ldots, n$ and associated with price processes $\{Y_t^i\}_{t\geq 0}$, each of them following a geometric Brownian motion [7, 8, 27, 28, 38]:

$$\frac{dY_t^i}{Y_t^i} = \mu_i\, dt + \sum_{j=1}^n \bar{\sigma}_{ij}\, dW_t^j.$$

The value of a portfolio with weights $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}(y, t)$ is denoted by $Y_t^{\tilde{\boldsymbol{\theta}}}$. It can be shown that $\{Y_t^{\tilde{\boldsymbol{\theta}}}\}_{t\geq 0}$ satisfies (2.3). The assumption $\boldsymbol{\theta} \in \mathcal{S}^n$ corresponds to the situation in which borrowing of assets is not allowed ($\theta_i \geq 0$) and $\sum_{i=1}^n \theta_i = 1$. We have $\mu(\boldsymbol{\theta}) = \boldsymbol{\mu}^T \boldsymbol{\theta}$ and $\sigma(\boldsymbol{\theta})^2 = \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta}$ with $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ and $\boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}} \bar{\boldsymbol{\Sigma}}^T$ where $\bar{\boldsymbol{\Sigma}} = (\bar{\sigma}_{ij})$. The terminal function U represents the predetermined terminal utility function of the investor.

REMARK 2.1. In the case of zero inflow $\varepsilon = 0$, the assumption (2.3) made on the stochastic process $\{Y_t^{\tilde{\boldsymbol{\theta}}}\}_{t\geq 0}$ is related to the well-known Merton model for optimal consumption and portfolio selection [27, 28]. However, for Merton's model, one has to consider a larger set of constraints for the control function $\boldsymbol{\theta}$. Namely, the simplex $\mathcal{S}^n$ has to be replaced by a larger set $\mathcal{S}_o^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta} \geq \mathbf{0}, \mathbf{1}^T \boldsymbol{\theta} \leq 1\} \subset \mathbb{R}^n$. It is worth noting that all results concerning $C^{1,1}$ smoothness of the value function α (see Theorem 4.1) as well as those regarding existence and uniqueness of classical solutions (see Theorem 5.3) and the numerical discretization scheme remain true when $\mathcal{S}^n$ is replaced by $\mathcal{S}_o^n$.

3. The Riccati transformation of the HJB equation to a quasilinear equation

Following the methodology of the Riccati transformation first proposed by Abe and Ishimura [1] and later studied by Ishimura et al. [16], Xia [40] and Macová and Ševčovič [26] for problems without inequality constraints, and further analysed by Ishimura and Ševčovič [17], we introduce the following transformation:

$$\varphi(x, t) = 1 - \frac{\partial_x^2 V(x, t)}{\partial_x V(x, t)}. \tag{3.1}$$

REMARK 3.1. The function $a(x, t) \equiv \varphi(x, t) - 1$ can be viewed as the coefficient of absolute risk aversion for the value function $V(x, t)$, representing the intermediate utility function of an investor at a time $t \in [0, T]$ [34]. In the original variable y, denoting $\widetilde{V}(y, t) = V(\ln y, t)$, we can deduce that the function $\widetilde{a}(y, t) \equiv \varphi(\ln y, t)$ is the coefficient of relative risk aversion of the intermediate utility function $\widetilde{V}(y, t)$, which is defined as the ratio $\widetilde{a}(y, t) = -y\partial_y^2 \widetilde{V}(y, t)/\partial_y \widetilde{V}(y, t)$.

REMARK 3.2. It is worth noting that the pension saving model based on the SDE (2.2) takes into account neither transaction costs nor consumption. It follows from recent papers by Dai et al. [10, 11] that a model incorporating these effects leads to an HJB equation in two spatial dimensions. In such a case, transformation based on a simple one-dimensional Riccati transformation (3.1) is not possible.

Suppose for a moment that $\varphi(x, t) > 0$ for all $x \in \mathbb{R}$ and $t \in [0, T]$. This assumption is clearly satisfied for $t = T$ if we consider a function $U(x)$ that is an increasing and concave function in the x variable. We discuss this assumption further in Section 5. Now, problem (2.4) can be rewritten as follows:

$$0 = \partial_t V + (\varepsilon e^{-x} + r - \alpha(\varphi))\partial_x V, \quad V(x, T) := U(x), \tag{3.2}$$

where $\alpha(\varphi)$ is the value function of the parametric optimization problem

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\mathcal{S}^n}\Big\{-\mu(\boldsymbol{\theta}) + \frac{\varphi}{2}\sigma(\boldsymbol{\theta})^2\Big\}. \tag{3.3}$$

If the variance function $\boldsymbol{\theta} \mapsto \sigma(\boldsymbol{\theta})^2$ is strictly convex and $\boldsymbol{\theta} \mapsto \mu(\boldsymbol{\theta})$ is linear (as discussed in Section 2), problem (3.3) belongs to a class of parametric convex optimization problems [4].

THEOREM 3.3. *Suppose that the value function V satisfies (3.2) and the function φ is defined as in (3.1). Then φ is a solution to the Cauchy problem for the quasilinear parabolic equation*

$$\begin{aligned} &\partial_t\varphi + \partial_x^2\alpha(\varphi) + \partial_x[(\varepsilon e^{-x} + r)\varphi + (1-\varphi)\alpha(\varphi)] = 0, \qquad x\in\mathbb{R}, t\in[0,T),\\ &\varphi(x,T) = 1 - U''(x)/U'(x), \qquad x\in\mathbb{R}. \end{aligned} \tag{3.4}$$

PROOF. The statement is proved by differentiating (3.1) with respect to t and calculating derivatives $\partial_t V$, $\partial_x\partial_t V$, $\partial_x^2\partial_t V$ from (3.2). Indeed, as $\partial_x^2 V = (1-\varphi)\partial_x V$,

$$\partial_t\varphi = -\frac{\partial_x^2\partial_t V}{\partial_x V} + \frac{\partial_x^2 V\partial_x\partial_t V}{(\partial_x V)^2} = -\frac{\partial_x^2\partial_t V}{\partial_x V} + (1-\varphi)\frac{\partial_x\partial_t V}{\partial_x V}.$$

Let us denote

$$g(x,t) = \alpha(\varphi(x,t)) - \varepsilon e^{-x} - r. \tag{3.5}$$

Then $\partial_t V = g\partial_x V$ and therefore

$$\begin{aligned} \partial_x\partial_t V &= \partial_x g\partial_x V + g\partial_x^2 V = [\partial_x g + g(1-\varphi)]\partial_x V,\\ \partial_x^2\partial_t V &= [\partial_x^2 g + \partial_x(g(1-\varphi)) + (\partial_x g + g(1-\varphi))(1-\varphi)]\partial_x V. \end{aligned}$$

Hence

$$\partial_t\varphi = -\partial_x(\partial_x g + g(1-\varphi)), \tag{3.6}$$

and $\partial_t\varphi = -\partial_x[\partial_x\alpha(\varphi) + (\varepsilon e^{-x} + r)\varphi + \alpha(\varphi)(1-\varphi)]$, as claimed.

Finally, we notice that $\partial_x^2\alpha(\varphi) = \partial_x(\alpha'(\varphi)\partial_x\varphi)$. Moreover, if α is strictly increasing then (3.4) is indeed a quasilinear parabolic PDE with terminal condition at $t = T$ [24, Ch. 1, (2.4)]. □

Conversely, one can construct a solution $V(x,t)$ to the HJB equation (3.2) using a solution φ satisfying (3.4). Indeed, suppose that the function φ satisfies (3.4). We define a function $V = V(x,t)$ as the unique solution to the first-order linear PDE satisfying the terminal condition

$$\partial_t V - g\partial_x V = 0, \qquad V(x,T) = U(x), \qquad x\in\mathbb{R}, t\in[0,T), \tag{3.7}$$

where the function $g = g(x,t)$ is given by (3.5). Let us introduce $\psi = \psi(x,t)$ as follows:

$$\psi = 1 - \frac{\partial_x^2 V}{\partial_x V}.$$

Then following derivation of (3.6) we end up with an equation for the function ψ:

$$\partial_t \psi = -\partial_x(\partial_x g + g(1-\psi)).$$

Hence the difference $h \equiv \psi - \varphi$ satisfies a linear PDE: $\partial_t h = \partial_x g(h)$. Since $\varphi(x,T) \equiv \psi(x,T)$, we deduce that $\varphi(x,t) = \psi(x,t)$ for all $x \in \mathbb{R}$ and $t \in [0,T]$. But this means that V satisfies the fully nonlinear equation

$$\partial_t V - [\alpha(1 - \partial_x^2 V/\partial_x V) - \varepsilon e^{-x} - r]\partial_x V = 0, \quad V(x,T) = U(x). \tag{3.8}$$

In other words, $V = V(x,t)$ satisfies the HJB equation (3.2). Consequently, it is a solution to the HJB equation (2.4). Moreover, (3.8) is a fully nonlinear parabolic equation which is monototone in its principal part $\partial_x^2 V$. This way one deduces that the solution V to (3.8) is unique. In summary, we have shown that we can replace solving the HJB equation (2.4) by solving the auxiliary quasilinear equation (3.4).

