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Analysis of the Nonlinear Option Pricing Model Under Variable Transaction Costs

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Abstract In this paper we analyze a nonlinear Black–Scholes model for option pricing under variable transaction costs. The diffusion coefficient of the nonlinear parabolic equation for the price V is assumed to be a function of the underlying asset price and the Gamma of the option. We show that the generalizations of the classical Black–Scholes model can be analyzed by means of transformation of the fully nonlinear parabolic equation into a quasilinear parabolic equation for the second derivative of the option price. We show existence of a classical smooth solution and prove useful bounds on the option prices. Furthermore, we construct an effective numerical scheme for approximation of the solution. The solutions are obtained by means of the efficient numerical discretization scheme of the Gamma equation. Several computational examples are presented.

Keywords Black–Scholes equation with nonlinear volatility · Quasilinear parabolic equation · Variable transaction costs

Mathematics Subject Classification 35K15 · 35K55 · 90A09 · 91B28

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1 Introduction

The classical linear Black–Scholes option pricing model with a constant historical volatility was proposed in Black and Scholes (1973). The model was derived under several restrictive assumptions, such as assumption on market completeness, continuous trading and zero transaction costs. According to this option pricing theory the price $V(S, t)$ of a contingent claim written on the underlying asset $S > 0$ at the time $t \in [0, T]$ is a solution to the linear parabolic equation

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0, \tag{1}$$

where $r > 0$ is the risk-free interest rate of a zero-coupon bond and σ is the historical volatility of the underlying asset which is assumed to follow a stochastic differential equation of the geometric Brownian motion, i.e.

$$dS = \rho S\,dt + \sigma S\,dW, \tag{2}$$

with a drift ρ (cf. Kwok 1998, Wilmott et al. 1995a, b).

However, practical analysis of market data shows the need for more realistic models taking into account the aforementioned drawbacks of the classical Black–Scholes theory. It stimulated development of various nonlinear option pricing models in which the volatility function is no longer constant, but is a function of the solution V itself. We focus on the case where the volatility depends on the second derivative $\partial_S^2 V$ of the option price with respect to the underlying asset price S.

$$\partial_t V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0, \tag{3}$$

where $\hat{\sigma}(S\partial_S^2 V)$ is a function of the product of the asset price and Gamma of the option (Gamma is the second derivative of V with respect to S).

Motivation for studying the nonlinear extensions of the classical Black-Scholes Eq. (3) with a volatility depending on $S\partial_S^2 V$ arises from classical option pricing models taking into account non–trivial transaction costs due to buying and selling assets (cf. Leland 1985), market feedbacks effects due to large traders choosing given stock–trading strategies (cf. Frey and Patie 2002; Frey and Stremme 1997; Schönbucher and Wilmott 2000), risk from a volatile and unprotected portfolio (see Jandačka and Ševčovič 2005) or investors preferences (cf. Barles and Soner 1998).

One of the first nonlinear models taking into account transaction costs is the Leland model Leland (1985) for pricing the call and put options. This model was further extended by Hoggard et al. (1994) for general type of derivatives. Qualitative and numerical properties of this model were analyzed by Grandits and Schachinger (2001), Imai et al. (2007), Ishimura (2010), Grossinho and Morais (2009) and others.

In this model the variance $\hat{\sigma}^2$ is given by

$$\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2 \left(1 - \mathrm{Le}\,\mathrm{sgn}\left(S\partial_S^2 V\right)\right) = \begin{cases} \sigma^2(1 - \mathrm{Le}), & \text{if } \partial_S^2 V > 0, \\ \sigma^2(1 + \mathrm{Le}), & \text{if } \partial_S^2 V < 0, \end{cases} \tag{4}$$

where Le $= \sqrt{\frac{2}{\pi}} \frac{C_0}{\sigma \sqrt{\Delta t}}$ is the so-called Leland number, σ is a constant historical volatility, $C_0 > 0$ is a constant transaction costs per unit dollar of transaction in the underlying asset market and Δt is the time-lag between consecutive portfolio adjustments. The nonlinear model (3) with the volatility function given as in (4) can be also viewed as a jumping volatility model investigated by Avellaneda et al. (1995).

The important contribution in this direction has been presented in the paper Amster et al. (2005) by Amster, Averbuj, Mariani and Rial, where the transaction costs are assumed to be a nonincreasing linear function of the form $C(\xi) = C_0 - \kappa \xi$, $(C_0, \kappa > 0)$, depending on the volume of trading stock $\xi \geq 0$ needed to hedge the replicating portfolio. A disadvantage of such a transaction costs function is the fact that it may attain negative values when the amount of transactions exceeds the critical value $\xi = C_0/\kappa$. In the model studied by Amster et al. (2005) (see also Averbuj 2012, Mariani et al. 2011) volatility function has the following form:

$$\hat{\sigma}(S \partial_S^2 V)^2 = \sigma^2 \left(1 - \text{Le sgn} \left(S \partial_S^2 V \right) + \kappa S \partial_S^2 V \right). \tag{5}$$

In Bakstein et al. (2004) Bakstein and Howison investigated a parametrized model for liquidity effects arising from the asset trading. In their model $\hat{\sigma}$ is a quadratic function of the term $H = S \partial_S^2 V$:

$$\hat{\sigma}(S \partial_S^2 V)^2 = \sigma^2 \left(1 + \bar{\gamma}^2 (1-\alpha)^2 + 2\lambda S \partial_S^2 V + \lambda^2 (1-\alpha)^2 \left(S \partial_S^2 V \right)^2 \right.$$
$$\left. + 2\sqrt{\frac{2}{\pi}} \bar{\gamma} \, \text{sgn} \left(S \partial_S^2 V \right) + 2\sqrt{\frac{2}{\pi}} \lambda (1-\alpha)^2 \bar{\gamma} \left| S \partial_S^2 V \right| \right). \tag{6}$$

The parameter λ corresponds to a market depth measure, i.e. it scales the slope of the average transaction price. Next, the parameter $\bar{\gamma}$ models the relative bid–ask spreads and it is related to the Leland number through relation $2\bar{\gamma}\sqrt{2/\pi} = \text{Le}$. Finally, α transforms the average transaction price into the next quoted price, $0 \leq \alpha \leq 1$.

The risk adjusted pricing methodology (RAPM) model takes into account risk from the unprotected portfolio was proposed by Kratka (1998). It was generalized and analyzed by Jandačka and Ševčovič (2005). In this model the volatility function has the form:

$$\hat{\sigma}(S \partial_S^2 V)^2 = \sigma^2 \left(1 - \mu \left(S \partial_S^2 V \right)^{\frac{1}{3}} \right), \tag{7}$$

where $\sigma > 0$ is a constant historical volatility of the asset price return and $\mu = 3(C_0^2 R/2\pi)^{\frac{1}{3}}$, where C_0, $R \geq 0$ are non-negative constants representing cost measure and the risk premium measure, respectively.

The structure of the paper is as follows. In the next section we present a nonlinear option pricing model under variable transaction cost. It turns out that the volatility function depends on $S \partial_S^2 V$. In particular case of constant or linearly decreasing transaction costs it is a generalization of the Leland (1985) and Amster et al. model Amster et al. (2005), respectively. Section 3 is devoted to transformation of the fully nonlinear option pricing equation into a quasilinear Gamma equation. We prove existence of

classical Hölder smooth solutions and we derive useful bounds on the solution. In Sect. 4 we propose a numerical scheme for solving the Gamma equation based on finite volume method. We also present several numerical examples of computation of option prices based on a solution to the nonlinear Black–Scholes equation under variable transaction costs.

