Computers and Mathematics with Applications 76 (2018) 725–740

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Total value adjustment for European options with two stochastic factors. Mathematical model, analysis and numerical simulation

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ARTICLE INFO

Article history:
Received 6 February 2018
Received in revised form 10 May 2018
Accepted 13 May 2018
Available online 29 May 2018

Keywords:
(Non)linear PDEs
Option pricing
Counterparty risk
Credit value adjustment
Method of characteristics
Finite element method

ABSTRACT

In the present paper we derive novel (non)linear PDE models for pricing European options and the associated total value adjustment (XVA), when incorporating the counterparty risk. The main innovative aspect is the consideration of stochastic spreads instead of less realistic constant spreads previously used in the literature. For the nonlinear model, a rigorous mathematical analysis based on sectorial differential operators allows to state the existence and uniqueness of a solution. Moreover, for the numerical solution we propose an appropriate set of techniques based on the method of characteristics for time discretization, finite element for spatial discretization and fixed point iteration for the nonlinear terms. Finally, numerical examples illustrate the expected behaviour of the option prices and the total value adjustment.

1. Introduction

Counterparty risk can be understood as the risk to each party of a contract from a future situation in which one of the counterparties cannot live up its contractual obligations. Since the bankrupt of several institutions during the last financial crisis, a relevant effort in quantitative finance research concerns to the consideration of counterparty risk in financial contracts, specially in the pricing of derivatives (see [1–3], for example). As a consequence, different adjustments on the value of the derivative without counterparty risk (hereafter referred as risk-free derivative) are being included in the derivative pricing. For example, the credit value adjustment (CVA) refers to the variation on the price of a contract due to the possibility of default of one (or both) of the counterparties. Adjustments on debit (DVA) and funding (FVA) are also important issues included in the so called total value adjustment (XVA). The XVA incorporates the sum of all the adjustments related to counterparty risk.

Among the methodologies to include the ingredients involved in the XVA, one of them leads to partial differential equation (PDE) formulations. More precisely, this approach starts from the works by Piterbarg [4], where funding costs are included, and Burgard and Kjaer [5], that considers funding costs and bilateral counterparty risk. In both works, for the case of a European style option the PDE formulation is based on the use of appropriate hedging arguments combined with Itô's lemma for jump–diffusion processes [6]. This is the approach we follow here. Recently, in [7] a one factor model to price the adjustments associated to European and American options has been analysed and numerically solved. In particular,

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https://doi.org/10.1016/j.camwa.2018.05.012
0898-1221/

funding value adjustment (FVA), debit value adjustment (DVA) and credit value adjustment (CVA) have been considered. Furthermore, the model in [7] is extended to incorporate the collateral value adjustment(CollVA), in case that a collateral is used to guarantee the obligations related to the options contract. In [7], constant default intensities for both counterparties have been considered, so that a model depending on just one underlying stochastic factor (the underlying asset) is posed.

However, counterparties default intensities do not always exhibit constant behaviours. In a general framework, intensities might follow a stochastic process [8]. In the present work we focus on the European options pricing and the corresponding XVA adjustments when stochastic intensities are assumed. More precisely, we state PDE models for the derivative value from the point of view of an investor, when the trade takes place between two counterparties: an investor and a hedger. If we consider stochastic intensities of default for both counterparties then a model with three stochastic factors is obtained [8]. Our approach is based on the same framework and assumptions as in [8], although with the additional hypothesis of a zero default intensity for a hedger, thus leading to a two stochastic factors model. However, a case of an additional stochastic intensity only increases the spatial dimension of the PDE, so that we trust that the theoretical analysis and numerical methods here used can be extended to this more general setting.

As in [8], we include all the components in the pricing of uncollateralized derivatives with counterparty risk, with the following assumptions:

- The price of a derivative should reflect all of its hedging costs.
- Since in a high percentage of uncollateralized transactions the presence of an investor (risk taker) and a hedger (risk hedger) is implied, the price of the derivative should just reflect the hedging costs transmitted by the hedger.
- The hedger will only be willing to hedge the fluctuations in the price of the derivative that he will experience while not having defaulted.
- There is neither CVA nor FVA to be made to fully collateralized derivatives (with continuous collateral margining in cash, symmetrical collateral mechanism and no threshold, minimum transfer amount, etc.).

Moreover, we will consider the following market assumptions:

- There is a liquid CDS (credit default swap) curve for the investor.
- There is a liquid curve of bonds issued by the hedger.
- Continuous hedging, unlimited liquidity, no bid-offer spreads, no trading costs.
- Recovery rates are either deterministic or there are recovery locks available so that recovery risk is not a concern,

as well as the following model assumptions:

- Only the investor is defaultable.
- The underlying asset follows a diffusion process under the real world measure.
- The underlying asset of a derivative is unaffected by a default event of the investor.
- The investor credit spread is stochastic and follows a diffusion process correlated with the asset price under the real world measure.

Keeping in mind these assumptions, in the present paper we state a PDE formulation by means of suitable hedging arguments and the use of Itô's Lemma for jump–diffusion processes [6]. After arguing the hedging strategy, different linear or nonlinear PDEs arise depending on the choice of the mark-to-market value at the default. If the mark-to-market value is equal to the risk-free derivative price then a linear PDE involving the riskless derivative price is obtained. Alternatively, if the mark-to-market value is equal to the risky derivative price, then a nonlinear PDE arises. For the nonlinear PDE formulation we develop the mathematical analysis of the model to obtain existence and uniqueness of a solution in the appropriate functional space on a bounded domain. For this purpose, we use the tools of nonlinear parabolic PDEs involving sectorial operators [9].

In addition, we propose a set of numerical methods to solve the PDEs for both choices of the mark-to-market value. First, we truncate the unbounded domain and formulate suitable boundary conditions at the boundaries of the resulting bounded domain, following some ideas in [10]. Next, we propose a time discretization based on the method of characteristics combined with a finite element discretization in the asset and spread variables. The method of characteristics has been proposed in [11] in the context of fluid mechanics problems and used in finance in [12] for vanilla options and in [10,13] for Asian options or in [14] for pension plans. For the nonlinear PDE a fixed point iteration algorithm is additionally proposed.

The plan of the article is as follows. In Section 2 we propose the mathematical model. Section 3 is devoted to the mathematical analysis of the nonlinear PDE problem that models the price of the XVA. Furthermore, we prove the existence and uniqueness of solution. In Section 4 we describe numerical methods we propose to compute a solution of our models. In Section 5, we show and discuss the numerical results for some illustrative examples. Finally, Section 6 contains some conclusions.

2. Two stochastic factors models

In this section, we obtain the models for European options and their associated XVA pricing when the counterparty risk and funding costs are taken into account. The main difference with the one factor model presented in [7] comes from the consideration of stochastic default intensities instead of constant ones. As previously indicated, we assume an investor as a risky counterparty and consider that issuer's intensity of default is null. Thus, the underlying asset price S, and the short term CDS spread of the investor h, are modelled by means of stochastic processes satisfying the following stochastic differential equations (SDEs):

$$dS_t = (r(t) - q(t))\,S_t\,dt + \sigma^S(t)\,S_t\,dW_t^S, \tag{1}$$
$$dh_t = (\mu^h(t) - M^h(t)\sigma^h(t))\,dt + \sigma^h(t)\,dW_t^h, \tag{2}$$

where $(r(t) - q(t))$ and $(\mu^h(t) - M^h(t)\sigma^h(t))$ are the (respective) drifts of the processes. Moreover, $r(t)$ denotes the risk-free interest rate, $q(t)$ is the asset dividend yield rate, $M^h(t)$ is the market price of investor's credit risk, $\sigma^S(t, S)$ and $\sigma^h(t, h)$ are the volatility functions, and W_t^S and W_t^h are two correlated Wiener processes (i.e., $\rho\,dt = dW_t^S\,dW_t^h$) so that ρ is the instantaneous correlation between S_t and h_t.

In terms of the spread, the default intensity of the investor, λ_t, is defined as:

$$\lambda_t = \frac{h_t}{1 - R}, \tag{3}$$

where $0 \le R < 1$ denotes the investor recovery rate.

We consider a derivative trade between a hedger and an investor, where only the last one is defaultable. The main risk factors in the trading are the market risk due to changes produced in the asset value, investor spread risk and investor default instant. Thus, from the point of view of the investor, the derivative value at time t is denoted by $\widehat{V}_t = \widehat{V}(t, S_t, h_t, J_t^I)$ and depends on the spot value of the asset (S_t), on the spread of the investor (h_t) and on the investor default state at time t (J_t^I). Note that $J_t^I = 1$ in case of default before or at time t, otherwise $J_t^I = 0$. The price of the same derivative between two default-free counterparties (risk-free derivative) is denoted by $V_t = V(t, S_t)$. The risky derivative price $\widehat{V}_t$ includes adjustments (such as DVA, FCA and/or CollVA) into valuation, whereas the risk-free derivative price V_t does not include any counterparty risk adjustment.

