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Matthias Ehrhardt
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Novel Methods in Computational Finance

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Chapter 1

Nonlinear Parabolic Equations Arising in Mathematical Finance

Daniel Ševčovič

Abstract This survey chapter is focused on qualitative and numerical analyses of fully nonlinear partial differential equations of parabolic type arising in financial mathematics. The main purpose is to review various non-linear extensions of the classical Black-Scholes theory for pricing financial instruments, as well as models of stochastic dynamic portfolio optimization leading to the Hamilton-Jacobi-Bellman (HJB) equation. After suitable transformations, both problems can be represented by solutions to nonlinear parabolic equations. Qualitative analysis will be focused on issues concerning the existence and uniqueness of solutions. In the numerical part we discuss a stable finite-volume and finite difference schemes for solving fully nonlinear parabolic equations.

1.1 Nonlinear Generalization of the Black-Scholes Equation for Pricing Financial Instruments

According to the classical theory developed by Black, Scholes and Merton the value $V(S, t)$ of an option in the idealized financial market can be computed from a solution to the well-known Black-Scholes (BS) linear parabolic equation:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (r - q)S \partial_S V - rV = 0, \quad t \in [0, T], \quad S > 0, \quad (1.1)$$

derived by Black and Scholes and, independently by Merton (cf. [29, 38]). Here $\sigma > 0$ is the volatility of the underlying asset driven by the geometric Brownian motion, $r > 0$ is the risk-free interest rate of zero-coupon bond and $q \geq 0$ is the dividend rate. Similarly, as in the case of the HJB equation the solution is subject to the terminal condition $V(S, T) = \bar{V}(S)$ at $t = T$.

The linear Black-Scholes equation with constant volatility σ has been derived under several restrictive assumptions like e.g., frictionless, liquid and complete

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markets, etc. We also recall that the linear Black-Scholes equation provides a solution corresponding to a perfectly replicated hedging portfolio which need not be a desirable property. In the last decades some of these assumptions have been relaxed in order to model, for instance, the presence of transaction costs (see e.g. Leland [18, 29] and Avellaneda and Paras [5]), feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Schönbucher and Willmott [40], Frey and Patie [16], Frey and Stremme [15], imperfect replication and investor's preferences (Barles and Soner [8]), risk from the unprotected portfolio (Jandačka and Ševčovič [22]). Another nonlinear model in which transaction costs are described by a decreasing function of the number of shares has been derived by Amster et al. [2]. In all aforementioned generalizations of the linear BS equation (1.1) the constant volatility σ is replaced by a nonlinear function:

$$\sigma = \sigma(S\partial_S^2 V) \quad (1.2)$$

depending on the second derivative $\partial_S^2 V$ of the option price itself.

One of the first nonlinear models taking into account transaction costs is the Leland model for pricing the call and put options. This model was further extended by Hoggard et al. [18] for general type of derivatives. In this model the variance σ^2 is given by

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 (1 - \text{Le} \operatorname{sgn}(S\partial_S^2 V)) = \begin{cases} \sigma^2(1 - \text{Le}), & \text{if } \partial_S^2 V > 0, \\ \sigma^2(1 + \text{Le}), & \text{if } \partial_S^2 V < 0, \end{cases} \quad (1.3)$$

where $\text{Le} = \sqrt{\frac{2}{\pi}} \frac{C_0}{\sigma\sqrt{\Delta t}}$ is the so-called Leland number, σ_0 is a constant historical volatility, $C_0 > 0$ is a constant transaction costs per unit dollar of transaction in the underlying asset market and Δt is the time-lag between consecutive portfolio adjustments. The nonlinear model with the volatility function given as in (1.3) can be also viewed as a jumping volatility model investigated by Avellaneda and Paras [5].

The important contribution in this direction has been presented in the work by Amster et al. [2], where the transaction costs are assumed to be a non-increasing linear function of the form $C(\xi) = C_0 - \kappa\xi$, ($C_0, \kappa > 0$), depending on the volume of trading stock $\xi \geq 0$ needed to hedge the replicating portfolio. A disadvantage of such a transaction costs function is the fact that it may attain negative values when the amount of transactions exceeds the critical value $\xi = C_0/\kappa$. In the model studied by Amster et al. [2] (see also Averbuj [4], Mariani et al. [33]) the volatility function has the following form:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 (1 - \text{Le} \operatorname{sgn}(S\partial_S^2 V) + \kappa S\partial_S^2 V). \quad (1.4)$$

In the recent paper [39] Ševčovič and Žitnanská investigated a model for pricing option under variable transaction costs.