Proposition 3.4. *Let $\varphi(x,t)$ be a solution to the Cauchy problem (3.4). Then the function $V(x,t)$ given by (3.7) is a solution to the HJB equation (2.4). Moreover, $\varphi = 1 - \partial_x^2 V/\partial_x V$.*

Remark 3.5. The advantage of transforming (2.4) to (3.2)–(3.3) is that we can define and compute the function $\alpha(\varphi)$ in advance as a result of the underlying parametric optimization problem (either analytically or numerically). This can then be put into the quasilinear equation (3.4) which can be solved for φ, instead of solving the original fully nonlinear HJB equation (3.2) as well as (2.4). In this way we do not calculate the value function V itself. On the other hand, it is only the optimal feedback strategy $\boldsymbol{\theta}$ that is of interest to the investor and therefore V is not important. The optimal strategy $\boldsymbol{\theta} = \boldsymbol{\theta}(x,t)$ can be computed as the unique optimal solution to the quadratic optimization problem (3.3) for the parameter values $\varphi = \varphi(x,t)$.

4. A parametric quadratic programming problem

In the case of a portfolio consisting of n assets, we denote by $\boldsymbol{\mu}$ the vector of expected asset returns and by $\boldsymbol{\Sigma}$ the covariance matrix of returns which we assume to be symmetric and positive definite. For the portfolio return and variance we have $\mu(\boldsymbol{\theta}) = \boldsymbol{\mu}^T\boldsymbol{\theta}$ and $\sigma(\boldsymbol{\theta})^2 = \boldsymbol{\theta}^T\boldsymbol{\Sigma}\,\boldsymbol{\theta}$. For $\varphi > 0$, (3.3) becomes a problem of parametric quadratic convex programming,

$$\alpha(\varphi) = \min_{\theta \in \mathcal{S}^n} \left\{ -\boldsymbol{\mu}^T\boldsymbol{\theta} + \frac{\varphi}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} \right\}, \tag{4.1}$$

over the compact convex simplex $\mathcal{S}^n$. In this section, we discuss qualitative properties of the value function $\alpha = \alpha(\varphi)$ for this case. By $C^{k,1}(\mathbb{R}^+)$ we denote the space of all functions defined on $(0,\infty)$ whose kth derivative is Lipschitz continuous. By $\alpha'(\varphi)$ we denote the derivative of $\alpha(\varphi)$ with respect to φ.

Theorem 4.1. *Let* $\boldsymbol{\Sigma} > 0$ *be positive definite and* $\boldsymbol{\mu} \in \mathbb{R}^n$. *Then the optimal value function* $\alpha(\varphi)$ *defined as in* (4.1) *is* $C^{1,1}$ *continuous. Moreover,* $\varphi \mapsto \alpha(\varphi)$ *is strictly increasing and*

$$\alpha'(\varphi) = \tfrac{1}{2}\hat{\boldsymbol{\theta}}^T \boldsymbol{\Sigma} \hat{\boldsymbol{\theta}}, \tag{4.2}$$

where $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\varphi) \in \mathcal{S}^n$ *is the unique minimizer of* (4.1) *for* $\varphi > 0$. *The function* $(0, \infty) \ni \varphi \mapsto \hat{\boldsymbol{\theta}}(\varphi) \in \mathbb{R}^n$ *is locally Lipschitz continuous.*

Proof. First, we notice that the mapping $(0, \infty) \ni \varphi \mapsto \hat{\boldsymbol{\theta}}(\varphi) \in \mathcal{S}^n$ is continuous, which can be deduced directly from basic properties of strictly convex functions minimized over the compact convex set $\mathcal{S}^n$.

Let us denote by $f(\boldsymbol{\theta}, \varphi) := -\boldsymbol{\mu}^T \boldsymbol{\theta} + \varphi \frac{1}{2} \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta}$ the objective function in problem (4.1). Since $|\partial_\varphi f(\boldsymbol{\theta}, \varphi)|$ is a continuous function on the compact set $\mathcal{S}^n$, we have $\sup_{\boldsymbol{\theta} \in \mathcal{S}^n} |\partial_\varphi f(\boldsymbol{\theta}, \varphi)| = C(\varphi) < \infty$. Strict convexity of f in $\boldsymbol{\theta}$ implies the existence of a unique minimizer $\hat{\boldsymbol{\theta}} \equiv \hat{\boldsymbol{\theta}}(\varphi)$ in (4.1). Moreover, $\partial_\varphi f(\hat{\boldsymbol{\theta}}(\varphi), \varphi) \equiv \frac{1}{2} \hat{\boldsymbol{\theta}}(\varphi)^T \boldsymbol{\Sigma} \hat{\boldsymbol{\theta}}(\varphi)$ is continuous in φ due to continuity of $\hat{\boldsymbol{\theta}}(\varphi)$. Applying the general envelope theorem due to Milgrom and Segal [29, Theorem 2], the function $\alpha(\varphi)$ is differentiable on the set $(0, \infty)$.

Next, we prove that $\alpha'(\varphi) > 0$. The function $f(\boldsymbol{\theta}, \varphi)$ is linear in φ for any $\boldsymbol{\theta} \in \mathcal{S}^n$. Therefore it is absolutely continuous in φ for any $\boldsymbol{\theta}$. Again, applying the aforementioned theorem [29, Theorem 2], we obtain

$$\alpha(\varphi) = \alpha(0) + \int_0^\varphi \partial_\varphi f(\hat{\boldsymbol{\theta}}(\xi), \xi)\, d\xi.$$

Therefore $\alpha'(\varphi) = \partial_\varphi f(\hat{\boldsymbol{\theta}}(\varphi), \varphi) = \frac{1}{2} \hat{\boldsymbol{\theta}}(\varphi)^T \boldsymbol{\Sigma} \hat{\boldsymbol{\theta}}(\varphi)$, which is strictly positive on $\mathcal{S}^n$. Hence $\varphi \mapsto \alpha(\varphi)$ is a C^1 continuous and increasing function for $\varphi > 0$.

Local Lipschitz continuity of $\alpha'(\varphi)$ now follows from the general result proved by Klatte [20] (see also the paper by Aubin [2]). Indeed, according to Klatte [20, Theorem 2] the minimizer function $\hat{\boldsymbol{\theta}}(\varphi)$ is locally Lipschitz continuous in φ. Hence the derivative $\alpha'(\varphi) = \frac{1}{2} \hat{\boldsymbol{\theta}}(\varphi)^T \boldsymbol{\Sigma} \hat{\boldsymbol{\theta}}(\varphi)$ is locally Lipschitz as well. □

Corollary 4.2. *Equation* (3.4) *is a strictly parabolic PDE, that is, there exist positive real numbers* $\lambda^-, \lambda^+ \in (0, \infty)$ *such that for the diffusion coefficient* $\alpha'(\varphi)$ *of equation* (3.4) *the following inequalities hold:*

$$0 < \lambda^- \le \alpha'(\varphi) \le \lambda^+ < \infty \quad \text{for all } \varphi > 0. \tag{4.3}$$

Proof. These inequalities follow directly from (4.2), which is a quadratic positive definite form on a compact set $\mathcal{S}^n$. With regard to (4.2), the function $\alpha'(\varphi)$ attains its maximum λ^+ and minimum λ^-. □

Example 4.1. An illustrative example of the value function α having discontinuous second derivative α'' based on real market data is depicted in Figure 1. In this example we consider the German DAX index consisting of 30 stocks. Based on historical data from August 2010 to April 2012, we have computed the covariance matrix $\boldsymbol{\Sigma}$

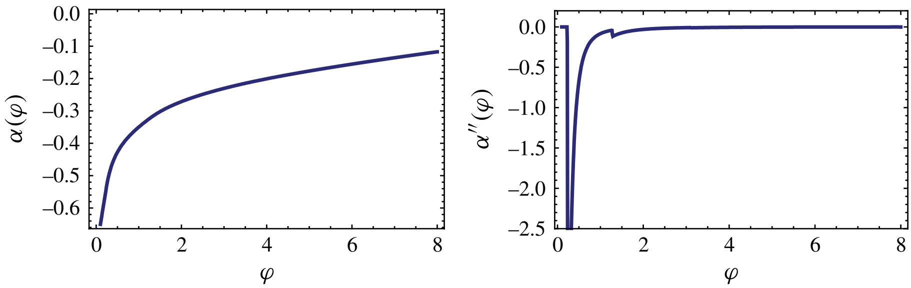

FIGURE 1. The value function α and its second derivative α'' for the portfolio of the German DAX 30 index, computed from historical data, August 2010–April 2012. Source: finance.yahoo.com.

and the vector of mean returns $\boldsymbol{\mu}$. One observes that there are at least two points of discontinuity of the second derivative $\alpha''(\varphi)$.

4.1. Higher smoothness of the value function In this section we discuss further smoothness properties of the value function $\alpha = \alpha(\varphi)$ in the variable φ, for the case specified at the beginning of Section 4. Furthermore, we show that α is locally a rational function that is concave on an open set.

Let us denote

$$\mathcal{I}_\emptyset = \{\varphi > 0 \,|\, \hat{\theta}_i(\varphi) > 0 \text{ for all } i = 1, \dots, n\}.$$

Then

$$(0, \infty) = \mathcal{I}_\emptyset \cup \bigcup_{|M| \le n-1} \mathcal{I}_M \quad \text{where } \mathcal{I}_M = \{\varphi > 0 \,|\, \hat{\theta}_i(\varphi) = 0 \text{ if and only if } i \in M\},$$

and M varies over all subsets of active indices, $M \subset \{1, \dots, n\}$. Here $|M|$ denotes the number of elements of the set M. Since $\varphi \mapsto \hat{\boldsymbol{\theta}}(\varphi)$ is continuous, the set $\mathcal{I}_\emptyset$ is open.