The price of the risky derivative upon default of the investor is given by:

$$\widehat{V}(t, S_t, h_t, 1) = RM^+(t, S_t, h_t) + M^-(t, S_t, h_t), \tag{4}$$

where $M(t, S_t, h_t)$ denotes the mark-to-market price, $M^+ = \max(M, 0)$ and $M^- = \min(0, M)$.

In terms of the mark-to-market condition (4), we introduce $\Delta\widehat{V}$ as the variation of $\widehat{V}$ at default, which is given by:

$$\Delta\widehat{V}_t = RM_t^+ + M_t^- - \widehat{V}_t, \tag{5}$$

where $M_t = M(t, S_t, h_t)$. As it is usually assumed in the literature [5], we only consider two possible choices for M_t: either the risk-free or the risky derivative value. In order to state the pricing model of the risky derivative, this one is hedged by a self-financing portfolio, Π_t, which is designed to hedge all underlying risk factors.

With this aim, the hedger will trade with different financial instruments in order to hedge the following risk factors:

- Market risk: a fully collateralized derivative is employed to hedge this kind of risk. We denote by H_t the net present value associated to that derivative, from the point of view of the hedger.
- Spread risk and default risk of the investor: the hedger will trade with two credit default swaps with different maturity times. The first one, CDS$(t, t+dt)$, for which the buyer pays a premium $h(t)dt$ at time $t+dt$, presents a short maturity date. If the default time takes place before the maturity time $t+dt$, the buyer of the protection receives $(1-R)$, where R denotes the recovery rate at time $t+dt$. Moreover, the premium $h(t)dt$ is such that $CDS(t, t+dt) = 0$. The second credit default swap, CDS(t, T), represents the amount of money guaranteed until a longer maturity time, $T > t$.

Thus, from no arbitrage arguments we have $\widehat{V}_t = \Pi_t$.

Let us assume that the portfolio at time t, Π_t, is made up of:

- $\alpha(t)$ units of the net present value of a fully collateralized derivative H_t,
- $\beta(t)$ units of cash in collateral accounts,
- $\gamma(t)$ units of a long term credit default swap,
- $\varepsilon(t)$ units of a short term credit default swap,
- $\Omega(t)$ units of a short term bond,

such that:

$$\Pi_t = \alpha(t)H(t) + \beta(t) + \gamma(t)\mathrm{CDS}(t, T) + \varepsilon(t)\mathrm{CDS}(t, t+dt) + \Omega(t)B(t, t+dt). \tag{6}$$

The hedger trades on bonds that mature on $t+dt$ to match the spread duration of the uncollateralized derivative, imposing that the net buyback is equal to $\widehat{V}_t$. This is known as a self-financing condition of the replication strategy, so that

$$\widehat{V}_t = \Omega(t)B(t, t+dt),$$

which implies that the number of units of $B(t, t+dt)$ is given by:

$$\Omega(t) = \frac{\widehat{V}_t}{B(t, t+dt)}.$$

Therefore, as a consequence of the self-financing condition, the portfolio evolution comes from the changes in each component:

$$\begin{aligned} d\widehat{V}_t &= \alpha(t) dH(t) + d\beta(t) + \gamma(t) d\mathrm{CDS}(t, T) \\ &\quad + \varepsilon(t) d\mathrm{CDS}(t, t+dt) + \frac{\widehat{V}_t}{B(t, t+dt)} dB(t, t+dt). \end{aligned} \tag{7}$$

Applying Itô's lemma for jump–diffusion processes [6], the change $d\widehat{V}_t$ of $\widehat{V}_t$ from t to $t+dt$ is given by:

$$\begin{aligned} d\widehat{V}_t \;&=\; \frac{\partial \widehat{V}}{\partial t}(t, S_t, h_t)\, dt + \frac{\partial \widehat{V}}{\partial S}(t, S_t, h_t)\, dS_t + \frac{\partial \widehat{V}}{\partial h}(t, S_t, h_t)\, dh_t \\ &\quad + \left(\frac{(\sigma^S)^2 S^2}{2} \frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2} \frac{\partial^2 \widehat{V}}{\partial h^2} + \rho \sigma^S \sigma^h S \frac{\partial^2 \widehat{V}}{\partial S \partial h} \right)(t, S_t, h_t)\, dt \\ &\quad + \Delta \widehat{V}(t, S_t, h_t)\, dJ_t^I \\ &= \left(\frac{\partial \widehat{V}}{\partial t} + (r-q) S \frac{\partial \widehat{V}}{\partial S} + (\mu^h - M^h \sigma^h) \frac{\partial \widehat{V}}{\partial h} + \frac{(\sigma^S)^2 S^2}{2} \frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2} \frac{\partial^2 \widehat{V}}{\partial h^2} \right. \\ &\quad \left. + \rho \sigma^S \sigma^h S \frac{\partial^2 \widehat{V}}{\partial S \partial h} \right)(t, S_t, h_t)\, dt + \sigma^S S \frac{\partial \widehat{V}}{\partial S}(t, S_t, h_t)\, dW_t^S + \sigma^h \frac{\partial \widehat{V}}{\partial h}(t, S_t, h_t)\, dW_t^h \\ &\quad + \Delta \widehat{V}(t, S_t, h_t)\, dJ_t^I. \end{aligned}$$

Then, let us show the evolution of the rest of financial instruments in the portfolio. The cash amount $\beta(t)$ is a sum of $-\alpha(t)H(t)$ and $-\gamma(t)\mathrm{CDS}(t, T)$, that has been posted to the hedger. Thus, the change in $\beta(t)$ is given by

$$d\beta(t) = (-\alpha(t) H_t - \gamma(t) \mathrm{CDS}(t, T))\, c(t)\, dt,$$

where $c(t)$ represents the accrual rate, that is the rate of interest that is added to the principal of a financial instrument between cash payments of that interest.

Applying Itô's lemma [15],

$$dH_t = \left(\frac{\partial H}{\partial t} + (r-q) S \frac{\partial H}{\partial S} + \frac{(\sigma^S)^2 S^2}{2} \frac{\partial^2 H}{\partial S^2} \right)(t, S_t, h_t)\, dt + \sigma^S S_t \frac{\partial H}{\partial t}(t, S_t, h_t)\, dW_t^S. \tag{8}$$

The differential change in the short term CDS and bond are respectively given by:

$$d\mathrm{CDS}(t, t+dt) = h(t)\, dt - (1-R)\, dJ_t^I, \tag{9}$$

$$dB(t, t+dt) = f(t) B(t, t+dt)\, dt, \tag{10}$$

where $f(t)$ represents the EONIA rate, i.e. the weighted average of overnight euro interbank offer rates (EURIBOR) for interbank loans.

Applying Itô's lemma for jump–diffusion, the change of the long term CDS is

$$\begin{aligned} d\mathrm{CDS}(t, T) &= \left(\frac{\partial \mathrm{CDS}(t, T)}{\partial t} + (\mu^h - M^h(t) \sigma^h) \frac{\partial \mathrm{CDS}(t, T)}{\partial h} \right. \\ &\quad \left. + \frac{(\sigma^h)^2}{2} \frac{\partial^2 \mathrm{CDS}(t, T)}{\partial h^2} \right) dt + \sigma^h \frac{\partial \mathrm{CDS}(t, T)}{\partial h}\, dW_t^h + \Delta \mathrm{CDS}(t, T)\, dJ_t^I, \end{aligned} \tag{11}$$

where $\Delta\mathrm{CDS}$ represents the variation of the CDS price at default.