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \left(1 - \sqrt{\frac{2}{\pi}} \tilde{C}(\sigma S |\partial_S^2 V| \sqrt{\Delta t}) \frac{\text{sgn}(S\partial_S^2 V)}{\sigma \sqrt{\Delta t}} \right) \quad (1.5)$$

where \tilde{C} is the mean value modification of the transaction cost function $C = C(\xi)$ defined as follows: $\tilde{C}(\xi) = \int_0^\infty C(\xi x) x e^{-x^2/2} dx$. As an example one can consider the piecewise linear transaction cost function of the form:

$$C(\xi) = \begin{cases} C_0, & \text{if } 0 \leq \xi \leq \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if } \xi_- \leq \xi \leq \xi_+, \\ \underline{C}_0, & \text{if } \xi \geq \xi_+. \end{cases} \quad (1.6)$$

Bakstein and Howison [7] investigated a parametrized model for liquidity effects arising from the asset trading. In their model σ is a quadratic function of the term $H = S\partial_S^2 V$:

$$\begin{aligned} \sigma(S\partial_S^2 V)^2 = & \sigma_0^2 \left(1 + \bar{\gamma}^2(1 - \alpha)^2 + 2\lambda S\partial_S^2 V + \lambda^2(1 - \alpha)^2 (S\partial_S^2 V)^2 \right. \\ & \left. + 2\sqrt{\frac{2}{\pi}} \bar{\gamma} \text{sgn}(S\partial_S^2 V) + 2\sqrt{\frac{2}{\pi}} \lambda(1 - \alpha)^2 \bar{\gamma} |S\partial_S^2 V| \right). \end{aligned} \quad (1.7)$$

The parameter λ corresponds to a market depth measure, i.e. it scales the slope of the average transaction price. Next, the parameter $\bar{\gamma}$ models the relative bid-ask spreads and it is related to the Leland number through relation $2\bar{\gamma} \sqrt{2/\pi} = \text{Le}$. Finally, α transforms the average transaction price into the next quoted price, $0 \leq \alpha \leq 1$.

The risk adjusted pricing methodology (RAPM) model takes into account the risk from the unprotected portfolio was proposed by Kratka [28]. It was generalized and analyzed by Jandačka and Ševčovič [22]. In this model the volatility function has the form:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \left(1 + \mu (S\partial_S^2 V)^{\frac{1}{3}} \right), \quad (1.8)$$

where $\sigma_0 > 0$ is the constant historical volatility of the asset price return and $\mu = 3(C_0^2 R / 2\pi)^{\frac{1}{3}}$, where $C_0, R \geq 0$ are non-negative constants representing the transaction cost measure and the risk premium measure, respectively.

If transaction costs are taken into account perfect replication of the contingent claim is no longer possible and further restrictions are needed in the model. By assuming that investor's preferences are characterized by an exponential utility

function Barles and Soner (cf. [8]) derived a nonlinear Black-Scholes equation with the volatility σ given by

$$\sigma(S\partial_S^2 V, S, t)^2 = \sigma_0^2 (1 + \Psi(a^2 e^{r(T-t)} S^2 \partial_S^2 V)) \quad (1.9)$$

where Ψ is a solution to the ODE:

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x) - x}}, \quad \Psi(0) = 0,$$

and $a > 0$ is a given constant representing risk aversion. Notice that $\Psi(x) = O(x^{\frac{1}{3}})$ for $x \rightarrow 0$ and $\Psi(x) = O(x)$ for $x \rightarrow \infty$.

All the nonlinear volatility models mentioned in this section can be written in the form of a solution to the fully nonlinear parabolic equation:

$$\partial_t V + \frac{1}{2}\sigma(\partial_S^2 V)^2 S^2 \partial_S^2 V + (r - q)S\partial_S V - rV = 0, \quad t \in [0, T], S > 0. \quad (1.10)$$

Jandačka and Ševčovič [22] proposed the method of transformation of equation (1.10) into a quasi-linear parabolic equation for the second derivative $\partial_S^2 V$ (the so-called Gamma of an option) of a solution. Indeed, if we introduce the new variables $H(x, \tau) = S\partial_S^2 V(S, t)$, $x = \ln S$ and $\tau = T - t$ then Eq. (1.10) can be transformed into the so-called Gamma equation:

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + (r - q)\partial_x H - qH, \quad x \in \mathbb{R}, \tau \in (0, T), \quad (1.11)$$

where

$$\beta(x, H) = \frac{1}{2}\sigma(H)^2 H$$

(cf. [10, 22]). Recall that the Gamma equation can be obtained by twice differentiation with respect to x of the Black-Scholes equation (1.18) with the volatility of the general type (1.2). A solution $H(x, \tau)$ to the Cauchy problem for (1.11) is subject to the initial condition $H(x, 0) = H_0(x)$.

1.2 Nonlinear Hamilton-Jacobi-Bellman Equation and Optimal Allocation Problems

Optimal allocation and optimal investment problems with state constraints attracted a lot of attention from both theoretical as well as application point of view. The main purpose is to maximize the total expected discounted utility of consumption for the optimal portfolio investment consisting of several stochastic assets, over infinite or

finite time horizons. It is known that the value function of the underlying stochastic control problem is the unique smooth solution to the corresponding Hamilton-Jacobi-Bellman (HJB) equation and the optimal consumption and portfolio are presented in feedback form (Zariphopoulou [44]).