2 Option Pricing Model Under Variable Transaction Costs

One of the key assumptions of the classical Black–Scholes theory is the possibility of continuous adjustment (or hedging) of the portfolio consisting of options and underlying assets. In the context of transaction costs needed for buying and selling the underlying asset, continuous hedging leads to an infinite number of transactions and the unbounded total transaction costs. The Leland model Leland (1985) (see also Hoggard et al. 1994) is based on a simple, but very important modification of the Black–Scholes model, which includes transaction costs and possibility of rearranging the portfolio at discrete times. Since the portfolio is maintained at regular intervals, it means that the total transaction costs are limited.

Our derivation of the variable transaction costs option pricing model follows ideas proposed by Leland (1985). Therefore we recall crucial steps of derivation of Leland's approach for modeling transaction costs. We assume that the cost C_0 per one transaction is constant, i.e. it does not depend on the volume of transactions. The underlying asset is purchased at a higher *ask* price S_{ask} and it is sold for a lower *bid* price S_{bid}. The price of S is then computed as an average of *ask* and *bid* prices, i.e., $S = (S_{ask} + S_{bid})/2$. Then $C_0 \geq 0$ represents a constant percentage of the cost of the sale and purchase of a share relative to the price S, i.e.

$$C_0 = \frac{S_{ask} - S_{bid}}{S} = 2 \frac{S_{ask} - S_{bid}}{S_{ask} + S_{bid}}. \tag{8}$$

The value $\Pi = V + \delta S$ of the synthesized portfolio consisting of one option in a long position at the price V and δ underlying assets at the price S changes over the time interval $[t, t + \Delta t]$ by selling $\Delta \delta < 0$ or buying $\Delta \delta > 0$ short positioned assets. It means that the purchase or selling $\Delta \delta$ assets at a price of S yields the additional cost ΔTC for the option holder

$$\Delta TC = \frac{S}{2} C_0 |\Delta \delta|. \tag{9}$$

Consequently, the value of the portfolio changes to:

$$\Delta \Pi = \Delta(V + \delta S) - \Delta TC \tag{10}$$

during the time interval $[t, t + \Delta t]$. The key step in derivation of the Leland model consists in approximation of the change ΔTC of transaction costs by its expected value $\mathbb{E}[\Delta TC]$, i.e. $\Delta TC \approx r_{TC} S \Delta t$, where the transaction costs measure r_{TC} is defined as the expected value of the change of the transaction costs per unit time interval Δt and price S:

$$r_{TC} = \frac{\mathbb{E}[\Delta TC]}{S\Delta t}. \tag{11}$$

Hence Eq. (10) describing the change in the portfolio has the form:

$$\Delta\Pi = \Delta(V + \delta S) - r_{TC}S\Delta t. \tag{12}$$

Since the underlying asset follows the geometric Brownian motion we have

$$\Delta S = \rho S\Delta t + \sigma S\Delta W, \tag{13}$$

where $\Delta W = W_{t+\Delta t} - W_t$ is the increment of the Wiener process. Now, assuming the change $\Delta\Pi$ in the portfolio is balanced by a bond with the risk-free rate $r \geq 0$, i.e. $\Delta\Pi = r\Pi\Delta$, using Itō's lemma for ΔV and applying the delta hedging strategy $\delta = -\partial_S V$ we obtain generalization of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + rS\partial_S V - rV - r_{TC}S = 0. \tag{14}$$

Furthermore, applying Itō's formula for the function $\delta = -\partial_S V$ we obtain

$$\Delta\delta = -\sigma S\partial_S^2 V\Delta W = -\sigma S\partial_S^2 V\Phi\sqrt{\Delta t}$$

plus higher order terms in $\sqrt{\Delta t}$. Here $\Phi \sim N(0,1)$ a normally distributed random variable. Hence, in the lowest order $O(\sqrt{\Delta t})$ we have that

$$|\Delta\delta| = \alpha|\Phi|, \quad \text{where} \quad \alpha := \sigma S\left|\partial_S^2 V\right|\sqrt{\Delta t}. \tag{15}$$

For the case of constant transaction costs given by (9), using the fact that $\mathbb{E}[|\Phi|] = \sqrt{2/\pi}$ we obtain

$$r_{TC}S = \frac{\mathbb{E}[\Delta TC]}{\Delta t} = \frac{1}{2}C_0 S\frac{\mathbb{E}[|\Delta\delta|]}{\Delta t} = \frac{1}{2}S^2\sqrt{\frac{2}{\pi}}\frac{C_0}{\sqrt{\Delta t}}\sigma|\partial_S^2 V| = \frac{1}{2}\sigma^2 S^2\mathrm{Le}|\partial_S^2 V|,$$

where $\mathrm{Le} = \sqrt{\frac{2}{\pi}}\frac{C_0}{\sigma\sqrt{\Delta t}}$ is the Leland number. Inserting the term $r_{TC}S$ into (14) we obtain the Leland Eq. (3) with the volatility function given by (4).

Following derivation of the Leland model we present our approach on modeling variable transaction costs. Large investors can expect a discount due to large amount of transactions. The more they purchase the less they will pay per one traded underlying asset. In general, we will assume that the cost C per one transaction is a nonincreasing function of the amount of transactions, $|\Delta\delta|$, per unit of time Δt, i.e.

$$C = C(|\Delta\delta|). \tag{16}$$

It means that the purchase of $\Delta\delta > 0$ or sales of $\Delta\delta < 0$ shares at a price of S, we calculate the additional transaction costs ΔTC per unit of time Δt:

$$\Delta TC = \frac{S}{2}C(|\Delta\delta|)|\Delta\delta|. \tag{17}$$

Hence the transaction costs measure r_{TC} can be expressed as

$$r_{TC} = \frac{\mathbb{E}[\Delta TC]}{S\Delta t} = \frac{1}{2}\frac{\mathbb{E}[C(|\Delta\delta|)|\Delta\delta|]}{\Delta t}, \tag{18}$$

where C is the transaction costs function and $\Delta\delta$ is the number of purchased $\Delta\delta > 0$ or sold $\Delta\delta < 0$ shares per unit of time Δt.

In order to simplify notation we introduce the so-called mean value modification of the transaction costs function $\tilde{C}(\alpha)$ defined as follows:

Definition 2.1 Let $C = C(\xi)$, $C : \mathbb{R}_0^+ \to \mathbb{R}$, be a transaction costs function (Fig. 1). The integral transformation $\tilde{C} : \mathbb{R}_0^+ \to \mathbb{R}$ of the function C defined as follows:

$$\tilde{C}(\xi) = \sqrt{\frac{\pi}{2}}\mathbb{E}[C(\xi|\Phi|)|\Phi|] = \int_0^\infty C(\xi x)x\,e^{-x^2/2}dx, \tag{19}$$

is called the mean value modification of the transaction costs function. Here Φ is the random variable with a standardized normal distribution, i.e., $\Phi \sim N(0,1)$.