Next, replacing (8)–(11) into Eq. (7), the latter can be written as:

$$\begin{aligned} &\left(\frac{\partial \widehat{V}}{\partial t} + (r-q) S \frac{\partial \widehat{V}}{\partial S} + (\mu^h - M^h \sigma^h) \frac{\partial \widehat{V}}{\partial h} + \frac{(\sigma^S)^2 S^2}{2} \frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2} \frac{\partial^2 \widehat{V}}{\partial h^2} \right. \\ &\quad \left. + \rho \sigma^S \sigma^h S \frac{\partial^2 \widehat{V}}{\partial S \partial h} \right) dt + \sigma^S S \frac{\partial \widehat{V}}{\partial S}\, dW_t^S + \sigma^h \frac{\partial \widehat{V}}{\partial h}\, dW_t^h + \Delta \widehat{V} dJ_t^I \\ &= \alpha(t) \left(\left(\frac{\partial H}{\partial t} + (r-q) S_t \frac{\partial H}{\partial S} + \frac{(\sigma^S)^2 S^2}{2} \frac{\partial^2 H}{\partial S^2} \right) dt + \sigma^S S \frac{\partial H}{\partial S}\, dW_t^S \right) \end{aligned} \tag{12}$$

$$
\begin{aligned}
&- \left(\alpha(t)H(t) + \gamma(t)CDS(t,T)\right) c(t)\,dt + \gamma(t)\Bigg[\Bigg(\frac{\partial \mathrm{CDS}(t,T)}{\partial t} \\
&+ \left(\mu^h - M^h\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \mathrm{CDS}(t,T)}{\partial h^2}\Bigg)\,dt \\
&+ \sigma^h \frac{\partial \mathrm{CDS}(t,T)}{\partial h}\,dW_t^h + \Delta \mathrm{CDS}(t,T)\,dJ_t^I\Bigg] \\
&+ \varepsilon(t)\left(h(t)dt - (1-R)dJ_t^I\right) + \frac{\widehat{V}_t}{B(t,t+dt)} f(t)B(t,t+dt)\,dt.
\end{aligned}
$$

In order to obtain a risk-free portfolio, we remove the risky terms in (12) with

$$
\alpha(t) = \frac{\partial \widehat{V}/\partial S}{\partial H/\partial S},\quad \gamma(t) = \frac{\partial \widehat{V}/\partial h}{\partial \mathrm{CDS}(t,T)/\partial h},\quad \varepsilon(t) = \gamma(t)\frac{\Delta \mathrm{CDS}(t,T)}{1-R} - \frac{\Delta \widehat{V}}{1-R}. \tag{13}
$$

Moreover, as in [7] we consider the Black–Scholes equations modelling $H(t)$ and $\mathrm{CDS}(t,T)$:

$$
\frac{\partial H}{\partial t} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + (r-q)S\frac{\partial H}{\partial S} - cH = 0, \tag{14}
$$

$$
\begin{aligned}
&\frac{\partial \mathrm{CDS}(t,T)}{\partial t} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \mathrm{CDS}(t,T)}{\partial h^2} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h} \\
&\qquad + \frac{h_t}{1-R}\Delta \mathrm{CDS}(t,T) - c\,\mathrm{CDS}(t,T) = 0.
\end{aligned} \tag{15}
$$

Next, by using (14)–(15), the hedging equation (12) in $[0,T)\times(0,\infty)\times(0,\infty)$ is simplified to:

$$
\begin{aligned}
&\frac{\partial \widehat{V}}{\partial t} + \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 \widehat{V}}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 \widehat{V}}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 \widehat{V}}{\partial S\partial h} \\
&\quad = \alpha\left(cH - (r-q)S\frac{\partial H}{\partial S}\right) - \alpha cH \\
&\quad + \gamma\left(-\frac{h}{1-R}\Delta \mathrm{CDS}(t,T) - \left(\mu^h - M^h\sigma^h\right)\frac{\partial \mathrm{CDS}(t,T)}{\partial h}\right) + \varepsilon h + f\widehat{V}.
\end{aligned} \tag{16}
$$

Thus, the derivative price is modelled by the following final value PDE problem:

$$
\begin{cases}
\dfrac{\partial \widehat{V}}{\partial t} + \widetilde{\mathcal{L}}_{Sh}\widehat{V} + \dfrac{\Delta \widehat{V}}{1-R}h - f\widehat{V} = 0, \text{ in } [0,T)\times(0,\infty)\times(0,\infty), \\
\widehat{V}(T,S,h) = G(S),
\end{cases} \tag{17}
$$

where $G(S)$ represents the option payoff and the differential operator $\widetilde{\mathcal{L}}_{Sh}$ is

$$
\begin{aligned}
\widetilde{\mathcal{L}}_{Sh}V \equiv\ & \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 V}{\partial h\partial S} \\
&+ (r-q)S\frac{\partial V}{\partial S} + \left(\mu^h - M^h\sigma^h\right)\frac{\partial V}{\partial h}.
\end{aligned} \tag{18}
$$

In order to write $\widetilde{\mathcal{L}}_{Sh}$ in terms of the spread h, we use the relationship between the drift of the spread $(\mu^h - M^h\sigma^h)$ and investor's intensity of default λ:

$$
\mu^h - M^h\sigma^h = -\kappa\lambda. \tag{19}
$$

Thus, using the relationship (3) between h_t and λ_t in (19), we get

$$
\mu^h - M^h\sigma^h = -\kappa\frac{h}{1-R}.
$$

Therefore, the differential operator (18) turns into:

$$
\mathcal{L}_{Sh}V \equiv \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{(\sigma^h)^2}{2}\frac{\partial^2 V}{\partial h^2} + \rho\sigma^S\sigma^h S\frac{\partial^2 V}{\partial h\partial S} + (r-q)S\frac{\partial V}{\partial S} - \kappa\frac{h}{1-R}\frac{\partial V}{\partial h}. \tag{20}
$$

According to expression (5) and the possible choices for the mark-to-market value at default, different kinds of PDEs arise: the risk-free derivative value leads to a linear PDE, while the risky one gives rise to a nonlinear PDE. Therefore, two alternative problems are posed:

- If $M = \widehat{V}$, a nonlinear PDE model for the risky derivative value is obtained:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - f\widehat{V} = h\,\widehat{V}^{+}, \text{ in } [0,T) \times (0,\infty) \times (0,\infty), \\ \widehat{V}(T,S,h) = G(S). \end{cases} \tag{21}$$

- If $M = V$, we obtain a linear PDE problem:

$$\begin{cases} \dfrac{\partial \widehat{V}}{\partial t} + \mathcal{L}_{Sh}\widehat{V} - \left(\dfrac{h}{1-R} + f\right)\widehat{V} = hV^{+} - \dfrac{h}{1-R}V, \text{ in } [0,T) \times (0,\infty) \times (0,\infty) \\ \widehat{V}(T,S,h) = G(S). \end{cases} \tag{22}$$

We have used the notation $V^{+} = \max(V, 0)$ and analogously for $\widehat{V}$. Next, in order to pose the PDEs modelling the XVA, the risky derivative value can be split up into $\widehat{V} = V + U$, where U represents the XVA price and V is the value of the risk-free derivative. Thus V satisfies:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}_{S}V - fV = 0, \text{ in } [0,T) \times (0,\infty) \times (0,\infty), \\ V(T,S) = G(S), \end{cases} \tag{23}$$

where the classical linear Black–Scholes operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S V \equiv \frac{(\sigma^S)^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S}.$$

Therefore, the XVA price U satisfies either a linear or a nonlinear problem:

- If $M = \widehat{V}$, we obtain the nonlinear problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - fU = h(V+U)^{+}, \text{ in } [0,T) \times (0,\infty) \times (0,\infty), \\ U(T,S,h) = 0. \end{cases} \tag{24}$$

- If $M = V$, we obtain the linear problem:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{L}_{Sh}U - \left(\dfrac{h}{1-R} + f\right)U = hV^{+}, \text{ in } [0,T) \times (0,\infty) \times (0,\infty), \\ U(T,S,h) = 0. \end{cases} \tag{25}$$

As our goal is to solve numerically problems (24) and (25) by a finite element method, we first proceed to localize the problems on a bounded domain. For this purpose, let us consider $\Omega = (0, S_\infty) \times (0, h_\infty)$ for large enough values of S_∞ and h_∞, so that the choice of these values does not affect the solution in the domain of financial interest. In the bounded domain we need to impose appropriate boundary conditions to be satisfied by U. For this purpose, first we consider the conditions satisfied by the risky value $\widehat{V}$ and the risk-free value V at $S = 0$ and $S = S_\infty$, that is

$$\begin{cases} \widehat{V}(t, S_\infty, h) = V(t, S_\infty) = V_\infty(t), \\ \widehat{V}(t, 0, h) = V(t, 0) = V_0(t), \end{cases} \tag{26}$$

where the values of $V_\infty(t)$ and $V_0(t)$ are given by

$$V_\infty(t) = \begin{cases} S_\infty - K, & \text{for a call option,} \\ 0, & \text{for a put option,} \end{cases}$$

$$V_0(t) = \begin{cases} 0, & \text{for a call option,} \\ K\exp(-f(T-t)), & \text{for a put option.} \end{cases}$$

Moreover, the boundary $h = 0$ corresponds to zero spread, which is equivalent to intensity of default $\lambda = 0$. Therefore, when $h = 0$ the derivative has no counterparty risk and behaves like the risk-free derivative, so that we impose the reasonable condition $\widehat{V}(t, S, 0) = V(t, S)$.