Let us consider the stylized financial market in which the aim of a portfolio manager is to maximize the expected value of the terminal wealth of a portfolio, measured by a prescribed utility function U . In particular, if n is the number of assets entering the portfolio, T the investment horizon, the goal is to find an optimal trading strategy $\{\theta\} = \{\theta_t \in \mathbb{R}^n \mid t \in [0, T]\}$ belonging to a set $A = A_{0,T}$ of strategies $A_{t,T} = \{\{\theta\} \mid \theta_s \in S^n, s \in [t, T]\}$, where $S^n = \{\theta_t \in \mathbb{R}^n \mid \theta_t \in [0, 1]^n, 1^T \theta_t = 1\}$ is a convex compact simplex such that $\{\theta\}$ maximizes the expected terminal utility from the portfolio:

$$\max_{\{\theta\} \in A} \mathbb{E} [U(X_T^\theta) \mid X_0^\theta = x_0]. \quad (1.12)$$

Here $X_t = \ln Y_t$ represents a stochastic process governed by the following stochastic differential equation

$$dX_t^\theta = \left(\mu(\theta) - \frac{1}{2} \sigma(\theta)^2 \right) dt + \sigma(\theta) dW_t$$

for a logarithmic portfolio value, where x_0 is its initial value at the time $t = 0$. Here $\mu(\theta)$ and $\sigma(\theta)$ are the expected return and volatility of the portfolio. As a typical example, one can consider functions $\mu(\theta) = \mu^T \theta$ and $\sigma^2(\theta) = \theta^T \Sigma \theta$, where μ is a vector of mean returns and Σ is a covariance matrix. It is known from the theory of stochastic dynamic programming that the so-called value function

$$V(x, t) := \sup_{\{\theta\} \in A_{t,T}} \mathbb{E} [U(X_T^\theta) \mid X_t^\theta = x] \quad (1.13)$$

subject to the terminal condition $V(x, T) := U(x)$ can be used for solving the stochastic dynamic optimization problem (1.12) (cf. Bertsekas [9], Fleming and Soner [14]). Moreover, it is also known, that the value function $V = V(x, t)$ satisfies the following Hamilton-Jacobi-Bellman equation:

$$\partial_t V + \max_{\theta \in S^n} \left\{ \left(\mu(\theta) - \frac{1}{2} \sigma(\theta)^2 \right) \partial_x V + \frac{1}{2} \sigma(\theta)^2 \partial_x^2 V \right\} = 0, \quad (1.14)$$

for all $x \in \mathbb{R}$, $t \in [0, T)$ and it satisfies the terminal condition $V(., T) := U(.)$ (see e.g. [20, 32]).

In general, explicit solutions to HJB equations are not available and this is why various numerical approaches have to be adopted. Regarding numerical approaches for solving HJB equations associated with portfolio optimization, we can mention and refer to finite difference methods for approximating its viscosity solution developed and analyzed by Tourin and Zariphopoulou [42], Crandall

et al. [12], Nayak and Papanicolaou [36]. Other approach based on Markov chain approximation techniques was investigated by Song [41] and Fleming and Soner [14]. Classical methods for solving HJB equations are discussed by Benton [44]. In [34], Musiela and Zariphopoulou applied the power-like transformation in order to linearize the non-linear PDE for the value function in the case of an exponential utility function. Muthamaran and Sunil [35] solved a multi-dimensional portfolio optimization problem with transaction costs. They used finite element method and iterative procedure that converts a free-boundary problem into a sequence of fixed boundary problems. Peyrl et al. [37] applied a successive approximation algorithm for solving the corresponding HJB equation. The fixed point-policy iteration scheme for solving discretized HJB equations is discussed in Huang et al. [19]. Witte and Reisinger [43] presented a penalty approach for the numerical solution of discrete continuously controlled HJB equations.

In the recent paper [23] Kilianová and Ševčovič transformed the fully nonlinear HJB equation (1.14) into the Cauchy problem for the quasi-linear parabolic equation:

$$\partial_t \varphi + \partial_x^2 \beta(\varphi) + \partial_x [(1 - \varphi)\beta(\varphi)] = 0, \quad x \in R, t \in [0, T), \quad (1.15)$$

$$\varphi(x, T) = 1 - \frac{U''(x)}{U'(x)}, \quad x \in R. \quad (1.16)$$

To this aim we introduced the following transformation:

$$\varphi(x, t) = 1 - \frac{\partial_x^2 V(x, t)}{\partial_x V(x, t)}.$$

It is referred to as the Riccati transformation and it has been proposed and studied in [1, 32] and further analyzed by Ishimura and Ševčovič in [20]. The resulting equation was solved numerically by an iterative method based on the finite volume approximation. Furthermore, it follows from the analysis [23] by Kilianová and Ševčovič that the diffusion function $\beta(\varphi)$ is the value function of the following parametric optimization problem:

$$\beta(\varphi) = \min_{\theta \in S^n} \{-\mu(\theta) + \frac{\varphi}{2} \sigma(\theta)^2\}. \quad (1.17)$$