Applying definition (19) to Eq. (18) we obtain the following expression for the transaction costs measure:

$$r_{TC} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\tilde{C}(\alpha)\alpha}{\Delta t}, \quad \text{where} \quad \alpha = \sigma S\left|\partial_S^2 V\right|\sqrt{\Delta t}. \tag{20}$$

Inserting the transaction costs measure r_{TC} into (14) we obtain generalization of the Leland model for the case of arbitrary transaction costs function $C(\xi)$.

Proposition 2.1 Let $C : \mathbb{R}_0^+ \to \mathbb{R}$ be a measurable and bounded transaction costs function. Then the nonlinear Black–Scholes equation for pricing option under variable transaction costs is given by the nonlinear parabolic equation

$$\partial_t V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2\partial_S^2 V + rS\partial_S V - rV = 0, \tag{21}$$

where

$$\hat{\sigma}(S\partial_S^2 V)^2 = \sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S|\partial_S^2 V|\sqrt{\Delta t})\frac{\mathrm{sgn}(S\partial_S^2 V)}{\sigma\sqrt{\Delta t}}\right). \tag{22}$$

In the next two propositions we summarize several useful properties of the mean value modification of a transaction costs function.
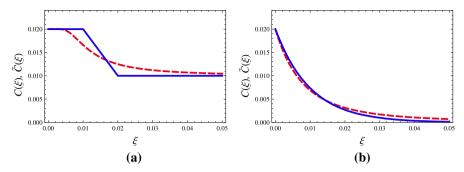
Fig. 1 Various types of transaction costs functions $C(\xi)$ (*solid line*) and their mean value modification $\tilde{C}(\xi)$ (*dashed line*). **a** Piecewise linear transaction costs function with $C_0 = 0.02, \kappa = 1, \xi_- = 0.01, \xi_+ = 0.02$; b) Exponentially decreasing transaction costs function with $C_0 = 0.02, \kappa = 100$

Proposition 2.2 *Let $C(\xi)$ be a measurable bounded transaction costs function such that $\underline{C}_0 \leq C(\xi) \leq C_0$ for all $\xi \geq 0$. Then its mean value modification $\tilde{C}(\xi)$ is a C^∞ smooth function for $\xi > 0$. Furthermore, it has the following properties:*

1. *$\tilde{C}(0) = C(0)$;*
2. *if $C(+\infty) = \lim_{\xi \to \infty} C(\xi)$ then $\tilde{C}(+\infty) = C(+\infty)$;*
3. *$\underline{C}_0 \leq \tilde{C}(\xi) \leq C_0$ for all $\xi \geq 0$;*
4. *if C is nonincreasing (nondecreasing) then $\tilde{C}$ is a non- increasing (nondecreasing) function as well;*
5. *if C is a (non-constant) convex function then $\tilde{C}$ is a (strictly) convex function.*

Proposition 2.3 *Let $C(\xi)$ be a measurable and bounded transaction costs function which is nonincreasing for $\xi \geq 0$.*

- *If $C(0) > 0$ then the function $\xi \mapsto \frac{\tilde{C}(\xi)}{\xi}$ is strictly convex for $\xi > 0$.*
- *If $\underline{C}_0 \leq C(\xi) \leq C_0$ for all $\xi \geq 0$ then*

$$\tilde{C}(\xi) + \xi \tilde{C}'(\xi) \geq 2\underline{C}_0 - C_0.$$

Proof (a) Since $\tilde{C}(\xi) = \sqrt{2\pi} \int_0^\infty C(\xi x) x f(x) dx = \sqrt{2\pi} \xi^{-2} \int_0^\infty C(y) y f(y/\xi) dy$ where $f(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $f'(x) = -xf(x)$, $f''(x) = (x^2 - 1)f(x)$ then, by using substitution $y = \xi x$, we obtain the following identity:

$$\frac{d^2}{d\xi^2}\left(\frac{\tilde{C}(\xi)}{\xi}\right) = \sqrt{2\pi} \int_0^\infty C(y) y \left(\frac{12}{\xi^5} f(y/\xi) + \frac{3y}{\xi^6} f'(y/\xi) + \frac{y^2}{\xi^7} f''(y/\xi)\right) dy$$

$$= \frac{\sqrt{2\pi}}{\xi^3} \int_0^\infty C(\xi x) x \left(12 f(x) + 8xf'(x) + x^2 f''(x)\right) dx$$

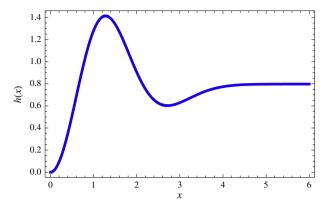$$= \frac{\sqrt{2\pi}}{\xi^3} \int_0^\infty C(\xi x) x \left(x^4 - 9x^2 + 12\right) f(x) dx$$

Fig. 2 Plot of the auxiliary function $h(x)$

$$= \frac{\sqrt{2\pi}}{\xi^3} \int_0^\infty C(\xi x) h'(x) dx$$

$$= \frac{2C(+\infty)}{\xi^3} - \frac{\sqrt{2\pi}}{\xi^2} \int_0^\infty C'(\xi x) h(x) dx > 0$$

because $C(+\infty) \geq 0$, $C'(x) \leq 0$ (if $C(+\infty) = 0$ then $C' \not\equiv 0$) and $h(x) > 0$ for $x > 0$ where

$$h(x) = \int_0^x t(t^4 - 9t^2 + 12) f(t) dt = \sqrt{\frac{2}{\pi}} - (2 - 5x^2 + x^4) f(x)$$

is a strictly positive function for $x > 0$ depicted in Fig. 2. It has a unique positive minimum at $x^* = \sqrt{(9 + \sqrt{33})/2} \simeq 2.715$.

(b) Again, as $\tilde{C}(\xi) = \xi^{-2} \sqrt{2\pi} \int_0^\infty C(y) y f(y/\xi) dy$, we obtain

$$\xi \tilde{C}'(\xi) = -2\tilde{C}(\xi) - \xi^{-3} \sqrt{2\pi} \int_0^\infty C(y) y^2 f'(y/\xi) dy$$

$$= -2\tilde{C}(\xi) - \sqrt{2\pi} \int_0^\infty C(\xi x) x^2 f'(x) dx$$

$$= -2\tilde{C}(\xi) + \sqrt{2\pi} \int_0^\infty C(\xi x) x^3 f(x) dx.$$

Since $C(\xi x) \geq \underline{C}_0$, $\tilde{C}(\xi) \leq C_0$ and $\sqrt{2\pi} \int_0^\infty x^3 f(x) dx = 2$ we obtain $\tilde{C}(\xi) + \xi \tilde{C}'(\xi) \geq -C_0 + 2\underline{C}_0$, as claimed. $\qquad\square$

2.1 Piecewise Linear Nonincreasing Transaction Costs Function

We present an example of a realistic transaction costs function which is nonincreasing with respect to the amount of transactions as in model studied by Amster et al. (2005),

Averbuj (2012) Mariani et al. (2011). The benefit is the elimination of the problem of negative values of the linear decreasing costs function. We define the following piecewise linear function.

Definition 2.2 We define a piecewise linear nonincreasing transaction costs function as follows:

$$C(\xi) = \begin{cases} C_0, & \text{if } 0 \leq \xi \leq \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if } \xi_- \leq \xi \leq \xi_+, \\ \underline{C_0}, & \text{if } \xi \geq \xi_+, \end{cases} \qquad (23)$$

where we assume $C_0, \kappa > 0$, and $0 \leq \xi_- < \xi_+ \leq \infty$ are given constants and $\underline{C_0} = C_0 - \kappa(\xi_+ - \xi_-) > 0$.