First, let us consider the case $\varphi \in \mathcal{I}_\emptyset$. If we introduce the Lagrange function $L(\boldsymbol{\theta}, \lambda) = (\varphi/2)\boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta} - \boldsymbol{\mu}^T \boldsymbol{\theta} - \lambda \mathbf{1}^T \boldsymbol{\theta}$ then the optimal solution $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\varphi)$ and the Lagrange multiplier $\lambda = \lambda(\varphi)$ are given by

$$\hat{\boldsymbol{\theta}} = \frac{1}{\varphi}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \lambda \boldsymbol{\Sigma}^{-1}\mathbf{1}), \qquad \lambda = \frac{\varphi - \mathbf{1}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Hence

$$\hat{\boldsymbol{\theta}}(\varphi) = \mathbf{a} - \frac{1}{\varphi}\mathbf{b} \quad \text{and} \quad \alpha(\varphi) = a\varphi - \frac{b}{\varphi} + c, \tag{4.4}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ can be expressed as follows:

$$\mathbf{a} = \frac{1}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}} \boldsymbol{\Sigma}^{-1}\mathbf{1}, \quad \mathbf{b} = -\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}} \boldsymbol{\Sigma}^{-1}\mathbf{1}. \tag{4.5}$$

TABLE 1. Sets of active indices for the German DAX 30 index. The assets are labelled: 1 – Adidas, 15 – Fresenius, 16 – Fres Medical, 21 – Linde, 23 – Merck, 27 – SAP, 30 – Volkswagen.

$\mathcal{I}_M$	M
(0, 0.23)	{23}
(0.23, 1.27)	{23, 30}
(1.27, 3.15)	{16, 23, 30}
(3.15, 6.62)	{16, 23, 27, 30}
(6.62, 7.96)	{16, 21, 23, 27, 30}
(7.96, 8.98)	{15, 16, 21, 23, 27, 30}
(8.98, . . .)	{1, 15, 16, 21, 23, 27, 30}

After straightforward calculations we conclude that

$$a = \frac{1}{2}\frac{1}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}} > 0, \quad b = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{2}\frac{(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}} \geq 0, \quad c = -\frac{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{4.6}$$

The inequality $b \geq 0$ follows from the Cauchy–Schwartz inequality. Notice that $b > 0$ unless the vectors $\boldsymbol{\mu}$ and $\mathbf{1}$ are linearly dependent.

Now if $\varphi \in \mathcal{I}_M$ for some subset $M \subset \{1, \dots, n\}$ of active indices, then the quadratic minimization problem (3.3) can be reduced to a lower-dimensional simplex $\mathcal{S}^{n-|M|}$. Hence the function $\alpha(\varphi)$ is smooth on $\mathrm{int}(\mathcal{I}_M)$ and therefore $\hat{\boldsymbol{\theta}}(\varphi)$ and $\alpha(\varphi)$ are given by

$$\hat{\boldsymbol{\theta}}(\varphi) = \mathbf{a}_M - \frac{1}{\varphi}\mathbf{b}_M, \quad \alpha(\varphi) = a_M\varphi - \frac{b_M}{\varphi} + c_M, \tag{4.7}$$

for any $\varphi \in \mathrm{int}(\mathcal{I}_M)$, where $\mathbf{a}_M, \mathbf{b}_M \in \mathbb{R}^n$ and $a_M > 0$, $b_M \geq 0$ and $c_M \in \mathbb{R}$ are constants calculated using the same formulas as in (4.5) and (4.6), where data (columns and rows) from $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ corresponding to the active indices in the particular set M are removed.

PROPOSITION 4.3. *The function $\varphi \mapsto \alpha(\varphi)$ defined in (3.3) is a C^∞ smooth function on the open set $\mathcal{J} = \mathcal{I}_\emptyset \cup \bigcup_{|M|\leq n-1} \mathrm{int}(\mathcal{I}_M) \subset (0, \infty)$. It is given by (4.4) for $\varphi \in \mathcal{I}_\emptyset$ and by (4.7) for $\varphi \in \mathrm{int}(\mathcal{I}_M)$, where $M \subset \{1, \dots, n\}$, respectively.*

4.2. Information gained from the second derivative of the value function There is useful information that can be extracted from the shape of $\alpha''(\varphi)$. For illustration, let us observe the points of discontinuity of $\alpha''(\varphi)$ depicted in Figure 1 for the example of the German DAX 30 index. The intervals between the points of discontinuity correspond to the sets $\mathcal{I}_M$. For the portfolio of the German DAX 30 index we obtain the sets of active indices corresponding to the continuity intervals as summarized in Table 1.

High values of φ represent high risk aversion of the investor. There is only one single asset present with a nonzero weight (equal to one) in the first interval.

This asset is the riskiest one and has highest expected return. Indeed, for lower values of φ, the investor's risk aversion is low and therefore they do not hesitate to undergo high risk for the sake of gaining high return.

Hence, if we were able to bound the parameter φ (see Section 5) by a constant $\varphi^+ < \infty$, that is, $\varphi(x,t) \leq \varphi^+$, it would be possible to identify the intervals of continuity of $\alpha''(\varphi)$ on the interval $(0,\varphi^+]$ and the corresponding sets of active indices. This would provide the investor with information about which assets enter the portfolio with zero weight. As is confirmed in Section 8, in the numerical example of the German DAX 30 index it is only the assets in Table 1, out of the overall number of 30, that enter the portfolio with a nonzero weight at some time in $[0,T]$; that is, the rest of the assets stay inactive for the whole time horizon considered.

4.3. Example: explicit form of the value function for the 2D problem The goal of this section is to present an explicit form of the value function α for the two-dimensional problem. Furthermore, we show that the result obtained in Theorem 4.1 is optimal in the sense that the function $\varphi \mapsto \alpha(\varphi)$ is only $C^{1,1}$ smooth but not C^2 smooth. Finally, we show that in the case $n = 2$ we are able to explicitly determine the sets $\mathcal{I}_\emptyset$ and $\mathcal{I}_M$.

A vector $\boldsymbol{\theta} \in \mathcal{S}^2$ can be written as $\boldsymbol{\theta} = (\theta, 1-\theta)^T$ where $\theta \in [0,1]$ is a real number. We denote by μ^s, μ^b the mean returns on riskier stocks and less risky bonds and by $\sigma^s, \sigma^b > 0$ their standard deviations. We assume that $\mu^s \geq \mu^b \geq 0$ and $\sigma^b - \varrho\sigma^s \geq 0$ where $\varrho \in [-1,1]$ is the correlation between returns on stocks and bonds. The mean return $\mu(\theta)$ and variance $\sigma(\theta)^2$ of the portfolio can be expressed as

$$\mu(\theta) = \theta\mu^s + (1-\theta)\mu^b, \qquad \sigma(\theta)^2 = \theta^2\gamma - 2\theta\delta + (\sigma^b)^2,$$

where $\gamma = (\sigma^s)^2 + (\sigma^b)^2 - 2\sigma^s\sigma^b\varrho$ and $\delta = (\sigma^b)^2 - \sigma^s\sigma^b\varrho$.

For a given $\varphi > 0$, the objective function in (3.3) is quadratic in θ with the coefficient of the quadratic term equal to $\frac{1}{2}\gamma\varphi$. If we relax the inequality constraints $0 \leq \theta \leq 1$ then it is a straightforward calculation to verify that the unconstrained minimizer $\hat{\theta}^{\mathrm{uc}}$ is given by $\hat{\theta}^{\mathrm{uc}}(\varphi) = \omega/\varphi + \delta/\gamma \geq 0$, where $\omega = (\mu^s - \mu^b)/\gamma \geq 0$. Consequently, the optimal solution $\hat{\theta} = \hat{\theta}(\varphi)$ for the constrained problem over $\theta \in [0,1]$ can be written in the form $\hat{\theta}(\varphi) = \min\{\omega/\varphi + \delta/\gamma, 1\}$. Therefore,

$$\alpha(\varphi) = \begin{cases} -\mu^b - \omega\delta - \dfrac{\omega^2\gamma}{2\varphi} + \dfrac{\varphi}{2}(1-\varrho^2)(\sigma^s\sigma^b)^2 & \text{if } \dfrac{1}{\varphi} < \dfrac{1}{\omega}\left(1 - \dfrac{\delta}{\gamma}\right) \\ \dfrac{(\sigma^s)^2}{2}\varphi - \mu^s & \text{if } \dfrac{1}{\varphi} \geq \dfrac{1}{\omega}\left(1 - \dfrac{\delta}{\gamma}\right). \end{cases}$$

In terms of the sets $\mathcal{I}_\emptyset$ and $\mathcal{I}_M$ we have $(0,\infty) = \mathcal{I}_\emptyset \cup \mathcal{I}_{\{1\}}$ where

$$\begin{cases} \mathcal{I}_\emptyset = (\omega\gamma/(\gamma-\delta), \infty), \mathcal{I}_{\{1\}} = (0, \omega\gamma/(\gamma-\delta)] & \text{if } \gamma > \delta \\ \mathcal{I}_\emptyset = \emptyset, \mathcal{I}_{\{1\}} = (0,\infty) & \text{if } \gamma \leq \delta. \end{cases}$$

By Proposition 4.3, the function $\varphi \mapsto \alpha(\varphi)$ is $C^{1,1}$ smooth for $\varphi > 0$ and it is C^∞ smooth on the set $\mathcal{J} = (0, \infty) \setminus \{\omega\gamma/(\gamma - \delta)\}$, if $\gamma > \delta$. Notice that $\gamma > \delta$ if and only if $\sigma^b - \varrho\sigma^s > 0$. The latter condition is automatically satisfied for nonpositive correlation $\varrho \le 0$ between returns on stocks and bonds.