The dispersion function $\theta \mapsto \sigma(\theta)^2$ is assumed to be strictly convex and $\theta \mapsto \mu(\theta)$ is a linear function. Therefore problem (1.17) belongs to a class of parametric convex optimization problems (cf. Bank et al. [6], Hamala and Trnovská [17]). Useful generalization of the HJB equation (1.14) in case the covariance matrix Σ belongs to some set P of (e.g. ellipsoidal sets) of covariance matrices was studied by Kilianová and Trnovská in [24] with regard to application to the so-called “worst case variance” portfolio model in which the diffusion function (1.17) has the form:

$$\beta(\varphi) = \min_{\theta \in S^n} \max_{\Sigma \in P} -\mu^T \theta + \frac{\varphi}{2} \theta^T \Sigma \theta.$$

They showed this problem can be analyzed by the methods of semidefinite programming. The value function $\beta(\varphi)$ need not be sufficiently smooth and its second derivative can have jumps.

In fact, the Riccati transformation is the logarithmic derivative of the derivative of the value function. In the context of a class of HJB equations with range constraints, the Riccati transformation has been analyzed recently by Ishimura and Ševčovič in [20] where a traveling wave solution to the HJB equation was constructed. Concerning numerical methods for solving the quasi-linear parabolic PDE obtained from the HJB equation by means of the Riccati transformation we mention recent papers by Ishimura et al. [21, 25]. In [25], Koleva considered a similar nonlinear parabolic equation, obtained by means of a Riccati-like transformation of the HJB equation, arising in pension saving management. In contrary to our model problem, she considered a problem without constraints on the optimal decision. She applied two iterative numerical methods, namely the fully implicit Picard method and the mixed Picard-Newton method and discussed their accuracy and effectiveness.

In summary, the nonlinear volatility generalization of the Black-Scholes equation as well as the Hamilton-Jacobi-Bellman equation can be transformed into the quasilinear parabolic equation for the unknown function $H = H(x, \tau)$ representing either the Gamma of the portfolio $H = S\partial_x^2 V$ (nonlinear volatility Black-Scholes models) or the relative risk aversion function $H = 1 - \partial_x^2 V / \partial_x V$ (Hamilton-Jacobi-Bellman equation). The resulting quasilinear parabolic equation has the form:

$$\partial_\tau H = \partial_x^2 \beta(H) + f(x, H, \partial_x H), \quad x \in \mathbb{R}, \tau \in (0, T), \quad (1.18)$$

where β is a suitable nonlinear function.

1.3 Existence of Classical Solutions, Comparison Principle

In this section we recall results on existence of classical smooth solutions to the Cauchy problem for the quasilinear parabolic equation (1.18). Following the methodology based on the so-called Schauder's type of estimates (cf. Ladyzhenskaya et al. [30]), we shall proceed with a definition of function spaces we will work with. Let $\Omega = (x_L, x_R) \subset \mathbb{R}$ be a bounded interval. We denote $Q_T = \Omega \times (0, T)$ the space-time cylinder. Let $0 < \lambda < 1$. By $\mathcal{H}^\lambda(\Omega)$ we denote the Banach space consisting of all continuous functions H defined on $\bar{\Omega}$ which are λ -Hölder continuous. It means that their Hölder semi-norm $\langle H \rangle^{(\lambda)} = \sup_{x, y \in \Omega, x \neq y} |H(x) - H(y)| / |x - y|^\lambda$ is finite. The norm in the space $\mathcal{H}^\lambda(\Omega)$ is then the sum of the maximum norm of H and the semi-norm $\langle H \rangle^{(\lambda)}$. The space $\mathcal{H}^{2+\lambda}(\Omega)$ consists of all twice continuously differentiable functions H in $\bar{\Omega}$ whose second derivative $\partial_x^2 H$ belongs to $\mathcal{H}^\lambda(\Omega)$. The space $\mathcal{H}^{2+\lambda}(\mathbb{R})$ consists of all functions $H : \mathbb{R} \rightarrow \mathbb{R}$ such that $H \in \mathcal{H}^{2+\lambda}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}$.

The parabolic Hölder space $\mathcal{H}^{\lambda, \lambda/2}(Q_T)$ of functions defined on a bounded cylinder Q_T consists of all continuous functions $H(x, \tau)$ in \bar{Q}_T such that H is λ -Hölder continuous in the x -variable and $\lambda/2$ -Hölder continuous in the t -variable. The norm is defined as the sum of the maximum norm and corresponding Hölder semi-norms. The space $\mathcal{H}^{2+\lambda, 1+\lambda/2}(Q_T)$ consists of all continuous functions on \bar{Q}_T such that $\partial_\tau H, \partial_x^2 H \in \mathcal{H}^{\lambda, \lambda/2}(Q_T)$. Finally, the space $\mathcal{H}^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T])$ consists of all functions $H : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ such that $H \in \mathcal{H}^{2+\lambda, 1+\lambda/2}(Q_T)$ for any bounded cylinder Q_T (cf. [30, Chap. I]).

