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Numerical methods for nonlinear option pricing models with variable transaction costs

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Abstract

The classical Black-Scholes equation for options pricing exhibits several limitations when applied to real markets in certain conditions. In many settings, the consideration of a constant volatility is no more realistic. In the present paper, we consider the case where the volatility is assumed to depend on the product of the asset price and the second derivative of the option with respect to the asset price (an option Greek which is known as Gamma). This hypothesis has been made in models that incorporate transaction costs, market feedback effects related to stocks trading strategies or illiquid markets, risks related to unprotected portfolios, etc. In these settings, the corresponding nonlinear Black-Scholes equation can be transformed into a quasilinear equation (Gamma equation) in a new unknown variable related to the Gamma of the option.

Once this semilinear Gamma equation has been obtained, we propose a duality method, combined with a characteristics scheme and finite elements methods. The duality method is applied to the maximal monotone operator that governs the nonlinear term in the Gamma equation. By a suitable numerical integration technique the value of the European option can be recovered. Finally, we present some examples of European options to show the good performance of the new numerical global strategy.

Key words: option pricing, option gamma, nonlinear Black–Scholes, duality methods, finite elements

1 Introduction

The classical linear Black–Scholes presented in 1973 establishes that the price V of an option can be obtained as the solution of the parabolic equation:

$$\partial_t V + \frac{\sigma^2}{2} S^2 \, \partial_S^2 V + rS\partial_S V - rV = 0 \,, \tag{1}$$

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where $r > 0$ denotes the risk–free interest rate and σ is the (constant) volatility of the underlying asset, the price of which is assumed to be a stochastic process that follows the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \,,$$

the constant μ being the drift and the process W_t a geometric Brownian motion. Note that the option price, V_t, is a process that is obtained from the solution V of (1), by the expression $V_t = V(t, S_t)$. Equation (1) has been derived under several restrictive assumptions.

However, from the analysis of market data, the need of more realistic models arises. For example, in several setting, different models assume nonconstant volatility expressions that depend on the gamma of the option in the form:

$$\sigma = \hat{\sigma}(S \, \partial_S^2 V) \,,$$

so that the following nonlinear Black–Scholes equation is posed:

$$\partial_t V + \frac{1}{2} \hat{\sigma}(S \partial_S^2 V)^2 S^2 \, \partial_S^2 V + rS \partial_S V - rV = 0. \tag{2}$$

For example, this kind of dependency arises in option pricing models that take into account non-trivial variable transaction costs related to assets buying and selling [1, 2, 10], market feedback effects due to large traders choosing given stock–trading strategies [6, 7], risk from volatile and unprotected portfolios [8], or investor preferences [4], among others.

2 Mathematical model

In this section, we just remind a result by Ševčovič and Žitňanská [11] that establishes the equivalence between the nonlinear Black–Scholes equation (2) and a quasilinear parabolic equation. For this purpose, we introduce the function

$$\beta(H) = \frac{\hat{\sigma}(H)^2}{2} H \,.$$

Proposition (Ševčovič–Žitňanská, [11]) *Assume the function $V = V(S, t)$ is a solution to the nonlinear Black–Scholes equation*

$$\partial_t V + S \, \beta(S \partial_S^2 V) + rS \partial_S V - rV = 0 \,, \qquad S > 0 \,, \quad t \in (0, T) \,. \tag{3}$$

Then the transformed function $H = H(x, \tau) = S \partial_S^2 V(S, t)$, where $x = \ln(S/E)$, $\tau = T - t$, is a solution to the quasilinear parabolic (Gamma) equation:

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r \partial_x H \,. \tag{4}$$

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On the other hand, if H is a solution to (4) such that $H(-\infty,\tau) = \partial_x H(-\infty,\tau) = 0$ and $\beta'(0)$ is finite, then the function

$$V(S,t) = aS + b\,exp(-r(T-t)) + \int_{-\infty}^{+\infty} (S - E\,exp(\xi))^+ \, H(\xi, T-t)\, d\xi \tag{5}$$

is a solution to the nonlinear Black–Scholes equation (3) for any $a, b \in \mathbb{R}$.