In order to impose the boundary condition on U at $h = h_\infty$, we introduce the matrix

$$A = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 S^2 & \rho\sigma^S\sigma^h S \\ \rho\sigma^S\sigma^h S & (\sigma^h)^2 \end{pmatrix} \tag{27}$$

and we assume that U satisfies the Neumann condition $(A\nabla U, \vec{n}) = 0$ for $h = h_\infty$, where $\vec{n}$ represents the unit outer normal vector on $\partial\Omega$.

Next, recalling that $\widehat{V} = V + U$ and the previous considerations, we can rewrite the nonlinear initial–boundary value problem associated to (24) in terms of the new time variable $\tau = T - t$ as follows:

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \mathcal{L}_{Sh}U + fU = -h(V+U)^+, \qquad (S,h) \in \Omega, \\ U(\tau, S_\infty, h) = 0, \\ U(\tau, 0, h) = 0, \\ U(\tau, S, 0) = 0, \\ (A\nabla U, \vec{n})(\tau, S, h_\infty) = 0, \\ U(0, S, h) = 0. \end{cases} \tag{28}$$

For the linear equation (25), we consider the same boundary conditions.

In Section 4 (Numerical methods) we consider a bounded computational domain and, using the properties of the differential operator, we show that prescribing a boundary condition at the boundary $S = 0$ is neither necessary for the analytical nor numerical solution.

3. Existence and uniqueness of solution

For the mathematical analysis of the model (28), we transform the associated PDE into an equivalent one governed by a sectorial operator. Thus, we introduce in (28) the following change of variable:

$$x = \ln\left(\frac{S}{K}\right), \qquad u(\tau, x, h) = U(t, S, h).$$

Note that $x \in (-\infty, x_\infty)$. Therefore, we introduce a new truncation by considering the bounded domain $\widehat{\Omega} = (x_0, x_\infty) \times (0, h_\infty)$ and the following problem is posed:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + \mathcal{A}u = H(\tau, u), \\ u(\tau, x_\infty, h) = 0, \\ u(\tau, x_0, h) = 0, \\ u(\tau, x, 0) = 0, \\ (\widehat{A}\nabla u, \vec{n})(\tau, x, h_\infty) = 0, \\ u(0, x, h) = 0, \end{cases} \tag{29}$$

where $\mathcal{A}u = -\mathrm{div}\,(\widehat{A}\nabla u)$, with the constant matrix $\widehat{A}$ given by:

$$\widehat{A} = \frac{1}{2}\begin{pmatrix} (\sigma^S)^2 & \rho\sigma^S\sigma^h \\ \rho\sigma^S\sigma^h & (\sigma^h)^2 \end{pmatrix}.$$

The matrix $\widehat{A}$ is positive definite if and only if $|\rho| < 1$. Moreover, H is given by:

$$\begin{aligned} H(\tau, \varphi)(x, h) &= (V(\tau, Ke^x) + \varphi(x,h))^+ - c_0\,\varphi(x,h) \\ &\quad - c_1\frac{\partial \varphi}{\partial x}(x,h) - c_2(h)\frac{\partial \varphi}{\partial h}(x,h), \quad \text{for all } \tau \in [0,T],\ \varphi \in H^1_\Gamma(\widehat{\Omega}), \end{aligned} \tag{30}$$

with

$$c_0 = f, \qquad c_1 = \frac{(\sigma^S)^2}{2} - (r-q), \qquad c_2(h) = \frac{\kappa}{1-R}h.$$

In the definition of H we use the notation $\Gamma = \{(x,h) \in \partial\widehat{\Omega} / h \neq h_\infty\}$ and $H^1_\Gamma(\widehat{\Omega}) = \{v \in H^1(\widehat{\Omega}) / v = 0 \text{ on } \Gamma\}$. In $H^1_\Gamma(\widehat{\Omega})$ we consider the norm:

$$\|v\|^2_{H^1_\Gamma(\widehat{\Omega})} = \int_{\widehat{\Omega}} |\nabla v|^2\,dx\,dh, \tag{31}$$

which is equivalent to the usual norm in $H^1(\widehat{\Omega})$ (see [16], for example).

Next, we recall the definition of a sectorial operator and a theorem that establishes the conditions for the existence and uniqueness of solution for a nonlinear PDE problem associated to a sectorial operator (see [9]).

Definition 3.1. A linear operator $\mathcal{B}$ in a Banach space X is a sectorial operator if it is a closed densely defined operator such that, for some $\phi \in (0, \pi/2)$, $M \geq 1$ and a real a, the sector $S_{a,\phi} = \{\lambda \mid \phi \leq | \, arg(\lambda - a) \, | \leq \pi, \lambda \neq a\}$ is in the resolvent set of $\mathcal{B}$, and

$$\|(\lambda - \mathcal{B})^{-1}\| \leq M/\mid \lambda - a \mid, \quad \text{for any } \lambda \in S_{a,\phi}.$$

Recall that for a sectorial operator $\mathcal{B}$ one can introduce a scale of fractional power spaces $X^\alpha = Range(\mathcal{B}^{-\alpha})$, such that $X = X^0$ and $X^1 = Dom(\mathcal{B})$, equipped with the norm $\| y \| = \| \mathcal{B}^\alpha y \|$, where $\mathcal{B}^\alpha$ is a fractional power of $\mathcal{B}$ $(\alpha > 0)$.

Theorem 3.2. *Assume that $\mathcal{B}$ is a sectorial operator in the Hilbert space X, $0 \leq \alpha < 1$ and $f : \mathcal{U} \to X$, with $\mathcal{U}$ an open subset of $\mathbb{R} \times X^\alpha$ and $f(\tau, y)$ a function locally Hölder continuous in τ and locally Lipschitzian in y. Then, for any $(\tau_0, y_0) \in \mathcal{U}$ there exists $T_0 = T_0(\tau_0, y_0) > 0$, such that the initial value nonlinear PDE problem:*

$$\begin{cases} \dfrac{dy}{d\tau} + \mathcal{B}y = f(\tau, y), & \tau > \tau_0, \\ y(\tau_0) = y_0, \end{cases} \tag{32}$$

has a unique solution $y \in \mathcal{C}([\tau_0, \tau_0 + T_0], X^\alpha) \cap \mathcal{C}^1([\tau_0, \tau_0 + T_0], X)$ on $(\tau_0, \tau_0 + T_0)$.

In order to apply the theorem, we will consider $X = L^2(\widehat{\Omega})$, $X^\alpha = H^1_\Gamma(\widehat{\Omega})$ with $\alpha = 1/2$, and $\mathcal{U} = (0, T) \times H^1_\Gamma(\widehat{\Omega})$. We will prove that the operator $\mathcal{A}$ in (29) is a sectorial operator and that the function H satisfies the conditions assumed for f in the previous theorem. For the first purpose, we first recall a lemma from [9].

Lemma 3.3. *If $\mathcal{B}$ is a bounded below, self-adjoint densely defined closed operator in a Hilbert space X, then $\mathcal{B}$ is sectorial.*

Proposition 3.4 (*[9, Section 1.3]*). *The operator $\mathcal{A}$ in (29) is a self-adjoint closed operator bounded from below. Therefore, $\mathcal{A}$ is sectorial.*

Proof. In order to prove that $\mathcal{A}$ is self-adjoint, for all $\varphi, \ \chi \in H^1_\Gamma(\widehat{\Omega})$ we compute

$$\begin{aligned} \langle \mathcal{A}\varphi, \chi \rangle &= \int_{\widehat{\Omega}} (\mathcal{A}\varphi, \chi)\, dx\, dh = -\int_{\widehat{\Omega}} \operatorname{div}(\widehat{A}\nabla\varphi)\chi \, dx\, dh \\ &= \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\chi \, dx\, dh - \int_{\partial\widehat{\Omega}} (\widehat{A}\nabla\varphi \cdot n)\chi \, d\gamma = \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\chi \, dx\, dh, \end{aligned}$$

where the last equality holds due to the boundary conditions. Moreover, we have

$$\begin{aligned} \langle \varphi, \mathcal{A}\chi \rangle &= \int_{\widehat{\Omega}} (\varphi, \mathcal{A}\chi)\, dx\, dh = -\int_{\widehat{\Omega}} \varphi \operatorname{div}(\widehat{A}\nabla\chi)\, dx\, dh \\ &= \int_{\widehat{\Omega}} \nabla\varphi \cdot \widehat{A}\nabla\chi \, dx\, dh - \int_{\partial\widehat{\Omega}} \varphi(\widehat{A}\nabla\chi \cdot n)\, d\gamma = \int_{\widehat{\Omega}} \nabla\varphi \cdot \widehat{A}\nabla\chi \, dx\, dh. \end{aligned}$$

The matrix $\widehat{A}$ is symmetric and we proved that $\langle \mathcal{A}\varphi, \chi \rangle = \langle \varphi, \mathcal{A}\chi \rangle$. Hence, $\mathcal{A}$ is a self-adjoint operator in $H^1_\Gamma(\widehat{\Omega})$.

