On some nonlinear problems arising in derivative pricing

Algoritmy 2005, Podbanské

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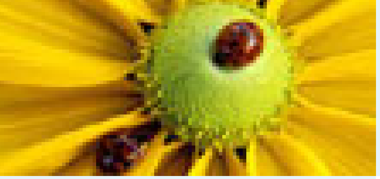


Goals

Present (incomplete) overview of nonlinear phenomena in derivative pricing models

Partial differential equations perspective

Calibration of real market data

Content

- 0. Review of basic linear derivative pricing theories

  Black-Scholes equation for valuing option prices and Vašíček equation for pricing a zero-coupon bond

- I. Nonlinear derivative pricing models

  Risk adjusted pricing methodology leading to a fully nonlinear parabolic equation for pricing an option in the presence of a risk from volatile portfolio and transaction costs

- II. Calibration of linear models leading to a nonlinear optimization problem

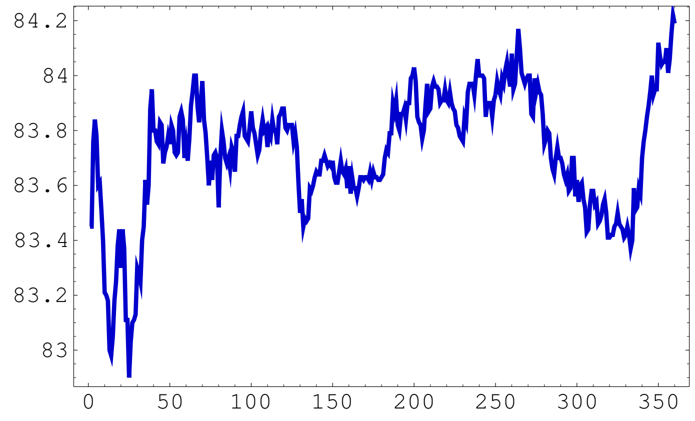
  Calibration of parameters of the Cox-Ingersoll-Ross linear equation

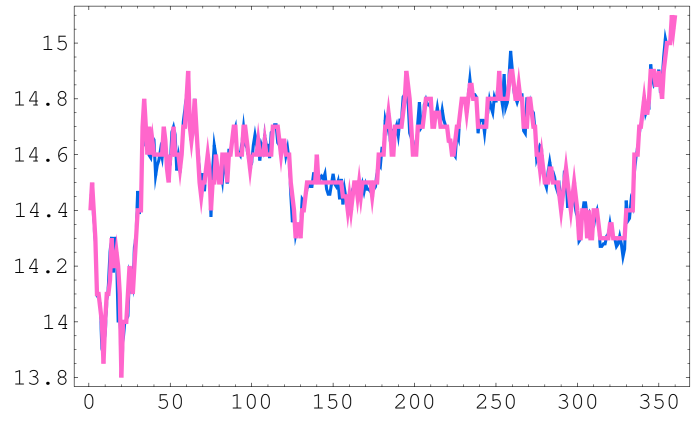
- III. Linear models which are "statistically nonlinear"

  Volatility averaging of the two factor Fong-Vašíček model.

Review of basic linear derivative pricing models

- Black-Scholes equation for pricing an option on assets
- Vašíček equation for pricing a zero-coupon bond

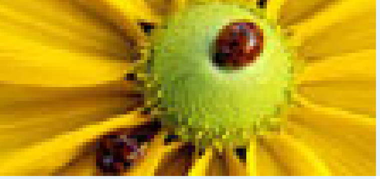




IBM stock price (22.5.2002) Call option price with $E = 70$

Call option is the right (not obligation) to buy an asset at expiration time T at expiration price E.

Question: what is the price $V = V(S, t)$ of the option as a function of the stock price S and time $t \in [0, T]$ to maturity T.



Assumptions

- the asset price follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dW$$

  where $W(t), t \geq 0$, is the Wiener process
- self-financing portfolio consisting of options, assets (stocks) and secure bonds ("mass balance")
- construction of risk-free portfolio (elimination of a "risky" stochastic term by hedging the portfolio)
- Itô's lemma