Moreover, if we consider the initial condition

$$H(x,0) = \delta(x)\,, \tag{6}$$

where $\delta(x)$ denotes the Dirac delta function with basis point x, then we can recover the payoffs of the European vanilla options with the choices:

- $a = b = 0$, for the call option (i.e. $V(S,T) = (S-E)^+$),
- $a = -1$, $b = E$, for the put option (i.e. $V(S,T) = (E-S)^+$),

the constant E being the strike price. As the analytical expression for the solution of (4) is not available, in next section we propose a set of numerical techniques for its approximation (see e.g. [8, 9, 11] for other numerical strategies).

3 Numerical solution of the quasilinear parabolic equation

In order to solve numerically the nonlinear equation (4) jointly with the initial condition (6), we note the main difficulties: the possibility of advection term dominating diffusion one, the nonlinear diffusion term, the presence of an unbounded domain and the Dirac delta function in the initial condition. First, as in other problems in which advection can dominate over diffusion, we propose the characteristics method for the time discretization. Secondly, as the nonlinear term can be related to maximal monotone operators [3], we make use of a duality method based on a result in [5]. In order to deal with the unbounded domain, as usually in financial problems, we propose a domain truncation in the asset variable by taking $S_\infty = 4E$, which corresponds to x_∞ as the upper finite boundary of the computational domain in x. Also we consider x_0 as the lower boundary of this domain $\Omega = (x_0, x_\infty)$. Concerning to the Dirac delta function, we approximate it by a Gaussian density. Finally, a finite element method is proposed for the discretization in the spatial-like variable x at each time step.

So, first following [3, 5], we introduce the parameter $\omega > 0$ and a new variable θ given in terms of the function β^ω by:

$$\theta = \beta^\omega(H) = \beta(H) - \omega H\,. \tag{7}$$

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As $\beta(H) = \theta + \omega H$, equation (4) can be equivalently written as:

$$H_\tau - (\omega + r)H_x - \omega H_{xx} = \theta_x + \theta_{xx}\,. \tag{8}$$

Next, in order to apply the method of characteristics, we introduce the material derivative of function H:

$$\frac{DH}{D\tau} = H_\tau - (\omega + r)\,H_x\,, \tag{9}$$

which represents the derivative associated to the constant scalar velocity field $-(\omega + r)$, so that (8) turns into:

$$\frac{DH}{D\tau} - \omega H_{xx} = \theta_x + \theta_{xx}\,. \tag{10}$$

Note that (10) is still a nonlinear problem, as θ and H are related by (7).

In order to discretize (10) in time by the characteristics (also known as semilagrangian) method, we introduce the time stepsize $\Delta\tau > 0$ and mesh points in time $\tau^n = n\Delta\tau$ for $n = 0, 1, 2, \ldots$, so that we consider the following final value problem:

$$\begin{cases} \dfrac{d\chi}{d\tau} = -(\omega + r)\,\chi(\tau) \\ \chi(\tau^{n+1}) = x\,, \end{cases}$$

that provides the characteristics curve (associated to the scalar velocity field) passing through the point x at time τ^{n+1}. Its analytical solution provides the position at time τ^n to be used in the characteristics method:

$$\chi^n(x) = \chi(x, \tau^{n+1}; \tau^n) = x \exp((\omega + r)\Delta\tau)\,.$$

We can now approximate the material derivative in (10) by a first order upwinded quotient. If we denote $H^n(\cdot) = H(\cdot, \tau^n)$, then (10) is approximated by:

$$\frac{H^{n+1} - H^n \circ \chi^n}{\Delta\tau} - \omega H^{n+1}_{xx} = \theta^{n+1}_x + \theta^{n+1}_{xx}\,. \tag{11}$$