5. Existence, uniqueness and boundedness of classical solutions

In this section, we investigate properties of classical smooth solutions to the Cauchy problem for the backward quasilinear parabolic equation (3.4) satisfying the terminal condition at $t = T$. In the first part, we introduce several function spaces that we work with. Then we provide useful upper and lower bounds on bounded smooth solutions. Finally, following the methodology based on the so-called Schauder type estimates [24], we prove existence and uniqueness of classical solutions to (3.4).

Let $\Omega = (x_L, x_R) \subset \mathbb{R}$ be a bounded interval. We denote by $Q_T = \Omega \times (0, T)$ the space–time cylinder. Let $0 < \lambda < 1$. By $H^\lambda(\Omega)$ we denote the Banach space consisting of all continuous functions φ on $\bar{\Omega}$ which are λ-Hölder continuous, that is, the Hölder seminorm $\langle\varphi\rangle^{(\lambda)} = \sup_{x,y\in\Omega, x\neq y} |\varphi(x) - \varphi(y)|/|x - y|^\lambda$ is finite. The norm in the space $H^\lambda(\Omega)$ is then the sum of the maximum norm of φ and the seminorm $\langle\varphi\rangle^{(\lambda)}$. The space $H^{2+\lambda}(\Omega)$ consists of all twice continuously differentiable functions φ in $\bar{\Omega}$ whose second derivative $\partial_x^2\varphi$ belongs to $H^\lambda(\Omega)$. The space $H^{2+\lambda}(\mathbb{R})$ consists of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \in H^{2+\lambda}(\Omega)$ for any bounded $\Omega \subset \mathbb{R}$.

Next, we define the parabolic Hölder space $H^{\lambda,\lambda/2}(Q_T)$ of functions defined on a bounded cylinder Q_T. It consists of all continuous functions $\varphi(x, t)$ in $\bar{Q}_T$ such that φ is λ-Hölder continuous in the x variable and $\lambda/2$-Hölder continuous in the t variable. The norm is defined as the sum of the maximum norm and corresponding Hölder seminorms. The space $H^{2+\lambda,1+\lambda/2}(Q_T)$ consists of all continuous functions on $\bar{Q}_T$ such that $\partial_t\varphi, \partial_x^2\varphi \in H^{\lambda,\lambda/2}(Q_T)$. Finally, the space $H^{2+\lambda,1+\lambda/2}(\mathbb{R} \times [0, T])$ consists of all functions $\varphi : \mathbb{R} \times [0, T] \to \mathbb{R}$ such that $\varphi \in H^{2+\lambda,1+\lambda/2}(Q_T)$ for any bounded cylinder Q_T. We also work with the Lebesgue and Sobolev spaces. By $L_p(Q_T)$, $1 \le p \le \infty$, we denote the Lebesgue space of all p-integrable functions (essentially bounded functions for $p = \infty$) defined on Q_T, equipped with the norm $\|\varphi\|_{L_p} = (\int_{Q_T} |\varphi|^p)^{1/p}$, $\|\varphi\|_{L_\infty} = \sup_{Q_T} |\varphi|$. The Sobolev space $W_2^1(Q_T)$ consists of all functions $\varphi \in L_2(Q_T)$ such that distributional derivatives $\partial_x\varphi, \partial_t\varphi \in L_2(Q_T)$. The norm is defined as $\|\varphi\|_{W_2^1} = \|\varphi\|_{L_2} + \|\partial_t\varphi\|_{L_2} + \|\partial_x\varphi\|_{L_2}$. Finally, the parabolic Sobolev space $W_2^{2,1}(Q_T)$ consists of all functions $\varphi \in L_2(Q_T)$ such that $\partial_x\varphi, \partial_x^2\varphi, \partial_t\varphi \in L_2(Q_T)$, $\|\varphi\|_{W_2^{2,1}} = \|\varphi\|_{L_2} + \|\partial_t\varphi\|_{L_2} + \|\partial_x\varphi\|_{L_2} + \|\partial_x^2\varphi\|_{L_2}$ [24, Ch. I].

We first derive lower and upper bounds of a solution φ to the Cauchy problem (3.4). The idea of proving upper and lower estimates for $\varphi(x, t)$ is based on construction of suitable sub- and super-solutions to the parabolic equation (3.4) [24, 35].

Remark 5.1. Recall that the value $\varphi(x, t) - 1$ can be interpreted as the coefficient of absolute risk aversion for the intermediate utility (value) function $V(x, t)$. Therefore, upper and lower bounds for the solution $\varphi(x,t)$ can also be used in estimation of the absolute risk aversion from above and below.

PROPOSITION 5.2. *Suppose that the terminal condition $\varphi(x,T)$ is positive and uniformly bounded from above, that is, there exists a constant φ^+ such that $0<\varphi(x,T)\le\varphi^+$ for any $x\in\mathbb{R}$. Assume that $\alpha=\alpha(\varphi)$ is a smooth function satisfying (4.3). If $\varphi\in H^{2+\lambda,1+\lambda/2}(\mathbb{R}\times[0,T])\cap L_\infty(\mathbb{R}\times(0,T))$, for some $0<\lambda<1$, is a bounded solution to the Cauchy problem for the quasilinear parabolic equation (3.4) then it satisfies the following inequalities:*

$$0<\varphi(x,t)\le\varphi^+\quad \textit{for any } t\in[0,T) \textit{ and } x\in\mathbb{R}.$$

PROOF. Equation (3.4) can be rewritten as a fully nonlinear parabolic equation of the form

$$\partial_\tau\varphi=\mathcal{H}(x,t,\varphi,\partial_x\varphi,\partial_x^2\varphi), \tag{5.1}$$

where $\tau=T-t\in(0,T)$ and $\mathcal{H}\equiv\partial_x^2\alpha(\varphi)+\partial_x[\alpha(\varphi)+(\varepsilon e^{-x}+r)\varphi-\alpha(\varphi)\varphi]$. Notice that the right-hand side of (5.1) is a strictly parabolic operator such that

$$0<\lambda^-\le\partial_q\mathcal{H}(x,t,\varphi,p,q)\equiv\alpha'(\varphi)\le\lambda_+<\infty,$$

for all $\varphi>0$. Let us define constant sub- and super-solutions $\underline{\varphi}$ and $\overline{\varphi}$ as follows:

$$\underline{\varphi}(x,t)\equiv0,\qquad \overline{\varphi}(x,t)\equiv\varphi^+\quad \text{for all } x\in\mathbb{R},\ t\in(0,T).$$

We have $\mathcal{H}(x,t,\underline{\varphi},\partial_x\underline{\varphi},\partial_x^2\underline{\varphi})\equiv0$ and $\mathcal{H}(x,t,\overline{\varphi},\partial_x\overline{\varphi},\partial_x^2\overline{\varphi})=-(\varepsilon e^{-x}+r)\varphi^+<0$. Therefore $\underline{\varphi},\overline{\varphi}$ are indeed sub- and super-solutions to the strictly parabolic nonlinear equation (5.1), that is,

$$\partial_\tau\underline{\varphi}\le\mathcal{H}(t,x,\underline{\varphi},\partial_x\underline{\varphi},\partial_x^2\underline{\varphi}),\qquad \partial_\tau\overline{\varphi}\ge\mathcal{H}(t,x,\overline{\varphi},\partial_x\overline{\varphi},\partial_x^2\overline{\varphi}),$$

satisfying the inequality $\underline{\varphi}(x,T)<\varphi(x,T)\le\overline{\varphi}(x,T)$ for any $x\in\mathbb{R}$. The inequality $0<\varphi(x,t)\le\varphi^+$, $x\in\mathbb{R}$, $t\in(0,T)$, is therefore a consequence of the parabolic comparison principle for strongly parabolic equations ([24, Ch. V, (8.2)], [35]). □

THEOREM 5.3. *Suppose that $\mathbf{\Sigma}$ is positive definite, $\boldsymbol{\mu}\in\mathbb{R}^n$, $\varepsilon,r\ge0$, and the optimal value function $\alpha(\varphi)$ is given by (4.1). Assume that the terminal condition $\varphi(x,T)=1-U''(x)/U'(x)$, $x\in\mathbb{R}$, is positive and uniformly bounded for $x\in\mathbb{R}$ and belongs to the Hölder space $H^{2+\lambda}(\mathbb{R})$ for some $0<\lambda<1/2$. Then there exists a unique classical solution $\varphi(x,t)$ to the backward quasilinear parabolic equation (3.4) satisfying the terminal condition $\varphi(x,T)$. The function $t\mapsto\partial_t\varphi(x,t)$ is $\lambda/2$-Hölder continuous for all $x\in\mathbb{R}$, whereas $x\mapsto\partial_x\varphi(x,t)$ is Lipschitz continuous for all $t\in[0,T]$. Moreover, $\alpha(\varphi(\cdot,\cdot))\in H^{2+\lambda,1+\lambda/2}(\mathbb{R}\times[0,T])$ and $0<\varphi(x,t)\le\sup_{x\in\mathbb{R}}\varphi(x,T)$ for all $(x,t)\in\mathbb{R}\times[0,T)$.*