In the nonlinear models discussed in the previous sections one can derive useful lower and upper bounds of a solution H to the Cauchy problem (1.18). The idea of proving upper and lower estimates for $H(x, \tau)$ is based on construction of suitable sub- and super-solutions to the parabolic equation (1.18) (cf. [30]).

$$\lambda_- \leq \beta'(H) \leq \lambda_+$$

for any $H \geq 0$ where $\lambda_\pm > 0$ are constants. This implies strong parabolicity of the governing nonlinear parabolic equation.

Theorem 1.1 ([39, Theorem 3.1]) *Suppose that the initial condition $H(\cdot, 0) \geq 0$ belongs to the Hölder space $\mathcal{H}^{2+\lambda}(\mathbb{R})$ for some $0 < \lambda < \min(1/2, \varepsilon)$ and $\bar{H} = \sup_{x \in \mathbb{R}} H(x, 0) < \infty$. Assume that $\beta, f \in C^{1,\varepsilon}$ and β satisfies $\lambda_- \leq \beta'(H) \leq \lambda_+$ for any $0 \leq H \leq \bar{H}$ where $\lambda_\pm > 0$ are constants.*

Then there exists a unique classical solution $H(x, \tau)$ to the quasilinear parabolic equation (1.18) satisfying the initial condition $H(x, 0)$. The function $\tau \mapsto \partial_\tau H(x, \tau)$ is $\lambda/2$ -Hölder continuous for all $x \in \mathbb{R}$ whereas $x \mapsto \partial_x H(x, \tau)$ is Lipschitz continuous for all $\tau \in [0, T]$. Moreover, $\beta(H(\cdot, \cdot)) \in \mathcal{H}^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T])$ and $0 < H(x, \tau) \leq \bar{H}$ for all $(x, \tau) \in \mathbb{R} \times [0, T]$.

The proof is based on the so-called Schauder's theory on existence and uniqueness of classical Hölder smooth solutions to a quasi-linear parabolic equation of the form (1.18). It follows the same ideas as the proof of [23, Theorem 5.3] where Kilianová and Ševčovič investigated a similar quasilinear parabolic equation obtained from a nonlinear HJB equation in which a stronger assumption $\beta \in C^{1,1}$ is assumed.

1.4 Numerical Full Space-Time Discretization Scheme for Solving the Gamma Equation

In this section we present an efficient numerical scheme for solving the Gamma equation. The construction of numerical approximation of a solution H to (1.18) is based on a derivation of a system of difference equations corresponding to (1.18) to be solved at every discrete time step. We make use of the numerical scheme adopted from the paper by Jandačka and Ševčovič [22] in order to solve the Gamma equation (1.18) for a general function $\beta = \beta(H)$ including, in particular, the case of the

model with variable transaction costs. The efficient numerical discretization is based on the finite volume approximation of the partial derivatives entering (1.18). The resulting scheme is semi-implicit in a finite-time difference approximation scheme.

Other finite difference numerical approximation schemes are based on discretization of the original fully nonlinear Black-Scholes equation in non-divergence form. We refer the reader to recent publications by Ankudinova and Ehrhardt [3], Company et al. [11], Düring et al. [13], Liao and Khaliq [31], Zhou et al. [45]. Recently, a quasilinearization technique for solving the fully nonlinear parabolic equation was proposed and analyzed by Koleva and Vulkov [26]. Our approach is based on a solution to the quasilinear Gamma equation written in the divergence form, so we can use existing finite volume based numerical scheme to solve the problem efficiently (cf. Jandačka and Ševčovič [22], Kútík and Mikula [27]).

For numerical reasons we restrict the spatial interval to $x \in (-L, L)$ where $L > 0$ is sufficiently large. Since $S = Ee^x \in (Ee^{-L}, Ee^L)$ it is sufficient to take $L \approx 2$ in order to include the important range of values of S . For the purpose of construction of a numerical scheme, the time interval $[0, T]$ is uniformly divided with a time step $k = T/m$ into discrete points $\tau_j = jk$, where $j = 0, 1, \dots, m$. We consider the spatial interval $[-L, L]$ with uniform division with a step $h = L/n$, into discrete points $x_i = ih$, where $i = -n, \dots, n$.

The proposed numerical scheme is semi-implicit in time. Notice that the term $\partial_x^2 \beta$, can be expressed in the form $\partial_x^2 \beta = \partial_x (\beta'(H) \partial_x H)$, where β' is the derivative of $\beta(H)$ with respect to H . In the discretization scheme, the nonlinear terms $\beta'(H)$ are evaluated from the previous time step τ_{j-1} whereas linear terms are solved at the current time level.