Next, we prove that the operator $\mathcal{A}$ is bounded from below.

$$\begin{aligned} \langle \varphi, \mathcal{A}\varphi \rangle &= -\int_{\widehat{\Omega}} \varphi \operatorname{div}(\widehat{A}\nabla\varphi)\, dx\, dh = -\int_{\widehat{\Omega}} \operatorname{div}(\widehat{A}\nabla\varphi)\varphi \, dx\, dh \\ &= \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\varphi \, dx\, dh - \int_{\partial\widehat{\Omega}} \varphi(\widehat{A}\nabla\varphi)\, d\gamma = \int_{\widehat{\Omega}} \widehat{A}\nabla\varphi \cdot \nabla\varphi \, dx\, dh \\ &= \int_{\widehat{\Omega}} (\widehat{A}\nabla\varphi, \nabla\varphi)\, dx\, dh \geq \lambda_{min} \int_{\widehat{\Omega}} |\nabla\varphi|^2 \, dx\, dh = \lambda_{min} \|\varphi\|^2_{H^1_\Gamma(\widehat{\Omega})}, \end{aligned}$$

where we have used that $\widehat{A}$ is a positive definite matrix and $\lambda_{min} = \min(\sigma(\widehat{A})) > 0$ is the minimum of the eigenvalues of $\widehat{A}$.

Thus, from previous lemma, we have shown that $\mathcal{A}$ is a sectorial operator. $\square$

Proposition 3.5. *The function $H : \mathcal{U} \to X$ given by (30) is well defined, locally Lipschitz continuous in τ and locally Lipschitzian in φ.*

Proof. First, note that the function c_2 belongs to $L^\infty(\widehat{\Omega})$. Moreover, the function V is given by the classical Black–Scholes formula for European call or put options, so that $x \mapsto V(\tau, K e^x) \in L^2(\widehat{\Omega})$. Therefore, $(V(\tau, \cdot) + \varphi)^+ \in L^2(\widehat{\Omega})$ for any function $\varphi \in H^1_\Gamma(\widehat{\Omega})$, thus implying that $H(\tau, \varphi) \in L^2(\widehat{\Omega})$ so that $H(\tau, .) : L^2(\widehat{\Omega}) \to L^2(\widehat{\Omega})$ is well defined.

Next, we will prove that H is locally Lipschitzian in φ, i.e.

$$\|H(\tau, \varphi_1) - H(\tau, \varphi_2)\|_{L^2(\widehat{\Omega})} \leq L_H \, \|\varphi_1 - \varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}, \quad \text{for all } \varphi_1, \ \varphi_2 \in H^1_\Gamma(\widehat{\Omega}).$$

For this purpose, let us estimate the difference

$$\begin{aligned} |H(\tau,\varphi_1)-H(\tau,\varphi_2)| &\le |c_1| \left|\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_2}{\partial x}\right| + |c_2(h)| \left|\frac{\partial\varphi_1}{\partial h}-\frac{\partial\varphi_2}{\partial h}\right| + |c_0|\,|\varphi_1-\varphi_2| \\ &\quad + |h| \left|(V(\tau,\cdot)+\varphi_1)^+ - (V(\tau,\cdot)+\varphi_2)^+\right| \\ &\le |c_1| \left|\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_2}{\partial x}\right| + |c_2(h)| \left|\frac{\partial\varphi_1}{\partial h}-\frac{\partial\varphi_2}{\partial h}\right| + |c_0+h|\,|\varphi_1-\varphi_2|\,, \end{aligned}$$

where we have used the fact that $\left|\chi_1^+-\chi_2^+\right| \le |\chi_1-\chi_2|$, with $\chi_i = V(\tau,\cdot)+\varphi_i$. Then, by integration we get

$$\begin{aligned} \int_{\widehat{\Omega}} |H(\tau,\varphi_1)-H(\tau,\varphi_2)|^2\,dx\,dh &\le |c_1|^2 \int_{\widehat{\Omega}} \left|\frac{\partial\varphi_1}{\partial x}-\frac{\partial\varphi_2}{\partial x}\right|^2 dx\,dh \\ &\quad + \bar{c}_2^2 \int_{\widehat{\Omega}} \left|\frac{\partial\varphi_1}{\partial h}-\frac{\partial\varphi_2}{\partial h}\right|^2 dx\,dh \\ &\quad + \bar{c}_0^2 \int_{\widehat{\Omega}} |\varphi_1-\varphi_2|^2\,dx\,dh \end{aligned}$$

and, in terms of the norm, we get

$$\|H(\tau,\varphi_1)-H(\tau,\varphi_2)\|_{L^2(\widehat{\Omega})} \le L_H\,\|\varphi_1-\varphi_2\|_{H^1_\Gamma(\widehat{\Omega})}\,, \tag{33}$$

with $L_H = \max\{|c_1|\,,\,\bar{c}_2,\,C_0\,\bar{c}_0\}$, where the new constants are $\bar{c}_2 = \max\{|c_2(h)|\,/h \in [0,h_\infty]\}$, $\bar{c}_0 = \max\{|c_0+h|\,/h \in [0,h_\infty]\}$ and $C_0>0$ is the constant associated to the Poincaré – Friedrichs inequality.

Next, we prove that H is locally Lipschitz continuous in τ. Thus, for $\tau_1,\ \tau_2 \in [0,T]$ we compute

$$\begin{aligned} |H(\tau_1,\varphi)-H(\tau_2,\varphi)| &\le |h|\,\left|(V(\tau_1,\cdot)+\varphi)^+ - (V(\tau_2,\cdot)+\varphi)^+\right| \\ &\le h_\infty\,|V(\tau_1,\cdot)-V(\tau_2,\cdot)|\,, \end{aligned}$$

where we have used that $|\ \chi_1^+-\chi_2^+\ |\le|\ \chi_1-\chi_2\ |$, with $\chi_i = V(\tau_i,\cdot)+\varphi$. Therefore, in terms of norms we have

$$\|H(\tau_1,\varphi)-H(\tau_2,\varphi)\|^2_{L^2(\widehat{\Omega})} \le h_\infty^2\,\|V(\tau_1,\cdot)-V(\tau_2,\cdot)\|^2_{L^2(\widehat{\Omega})}\,. \tag{34}$$

Next, using that $V \in \mathcal{C}^1((0,T),X)$, we can apply that V is Lipschitz continuous in τ to obtain that $H(\tau,u)$ is a Lipschitz function in τ. □

Corollary 3.6. *For any initial condition $u_0 \in H^1_\Gamma(\widehat{\Omega})$ there exists $T_0 = T(0,u_0) > 0$ so that the initial value problem (29) has a unique solution in $(0,T_0)$.*

The previous corollary follows from Theorem 3.2 and provides an existence and uniqueness of a local solution, as $T_0 = T(0,u_0)$ is a local time. In order to extend it to any interval $(0,T)$ for a given $T > 0$ we need to apply Corollary 3.3.5 in [9].

Proposition 3.7. *The following inequality holds:*

$$\|H(\tau,\varphi)\|_{L^2(\widehat{\Omega})} \le K(\tau)(1+\|\varphi\|_{H^1_\Gamma(\widehat{\Omega})}),\ \textit{for all}\ (\tau,\varphi)\in(0,\infty)\times H^1_\Gamma(\widehat{\Omega}),$$

where K is continuous in $(0,\infty)$. Therefore, there exists a unique solution of problem (29) defined on the entire time interval $(0,T)$.

Proof. First, we note that the Lipschitz continuity properties also hold for $\tau\in(0,\infty)$ and prove the stated inequality. Thus, for any $(\tau,\varphi)\in(0,\infty)\times H^1_\Gamma(\widehat{\Omega})$ we have

$$\begin{aligned} \|H(\tau,\varphi)\|_{L^2(\widehat{\Omega})} &\le \|H(\tau,\varphi)-H(\tau,0)\|_{L^2(\widehat{\Omega})} + \|H(\tau,0)\|_{L^2(\widehat{\Omega})} \\ &\le L_H\,\|\varphi-0\|_{H^1_\Gamma(\widehat{\Omega})} + \|H(\tau,0)\|_{L^2(\widehat{\Omega})} \\ &\le \left(L_H + \|H(\tau,0)\|_{L^2(\widehat{\Omega})}\right)\left(\|\varphi\|_{H^1_\Gamma(\widehat{\Omega})}+1\right), \end{aligned}$$

where L_H is the Lipschitz constant for H, so that we can take

$$K(\tau) = L_H + \|H(\tau,0)\|_{L^2(\widehat{\Omega})}\,,$$

which is continuous in τ on $(0,\infty)$.