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

$$\Downarrow$$

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dx$$

Linear models - Black-Scholes-Merton equati

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

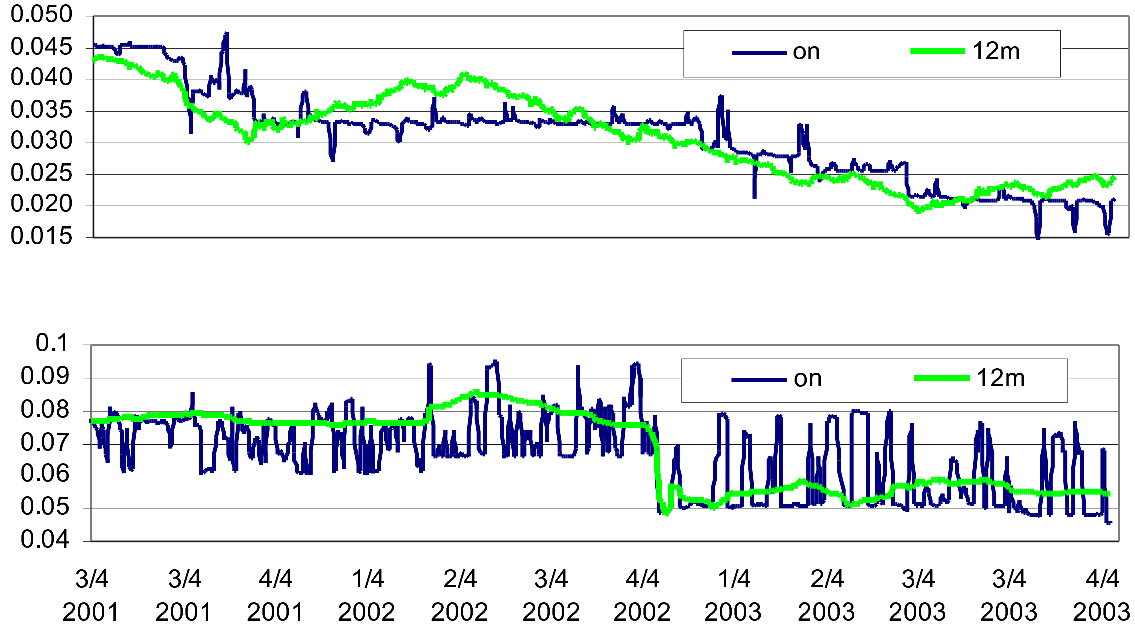
$$S \geq 0, t \in (0, T)$$

$$V(S,T) = \max(S - E, 0) \qquad \text{(Call option)}$$

This is a linear parabolic equation in V variable

M.S. Scholes and R.C. Merton
Swedish bank price 1997 (Nobel price for economy)

Linear models - Vašíček equation



Short-rate (overnight) and 1 year interest rates for EUROLIBOR (up) and BRIBOR (down)

Question: what is the price $P = P(r,t)$ of the price of zero-coupon bond with maturity T as a function of the short-rate (overnight) r and time $t \in [0,T]$

Linear models - Vašíček equation

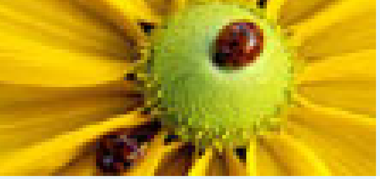
$$\frac{\partial P}{\partial t} + k(\theta - r)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} = rP + \lambda\frac{\partial P}{\partial r}$$

$$P(r,T) = 1$$

$$r \geq 0, t \in (0,T)$$

This is a linear parabolic equation in P variable

Oldřich Alfons Vašíček



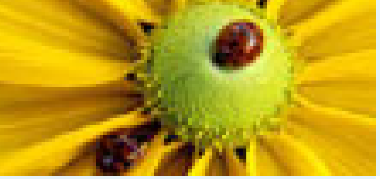
I. Nonlinear derivative pricing models

Joint work with M. Jandačka

Classical Black-Scholes theory does not take into account

- transaction costs (buying or selling assets, bid-ask spreads)
- risk from unprotected (non hedged) portfolio

Question: how to incorporate both transaction costs and risk arising from a volatile portfolio into the Black-Scholes equation framework?



Risk adjusted pricing methodology

The model is based on the Black-Scholes parabolic PDE in which:

- transaction costs are described following the Hoggard, Whalley and Wilmott approach (Leland's model)
- the risk from the unprotected volatile portfolio is described by the variance of the synthetised portfolio.

$$\Downarrow$$

1. Transaction costs as well as the volatile portfolio risk depend on the time-lag between two consecutive transactions.
2. Minimizing their sum yields the optimal length of the hedge interval.
3. It leads to a fully nonlinear PDE - Risk adjusted pricing methodology model proposed by Kratka (1998).

Modeling transaction costs

Transaction costs are described following the Hoggard, Whalley and Wilmott approach (1994) (also referred to as Leland's model (1985))

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \left(1 - Le\, sign(\partial_S^2 V)\right) \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $Le = \sqrt{\frac{2}{\pi}}\frac{C}{\sigma\sqrt{\Delta t}}$ is the so-called Leland number depending on

- C - the round trip transaction cost per unit dollar of transaction, $C = (S_{ask} - S_{bid})/S$
- Δt - the lag between two consecutive portfolio adjustments (re-hedging)

For a plain vanilla option (either Call or Put) the sign of $\partial_S^2 V$ is constant and therefore the above model is just the Black-Scholes equation with lowered volatility.