This is a realistic transaction costs function. Indeed, for a small volume of traded assets the transaction costs rate equals C_0. When the volume is large enough then a discount is applied with a lower transaction costs rate $\underline{C_0} < C_0$.

Notice that this function also covers several special cases studied before. Namely, the constant transaction costs function studied by Leland (1985), Hoggard et al. (1994) ($\kappa = \xi_- = 0, \xi_+ = \infty$) as well as linearly decreasing transaction costs investigated by Amster *et al.* in Amster et al. (2005) ($\kappa > 0, \xi_- = 0, \xi_+ = \infty$).

Taking into account the fact that $C'(\xi) = -\kappa$ for $\xi \in (\xi_-, \xi_+)$ and $C'(\xi) = 0$, otherwise, and using integration by parts we can easily derive that the mean value modification of a piecewise transaction costs function (23) is given by:

$$\tilde{C}(\xi) = C_0 - \kappa\xi \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} e^{-x^2/2}dx, \quad \text{for } \xi \geq 0. \qquad (24)$$

2.2 Exponentially Decreasing Transaction Costs Function

As an another example of a transaction costs function one can consider the following exponential function of the form

$$C(\xi) = C_0 \exp(-\kappa\xi), \quad \text{for } \xi \geq 0, \qquad (25)$$

where $C_0 > 0$ and $\kappa > 0$ are given constants. Its mean value modification $\tilde{C}$ can be derived by expanding the function $\xi \mapsto C(\xi|\Phi|)\xi$ into power series:

$$\tilde{C}(\xi) = \sqrt{\frac{\pi}{2}}\mathbb{E}[C(\xi|\Phi|)|\Phi|] = \int_0^\infty C(\xi x)x\, e^{-x^2/2}dx = C_0 \int_0^\infty e^{-\kappa\xi x}x\, e^{-x^2/2}dx$$

$$= C_0\left(\left[-e^{-\kappa\xi x}e^{-x^2/2}\right]_0^\infty - \int_0^\infty \left(e^{-\kappa\xi x}\right)'(-e^{-x^2/2})dx\right)$$

$$= C_0\left(1 - \kappa\xi \int_0^\infty e^{-\kappa\xi x - x^2/2}dx\right) = C_0\left(1 - \kappa\xi e^{\kappa^2\xi^2/2}\int_{\kappa\xi}^\infty e^{-t^2/2}dt\right)$$

$$= C_0\phi(-\sqrt{2}\kappa\xi), \quad \text{where} \quad \phi(x) = 1 + xe^{\frac{x^2}{4}}\left(\text{erf}(x/2) + 1\right)\frac{\sqrt{\pi}}{2},$$

and $\text{erf}(x/2) = \frac{2}{\sqrt{\pi}}\int_0^{x/2} e^{-s^2}ds$ is the error function.

3 Transformation of the Fully Nonlinear Black–Scholes Equation to the Quasilinear Gamma equation

The goal of this section is to study transformation of the nonlinear Black–Scholes equation into a quasilinear parabolic equation—the so-called Gamma introduced and investigated by Jandačka and Ševčovič (2005) (see also [Ševčovič et al. 2011, Chapter 11]).

In what follows, we will use the notation

$$\beta(H) = \frac{1}{2}\hat{\sigma}(H)^2 H, \tag{26}$$

where $\hat{\sigma}$ is the volatility function depending on the term $H = S\partial_S^2 V$. Let $E > 0$ be a numeraire for the underlying asset price, e.g. E is the expiration price for a call or put option.

Proposition 3.1 *Assume the function $V = V(S, t)$ is a solution to the nonlinear Black-Scholes equation*

$$\partial_t V + S\beta(S\partial_S^2 V) + rS\partial_S V - rV = 0, \quad S > 0, t \in (0, T). \tag{27}$$

Then the transformed function $H = H(x, \tau) = S\partial_S^2 V(S, t)$, where $x = \ln(S/E)$, $\tau = T - t$ is a solution to the quasilinear parabolic (Gamma) equation

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H. \tag{28}$$

On the other hand, if H is a solution to (28) such that $H(-\infty, \tau) = \partial_x H(-\infty, \tau) = 0$ and $\beta'(0)$ is finite then the function

$$V(S, t) = aS + be^{-r(T-t)} + \int_{-\infty}^{\infty} (S - Ee^x)^+ H(x, T - t)dx, \tag{29}$$

is a solution to the nonlinear Black–Scholes Eq. (27) for any $a, b \in \mathbb{R}$.

Proof The first part of the statement can be shown directly by taking the second derivative of (27) with respect to S. Indeed, as $\partial_x = S\partial_S$ we have

$$S\partial_S^2(S\beta) = S\partial_S(\beta + S\partial_S\beta) = \partial_x(\partial_x\beta + \beta),$$
$$S\partial_S^2(S\partial_S V) = S\partial_S(\partial_S V + S\partial_S^2 V) = H + \partial_x H.$$

Applying the operator $S\partial_S^2$ to Eq. (27) and taking into account $\partial_t = -\partial_\tau$ we conclude that H is the solution to Eq. (28) (see also Jandačka and Ševčovič 2005 and [Ševčovič et al. 2011, Chapter 11]).

On the other hand, if $V(S, t)$ is given by (29) then $S\partial_S^2 V(S, t) = H(x, \tau)$. Moreover, if H is a solution to (28) then

$$\partial_t V(S, t) = rbe^{-r(T-t)} - \int_{-\infty}^{\ln(S/E)} (S - Ee^x)\partial_\tau H(x, \tau)dx$$

$$= rbe^{-r(T-t)} - \int_{-\infty}^{\ln(S/E)} (S - Ee^x)\left[e^{-x}\partial_x(e^x\partial_x\beta(H(x, \tau)))\right.$$

$$\left. + r\partial_x H(x, \tau)\right]dx$$

$$= rbe^{-r(T-t)} - \left[(S - Ee^x)(\partial_x\beta(H(x, \tau)) + r\partial_x H(x, \tau)\right]_{x=-\infty}^{x=\ln(S/E)}$$

$$- S\int_{-\infty}^{\ln(S/E)} \partial_x\beta(H(x, \tau))dx - rE\int_{-\infty}^{\ln(S/E)} e^x H(x, \tau)dx$$

$$= rbe^{-r(T-t)} - S\beta(S\partial_S^2 V(S, t)) - rE\int_{-\infty}^{\ln(S/E)} e^x H(x, \tau)dx.$$

Here we have used the fact that $H(-\infty, \tau) = \partial_x H(-\infty, \tau) = 0$ and $\beta'(0^{\pm})$ is finite. Since

$$S\partial_S V(S, t) = aS + S\partial_S \int_{-\infty}^{\ln(S/E)} (S - Ee^x)H(x, \tau)dx$$

$$= aS + S\int_{-\infty}^{\ln(S/E)} H(x, \tau)dx$$

we finally obtain

$$\partial_t V + S\beta(S\partial_S^2 V) + rS\partial_S V$$

$$= r(aS + be^{-r(T-t)}) + r\int_{-\infty}^{\ln(S/E)} (S - Ee^x)H(x, \tau)dx = rV.$$

Therefore $V(S, t)$ solves the nonlinear Black–Scholes Eq. (27), as claimed. □

Remark 3.1 If the initial condition $H(x, 0) = \delta(x)$ is the Dirac δ-function then for the terminal pay-off diagram $V(S, T)$ given by (29) we obtain

1. $V(S, T) = (S - E)^+$ (call option) when $a = b = 0$,
2. $V(S, T) = (E - S)^+$ (put option) when $a = -1, b = E$.