Next, we can apply Corollary 3.3.5 in [9]. Thus, we consider $u(\tau_0,\cdot)$ as the unique solution of (29) at time $\tau_0 = T_0/2$ obtained from Corollary 3.6, so that from the Corollary 3.3.5 in [9] the unique solution of (29) through $(\tau_0,u(\tau_0,\cdot))$ exists for all $\tau\ge\tau_0$. Therefore, we obtain existence and uniqueness of solution of (29) in $(0,T)$. □

4. Numerical methods

In this section we describe some numerical techniques we propose to solve the nonlinear problem (28). The corresponding linear problem can be considered as a particular case and is solved by similar techniques.

The numerical approximation is mainly based on finite element for spatial discretization. As usually in European vanilla options, we choose the maximum for the asset price coordinate S_∞, equal to four times the strike price. Concerning the spread coordinate we consider the interval $[0, h_\infty]$, with $h_\infty = 0.2 = 20\%$ as a large enough value to not affect the numerical solution in the region of financial interest.

In order to solve it with a finite element method, we rewrite the PDE in (28) in a divergence form. Thus, we use matrix A from (27) and the vector

$$b = \begin{pmatrix} \left((\sigma^S)^2 - (r-q)\right) S \\ \dfrac{\rho\sigma^S\sigma^h}{2} + \kappa \dfrac{h}{1-R} \end{pmatrix}, \tag{35}$$

so that the PDE in (28) is transformed into a divergence form:

$$\frac{\partial U}{\partial \tau} - \operatorname{div}(A\nabla U) + b\cdot\nabla U + fU = -h(V+U)^+, \quad (S,h)\in\Omega. \tag{36}$$

4.1. Time discretization and the method of characteristics

For the time discretization we use a semi-Lagrangian method, also known as the method of characteristics, first used in finance in [12]. As in the one factor model [7], we introduce the material derivative of U, i.e.

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial S}\frac{\partial S}{\partial \tau} + \frac{\partial U}{\partial h}\frac{\partial h}{\partial \tau}$$

for given functions $S = S(\tau)$ and $h = h(\tau)$. Thus, in our problem the material derivative term is given by:

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \left((\sigma^S)^2 - (r-q)\right) S \frac{\partial U}{\partial S} + \left(\frac{\rho\sigma^S\sigma^h}{2} + \frac{\kappa}{1-R} h\right)\frac{\partial U}{\partial h} \tag{37}$$

and Eq. (36) becomes:

$$\frac{DU}{D\tau} - \operatorname{div}(A\nabla U) + fU = -h(V+U)^+. \tag{38}$$

Taking into account the advective term in (37), we introduce a constant time step $\Delta\tau > 0$, the time instants $\tau^n = n\Delta\tau$ $(n = 0, 1, \ldots)$ and the ODE problems associated to the computation of the characteristic curves:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \left(\left(\sigma^S\right)^2 - (r-q)\right)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h, \end{cases} \tag{39}$$

the solution of which, $\chi(\tau) = \chi((S,h), \tau^{n+1}; \tau)$ represents the characteristic curve associated to the vector field b passing through the point (S, h) at instant time τ^{n+1}.

Next, using the method of characteristics we approximate the material derivative in (38) and pose the semi-discrete problem:

$$\begin{cases} \dfrac{U^{n+1} - U^n\circ\chi^n}{\Delta\tau} - \operatorname{div}(A\nabla U^{n+1}) + fU^{n+1} = -h(V^{n+1} + U^{n+1})^+, \\ U^0(S,h) = 0, \end{cases} \tag{40}$$

where $\chi^n = \chi(\tau^n) = \chi((S,h), \tau^{n+1}; \tau^n)$ and $U^n(\cdot) \approx U(\tau^n, \cdot)$. The components of χ^n are deduced from (39):

$$\chi_1^n = S\exp\left(-((\sigma^S)^2 - r + q)\Delta\tau\right),$$
$$\chi_2^n = -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}\Delta\tau\right).$$

A piecewise bilinear interpolation method will be applied to evaluate $U^n\circ\chi^n$ in (40) at the nodes of the finite element mesh.

4.2. Fixed point iteration

Due to the nonlinearity of the problem (40), a fixed point scheme is proposed in each iteration of the method of characteristics. Thus, the global scheme is shown in Algorithm 1.

Algorithm 1

Let $N > 1$, $n = 0$, $\varepsilon > 0$ and U^0 given
While $n \leq N$:

(1) Let $U^{n+1,0} = U^n$, $k = 0$, $e = \varepsilon + 1$

(2) While $e \geq \varepsilon$:

(2.a) Search $U^{n+1,k+1}$ solution of:

$$(1 + \Delta\tau f)\, U^{n+1,k+1} - \Delta\tau \operatorname{div}(A\nabla U^{n+1,k+1}) = U^n \circ \chi^n - \Delta\tau\, h\,(V^{n+1} + U^{n+1,k})^+ \tag{41}$$

(2.b) $e = \dfrac{\|U^{n+1,k+1} - U^{n+1,k}\|}{\|U^{n+1,k+1}\|}$

4.3. Boundary conditions

In Section 3 we have considered appropriate boundary conditions in order to prove the existence and uniqueness of a solution of (29). We will now adapt them for the numerical solution of (28). First, we introduce the notation $x_0 = \tau, x_1 = S$ and $x_2 = h$, and the domain $\Omega^* = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$, where $x_0^\infty = T, x_1^\infty = S_\infty$ and $x_2^\infty = h_\infty$. The boundary of Ω^* is

$$\partial\Omega^* = \bigcup_{i=0}^{2} (\Gamma_i^{*,-} \cup \Gamma_i^{*,+}),$$

where we used the notation

$$\Gamma_i^{*,-} = \{(x_0, x_1, x_2) \in \partial\Omega^*/x_i = 0\}, \;\; \Gamma_i^{*,+} = \{(x_0, x_1, x_2) \in \partial\Omega^*/x_i = x_i^\infty\}. \tag{42}$$

Then, the PDE in problem (28) can be written in the form:

$$\sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{j=0}^{2} p_j \frac{\partial V}{\partial x_j} + c_0 V = g_0,$$

where the involved data are defined as follows

$$B(x_0, x_1, x_2) = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{(\sigma^S)^2}{2} x_1^2 & \dfrac{\rho\sigma^S\sigma^h}{2} x_1 \\ 0 & \dfrac{\rho\sigma^S\sigma^h}{2} x_1 & \dfrac{(\sigma^h)^2}{2} \end{pmatrix}, \quad c_0(x_0, x_1, x_2) = -f,$$

$$p(x_0, x_1, x_2) = (p_j) = \begin{pmatrix} -1 \\ (r-q)x_1 \\ -\kappa \dfrac{x_2}{1-R} \end{pmatrix}, \qquad g_0(x_0, x_1, x_2) = (V + U)^+ x_2.$$

Following [17], that includes the theory of Fichera [18], we introduce the following subsets of Γ^* in terms of the normal vector to the boundary pointing inwards Ω^*, $\vec{m} = (m_0, m_1, m_2)$:

$$\Sigma^0 = \{x \in \partial\Omega^*/\Sigma_{i,j=0}^2 b_{ij} m_i m_j = 0\}, \;\; \Sigma^1 = \partial\Omega^* - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(p_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

In our particular case, we have

$$\Sigma^0 = \Gamma_0^{*,-} \cup \Gamma_0^{*,+} \cup \Gamma_1^{*,-}, \quad \Sigma^1 = \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \cup \Gamma_2^{*,+}, \quad \Sigma^2 = \Gamma_0^{*,-}.$$

Thus, the boundary conditions must be imposed over the subset $\Sigma^1 \cup \Sigma^2$ [17], which matches with the set $\Gamma_0^{*,-} \cup \Gamma_1^{*,+} \cup \Gamma_2^{*,-} \cup \Gamma_2^{*,+}$.