Such a discretization scheme leads to a solution of a tridiagonal system of linear equations at every discrete time level. First, we replace the time derivative by the time difference, approximate H in nodal points by the average value of neighboring segments, then we collect all linear terms at the new time level τ_j and by taking all the remaining terms from the previous time level τ_{j-1} . We obtain a tridiagonal system for the solution vector $H^j = (H_{-n+1}^j, \dots, H_{n-1}^j)^T \in \mathbb{R}^{2n-1}$:

$$a_i^j H_{i-1}^j + b_i^j H_i^j + c_i^j H_{i+1}^j = d_i^j, \quad H_{-n}^j = 0, \quad H_n^j = 0, \quad (1.19)$$

where $i = -n + 1, \dots, n - 1$ and $j = 1, \dots, m$. The coefficients of the tridiagonal matrix are given by

$$a_i^j = -\frac{k}{h^2} \beta'_H(H_{i-1}^{j-1}) + \frac{k}{2h} r, \quad c_i^j = -\frac{k}{h^2} \beta'_H(H_i^{j-1}) - \frac{k}{2h} r, \quad b_i^j = 1 - (a_i^j + c_i^j),$$

$$d_i^j = H_i^{j-1} + \frac{k}{h} \left(\beta(H_i^{j-1}) - \beta(H_{i-1}^{j-1}) \right).$$

It means that the vector H^j at the time level τ_j is a solution to the system of linear equations $\mathbf{A}^{(j)} H^j = d^j$, where the $(2n-1) \times (2n-1)$ matrix $\mathbf{A}^{(j)} = \text{tridiag}(a^j, b^j, c^j)$.

In order to solve the tridiagonal system in every time step in a fast and effective way, we can use the efficient Thomas algorithm.

In [39] the authors showed that the option price $V(S, T - \tau_j)$ can be constructed from the discrete solution H_i^j by means of a simple integration scheme:

$$\text{(call option)} \quad V(S, T - \tau_j) = h \sum_{i=-n}^n (S - Ee^{x_i})^+ H_i^j, \quad j = 1, \dots, m,$$

$$\text{(put option)} \quad V(S, T - \tau_j) = h \sum_{i=-n}^n (Ee^{x_i} - S)^+ H_i^j, \quad j = 1, \dots, m.$$

1.5 Numerical Results for the Nonlinear Model with Variable Transaction Costs

In this section we present the numerical results for computation of the option price for the nonlinear volatility Black-Scholes model with variable transaction costs derived and analyzed by Ševčovič and Žitnanská in the recent paper [39]. As an example for numerical approximation of a solution we consider variable transaction costs described by the piecewise linear non-increasing function, depicted in Fig. 1.1. The function $\beta(H)$ corresponding to the variable transaction costs function $C(\xi)$ has the form

$$\beta(H) = \frac{\sigma_0^2}{2} \left(1 - \sqrt{\frac{2}{\pi}} \tilde{C}(\sigma|H|\sqrt{\Delta t}) \frac{\text{sgn}(H)}{\sigma\sqrt{\Delta t}} \right) H,$$

where \tilde{C} is the modified transaction costs function.

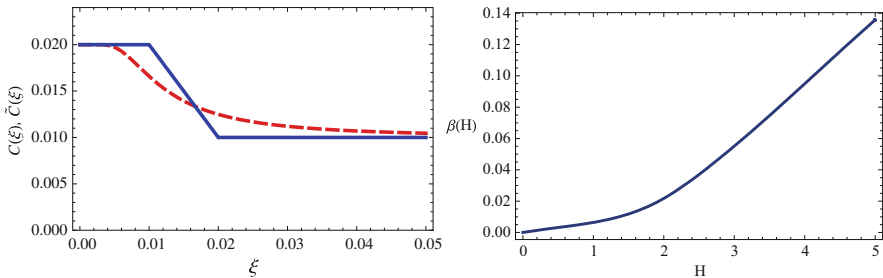


Fig. 1.1 Left: The piecewise linear transaction costs function C (solid line), its mean value modification \tilde{C} (dashed line). Right: the graph of the corresponding function $\beta(H)$. Source [39]

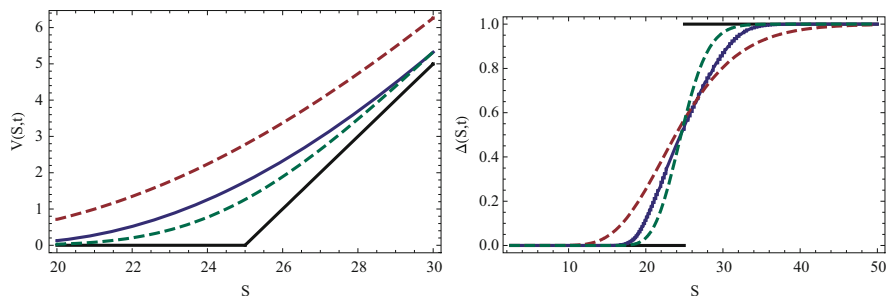


Fig. 1.2 The call option price $V(S, t)$ as a function of S for $t = 0$ (left) and its delta $\Delta(S, t) = \partial_S V(S, t)$. Source [39]