Remark 3.2 The initial Dirac δ-function can be approximated as follows:

$$H(x, 0) \approx f(d)/(\hat{\sigma}\sqrt{\tau^*}),$$

where $\tau^* > 0$ is sufficiently small, $f(d)$ is the PDF function of the normal distribution, $f(d) = e^{-d^2/2}/\sqrt{2\pi}$ and $d = \left(x + (r - \sigma^2/2)\tau^*\right)/\sigma\sqrt{\tau^*}$ where $\sigma = \hat{\sigma}(0)$. The idea behind such an approximation follows from observation that for a solution of the linear Black–Scholes equation with a constant volatility $\sigma > 0$ at the time $T - \tau^*$
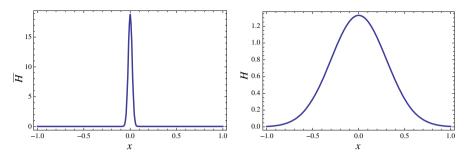
Fig. 3 Plots of the initial approximation of the function $H(x, 0)$ for τ^* sufficiently small (*left*) and the solution profile $H(x, T)$ at $\tau = T$ (*right*)

close to expiry T the value $H(x, \tau^*) = S\partial_S^2 V(S, T - \tau^*)$ is given by $H(x, \tau^*) = f(d)/(\hat{\sigma}\sqrt{\tau^*})$. An approximation of the initial condition for $0 < \tau^* \ll 1$ is shown in Fig. 3 (left).

3.1 Existence of Classical Solutions, Comparison Principle

The aim of this subsection is to analyze classical smooth solutions to the Cauchy problem for the quasilinear parabolic Eq. (28). Following the methodology based on the so-called Schauder's type of estimates (cf. Ladyženskaja et al. 1968), we shall prove existence and uniqueness of classical solutions to (28).

We proceed with a definition of function spaces we will work with. Let $\Omega = (x_L, x_R) \subset \mathbb{R}$ be a bounded interval. We denote $Q_T = \Omega \times (0, T)$ the space-time cylinder. Let $0 < \lambda < 1$. By $\mathcal{H}^\lambda(\Omega)$ we denote the Banach space consisting of all continuous functions H defined on $\bar{\Omega}$ which are λ-Hölder continuous. It means that their Hölder semi-norm $\langle H \rangle^{(\lambda)} = \sup_{x, y \in \Omega, x \neq y} |H(x) - H(y)|/|x - y|^\lambda$ is finite. The norm in the space $\mathcal{H}^\lambda(\Omega)$ is then the sum of the maximum norm of H and the semi-norm $\langle H \rangle^{(\lambda)}$. The space $\mathcal{H}^{2+\lambda}(\Omega)$ consists of all twice continuously differentiable functions H in $\bar{\Omega}$ whose second derivative $\partial_x^2 H$ belongs to $\mathcal{H}^\lambda(\Omega)$. The space $\mathcal{H}^{2+\lambda}(\mathbb{R})$ consists of all functions $H : \mathbb{R} \to \mathbb{R}$ such that $H \in \mathcal{H}^{2+\lambda}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}$.

The parabolic Hölder space $\mathcal{H}^{\lambda, \lambda/2}(Q_T)$ of functions defined on a bounded cylinder Q_T consists of all continuous functions $H(x, \tau)$ in $\bar{Q}_T$ such that H is λ-Hölder continuous in the x-variable and $\lambda/2$-Hölder continuous in the t-variable. The norm is defined as the sum of the maximum norm and corresponding Hölder semi-norms. The space $\mathcal{H}^{2+\lambda, 1+\lambda/2}(Q_T)$ consists of all continuous functions on $\bar{Q}_T$ such that $\partial_\tau H, \partial_x^2 H \in \mathcal{H}^{\lambda, \lambda/2}(Q_T)$. Finally, the space $\mathcal{H}^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T])$ consists of all functions $H : \mathbb{R} \times [0, T] \to \mathbb{R}$ such that $H \in \mathcal{H}^{2+\lambda, 1+\lambda/2}(Q_T)$ for any bounded cylinder Q_T (cf. [Ladyženskaja et al. 1968, Chapter I]).

We first derive useful lower and upper bounds of a solution H to the Cauchy problem (28). The idea of proving upper and lower estimates for $H(x, \tau)$ is based on construction of suitable sub- and super-solutions to the parabolic Eq. (28) (cf. Ladyženskaja et al. 1968).

Lemma 3.1 *Suppose that the initial condition $H(.,0) \in \mathcal{H}^{\lambda}(\mathbb{R})$ is non-negative and uniformly bounded from above, i.e., $\overline{H} = \sup_{x \in \mathbb{R}} H(x,0) < \infty$. Assume $\beta(H)$ is a $C^{1,\varepsilon}$ smooth function for $H \geq 0$ and satisfying the following strong parabolic inequalities:*

$$\lambda_- \leq \beta'(H) \leq \lambda_+$$

for any $H \geq 0$ where $\lambda_\pm > 0$ are constants. If the bounded solution $H(x,\tau)$ to the quasilinear parabolic Eq. (28) belongs to the function space $\mathcal{H}^{2+\lambda,1+\lambda/2}(\mathbb{R} \times [0,T]) \cap L_\infty(\mathbb{R} \times (0,T))$, for some $0 < \lambda < 1$, then it satisfies the following inequalities:

$$0 \leq H(x,\tau) \leq \overline{H}, \quad \text{for any } \tau \in [0,T) \text{ and } x \in \mathbb{R}.$$

Proof The quasilinear parabolic Eq. (28) can be rewritten in the form:

$$\partial_\tau H = \partial_x (\beta'(H)\partial_x H) + \beta'(H)\partial_x H + r\partial_x H. \tag{30}$$

Notice that the right-hand side of (30) is a strictly parabolic operator because $0 < \lambda_- \leq \beta'(H) \leq \lambda_+$. Since the constant functions $\underline{H} \equiv 0$ and $\overline{H}$ are solutions to (30) then the statement is a consequence of the parabolic comparison principle for strongly parabolic equations (see e.g. [Ladyženskaja et al. 1968, Chapter V,(8.2)]). $\qquad \square$

In the next Proposition we show that the diffusion function $\beta(H)$ corresponding to the nonlinear Black–Scholes equation for pricing options under the variable transaction costs satisfies the strong parabolicity assumption.

Proposition 3.2 *Let $C(\xi)$ be a measurable bounded transaction costs function which is nonincreasing and such that $\underline{C}_0 \leq C(\xi) \leq C_0$ for all $\xi \geq 0$. Let*

$$\hat{\sigma}(H)^2 = \frac{\sigma^2}{2}\left(1 - \sqrt{\frac{2}{\pi}}\frac{\tilde{C}(\sigma\sqrt{\Delta t}|H|)}{\sigma\sqrt{\Delta t}}sgn(H)\right).$$

Then for the diffusion function $\beta(H) = \hat{\sigma}(H)^2 H$ the following inequalities hold:

$$\frac{\sigma^2}{2}(1 - Le) \leq \beta'(H) \leq \frac{\sigma^2}{2}(1 - 2\underline{Le} + Le)$$

for all $H \geq 0$, where $\underline{Le} = \sqrt{\frac{2}{\pi}}\frac{\underline{C}_0}{\sigma\sqrt{\Delta t}}$ and $Le = \sqrt{\frac{2}{\pi}}\frac{C_0}{\sigma\sqrt{\Delta t}}$.