After studying the boundaries which need a boundary condition to be imposed in order to solve the problem, we proceed to obtain them. Let us remark that the condition imposed on the boundary $\Gamma_0^{*,-}$ corresponds with the initial condition. On the boundary $\Gamma_1^{*,+}$, corresponding with the nodes (S_∞, h), we divide equation (28) by S^2 and pass to the limit, so that the following condition is obtained [10,19]:

$$\lim_{S\to\infty} \frac{\partial^2 U}{\partial S^2} = 0. \tag{43}$$

Analogously to [19], we search a solution of the form

$$U(\tau, S, h) = H_1(\tau)S + H_2(\tau)h^2 + H_3(\tau)Sh + H_4(\tau)h + H_5(\tau), \tag{44}$$

where $H_1(\tau), H_2(\tau), H_3(\tau), H_4(\tau)$ and $H_5(\tau)$ are independent of S and h.

More precisely, assuming $S^2 \frac{\partial^2 U}{\partial S^2} \to 0$ when $S \to \infty$ in (28), we have:

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}(\sigma^h)^2 \frac{\partial^2 U}{\partial h^2} - \rho\sigma^S\sigma^h S \frac{\partial^2 U}{\partial h \partial S} - (r-q)S\frac{\partial U}{\partial S} + \kappa\frac{h}{1-R}\frac{\partial U}{\partial h} + fU = -h(V+U)^+.$$

This equation can be equivalently written as:

$$\frac{\partial U}{\partial \tau} - \operatorname{div}(\widetilde{A}\nabla U) + \widetilde{b}\cdot\nabla U + fU = -h(U+V)^+, \tag{45}$$

where the matrix $\widetilde{A}$ and vector $\widetilde{b}$ are defined as follows:

$$\widetilde{A} = \begin{pmatrix} 0 & \dfrac{\rho\sigma^S\sigma^h}{2}S \\ \dfrac{\rho\sigma^S\sigma^h}{2}S & \dfrac{(\sigma^h)^2}{2} \end{pmatrix}, \qquad \widetilde{b} = \begin{pmatrix} -(r-q)S \\ \dfrac{\rho\sigma^S\sigma^h}{2} + \kappa\dfrac{h}{1-R} \end{pmatrix}. \tag{46}$$

By using the method of characteristics in (45), we pose:

$$\frac{U^{n+1} - U^n\circ\chi^n}{\Delta\tau} - \operatorname{div}(\widetilde{A}\nabla U^{n+1}) + fU^{n+1} = -h(U+V)^+, \tag{47}$$

where $\chi^n \equiv \chi((S,h), \tau^{n+1}; \tau^n)$ is obtained from the solution of the problems:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = -(r-q)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \dfrac{d\chi_2}{d\tau} = \dfrac{\rho\sigma^S\sigma^h}{2} + \dfrac{\kappa}{1-R}\chi_2, \\ \chi_2(\tau^{n+1}) = h, \end{cases} \tag{48}$$

and its components are given by

$$\chi_1^n = S\exp((r-q)\Delta\tau),$$
$$\chi_2^n = -\frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa} + \left(h + \frac{(1-R)\sigma^S\sigma^h\rho}{2\kappa}\right)\exp\left(\frac{-\kappa}{1-R}\Delta\tau\right).$$

Now, replacing the solution (44) in each fixed point step of the discretized equation in (47), we obtain the following equation:

$$\begin{aligned} &(1+\Delta\tau f)H_1^{n+1,k+1}S + (1+\Delta\tau f)H_2^{n+1,k+1}h^2 \\ &- \Delta\tau\left((\sigma^h)^2 H_2^{n+1,k+1} + \rho\sigma^S\sigma^h H_2^{n+1,k+1}h\right) \\ &+ (1+\Delta\tau f)\,H_3^{n+1,k+1}Sh - \Delta\tau\frac{3\rho\sigma^S\sigma^h}{2}SH_3^{n+1,k+1} + (1+\Delta\tau f)\,H_4^{n+1,k+1}h \\ &- \Delta\tau\frac{\rho\sigma^S\sigma^h}{2}H_4^{n+1,k+1} + (1+\Delta\tau f)\,H_5^{n+1,k+1} \\ = &-\Delta\tau h(V^{n+1} + U^{n+1,k})^+ + U^n\circ\chi^n. \end{aligned}$$

If we choose $H_1^{n+1,k+1} = H_2^{n+1,k+1} = H_3^{n+1,k+1} = H_4^{n+1,k+1} = 0$, the following nonhomogeneous Dirichlet boundary condition is deduced:

$$U^{n+1,k+1}(S_\infty, h) = H_5^{n+1,k+1} = \frac{-\Delta\tau h(V^{n+1} + U^{n+1,k})^+ + U^n\circ\chi^n}{1+\Delta\tau f}. \tag{49}$$

Note that this Dirichlet condition on $\Gamma_1^{*,+}$ tends to the boundary condition proposed in (28) for $S = S_\infty$ when $\Delta\tau$ tends to zero.

Next, we analyse the boundary conditions on $\Gamma_2^{*,+}$ and $\Gamma_2^{*,-}$. First, note that on $\Gamma_2^{*,-}$ we have $h = 0$, which means that the probability of default is zero. Thus, the value with counterparty risk is equal to the risk-free value and then $U(\tau, S, 0) = 0$.

Table 1
Financial data for numerical tests.

$\sigma^S = 0.3$	$\sigma^h = 0.2$	$\rho = 0.2$	$E = 15$	$T = 0.5$
$r = 0.3$	$q = 0.24$	$R = 0.3$	$\kappa = 0.01$	$f = 0.04$

Thus, we will impose

$$U^{n,k}(S,0) = 0, \qquad \text{for } n = 0, 1, \ldots, \quad k = 0, 1, \ldots .$$

Following (28), for $h = h_\infty$ we impose $(A\nabla U^{n,k}, \vec{n}) = 0$.

4.4. Finite element method

For the spatial discretization of (41) a triangular mesh of Ω and the associated finite element space of piecewise linear Lagrange polynomials are considered. First, at each time step $n = 0, 1, 2, \ldots$ and each fixed point iteration $k = 0, 1, \ldots,$ by using Green's formula the following variational formulation is posed:

Find $U^{n+1,k+1} \in \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_2^-,\ \varphi = H_5^{n+1,k+1} \text{ on } \Gamma_1^+\}$, such that:

$$\int_\Omega (1+\Delta\tau f)U^{n+1,k+1}\varphi\, dS\, dh + \Delta\tau \int_\Omega A\nabla U^{n+1,k+1}\, \nabla\varphi\, dS\, dh$$
$$= \int_\Omega (U^n \circ \chi^n)\varphi\, dS\, dh - \Delta\tau \int_\Omega h(V^{n+1} + U^{n+1,k})^+ \varphi\, dS\, dh, \quad \forall \varphi \in H_*^1(\Omega),$$

where $H_*^1(\Omega) = \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-}\}$.

Next, for fixed natural numbers $M > 0$ and $L > 0$, we consider a uniform mesh of the computational domain Ω, the nodes of which are (S_i, h_j), with $S_i = i\Delta S$ $(i = 0, \ldots, M+1)$ and $h_j = j\Delta h$ $(j = 0, \ldots, L+1)$, where $\Delta S = S_\infty/(M+1)$ and $\Delta h = h_\infty/(L+1)$ denote the constant mesh steps in each coordinate. Associated to this uniform mesh, a piecewise linear Lagrange finite element discretization is considered. More precisely, we introduce the finite element spaces

$$\widehat{W} = \{\widehat{\varphi} \in \mathcal{C}(\overline{\Omega})/\widehat{\varphi}|_{T_j} \in \mathcal{P}_1,\ \forall T_j \in \mathcal{T}\}, \quad \widehat{W}_* = \{\widehat{\varphi} \in \widehat{V}/\widehat{\varphi} = 0 \text{ on } \Gamma_1^{*,+} \cup \Gamma_2^{*,-}\},$$

in order to find $\widehat{U}^{n+1,k+1} \in \widehat{W}$, satisfying the boundary conditions and such that:

$$\int_\Omega (1+\Delta\tau f)\widehat{U}^{n+1,k+1}\widehat{\varphi}\, dS\, dh + \Delta\tau \int_\Omega A\nabla \widehat{U}^{n+1,k+1}\, \nabla\widehat{\varphi}\, dS\, dh$$
$$= \int_\Omega (\widehat{U}^n \circ \chi^n)\widehat{\varphi}\, dS\, dh - \Delta\tau \int_\Omega h(V^{n+1} + \widehat{U}^{n+1,k})^+ \widehat{\varphi}\, dS\, dh, \quad \forall \widehat{\varphi} \in \widehat{W}_*.$$

Different quadrature formulae (Simpson, three nodes Gaussian, mid point and trapezoidal ones) have been used to obtain the coefficients of the matrix and the right hand side vector which define the linear system associated to the discretized problem. Moreover, such a system has been solved by a partial pivoting LU factorization method [20].