In our computations we chose the following model parameters describing the piecewise transaction costs function: $C_0 = 0.02$, $\kappa = 0.3$, $\xi_- = 0.05$, $\xi_+ = 0.1$. The length of the time interval between two consecutive portfolio rearrangements: $\Delta t = 1/261$. The maturity time $T = 1$, historical volatility $\sigma = 0.3$ and the risk-free interest rate $r = 0.011$. As for the numerical parameters we chose $L = 2.5$, $n = 250$, $m = 200$. The parameters C_0 , σ , κ , ξ_{\pm} and Δt correspond to the Leland numbers $\underline{Le} = 0.85935$ and $\underline{Le} = 0.21484$. In Fig. 1.2 we plot the solution $V_{vrc}(S, t)$ and the option price delta factor $\Delta(S, t) = \partial_S V(S, t)$, for $t = 0$. The upper dashed line corresponds to the solution of the linear Black-Scholes equation with the higher volatility

$$\hat{\sigma}_{max}^2 = \sigma^2 \left(1 - \underline{C}_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma \sqrt{\Delta t}} \right),$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-) > 0$, whereas the lower dashed line corresponds to the solution with a lower volatility

$$\hat{\sigma}_{min}^2 = \sigma^2 \left(1 - C_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma \sqrt{\Delta t}} \right).$$

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References

1. Abe, R., Ishimura, N.: Existence of solutions for the nonlinear partial differential equation arising in the optimal investment problem. Proc. Jpn. Acad. Ser. A. **84**, 11–14 (2008)
2. Amster, P., Averbuĳ, C.G., Mariani, M.C., Rial, D.: A Black-Scholes option pricing model with transaction costs. J. Math. Anal. Appl. **303**, 688–695 (2005)

3. Ankudinova, J., Ehrhardt, M.: On the numerical solution of nonlinear Black-Scholes equations. *Comput. Math. Appl.* **56**, 799–812 (2008)
4. Averbuj, C.G.: Nonlinear integral-differential evolution equation arising in option pricing when including transaction costs: a viscosity solution approach. *Rev. Bras. Econ. Empresas* **12**, 81–90 (2012)
5. Avellaneda, M., Levy, A., Paras, A.: Pricing and hedging derivative securities in markets with uncertain volatilities. *Appl. Math. Fin.* **2**, 73–88 (1995)
6. Bank, B., Guddat, J., Klatte, D., Kummer, B., Tammer, K.: *Nonlinear Parametric Optimization*. Birkhäuser Verlag, Basel/Boston, MA (1983)
7. Bakstein, D., Howison, S.: A non-arbitrage liquidity model with observable parameters. Working paper, <http://eprints.maths.ox.ac.uk/53/>
8. Barles, G., Soner, H.M.: Option Pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance Stochast.* **2**, 369–397 (1998)
9. Bertsekas, D.P.: *Dynamic Programming and Stochastic Control*. Academic, New York (1976)
10. Bordag, L.A., Frey, R.: Study of the risk-adjusted pricing methodology model with methods of Geometrical Analysis. *Stoch. Int. J. Probab. Stoch. Process* **83**, 333–345 (2011)
11. Company, R., Navarro, E., Pintos, J.R., Ponsoda, E.: Numerical solution of linear and nonlinear Black-Scholes option pricing equations. *Comput. Math. Appl.* **56**, 813–821 (2008)
12. Crandall, M.C., Ishii, H., Lions, P.L.: User’s guide to viscosity solutions of second order partial differential equations. *Bull. Am. Math. Soc.* **1**, 1–67 (1992)
13. Düring, B., Fournié, M., Jüngel, A.: High order compact finite difference schemes for a nonlinear Black-Scholes equation. *Int. J. Theor. Appl. Finance* **7**, 767–789 (2003)
14. Fleming, W.H., Soner, H.M.: *Controlled Markov Processes and Viscosity Solutions*, 2nd edn. Springer, New York (2005)
15. Frey, R., Patie, P.: Risk management for derivatives in illiquid markets: a simulation study. In: *Advances in Finance and Stochastics*, pp. 137–159. Springer, Berlin (2002)
16. Frey, R., Stremme, A.: Market volatility and feedback effects from dynamic hedging. *Math. Fin.* **4**, 351–374 (1997)
17. Hamala, M., Trnovská, M.: *Nonlinear Programming, Theory and Algorithms*. Epos, Bratislava (2013)
18. Hoggard, T., Whalley, A.E., Wilmott, P.: Hedging option portfolios in the presence of transaction costs. *Adv. Futures Opt. Res.* **7**, 21–35 (1994)
19. Huang, Y., Forsyth, P.A., Labahn, G.: Combined fixed point and policy iteration for HJB equations in finance. *SIAM J. Numer. Anal.* **50**, 1861–1882 (2012)
20. Ishimura, N., Ševčovič, D.: On traveling wave solutions to a Hamilton-Jacobi-Bellman equation with inequality constraints. *Jpn. J. Industr. Appl. Math.* **30**, 51–67 (2013)
21. Ishimura, N., Koleva, M.N., Vulkov, L.G.: Numerical solution via transformation methods of nonlinear models in option pricing. *AIP Conf. Proc.* **1301**, 387–394 (2010)
22. Jandačka, M., Ševčovič, D.: On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile. *J. Appl. Math.* **2005**, 235–258 (2005)
23. Kilianová, S., Ševčovič, D.: A method of solving Hamilton-Jacobi-Bellman equation for constrained optimal investment problem via Riccati transformation. *Anziam J.* **55**, 14–38 (2013)
24. Kilianová, S., Trnovská, M.: Robust portfolio optimization via solution to the Hamilton-Jacobi-Bellman equation. *Int. J. Comput. Math.* **93**, 725–734 (2016)
25. Koleva, M.: Iterative methods for solving nonlinear parabolic problem in pension saving management. *AIP Conf. Proc.* **1404**, 457–463 (2011)
26. Koleva, M.N., Vulkov, L.G.: Quasilinearization numerical scheme for fully nonlinear parabolic problems with applications in models of mathematical finance. *Math. Comput. Model.* **57**, 2564–2575 (2013)
27. Kútik, P., Mikula, K.: Finite volume schemes for solving nonlinear partial differential equations in financial mathematics. In: *Finite Volumes for Complex Applications VI Problems and Perspectives*. Springer Proceedings in Mathematics, vol. 4, pp. 643–651. Springer, Berlin (2011)