We will consider homogeneous Dirichlet boundary conditions on $\partial\Omega$, i.e. $H(x_0) = H(x_\infty) = 0$. Thus, the variational formulation of (11) consists in finding $H^{n+1} \in W^{1,2}_0(\Omega)$, such that:

$$\int_\Omega H^{n+1}\varphi - \Delta\tau\omega \int_\Omega H^{n+1}_{xx}\varphi = \int_\Omega (H^n \circ \chi^n)\varphi + \Delta\tau \int_\Omega \theta^{n+1}_x \varphi + \Delta\tau \int_\Omega \theta^{n+1}_{xx}\varphi\,, \quad \forall \varphi \in W^{1,2}_0(\Omega)$$

where $W^{1,2}_0(\Omega)$ stands for the classical notation of Sobolev spaces. Next, using Green's theorem, we get:

$$\begin{aligned} \int_\Omega H^{n+1}\varphi + \Delta\tau\omega \int_\Omega H^{n+1}_x \varphi_x = \int_\Omega (H^n \circ \chi^n)\varphi + \Delta\tau \int_\Omega \theta^{n+1}_x \varphi - \Delta\tau \int_\Omega \theta^{n+1}_x \varphi_x \\ + \Delta\tau\omega \int_{\partial\Omega} H^{n+1}_x \varphi + \Delta\tau \int_{\partial\Omega} \theta^{n+1}_x \varphi\,. \end{aligned}$$

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Taking into account the homogeneous boundary conditions, the two integrals on $\partial\Omega$ vanish and we get:

$$\int_\Omega H^{n+1}\varphi + \Delta\tau\omega \int_\Omega H_x^{n+1}\varphi_x = \int_\Omega (H^n \circ \chi^n)\varphi + \Delta\tau \int_\Omega \theta_x^{n+1}\varphi - \Delta\tau \int_\Omega \theta_x^{n+1}\varphi_x\,, \qquad (12)$$

jointly with the relation:

$$\theta^{n+1} = \beta^\omega(H^{n+1})\,. \qquad (13)$$

We propose the following fixed point algorithm to solve (12)-(13) at each time instant τ^{n+1}. Assume $(H^{n+1,0}, \theta^{n+1,0})$ is given. Then, for $k = 0, 1, \ldots$

- For given $(H^{n+1,k}, \theta^{n+1,k})$, we search $H^{n+1,k+1}$ as the solution of the linear problem

$$\int_\Omega H^{n+1,k+1}\varphi + \Delta\tau\omega \int_\Omega H_x^{n+1,k+1}\varphi_x = \int_\Omega (H^n\circ\chi^n)\varphi + \Delta\tau \int_\Omega \theta_x^{n+1,k}\varphi - \Delta\tau \int_\Omega \theta_x^{n+1,k}\varphi_x \qquad (14)$$

  for all $\varphi \in W_0^{1,2}(\Omega)$.

- We update $\theta^{n+1,k+1}$ by solving the nonlinear equation (13). As the exact solution is not available in most cases, we make use the theory of maximal monotone operators as in [5] and propose the updating:

$$\theta^{n+1,k+1} = \beta_\lambda^\omega(H^{n+1,k+1} + \lambda\theta^{n+1,k})\,, \qquad (15)$$

  where β_λ^ω denotes the Yosida regularization of function β^ω with parameter λ:

$$\beta_\lambda^\omega(H) = \inf_G \left(\beta^\omega(G) + \frac{(G-H)^2}{2\lambda}\right)\,.$$

  Moreover, for convergence reasons, we choose $\lambda = 1/(2\omega)$.

We note that Yosida regularization is strongly dependent on the function β and requires the computation of the inverse of an operator. Therefore, it is not always possible to get its analytical expression. This is the reason why we replace (15) by first order Taylor expansion:

$$\begin{aligned}
\theta^{n+1,k+1} &= \beta_\lambda^\omega(H^{n+1,k+1} + \lambda\theta^{n+1,k}) \\
&= \beta^\omega(H^{n+1,k+1} + \lambda\theta^{n+1,k} - \lambda\theta^{n+1,k+1}) \\
&= \beta^\omega\left(H^{n+1,k+1} + \lambda(\theta^{n+1,k} - \theta^{n+1,k+1})\right) \\
&= \beta^\omega\left(H^{n+1,k+1}\right) + (\beta^\omega)'\left(H^{n+1,k+1}\right)\lambda(\theta^{n+1,k} - \theta^{n+1,k+1}) \\
&\quad + o\left(\lambda^2(\theta^{n+1,k} - \theta^{n+1,k+1})^2\right)\,,
\end{aligned} \qquad (16)$$