Proof For $H \geq 0$ we have

$$\frac{2}{\sigma^2}\beta'(H) = 1 - \sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\left(\tilde{C}(\xi) + \xi\tilde{C}'(\xi)\right), \quad \text{where } \xi \equiv \sigma\sqrt{\Delta t}H.$$

Since C is a nonincreasing function then $\tilde{C}'(\xi) \leq 0$ and $\tilde{C}(\xi) \leq C_0$ (see Proposition 2.2) then the inequality $\frac{\sigma^2}{2}(1 - Le) \leq \beta'(H)$ easily follows. According to

Proposition 2.3 we have $\tilde{C}(\xi) + \xi \tilde{C}'(\xi) \geq 2\underline{C}_0 - C_0$ and so $\beta'(H) \leq \frac{\sigma^2}{2}(1 - 2\underline{\text{Le}} + \text{Le})$ and the proof of the statement follows. □

Theorem 3.1 *Suppose that the initial condition $H(.,0) \geq 0$ belongs to the Hölder space $\mathcal{H}^{2+\lambda}(\mathbb{R})$ for some $0 < \lambda < \min(1/2, \varepsilon)$ and $\overline{H} = \sup_{x \in \mathbb{R}} H(x, 0) < \infty$. Assume that $\beta \in C^{1,\varepsilon}$ satisfies $\lambda_- \leq \beta'(H) \leq \lambda_+$ for any $0 \leq H \leq \overline{H}$ where $\lambda_\pm > 0$ are constants.*

Then there exists a unique classical solution $H(x, \tau)$ to the quasilinear parabolic Eq. (28) satisfying the initial condition $H(x, 0)$. The function $\tau \mapsto \partial_\tau H(x, \tau)$ is $\lambda/2$-Hölder continuous for all $x \in \mathbb{R}$ whereas $x \mapsto \partial_x H(x, \tau)$ is Lipschitz continuous for all $\tau \in [0, T]$. Moreover, $\beta(H(., .)) \in \mathcal{H}^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T])$ and $0 < H(x, \tau) \leq \overline{H}$ for all $(x, \tau) \in \mathbb{R} \times [0, T]$.

Proof The proof is based on the so-called Schauder's theory on existence and uniqueness of classical Hölder smooth solutions to a quasi-linear parabolic equation of the form (28). It follows the same ideas as the proof of [Kilianová and Ševčovič (2013), Theorem 5.3] where Kilianová and Ševčovič investigated a similar quasilinear parabolic equation obtained from a nonlinear Hamilton–Jacobi–Bellman equation in which a stronger assumption $\beta \in C^{1,1}$ is assumed. Nevertheless, we sketch the key steps of the proof.

The Schauder theory requires that the diffusion coefficient of a quasi-linear parabolic equation is sufficiently smooth. Therefore the function β has to be regularized by a δ-parameterized family of smooth mollifier functions $\beta_{(\delta)}(H)$ such that $\beta_{(\delta)} \rightrightarrows \beta$, and $\beta'_{(\delta)} \rightrightarrows \beta'$ locally uniformly as $\delta \to 0$. For any $\delta > 0$, the existence of the unique classical bounded solution $H^\delta \in \mathcal{H}^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T]) \cap L_\infty(\mathbb{R} \times (0, T))$ to the Cauchy problem: $\partial_\tau H^\delta - \partial_x(\beta'_{(\delta)}(H^\delta)\partial_x H^\delta) = \partial_x f(H^\delta, \beta_{(\delta)}(H^\delta))$, $H^\delta(x, 0) = H(x, 0)$, $x \in \mathbb{R}, t \in [0, T]$, follows from [Ladyženskaja et al. 1968, Theor. 8.1 and Rem. 8.2, Ch. V, pp. 495–496]. Here $f(H, \beta(H)) := \beta(H) + rH$.

By virtue of Lemma 3.1, $H^\delta, 0 < \delta \ll 1$, is uniformly bounded in the space $L_\infty(Q_T)$ for any bounded cylinder Q_T. Using the inequality [Ladyženskaja et al. 1968, Chapter I,(6.6)] we can prove that $H^\delta, 0 < \delta \ll 1$, is also uniformly bounded in the Sobolev space $W_2^1(Q_T)$. It means that there exists a subsequence $H^{\delta_k} \rightharpoonup H$ weakly converging to a function $H \in W_2^1(Q_T)$ as $\delta_k \to 0$. As a consequence of the Rellich-Kondrashov compactness embedding theorem $W_2^1(Q_T) \hookrightarrow L_2(Q_T)$ (cf. [Ladyženskaja et al. 1968, Chapter II,Theorem2.1]) the limiting function $H \in W_2^1(Q_T)$ is a weak solution to the quasi-linear parabolic Eq. (28). Since $H, f \in W_2^1(Q_T)$ we obtain $\partial_x^2 \beta(H) \in L_2(Q_T)$. Furthermore, $\partial_\tau \beta(H) \in L_2(Q_T)$ because $\lambda_- < \beta'(H) < \lambda_+$. Hence, $\beta(H)$ belongs to the parabolic Sobolev space $W_2^{2,1}(Q_T)$ which is continuously embedded into the Hölder space $\mathcal{H}^{\lambda, \lambda/2}(Q_T)$ for any $0 < \lambda < \min(1/2, \varepsilon)$.

Finally, the transformed function $z(x, \tau) := \beta(H(x, \tau))$ is a solution to the quasi-linear parabolic equation in the non-divergent form:

$$\partial_\tau z = \zeta(z) \left[\partial_x^2 z + \partial_x f(\alpha(z), z) \right] = 0, \qquad z(x, 0) = \beta(H(x, 0)),$$

where $\zeta(z) = \beta'(\alpha(z))$ and $z \mapsto \alpha(z)$ is the inverse function to the increasing function $H \mapsto \beta(H)$. The function $z \mapsto \beta'(z)$ is ε-Hölder continuous. Thus $z \mapsto \zeta(z)$ is $\varepsilon-$ Hölder continuous as well. Now, we can apply a simple boot-strap argument to show that $z = z(x, \tau)$ is sufficiently smooth. Clearly, it is a solution to the linear parabolic equation in non-divergence form

$$\partial_\tau z = a(x, \tau)\partial_x^2 z + b(x, \tau)\partial_x z, \qquad z(x, 0) = \beta(H(x, 0)),$$

where $a(x, \tau) := \zeta(z(x, \tau)), b(x, \tau) = \zeta(z(x, \tau))\left(1 + r\alpha'(z(x, \tau))\right)$. The functions a and b belong to the Hölder space $H^{\lambda, \lambda/2}(Q_T)$ because $z \in H^{\lambda, \lambda/2}(Q_T)$. With regard to [Ladyženskaja et al. 1968, Theorem 12.2, Chapter III] we have $z = \beta(H) \in H^{2+\lambda, 1+\lambda/2}(Q_T)$ and the proof now follows because the domain $Q_T \subset \mathbb{R} \times (0, T)$ was arbitrary. $\qquad \square$

4 Numerical Full Space-Time Discretization Scheme for Solving the Gamma Equation

The purpose of this section is to derive an efficient numerical scheme for solving the Gamma equation. The construction of numerical approximation of a solution H to (28) is based on a derivation of a system of difference equations corresponding to (28) to be solved at every discrete time step. We make use of the numerical scheme adopted from the paper by Jandačka and Ševčovič (2005) in order to solve the Gamma Eq. (28) for a general function $\beta = \beta(H)$ including, in particular, the case of the model with variable transaction costs. The efficient numerical discretization is based on the finite volume approximation of the partial derivatives entering (28). The resulting scheme is semi–implicit in a finite–time difference approximation scheme.