PROOF. A key role in application of the so-called Schauder theory on existence and uniqueness of classical Hölder smooth solutions to a quasilinear parabolic equation is played by smoothness of the coefficients. Namely, this theory requires that the diffusion coefficient of a quasilinear parabolic equation is sufficiently smooth. Since $\partial_x^2\alpha(\varphi) = \partial_x(\alpha'(\varphi)\partial_x\varphi)$ and the diffusion coefficient $\alpha'(\varphi)$ is only Lipschitz continuous in φ, the backward quasilinear parabolic equation (3.4) should be regularized first. To this end, we construct a δ-parameterized family of smooth mollifier functions $\alpha_{(\delta)}(\varphi)$ such that

$$\alpha_{(\delta)}(\varphi) \rightrightarrows \alpha(\varphi) \quad \text{and} \quad \alpha'_{(\delta)}(\varphi) \rightrightarrows \alpha'(\varphi) \quad \text{as } \delta \to 0, \tag{5.2}$$

locally uniformly for $\varphi \in (0,\infty)$. Moreover, regularization can be constructed in such a way that $0 < \lambda^-/2 \le \alpha'_{(\delta)}(\varphi) \le 2\lambda^+ < \infty$ for all $\varphi > 0$, and all sufficiently small $0 < \delta \ll 1$.

Now, for any $\delta > 0$, by applying Theorem 8.1 and Remark 8.2 from the book by Ladyženskaja et al. [24, Ch. V, pp. 495–496] we deduce the existence of a unique classical bounded solution $\varphi^\delta \in H^{2+\lambda,1+\lambda/2}(\mathbb{R}\times[0,T]) \cap L_\infty(\mathbb{R}\times(0,T))$ to the Cauchy problem

$$\partial_t\varphi^\delta + \partial_x(\alpha'_{(\delta)}(\varphi^\delta)\partial_x\varphi^\delta) + \partial_x f(x,\varphi^\delta,\alpha_{(\delta)}(\varphi^\delta)) = 0, \quad \varphi^\delta(x,T) = \varphi(x,T), \tag{5.3}$$

$x \in \mathbb{R}$, $t \in [0,T)$, where $f(x,\varphi,\alpha(\varphi)) := (\varepsilon e^{-x} + r)\varphi + (1-\varphi)\alpha(\varphi)$.

Let $Q_T = (x_L, x_R) \times (0,T)$ be a bounded cylinder in $\mathbb{R}\times(0,T)$. By virtue of Proposition 5.2, φ^δ is bounded in the norm of the space $L_\infty(Q_T)$. More precisely,

$$\|\varphi^\delta\|_{L_\infty(Q_T)} \le \|\varphi(\cdot,T)\|_{L_\infty(\mathbb{R})},$$

for any $0 < \delta \ll 1$ (see also inequality (2.31) of Ladyženskaja et al. [24, Ch. I]). According to another inequality of Ladyženskaja et al. [24, Ch. I, (6.6)], φ^δ is also uniformly bounded in the space $W_2^1(Q_T)$, that is, there exists a constant $c_0 > 0$ such that

$$\varphi^\delta > 0, \quad \|\varphi^\delta\|_{W_2^1(Q_T)} \le c_0,$$

for any $0 < \delta \ll 1$. This means that there exists a subsequence $\varphi^{\delta_k} \rightharpoonup \varphi$ weakly converging to some element $\varphi \in W_2^1(Q_T)$ as $\delta_k \to 0$. Moreover, $\varphi^{\delta_k}(x,t) \to \varphi(x,t)$ for almost every (x,t). Notice that $\varphi^{\delta_k} \to \varphi$ strongly in $L_2(Q_T)$ because of the Rellich–Kondrashov compactness theorem on the embedding $W_2^1(Q_T) \hookrightarrow L_2(Q_T)$ [24, Ch. II, Theorem 2.1].

Hence $\alpha_{(\delta_k)}(\varphi^{\delta_k}) \to \alpha(\varphi)$ and $\alpha'_{(\delta_k)}(\varphi^{\delta_k}) \to \alpha'(\varphi)$ strongly in $L_2(Q_T)$. This is a consequence of the inequalities

$$\begin{aligned}
|\alpha_{(\delta)}(\varphi^\delta) - \alpha(\varphi)| &\le |\alpha_{(\delta)}(\varphi^\delta) - \alpha(\varphi^\delta)| + |\alpha(\varphi^\delta) - \alpha(\varphi)| \\
&\le |\alpha_{(\delta)}(\varphi^\delta) - \alpha(\varphi^\delta)| + \lambda^+|\varphi^\delta - \varphi|, \\
|\alpha'_{(\delta)}(\varphi^\delta) - \alpha'(\varphi)| &\le |\alpha'_{(\delta)}(\varphi^\delta) - \alpha'(\varphi^\delta)| + |\alpha'(\varphi^\delta) - \alpha'(\varphi)| \\
&\le |\alpha'_{(\delta)}(\varphi^\delta) - \alpha'(\varphi^\delta)| + L|\varphi^\delta - \varphi|,
\end{aligned}$$

where $L > 0$ is the Lipschitz constant of the function $\varphi \mapsto \alpha'(\varphi)$ (see Theorem 4.1 and (5.2)).

Multiplying (5.3) by a function $\eta \in W_2^1(Q_T)$ vanishing on the boundary ∂Q_T and integrating it over the domain Q_T yields the integral identity

$$\int_{Q_T} \partial_t \varphi^\delta \eta \, dx \, dt - \int_{Q_T} (\alpha'_{(\delta)}(\varphi)\partial_x \varphi^\delta + f(x, \varphi^\delta, \alpha_{(\delta)}(\varphi^\delta)))\partial_x \eta \, dx \, dt = 0.$$

Passing to the limit $\delta_k \to 0$ we conclude that $\varphi \in W_2^1(Q_T)$ is a weak solution to the backward quasilinear parabolic equation (3.4) satisfying the integral identity

$$\int_{Q_T} \partial_t \varphi \eta \, dx \, dt - \int_{Q_T} (\alpha'(\varphi)\partial_x \varphi + f(x, \varphi, \alpha(\varphi)))\partial_x \eta \, dx \, dt = 0$$

for any $\eta \in W_2^1(Q_T)$ vanishing on the boundary ∂Q_T. Since

$$\partial_t \varphi + \partial_x^2 \alpha(\varphi) + \partial_x f = 0 \tag{5.4}$$

and $\varphi, f \in W_2^1(Q_T)$, we have $\partial_x^2 \alpha(\varphi) \in L_2(Q_T)$. Furthermore, $\partial_t \alpha(\varphi) \in L_2(Q_T)$ because $\varphi \mapsto \alpha'(\varphi)$ is Lipschitz continuous (see Theorem 4.1), $\alpha'(\varphi) > \lambda^-$ and $\partial_t \varphi \in L_2(Q_T)$. Hence $\alpha(\varphi) \in W_2^{2,1}(Q_T)$.

Recall that the parabolic Sobolev space $W_2^{2,1}(Q_T)$ is continuously embedded into the Hölder space $H^{\lambda,\lambda/2}(Q_T)$ for any $0 < \lambda < 1/2$ [24, Lemma 3.3, Ch. II]. It follows from equation (5.4) that the transformed function $z(x, t) := \alpha(\varphi(x, t))$ is a solution to the quasilinear parabolic equation in the nondivergent form

$$\partial_t z + \zeta(z)[\partial_x^2 z + \partial_x f(x, \beta(z), z)] = 0, \quad z(x, T) = \alpha(\varphi(x, T)),$$

where $\zeta(z) = \alpha'(\beta(z))$ and $z \mapsto \beta(z)$ is the inverse function of the increasing function $\varphi \mapsto \alpha(\varphi)$, that is, $\alpha(\beta(z)) = z$ for any z. Since $z \mapsto \beta(z), \beta'(z)$ are Lipschitz continuous, $z \mapsto \zeta(z)$ is Lipschitz continuous as well. Next we make use of a simple bootstrap argument to show that $z = z(x, t)$ is sufficiently smooth. It is a solution to the linear parabolic equation in nondivergent form,

$$\partial_t z + a(x, t)\partial_x^2 z + b(x, t)\partial_x z = F(x, t), \quad z(x, T) = \alpha(\varphi(x, T)),$$

where $a(x, t) := \zeta(z)$, $b(x, t) = \zeta(z)((\varepsilon e^{-x} + r)\beta'(z) + 1 - \beta(z) - z\beta'(z))$ and $F(x, t) = (\varepsilon e^{-x} + r)\beta(z)$ with $z = z(x, t)$. All the coefficients a, b, F belong to the Hölder space $H^{\lambda,\lambda/2}(Q_T)$ because $z \in H^{\lambda,\lambda/2}(Q_T)$. With regard to a theorem given by Ladyženskaja et al. [24, Theorem 12.2, Ch. III] we have $z \in H^{2+\lambda,1+\lambda/2}(Q_T)$ and the result follows. □

Remark 5.4. Let us consider a utility function $U(x) = -(1/(a - 1)) \exp(-(a - 1)x)$ which represents an investor with constant coefficient $a > 1$ of absolute risk aversion. Then for the terminal condition $\varphi(x, T)$ we have that $\varphi(x, T) \equiv a$ is a constant function fulfilling all assumptions of Theorem 5.3 made on the terminal function $\varphi(\cdot, T)$.