The risk-free derivative value V is analytically given by the Black–Scholes formula for European options with a dividend yield [21].

5. Numerical results

In this section we present some examples to illustrate the performance of the models and the numerical methods in order to reproduce the expected behaviour of the risk-free value V, the risky value $\widehat{V}$, and the associated total value adjustment U, for different European options.

In all the tests we have used the same financial data, which are given in Table 1.

The XVA represents the amount that has to be discounted from the risk-free derivative value due to the investor probability of default. We have developed the model from the point of view of the investor, thus we expect the XVA to be negative, as we can observe in the following examples. Moreover, we have considered $M = \widehat{V}$ in all cases, so that a nonlinear problem is formulated and numerically solved.

In practice, due to the great difference in S and h range of values, we have scaled the equations and solved the problem in the computational dimensionless domain $\widetilde{\Omega} = [0, 1] \times [0, 1]$, with step sizes $\Delta\widetilde{S}$ and $\Delta\widetilde{h}$ in the respective directions.

5.1. Test 1: Convergence

Table 2 shows the order of convergence of the proposed algorithm when the XVA of a call option is computed. Following [10], we use the convergence ratio CR

$$CR = \frac{\left\| U_h - U_{h/2} \right\|_\infty}{\left\| U_{h/2} - U_{h/4} \right\|_\infty},$$

Table 2
Empirical illustration of the order of convergence (p) for Test 1.

$\Delta\widetilde{S} = \Delta\widetilde{h}$	$\Delta\tau$	CR	p
2^{-3}	1/10		
2^{-4}	1/20		
2^{-5}	1/40	2.02925134	1.02094757
2^{-6}	1/80	2.01447211	1.01040183
2^{-7}	1/160	2.00729238	1.00525078
2^{-8}	1/320	2.00367719	1.00265010
2^{-9}	1/640	2.00185420	1.00133690

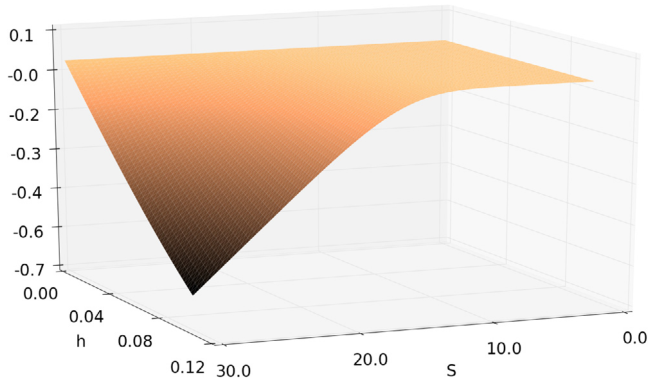

Fig. 1. XVA for a European call option (Test 2).

from which we compute the experimental order of convergence $p = \log_2(CR)$. In Table 2 we can see how the computed values of p tend to the expected order of convergence one. Note that the proposed method of characteristics converges with order one for a constant ratio between the time step and the mesh stepsize in space when combined with piecewise linear finite element here considered.

5.2. Test 2: European call options

In this example, we study a European call option sold by the investor. Fig. 1 shows the total value adjustment (XVA) for the European call option. In this and all forthcoming examples we consider $\Delta\widetilde{S} = \Delta\widetilde{h} = 2^{-7}$ and $\Delta\tau = 0.0025$.

We can observe that the XVA becomes more negative when the underlying asset price increases, that is, when the option is "in the money". In this framework, the buyer will be more interested in exercising the option and will be more exposed to seller's default. Moreover, when the spread is higher, the total value adjustment grows in absolute terms.

5.3. Test 3: European put options

In this example we assume that the investor sells a European put option. In Fig. 2 (left) the total value adjustment associated to this option is represented. In this case, the XVA is more negative when the asset price approaches to zero, that is when the put option is "in the money". Moreover, the XVA increases with the probability of default of the investor.

Next, the option value with counterparty risk is also shown in Fig. 2 (right). Note that the difference between functions represented in both figures provides the price of the European option without counterparty risk.

5.4. Test 4: European call and put options with the mark-to-market equal to risk-free value

In this test, we show the total value adjustment when the mark-to-market is chosen to be equal to the risk-free derivative. In Fig. 3 we show the XVA associated to European call and put options, respectively. Thus, if these values are compared with the computed XVA when mark-to-market is equal to risky derivative (see Figs. 1 and 2), we can conclude that there is not a significative difference between the choices of the mark-to-market close out.

6. Conclusions

In the present paper we have proposed different PDE models for pricing the total value adjustment (XVA) associated to the price of European options when counterparty risk is considered. The main innovative aspect comes from the consideration

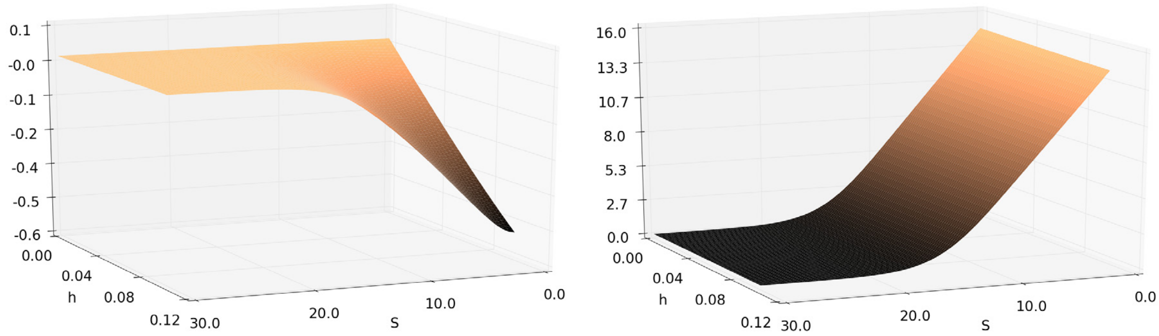

Fig. 2. XVA (left) and price (right) of a European put option (Test 3).

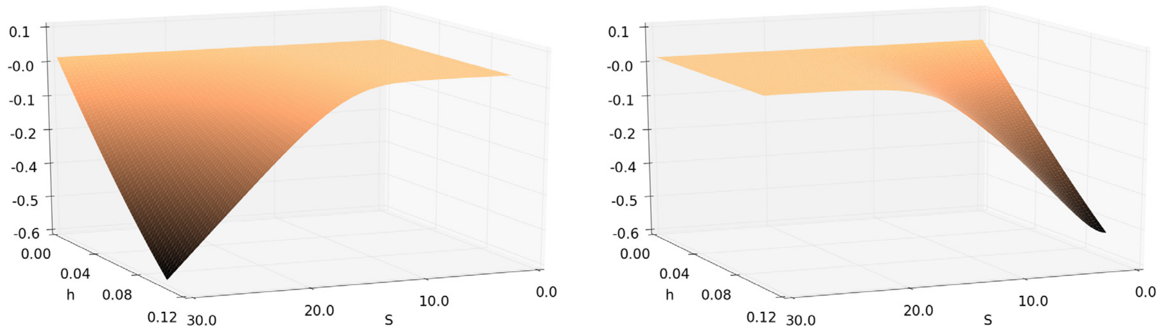

Fig. 3. XVA for European call (left) and put (right) options (Test 4).

of a stochastic spread. Linear and nonlinear PDE problems, depending on the choice of the mark-to-market close out, have been posed. For the nonlinear PDE, the existence and uniqueness of a solution is obtained through the theory of sectorial differential operators.

In order to solve these problems, a set of numerical methods based on the method of characteristics for time discretization, finite element and fixed point iteration techniques for nonlinearities were proposed and implemented. The linear problems at each step of the fixed point iteration were solved by a LU factorization.

The numerical examples show the expected behaviour for the XVA and the European option prices. Moreover, the first order convergence of the numerical method in practice is illustrated. Furthermore, the results are in agreement with those obtained for the one dimensional model in [7], when only the stochastic price dependence is analysed, assuming constant spreads. In a forthcoming work, the authors will try to extend the models, analysis and numerical techniques for American style options with a stochastic spread [22]. The extension of the mathematical analysis techniques and the numerical methods for the case of two stochastic spreads [8], thus involving PDEs with three spatial like variables, seems quite feasible.

Acknowledgements

First, second and fourth authors have been financially supported by Spanish Ministerio de Economía y Competitividad (grant MTM2016-76497-R) and Xunta de Galicia (grant GRC2014/044, including FEDER funds). Second author has also been supported by Spanish Predoc Fellowship BES-2014-070782. Third author has been supported by Slovakian VEGA 1/0062/18 grant.

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