Other finite difference numerical approximation schemes are based on discretization of the original fully nonlinear Black–Scholes equation in non-divergence form (3). We refer the reader to recent publications by Ankudinova and Ehrhardt (2008), Company et al. (2008), During et al. (2003), Liao and Khaliq (2009), Zhou et al. (2015). Recently, a quasilinearization technique for solving the fully nonlinear parabolic Eq. (3) was proposed and analyzed by Koleva et al. (2013). Our approach is based on a solution to the quasilinear Gamma equation written in the divergence form, so we can use existing finite volume based numerical scheme to solve the problem efficiently (cf. Jandačka and Ševčovič 2005; Kútik and Mikula 2011).

For numerical reasons we restrict the spatial interval to $x \in (-L, L)$ where $L > 0$ is sufficiently large. Since $S = Ee^x \in (Ee^{-L}, Ee^L)$ it is sufficient to take $L \approx 2$ in order to include the important range of values of S. For the purpose of construction of a numerical scheme, the time interval $[0, T]$ is uniformly divided with a time step $k = T/m$ into discrete points $\tau_j = jk$, where $j = 0, 1, \cdots, m$. We consider the spatial interval $[-L, L]$ with uniform division with a step $h = L/n$, into discrete points $x_i = ih$, where $i = -n, \cdots, n$.

The proposed numerical scheme is semi–implicit in time. Notice that the term $\partial_x^2 \beta$, can be expressed in the form $\partial_x^2 \beta = \partial_x \left(\beta'(H)\partial_x H\right)$, where β' is the derivative of $\beta(H)$ with respect to H. In the discretization scheme, the nonlinear terms $\beta'(H)$ are

evaluated from the previous time step τ_{j-1} where as linear terms are solved at the current time level.

Such a discretization scheme leads to a solution of a tridiagonal system of linear equations at every discrete time level. First, we replace the time derivative by the time difference, approximate H in nodal points by the average value of neighboring segments, then we collect all linear terms at the new time level τ_j and by taking all the remaining terms from the previous time level τ_{j-1}. We obtain a tridiagonal system for the solution vector $H^j = (H^j_{-n+1}, \cdots, H^j_{n-1})^T \in \mathbb{R}^{2n-1}$:

$$a^j_i H^j_{i-1} + b^j_i H^j_i + c^j_i H^j_{i+1} = d^j_i, \quad H^j_{-n} = 0, \ H^j_n = 0, \tag{31}$$

where $i = -n+1, \cdots, n-1$ and $j = 1, \cdots, m$. The coefficients of the tridiagonal matrix are given by

$$a^j_i = -\frac{k}{h^2} \beta'_H(H^{j-1}_{i-1}) + \frac{k}{2h} r \quad c^j_i = -\frac{k}{h^2} \beta'_H(H^{j-1}_i) - \frac{k}{2h} r, \quad b^j_i = 1 - (a^j_i + c^j_i),$$
$$d^j_i = H^{j-1}_i + \frac{k}{h} \left(\beta(H^{j-1}_i) - \beta(H^{j-1}_{i-1}) \right).$$

It means that the vector H^j at the time level τ_j is a solution to the system of linear equations $\mathbf{A}^{(j)} H^j = d^j$, where the $(2n-1) \times (2n-1)$ matrix $\mathbf{A}^{(j)} = \text{tridiag}(a^j, b^j, c^j)$. In order to solve the tridiagonal system in every time step in a fast and effective way, we can use the efficient Thomas algorithm.

Finally, with regard to Proposition 3.1 and Remark 3.1 the option price $V(S, T - \tau_j)$ can be constructed from the discrete solution H^j_i by means of a simple integration scheme:

$$\text{(call option)} \qquad V(S, T - \tau_j) = h \sum_{i=-n}^{n} (S - Ee^{x_i})^+ H^j_i, \quad j = 1, \cdots, m,$$

$$\text{(put option)} \qquad V(S, T - \tau_j) = h \sum_{i=-n}^{n} (Ee^{x_i} - S)^+ H^j_i, \quad j = 1, \cdots, m.$$

4.1 Numerical Results for the Nonlinear Model with Variable Transaction Costs

In this section we present the numerical results for computation of the option price. As an example for numerical approximation of a solution we consider the nonlinear Black–Scholes equation for pricing options under variable transaction costs described by the piecewise linear nonincreasing function, depicted in Fig. 1. The function $\beta(H)$ corresponding to the variable transaction costs function $C(\xi)$ has the form
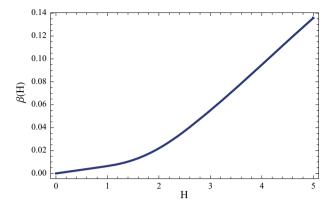
Fig. 4 A graph of the function $\beta(H)$ corresponding to the nonlinear model with a piecewise linear variable transaction costs function

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\text{sgn}(H)}{\sigma\sqrt{\Delta t}}\right)H,$$

where $\tilde{C}$ is the modified transaction costs function. We assume that the hedging time Δt is such that Le < 1 (Fig. 4).

In our computations we chose the following model parameters describing the piecewise transaction costs function: $C_0 = 0.02, \kappa = 0.3, \xi_- = 0.05, \xi_+ = 0.1$. The length of the time interval between two consecutive portfolio rearrangements: $\Delta t = 1/261$. The maturity time $T = 1$, historical volatility $\sigma = 0.3$ and the risk-free interest rate $r = 0.011$. As for the numerical parameters we chose $L = 2.5, n = 250, m = 200$, and the parameter $\tau^* = 0.005$ for approximation of the initial Dirac δ–function. The parameters $C_0, \sigma, \kappa, \xi_\pm$ and Δt correspond to the Leland numbers Le = 0.85935 and $\underline{\text{Le}} = 0.21484$. In Table 1 we present option values $V_{vtc}(S, 0)$ for different prices of the underlying asset achieved by a numerical solution. In Fig. 5 we plot the solution $V_{vtc}(S, t)$ and the option price delta factor $\Delta(S, t) = \partial_S V(S, t)$, for various times $t \in \{0, T/3, 2T/3\}$. The upper dashed line corresponds to the solution of the linear Black–Scholes equation with the higher volatility $\hat{\sigma}_{max}^2 = \sigma^2\left(1 - \underline{C}_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$, where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, whereas the lower dashed line corresponds to the solution with a lower volatility $\hat{\sigma}_{min}^2 = \sigma^2\left(1 - C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$.