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which does not require the computation of the Yosida regularization and is accurate enough if λ is small. From (16), we deduce:

$$\theta^{n+1,k+1}\left[1+(\beta^\omega)'(H^{n+1,k+1})\lambda\right]=\beta^\omega(H^{n+1,k+1})+(\beta^\omega)'(H^{n+1,k+1})\lambda\theta^{n+1,k}$$

so that:

$$\theta^{n+1,k+1}=\frac{\beta^\omega(H^{n+1,k+1})+(\beta^\omega)'(H^{n+1,k+1})\lambda\theta^{n+1,k}}{1+\lambda(\beta^\omega)'(H^{n+1,k+1})}\,.$$

Finally, taking into account that $\beta^\omega(H)=\beta(H)-\omega H$ we obtain:

$$\begin{aligned}\theta^{n+1,k+1}&=\frac{\beta(H^{n+1,k+1})-\omega H^{n+1,k+1}+\left[\beta'(H^{n+1,k+1})-\omega\right]\lambda\theta^{n+1,k}}{1+\lambda\left[\beta'(H^{n+1,k+1})-\omega\right]}\\&=\frac{\beta(H^{n+1,k+1})+\beta'(H^{n+1,k+1})\lambda\theta^{n+1,k}-\omega\left[H^{n+1,k+1}+\lambda\theta^{n+1,k}\right]}{1-\omega\lambda+\lambda\beta'(H^{n+1,k+1})}\,.\end{aligned}\tag{17}$$

The last expression is used instead of (15) to update θ^{n+1}. Let us remark that the first derivative of β is used in (14). In practice, it is approximated by a second order central differences formula. If the function β is not differentiable, it can be replaced by a regularized function $\hat{\beta}$.

For solving (14), we implement a finite element method. Thus, for a fixed natural number $M>0$, we consider a uniform mesh of the computational domain $\Omega=[x_0,x_\infty]$, the nodes of which are $x_j=x_0+j\Delta x$, $j=0,\ldots,M+1$, where $\Delta x=(x_\infty-x_0)/(M+1)$ denotes the constant mesh step. Associated to this uniform mesh a piecewise linear Lagrange finite elements discretization is considered.

More precisely, we search $H_h^{n+1,k+1}\in W_{0,h}$ such that:

$$\int_\Omega H_h^{n+1,k+1}\varphi+\Delta\tau\omega\int_\Omega H_{h,x}^{n+1,k+1}\varphi_x=\int_\Omega(H_h^n\circ\chi^n)\varphi+\Delta\tau\int_\Omega\theta_x^{n+1,k}\varphi-\Delta\tau\int_\Omega\theta_x^{n+1,k}\varphi_x\,,$$

for all $\varphi\in W_{0,h}$, where the space of finite elements is:

$$W_{0,h}=\left\{v_h:\Omega\to\mathbb{R}\,/\;\,v_h|_{[x_k,x_{k+1}]}\in\mathcal{P}_1\text{ for }k=0,1,\ldots,M,\;\;v_h=0\text{ on }\partial\Omega\right\}\,,$$

$\mathcal{P}_1$ being the space of polynomials of degree less or equal than one. The coefficients of the matrix and right hand side vector defining the linear system associated to the fully discretized problem are approximated by adequate quadrature formulae, when necessary. In particular, a five nodes Gaussian formula has been used. Finally, the system of linear equations is solved by a conjugate gradient method.

Once the function H is approximated at each time instant, we can recover the value of the derivative by means of (5), where $a=b=0$ for a call option and $a=-1,b=E$ for a put option.