Remark 5.5. It follows from the proof of Theorem 5.3 that its statement on the existence of a Hölder smooth solution φ to (3.4) remains true when the value function $\alpha(\varphi)$ is a general $C^{1,1}$ smooth function satisfying the estimates (4.3). This allows for consideration of a broader class of value functions defined as in (3.3) (see also Remark 2.1).

Combining Theorems 4.1 and 5.3, we obtain the following corollary.

Corollary 5.6. *Under the assumptions of Theorem 5.3, there exists a unique continuous optimal response function $\boldsymbol{\theta} = \boldsymbol{\theta}(x, t)$ to the HJB equation (2.4). It is given by $\boldsymbol{\theta}(x, t) = \hat{\boldsymbol{\theta}}(\varphi(x, t))$, where $\hat{\boldsymbol{\theta}}(\varphi)$ is the optimal solution to (4.1) for $\varphi = \varphi(x, t)$. The function $\mathbb{R} \ni x \mapsto \boldsymbol{\theta}(x, t) \in \mathbb{R}^n$ is Lipschitz continuous for all $t \in [0, T]$.*

6. A travelling wave solution

The aim of this section is to construct a semi-explicit travelling wave solution to the quasilinear equation (3.4). We utilize such a special solution for purposes of testing the numerical accuracy and estimating the convergence rate of the numerical scheme proposed in Section 7. In order to construct a travelling wave solution we assume that $\varepsilon = 0$, $r = 0$ and $\boldsymbol{\Sigma}$ is positive definite. In this case,

$$\partial_t \varphi + \partial_x^2 \alpha(\varphi) + \partial_x[\alpha(\varphi) - \alpha(\varphi)\varphi] = 0, \quad x \in \mathbb{R}, t \in [0, T). \tag{6.1}$$

In Theorem 4.1 we showed that the function $\alpha(\varphi)$ is a strictly increasing and locally $C^{1,1}$ smooth function in φ. Following the analysis and ideas due to Ishimura and Ševčovič [17] we construct a travelling wave solution to (6.1) of the form

$$\varphi(x, t) = v(x + c(T - t)), \quad x \in \mathbb{R}, t \in [0, T],$$

with wave speed $c \in \mathbb{R}$ and wave profile $v = v(\xi)$. Notice that the terminal condition $\varphi(x, T)$ to (6.1) is just the travelling wave profile $v(x)$.

Remark 6.1. In terms of the coefficient of absolute risk aversion $a(x) = -U''(x)/U'(x)$, we have $a(x) = v(x) - 1$. Hence, a decreasing travelling wave profile corresponds to a utility function with decreasing coefficient of absolute risk aversion $a(x)$. It might therefore be associated with an investor having higher risk preferences with increasing volume of the portfolio value x.

Putting $\varphi(x, t) = v(x + c(T - t))$ into (6.1), we deduce the existence of a constant $K_0 \in \mathbb{R}$ such that

$$\frac{d}{d\xi}\alpha(v(\xi)) = G(v(\xi)) \quad \text{where } G(v) = K_0 + cv - \alpha(v)(1 - v),$$

for any $\xi \in \mathbb{R}$. Let us define a new auxiliary variable $z = \alpha(v)$. Then the function $z = z(\xi)$ satisfies the ordinary differential equation (ODE)

$$z'(\xi) = F(z(\xi)), \quad \xi \in \mathbb{R}, \tag{6.2}$$

where $F(z) = G(\alpha^{-1}(z)) = K_0 + c\alpha^{-1}(z) - z + z\alpha^{-1}(z)$.

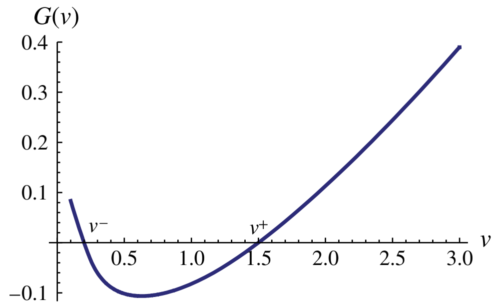

FIGURE 2. The function $G(v)$ calculated from α corresponding to the case of the German DAX 30 index. Its roots are prescribed as $v^- = 0.2$ and $v^+ = 1.5$.

Now, let us prescribe arbitrary limiting values $0 < v^- < v^+ < \infty$ for the travelling wave profile $v(\xi)$ corresponding to the limits $v^- = \lim_{\xi\to\infty} v(\xi)$, $v^+ = \lim_{\xi\to-\infty} v(\xi)$. We denote by $z^\pm$ the corresponding z values, that is, $z^\pm = \alpha(v^\pm)$. Thus $v^\pm$ are roots of the function G, $G(v^\pm) = 0$. Consequently, $F(z^\pm) = 0$.

Given $0 < v^- < v^+$, the travelling wave speed c and the intercept K_0 are uniquely determined from the equation $G(v^\pm) = 0$, that is,

$$c = \frac{\alpha(v^+)(1-v^+) - \alpha(v^-)(1-v^-)}{v^+ - v^-}, \qquad K_0 = -cv^+ + \alpha(v^+)(1-v^+). \tag{6.3}$$

According to Proposition 4.3, for any $v \in \mathcal{J} \subseteq (0,\infty)$, the function $v \mapsto \alpha(v)$ is C^∞ smooth and has the form $\alpha(v) = av - b/v + c$ for some constants $a > 0$, $b \geq 0$ and $c \in \mathbb{R}$. As a consequence, we obtain $h''(v) = -2a - 2b/v^3 < 0$ where $h(v) := \alpha(v)(1-v)$. Assume that $v^\pm \in \mathcal{J}$. Since $G'(v) = (h(v^+) - h(v^-))/(v^+ - v^-) - h'(v)$ and $h''(v^\pm) < 0$, we obtain $G'(v^-) < 0$, $G'(v^+) > 0$ and $G(v) < 0$ if and only if $v \in (v^-, v^+)$. In Figure 2 we plot the function $G(v)$ calculated from the function α corresponding to the case of the German DAX 30 index (see Figure 1 and Example 4.1). We prescribe the roots $v^- = 0.2$ and $v^+ = 1.5$.

Since $F(z) = G(\alpha^{-1}(z))$ and the function α is increasing, we obtain $F'(z^-) < 0$ and $F'(z^+) > 0$. Hence z^- is a stable and z^+ an unstable stationary solution to (6.2), that is, $\lim_{\xi\to\pm\infty} z(\xi) = z^\mp$ for any solution $z(\xi)$ to (6.2) such that $z(0) \in (z^-, z^+)$.

THEOREM 6.2. *Assume that $v^\pm \in \mathcal{J}$ are two limiting values $0 < v^- < v^+$. Up to a shift in the x variable, there exists a unique travelling wave solution $\varphi(x,t) = v(x + c(T-t))$ such that $\lim_{x\to-\infty} \varphi(x,t) = v^+$ and $\lim_{x\to\infty} \varphi(x,t) = v^-$. The travelling wave profile $v(\xi)$ is a decreasing function given by $v(\xi) = \alpha^{-1}(z(\xi))$, where $z = z(\xi)$ is a solution to the ODE (6.2). The travelling wave speed $c \in \mathbb{R}$ is given by (6.3).*

7. A numerical finite volume approximation scheme

This section is devoted to construction of a numerical approximation scheme for solving the Cauchy problem for the quasilinear parabolic equation (3.4). Recall that, instead of solving the fully nonlinear HJB equation (2.4) containing the maximal operator, we proposed its transformation to the quasilinear parabolic equation (3.4). In constructing the iterative numerical scheme we follow the method of a finite volume approximation scheme [25] combined with a nonlinear equation iterative solver proposed by Mikula and Kútik [23] for solving the generalized Black–Scholes equation with a volatility term depending nonlinearly on the second derivative of the option price.

Equation (3.4) belongs to a subclass of quasilinear parabolic equations of the general form

$$\partial_t\varphi + \partial_x^2 A(\varphi, x, t) + \partial_x B(\varphi, x, t) + C(\alpha, \varphi, x, t) = 0, \quad x \in \mathbb{R}, t \in [0, T),$$

satisfying the terminal condition at $t = T$ [24]. In our model we have

$$A(\varphi, x, t) = \alpha(\varphi), \quad B(\varphi, x, t) = (\varepsilon e^{-x} + r)\varphi + \alpha(\varphi)(1 - \varphi), \quad C \equiv 0.$$

In order to keep to standard PDE notation, we transform the equation from backward time to forward time via $\widetilde{\varphi}(x, \tau) := \varphi(x, T - t)$. We obtain $\partial_\tau\widetilde{\varphi} = -\partial_t\varphi$ and therefore

$$\partial_\tau\widetilde{\varphi} = \partial_x^2\widetilde{A}(\widetilde{\varphi}, x, \tau) + \partial_x\widetilde{B}(\widetilde{\varphi}, x, \tau) + \widetilde{C}(\alpha, \widetilde{\varphi}, x, \tau), \quad x \in \mathbb{R}, \tau \in (0, T], \tag{7.1}$$

with an initial condition $\widetilde{\varphi}(x, 0) = \widetilde{\varphi}_0(x) \equiv \varphi(x, T)$, where $\widetilde{A}(\widetilde{\varphi}, x, \tau) \equiv A(\varphi, x, T - \tau)$ is increasing in φ, and $\widetilde{B}(\widetilde{\varphi}, x, \tau) \equiv B(\varphi, x, T - \tau)$, $\widetilde{C}(\alpha, \widetilde{\varphi}, x, \tau) \equiv C(\alpha, \varphi, x, T - \tau)$. For convenience, we drop the $\widetilde{\ }$ sign in the following, but we keep in mind that we work with the transformed functions instead.