28. Kratka, M.: No mystery behind the smile. *Risk* **9**, 67–71 (1998)
29. Kwok, Y.K.: *Mathematical Models of Financial Derivatives*. Springer, New York (1998)
30. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: *Linear and Quasilinear Equations of Parabolic Type* (Translated from the Russian by S. Smith. *Translations of Mathematical Monographs*), vol. 23. American Mathematical Society, Providence, RI (1968)
31. Liao, W., Khaliq, A.Q.M.: High-order compact scheme for solving nonlinear Black-Scholes equation with transaction costs. *Int. J. Comput. Math.* **86**, 1009–1023 (2009)
32. Macová, Z., Ševčovič, D.: Weakly nonlinear analysis of the Hamilton-Jacobi-Bellman equation arising from pension saving management. *Int. J. Numer. Anal. Model.* **4**, 619–638 (2010)
33. Mariani, M.C., Ncheuguim, E., Sengupta, I.: Solution to a nonlinear Black-Scholes equation. *Electron. J. Differ. Equ.* **158**, 1–10 (2011)
34. Musiela, M., Zariphopoulou, T.: An example of indifference prices under exponential preferences. *Finance Stochast.* **8**, 229–239 (2004)
35. Muthuraman, K., Kumar, S.: Multi-dimensional portfolio optimization with proportional transaction costs. *Math Fin.* **16**, 301–335 (2006)
36. Nayak, S., Papanicolaou, G.: Market influence of portfolio optimizers. *Appl. Math. Fin.* **15**, 21–40 (2008)
37. Peyrl, H., Herzog, F., Geering, H.P.: Numerical solution of the Hamilton-Jacobi-Bellman equation for stochastic optimal control problems. In: *WSEAS International Conference on Dynamical Systems and Control*, Venice, Italy, 2–4 Nov 2005, pp. 489–497
38. Ševčovič, D., Stehlíková, B., Mikula, K.: *Analytical and Numerical Methods for Pricing Financial Derivatives*. Nova Science Publishers, Inc., Hauppauge (2011)
39. Ševčovič, D., Žitnanská, M.: Analysis of the nonlinear option pricing model under variable transaction costs. *Asia-Pacific Finan. Markets* **23**, 153–174 (2016)
40. Schönbucher, P., Wilmott, P.: The feedback-effect of hedging in illiquid markets. *SIAM J. Appl. Math.* **61**, 232–272 (2000)
41. Song, Q.S.: Convergence of Markov chain approximation on generalized HJB equation and its applications. *Automatica* **44**, 761–766 (2008)
42. Tourin, A., Zariphopoulou, T.: Numerical schemes for investment models with singular transactions. *Comput. Econ.* **4**, 287–307 (1994)
43. Witte, J.H., Reisinger, Ch.: Penalty methods for the solution of discrete HJB equations – continuous control and obstacle problems. *SIAM J. Numer. Anal.* **50**, 595–625 (2012)
44. Zariphopoulou, T.: Consumption-investment models with constraints. *SIAM J. Control. Optim.* **1**, 59–85 (1994)
45. Zhou, S., Han, L., Li, W., Zhang, Y., Han, M.: A positivity-preserving numerical scheme for option pricing model with transaction costs under jump-diffusion process. *Comput. Appl. Math.* **34**, 881–900 (2015)