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4 Numerical results

In this section we present a numerical result concerning Amster *et al* model [1, 2], in which the nonlinear function β is given by

$$\beta(H) = \frac{\sigma^2}{2}(H - \mathrm{Le}|H| + \kappa H^2)\,,$$

with $\sigma = 0.95$, $\kappa = 0.10$ and $\mathrm{Le} = 0.30$. As the function β is not differentiable due to the presence of the absolute value, we introduce the regularized function β_ϵ:

$$\beta_\epsilon(H) = \frac{\sigma^2}{2}(H - \mathrm{Le}\, f_\epsilon(H) + \kappa H^2)\,.$$

The function f_ϵ is a smooth approximation of the absolute value function and its first derivative is given by:

$$f'_\epsilon(H) = \begin{cases} -1\,, & \text{if } H < -\epsilon \\ s(H)\,, & \text{if } -\epsilon \le H \le \epsilon \\ 1\,, & \text{if } H > \epsilon \end{cases}$$

s being a cubic spline and $\epsilon = 10^{-3}$.

We have considered the case of European call and put options, the payoff of which is given in terms of the strike price $E = 100$. Moreover, we have taken the risk-free interest rate $r = 0.05$ and the maturity $T = 4$.

For the numerical solution, the time domain has been discretized in 800 steps, thus $\Delta\tau = 0.005$ and the spatial variable x is in $[-4, 1.4]$, for which we have considered a uniform mesh consisting of 1601 nodes. Figure 1 shows the payoff of the call option as well as the solution at time $t = 0$ (or $\tau = 4$), while Figure 2 shows analogous results for the put option.

5 Conclusions

A nonlinear model for derivatives pricing is solved by a numerical strategy including duality methods based on maximal monotone operators, characteristics methods for time discretization and finite elements. The method is independent of the nonlinear function β.

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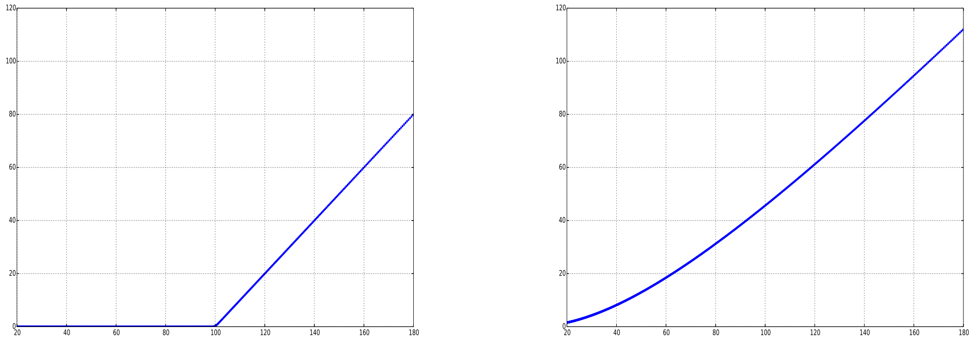

Figure 1: Call option. The terminal condition (payoff) and numerical solution

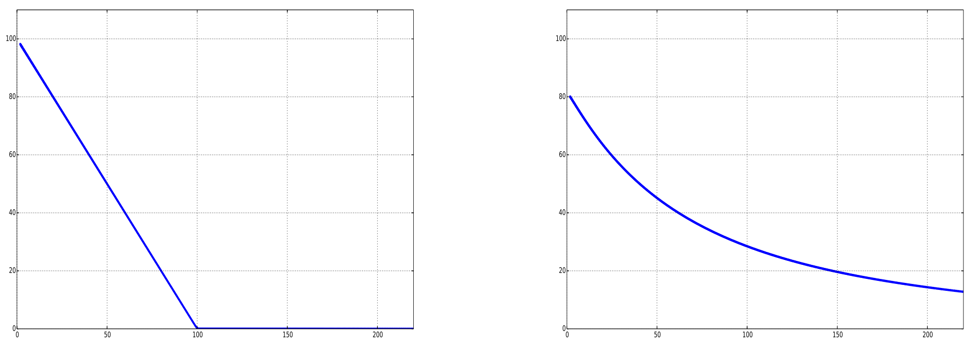

Figure 2: Put option. The terminal condition (payoff) and numerical solution

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