Let us consider a bounded computational domain $[x_L, x_R]$ and spatial discretization mesh points $x_i = x_L + ih$ for $i = 0, \ldots, n + 1$, where $h = (x_R - x_L)/(n + 1)$. Then $x_0 = x_L$ and $x_{n+1} = x_R$. The inner mesh points x_i, $i = 1, \ldots, n$, are the centres of the finite volume cells $(x_{i-1/2}, x_{i+1/2})$, denoted for simplicity by (x_{i-}, x_{i+}). We have $h = x_{i+} - x_{i-}$. Let us denote by $\tau^j = jk$, $j = 0, \ldots, m$, the time steps $k = T/m$. Integrating equation (7.1) over finite volumes, applying the midpoint rule on the left-hand-side integral and approximating the time derivative by forward finite difference with step k, we end up with a set of equations

$$\varphi_i^{j+1} = \frac{k}{h}(I_1 + I_2) + \varphi_i^j, \quad i = 1, \ldots, n, j = 0, \ldots, m,$$

where we denote

$$I_1 = \int_{x_{i-}}^{x_{i+}} \partial_x(\partial_x A(\varphi, x, \tau) + B(\varphi, x, \tau))\, dx, \quad I_2 = \int_{x_{i-}}^{x_{i+}} C(\alpha, \varphi, x, \tau)\, dx.$$

Depending on whether the above integrals are being computed on the jth or the $(j + 1)$th layer, we obtain different approximations. The symbol $\star$ is used below to denote either j or $j + 1$.

In order to compute the integral I_2 we apply the midpoint rule. We obtain

$$I_2^\star = hC(\alpha_i^\star, \varphi_i^\star, x_i, \tau^\star).$$

Concerning the integral I_1, we use the following notation:

$$D_{i\pm}^\star = \partial_\varphi A(\varphi, x, \tau)|_{\varphi_{i\pm}^\star, x_{i\pm}, \tau^\star}, \qquad E_{i\pm}^\star = \partial_x A(\varphi, x, \tau)|_{\varphi_{i\pm}^\star, x_{i\pm}, \tau^\star},$$
$$F_{i\pm}^\star = B(\varphi, x, \tau)|_{\varphi_{i\pm}^\star, x_{i\pm}, \tau^\star}, \qquad \partial_x\varphi|_{i\pm}^\star = \partial_x\varphi(x, \tau)|_{x_{i\pm}, \tau^\star}.$$

Using central spatial differences, we obtain the following numerical scheme for solving the general equation (7.1):

$$\varphi_i^{j+1} = \frac{k}{h}(D_{i+}^\star \partial_x\varphi|_{i+}^\star - D_{i-}^\star \partial_x\varphi|_{i-}^\star + E_{i+}^\star - E_{i-}^\star + F_{i+}^\star - F_{i-}^\star + I_2^\star) + \varphi_i^j \tag{7.2}$$

for $i = 1, \ldots, n$, with approximation of the derivatives

$$\partial_x\varphi|_{i+}^\star \approx \frac{\varphi(x_{i+1}, \tau^\star) - \varphi(x_i, \tau^\star)}{h}, \qquad \partial_x\varphi|_{i-}^\star \approx \frac{\varphi(x_i, \tau^\star) - \varphi(x_{i-1}, \tau^\star)}{h}.$$

We pay attention to the boundary values at x_0 and x_{n+1} later.

A simplified semi-implicit scheme To compute a solution at the new time layer $j+1$, we take the terms $D_{i\pm}^\star, E_{i\pm}^\star, F_{i\pm}^\star$ from the previous time layer with $\star = j$ and the term $\partial_x\varphi|_{i\pm}^\star$ from the new layer with $\star = j+1$. Reorganizing the new layer terms to the left-hand side and the old-layer terms to the right-hand side, we arrive at

$$-\frac{k}{h^2}D_+\varphi_{i+1}^{j+1} + \left(1 + \frac{k}{h^2}(D_{i+}^j + D_{i-}^j)\right)\varphi_i^{j+1} - \frac{k}{h^2}D_{i-}^j\varphi_{i-1}^{j+1}$$
$$= \frac{k}{h^2}(I_2^j + E_{i+}^j - E_{i-}^j + F_{i+}^j - F_{i-}^j) + \varphi_i^j,$$

which is a tridiagonal system that can be effectively solved by the Thomas algorithm.

An iterative fully implicit scheme We take $\star = j+1$ in all terms of (7.2) and φ_i^{j+1} is computed iteratively as follows. We denote by r_i^l the lth iterative approximation of φ_i^{j+1}, $i = 1, \ldots, n$, starting with $r_i^0 := \varphi_i^j$. In each iterate we solve the tridiagonal system for r_i^{l+1}, $i = 1, \ldots, n$, with the nonlinear terms $I_2^{\star,l}, D_{i\pm}^{\star,l}, E_{i\pm}^{\star,l}, F_{i\pm}^{\star,l}$ evaluated at $\tau^\star = \tau^{j+1}$ and $\varphi_i^{j+1} \approx r_i^l$. We update $r_i^l := r_i^{l+1}$ until an accuracy criterion is met and then we put $\varphi_i^{j+1} := r_i^l$ from the last iterate.

Boundary conditions We consider two classes of boundary conditions: inhomogeneous Dirichlet boundary conditions and mixed Robin type homogeneous boundary conditions, respectively,

$$\varphi(x_L, t) = \varphi_L(t), \qquad \varphi(x_R, t) = \varphi_R(t),$$
$$\partial_x\varphi(x, t) = d\varphi(x, t) \quad \text{at } x = x_L, x_R,$$

where the boundary functions $\varphi_L(t), \varphi_R(t)$ are prescribed for the Dirichlet boundary conditions and $d \in \mathbb{R}$ is constant for the Robin type boundary conditions. After discretization and using finite differences, we obtain the discrete boundary conditions

$$\varphi_0^j = L\varphi_L^j + M\varphi_1^j, \quad \varphi_{n+1}^j = R\varphi_R^j + N\varphi_n^j,$$

where $L = R = 1$, $M = N = 0$ for the case of Dirichlet boundary conditions and $L = R = 0$, $M = N = 1/(1 + dh)$ for the mixed Robin type boundary conditions.

In our numerical approximation of the quasilinear parabolic equation (3.4) we use the following boundary conditions:

$$\partial_x\varphi(x, t) - \varphi(x, t) = 0 \quad \text{at } x = x_L, \quad \partial_x\varphi(x, t) = 0 \quad \text{at } x = x_R,$$

for all $t \in [0, T]$. The boundary condition at $x = x_L$ is based on the following reasoning: if $\varepsilon > 0$ then, in the limit $x \to -\infty$, the dominant term in the equation $\partial_t\varphi + \partial_x^2\alpha(\varphi) + \partial_x[(\varepsilon e^{-x} + r)\varphi + (1 - \varphi)\alpha(\varphi)] = 0$ is equal to $\partial_x[(\varepsilon e^{-x} + r)\varphi(x, t)]$. To balance this term one has to assume that $\lim_{x\to-\infty} \partial_x(e^{-x}\varphi(x, t)) = 0$. This means that $\lim_{x\to-\infty} \partial_x\varphi(x, t) - \varphi(x, t) = 0$. The right boundary condition follows from the fact that, in the limit $x \to \infty$, equation (3.4) becomes $\partial_t\varphi + \partial_x^2\alpha(\varphi) + \partial_x[r\varphi + (1 - \varphi)\alpha(\varphi)] = 0$, having a constant solution, and so $\lim_{x\to+\infty} \partial_x\varphi(x, t) = 0$.

7.1. Numerical benchmark to a travelling wave solution We test the accuracy of the implicit scheme described above using the travelling wave analytical solution as described in Section 6 for the German DAX 30 index and for $\varepsilon = 0$, $r = 0$. We consider the time horizon $T = 10$ and the computational domain $[x_L, x_R] = [-4, 4]$. In order to compute the semi-analytical travelling wave solution $\varphi(x, t)$, we choose the limiting values $\nu^- = 0.2$, $\nu^+ = 1.5$. We solve equation (6.2) by means of the Merson method (Runge–Kutta method of fourth order) over the interval $[x_L, x_R + cT]$. In the numerical scheme we use Dirichlet boundary conditions on both ends, with values taken from the semi-analytical travelling wave solution. For clarification, we compute the function $\alpha(\varphi)$ numerically using the Matlab function *quadprog*, with a very fine discretization (of the order 10^{-5}) of the considered domain of φ, and so we consider it exact enough to substitute for the exact analytical solution. Having computed $\alpha(\varphi)$, we proceed with solution of the quasilinear PDE (3.4) by means of the iterative implicit finite volume numerical scheme. As the stopping criterion for the microiterates we choose the L_∞ norm of the difference of two consecutive iterates to be less than tolerance 10^{-9}. We solve equation (6.2) using the embedded Matlab function *ode45* with relative tolerance set to 10^{-8}.