The empirically observed fact (see Table 1) that

$$V_{\sigma_{min}}(S, t) \leq V_{vtc}(S, t) \leq V_{\sigma_{max}}(S, t) \quad \text{for all } S > 0, t \in [0, T],$$

can be proved analytically. It is a consequence of the parabolic comparison principle (cf. [Ladyženskaja et al. 1968, Chapter V,(8.2)]). Indeed, the fully nonlinear Black–Scholes Eq. (27) can be considered as a nonlinear strongly parabolic equation

$$\partial_\tau V = \mathcal{F}(S, V, \partial_S V, \partial_S^2 V), \tag{32}$$

Table 1 Call option values computed by means of the numerical solution of nonlinear Black–Scholes model in comparison to solutions $V_{\sigma max}$, $V_{\sigma min}$ computed by the linear Black–Scholes equation with constant volatility $\sigma = \sigma_{max}$ and $\sigma = \sigma_{min}$

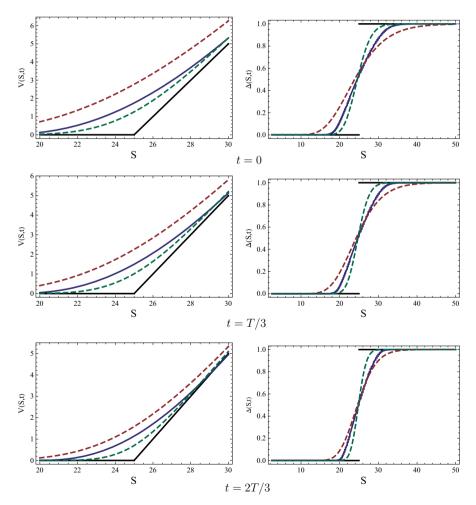| S	$V_{\sigma max}(S,0)$	$V_{vtc}(S,0)$	$V_{\sigma min}(S,0)$
20	0.709	0.127	0.029
23	1.752	0.844	0.421
25	2.768	1.748	1.258
28	4.723	3.695	3.474
30	6.256	5.321	5.327

Fig. 5 The call option price $V(S,t)$ as a function of S for $t \in \{0, T/3, 2T/3\}$ (*left*) and its delta $\Delta(S,t) = \partial_S V(S,t)$

for the option price $V = V(S, T - \tau)$, where

$$\mathcal{F}(S, V, \partial_S V, \partial_S^2 V) = \frac{1}{2}\sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S|\partial_S^2 V|\sqrt{\Delta t})\frac{\text{sgn}(S\partial_S^2 V)}{\sigma\sqrt{\Delta t}}\right)S^2\partial_S^2 V$$
$$+ rS\partial_S V - rV.$$

Recall that $0 < \lambda_- \leq \partial_Q \mathcal{F}(S, V, P, Q) \leq \lambda_+$ for all S, V, P and $Q \geq 0$ and so Eq. (32) is indeed a strongly parabolic equation. For the solution $V_{\sigma_{min}}(S, t)$ of the linear Black–Scholes equation with the constant volatility $\sigma_{min}^2 = \sigma^2(1 - \text{Le})$ we have

$$\partial_\tau V_{\sigma_{min}} = \frac{1}{2}\sigma^2(1 - \text{Le})S^2\partial_S^2 V_{\sigma_{min}} + rS\partial_S V_{\sigma_{min}} - rV_{\sigma_{min}}$$
$$\leq \mathcal{F}(S, V_{\sigma_{min}}, \partial_S V_{\sigma_{min}}, \partial_S^2 V_{\sigma_{min}}),$$

because $\tilde{C}(\xi) \leq C_0$ and so

$$\sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S|\partial_S^2 V_{\sigma_{min}}|\sqrt{\Delta t})\frac{1}{\sigma\sqrt{\Delta t}} \leq \sqrt{\frac{2}{\pi}}C_0\frac{1}{\sigma\sqrt{\Delta t}} = \text{Le}.$$

Hence $V_{\sigma_{min}}$ is a sub-solution to the strongly parabolic Eq. (32). Therefore, by the parabolic comparison principle, $V_{\sigma_{min}}(S, t) \leq V_{vtc}(S, t)$ for all $S > 0$ and $t \in [0, T]$. Analogously, the inequality $V_{vtc}(S, t) \leq V_{\sigma_{max}}(S, t)$ follows from the parabolic comparison principle because $\tilde{C}(\xi) \geq \underline{C_0}$.

The dependence of the call option price on time $t \in [0, T]$ for $S \in \{20, 23, 25\}$ with $E = 25$ is shown in Fig. 6. We can also see that the price converges to zero at expiration for $S \leq E$.

4.2 Some Practical Implications in Financial Portfolio Management

Numerical results obtained in the previous subsection can be interpreted from the financial portfolio management point of view. For example, temporal behavior of the call option price shown in Fig. 6 for the underlying asset price $S = E$ indicates that the option price given by the solution $V_{vtc}(S, t)$ of the nonlinear variable transaction cost model is closer to the lower bound $V_{\sigma_{min}}(S, t)$ for initial times $t \approx 0$. But in later times when $t \to T$ the price $V_{vtc}(S, t)$ is closer to the upper bound $V_{\sigma_{max}}(S, t)$. It can be interpreted as follows: at the beginning of the contract the portfolio manager need not perform many transaction in order to hedge the portfolio. The transaction costs per one transaction is equal to C_0. On the other hand, when the time t approaches expiration T then it is necessary to make frequent rearrangements of the portfolio and so the traded volume of assets increases. Hence the investor pays discounted lower transaction costs value $\underline{C_0}$ per one transaction of the short positioned underlying asset. Consequently, the option price is higher.

The comparison principle $V_{\sigma_{min}}(S, t) \leq V_{vtc}(S, t) \leq V_{\sigma_{max}}(S, t)$ (see also Table 1) has the following practical implication: if the transaction cost $C(\xi)$ per unit share
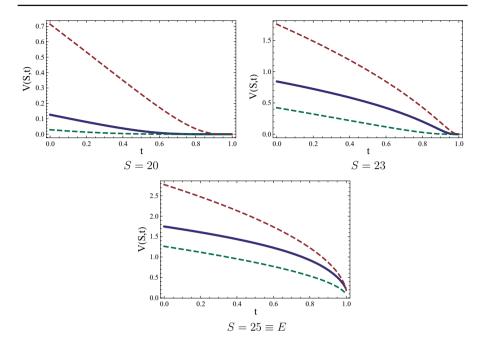
Fig. 6 The call option price $V(S, t)$ as a function of time $t \in [0, T]$ for $S \in \{20, 23, 25\}$

depends on the volume of traded shares ξ and it belongs to the interval $[\underline{C_0} \leq C \leq C_0]$ the option price V_{vtc} can be estimated from above (below) by the option price corresponding to the constant transaction costs $\underline{C_0}$ (C_0).

Conclusions

In this paper we have analyzed a nonlinear generalization of the Black–Scholes equations arising when options are priced under variable transaction costs for buying and selling underlying assets. The mathematical model is represented by the fully nonlinear parabolic equation with the diffusion coefficient depending on the second derivative of the option price. We have investigated properties of various realistic variable transaction costs functions. Furthermore, for a general class of nonlinear Black–Scholes equation we have developed a transformation technique, by means of which the fully nonlinear equation can be transformed into a quasilinear parabolic equation. We have proved existence and uniqueness of classical solutions to the transformed equation. Finally, we have presented a numerical approximation scheme and we computed option prices for pricing model under variable transaction costs.

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