Table 2 indicates that the scheme is empirically first-order accurate in the $L_\infty((0, T) : L_2)$ and $L_2((0, T) : W_2^1)$ norms when we restrict the time step k by $k = 0.1h$. It is of second order convergence when $k = 10h^2$: see Table 3. The so-called experimental order of convergence (EOC) corresponds to the order $r > 0$ of convergence such that $\text{err}(h) = O(h^r)$, where $\text{err}(h)$ is the norm of the difference of the numerical solution with the spatial step h and the exact travelling wave solution, that is,

$$r_i = \frac{\ln(\text{err}_i/\text{err}_{i-1})}{\ln(h_i/h_{i-1})}.$$

Figure 3 depicts the analytical travelling wave profile for times $t = 0$ and $t = T$.

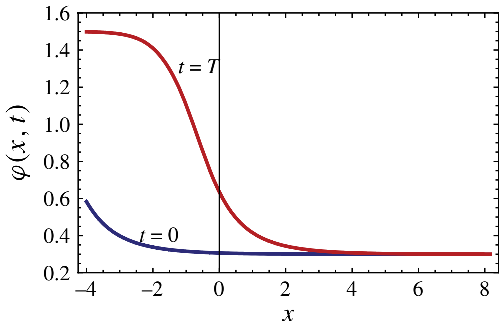

FIGURE 3. A travelling wave solution $\varphi(x, t)$ for $t = 0$ and $t = T$.

TABLE 2. The $L_\infty((0, T) : L_2)$ and $L_2((0, T) : W_2^1)$ norm of the error of the numerical solution with spatial step h and time–space step binding $k = 0.1h$ and the exact travelling wave solution, and the experimental order of convergence.

h	$L_\infty((0,T):L_2)$-err	$EOC_{k=0.1h}$	$L_2((0,T):W_2^1)$-err	$EOC_{k=0.1h}$
0.1	0.92313×10^{-3}	–	1.19224×10^{-3}	–
0.05	0.46046×10^{-3}	1.003	0.68451×10^{-3}	0.801
0.025	0.23194×10^{-3}	0.989	0.38057×10^{-3}	0.847
0.0125	0.11867×10^{-3}	0.967	0.20687×10^{-3}	0.879
0.00625	0.06004×10^{-3}	0.983	0.11737×10^{-3}	0.818

TABLE 3. The $L_\infty((0, T) : L_2)$ and $L_2((0, T) : W_2^1)$ norm of the error of the numerical solution with spatial step h and time–space step binding $k = 10h^2$ and the exact travelling wave solution, and the experimental order of convergence.

h	$L_\infty((0,T):L_2)$-err	$EOC_{k=10h^2}$	$L_2((0,T):W_2^1)$-err	$EOC_{k=10h^2}$
0.1	9.47564×10^{-3}	–	14.51654×10^{-3}	–
0.05	2.38427×10^{-3}	1.991	3.84091×10^{-3}	1.918
0.025	0.59656×10^{-3}	1.999	0.98843×10^{-3}	1.958
0.0125	0.14907×10^{-3}	2.001	0.25677×10^{-3}	1.945
0.00625	0.03725×10^{-3}	2.001	0.08456×10^{-3}	1.602

8. Application to portfolio optimization

In this section we present an example in which our goal is to optimize a portfolio consisting of $n = 30$ assets of the German DAX 30 index. The regular contribution to the portfolio is set to $\varepsilon = 1$ and $r = 0$. We consider the utility function of the form

$$U(x) = -\frac{1}{a-1}\exp(-(a-1)x), \tag{8.1}$$

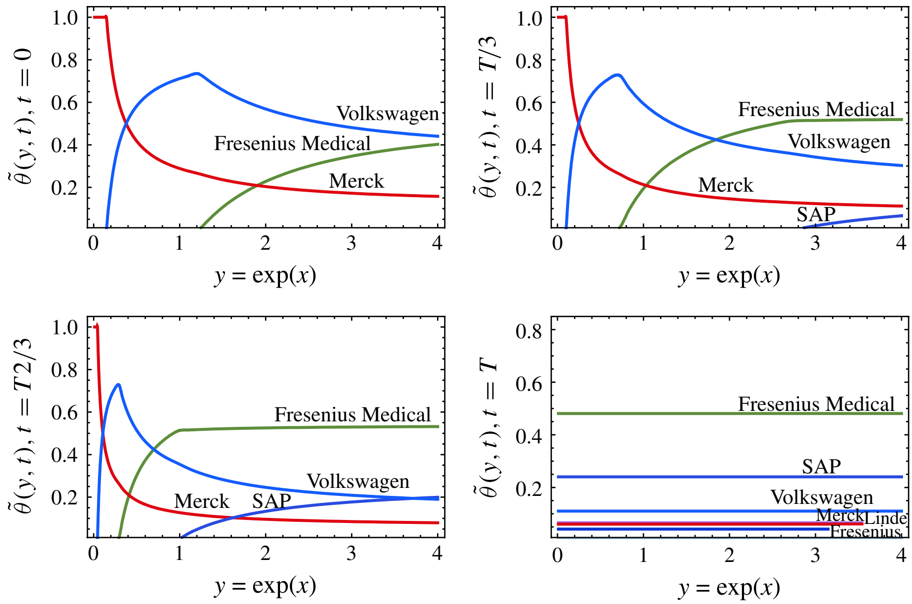

FIGURE 4. Optimal response strategy $\tilde{\theta}(y, t)$ for the DAX portfolio optimization, for time instances $t = 0$, $t = T/3$, $t = 2T/3$ and $t = T$, where $T = 10$.

where the coefficient of absolute risk aversion is set to $a = 9$. Notice that the constant absolute risk aversion (CARA) utility function (8.1) corresponds to the constant relative risk aversion (CRRA) function

$$\widetilde{U}(y) = -\frac{1}{a-1} y^{-a+1}$$

when expressed in the variable $y = e^x$. We consider the finite time horizon $T = 10$. Our guesses as to the minimal and maximal possible values of y are $y_L = 0.01$ and $y_R = 10$, respectively, so we consider $x \in [x_L, x_R]$ where $x_L = \ln y_L$, $x_R = \ln y_R$. Discretization steps are chosen as $h = 0.1$ and $k = 0.1h^2$. Concerning boundary conditions, we use the Robin boundary conditions with $d = 1$ on the left boundary and the von Neumann boundary conditions on the right boundary.

Figure 4 shows that there are only a few relevant assets out of the set of 30 assets entering the index. Table 4 summarizes historical average returns and the covariance matrix for these assets. The figure reveals the highest portion of Merck stocks for the early period of saving and for low account values y. It is indeed reasonable to invest in an asset with the highest expected return, although with the highest volatility, when the account value is low, in early times of saving. Evident fast decrement of the Merck weight can be observed for increasing account value. Fresenius Medical has the lowest volatility out of the considered five assets (and third lowest out of all 30 assets) and third-best mean return, which is reflected in its major representation in the portfolio.

TABLE 4. The covariance matrix Σ^{part} and mean returns for six stocks of the DAX 30 index: Merck, Volkswagen, SAP, Fresenius Medical, Linde, Fresenius. Based on historical data, August 2010–April 2012. Source: finance.yahoo.com.

Σ^{part}	Merck	VW	SAP	Fres Med	Linde	Fres	Mean return
Merck	1.6266	−0.0155	−0.0104	−0.0146	−0.0017	−0.0033	0.7315
VW	−0.0155	0.1584	0.0345	0.0292	0.0569	0.0238	0.3413
SAP	−0.0104	0.0345	0.0516	0.0183	0.0240	0.0143	0.1877
Fres Med	−0.0146	0.0292	0.0183	0.0434	0.0227	0.0248	0.2202
Linde	−0.0017	0.0569	0.0240	0.0227	0.0530	0.0201	0.1932
Fres	−0.0033	0.0238	0.01430	0.0248	0.0201	0.0386	0.1351

In Section 4.2 we showed that the sets of active indices can be identified directly from the function $\alpha''(\varphi)$. Moreover, based on Proposition 5.2, there is an upper bound on the investor's coefficient of absolute risk aversion $a(x,t)$ given by $\varphi^+ - 1$. When the utility function is given as in (8.1), we have $\varphi^+ = a + 1 = 10$ and so $\varphi(x,t) \leq 10$ for all x and t. Hence, only the interval $[0, \varphi^+]$ gives relevant information for the investor. Knowing the sets of active indices computed for $\varphi \in [0, \varphi^+]$, the investor knows the set $\bigcup_{\varphi\in(0,\varphi^+]}\{i \mid \hat{\theta}_i(\varphi) > 0\}$, namely the set of assets which will be entering the optimal portfolio with nonzero weight. To identify the set $\{i \mid \hat{\theta}_i(\varphi) > 0\}$ on a particular interval, it is enough to calculate the optimal $\boldsymbol{\theta}(\varphi)$ at one single point from the given interval.

Conclusions

We have proposed and analysed a method for the Riccati transformation for solving a class of Hamilton–Jacobi–Bellman equations arising from a problem of optimal portfolio construction. We derived a quasilinear backward parabolic equation for the coefficient of relative risk aversion corresponding to the value function: a solution to the original HJB equation. Using Schauder's theory, we showed existence and uniqueness of classical Hölder smooth solutions. We also derived useful qualitative properties of the value function of the auxiliary parametric quadratic programming problem after the transformation. A fully implicit iterative numerical scheme based on finite volume approximation has been proposed and numerically tested. We also provided a practical example on German DAX 30 index portfolio optimization.

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