Modeling risk from volatile portfolio

- a portfolio $\bar\Pi$ consists of options and assets

$$\bar\Pi = V + \delta S$$

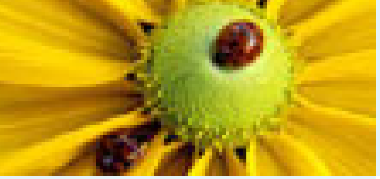
- is the portfolio $\bar\Pi$ is highly volatile an investor usually asks for a price compensation.

Volatility of a fluctuating portfolio can be measured by the variance of relative increments of the replicating portfolio

$$\Downarrow$$

introduce the measure r_{VP} of the portfolio volatility risk as follows:

$$r_{VP} = R\,\frac{var\left(\frac{\Delta\bar\Pi}{S}\right)}{\Delta t}\,.$$



Modeling risk from volatile portfolio

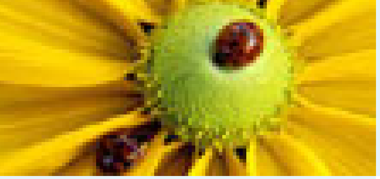
- Using Itô's formula the variance of $\Delta\bar{\Pi}$ can be computed as follows:

$$
\begin{aligned}
var(\Delta\bar{\Pi}) &= E\left[(\Delta\bar{\Pi} - E(\Delta\bar{\Pi}))^2\right] \\
&= E\left[\left((\partial_S V + \delta)\,\sigma S\phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2\Gamma\left(\phi^2 - 1\right)\Delta t\right)^2\right].
\end{aligned}
$$

  where $\phi \approx N(0,1)$ and $\Gamma = \partial_S^2 V$.

- assuming the δ-hedging of portfolio adjustments, i.e. we choose $\delta = -\partial_S V$. For the risk premium r_{VP} we have

$$r_{VP} = \frac{1}{2} R\sigma^4 S^2 \Gamma^2 \Delta t\,.$$



Balance equation for $\bar{\Pi} = V + \delta S$

- $d\bar{\Pi} = dV + \delta dS$
- $d\bar{\Pi} = r\bar{\Pi}dt + (r_{TC} + r_{VP})Sdt$

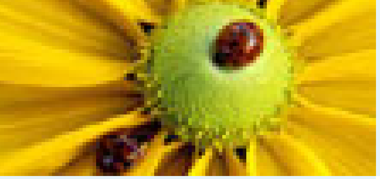
Using Itô's formula applied to $V = V(S,t)$ and eliminating stochastic term by taking $\delta = -\partial_S V$ hedge we obtain

$$\partial_t V + \frac{\sigma^2}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = (r_{TC} + r_{VP})S$$

where

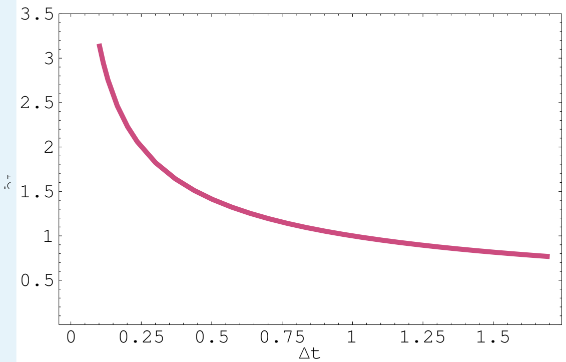
- $r_{TC} = \frac{C|\Gamma|\sigma S}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure
- $r_{VP} = \frac{1}{2} R\sigma^4 S^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure

and $\Gamma = \partial_S^2 V$.

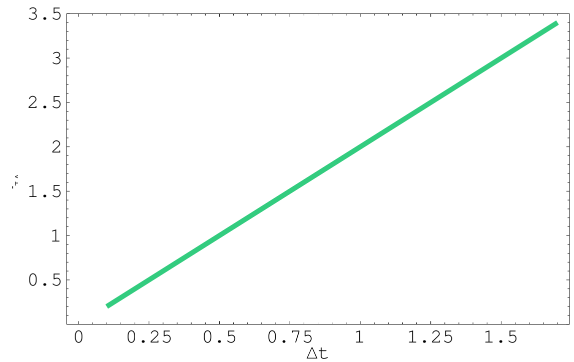


Minimizing the total risk

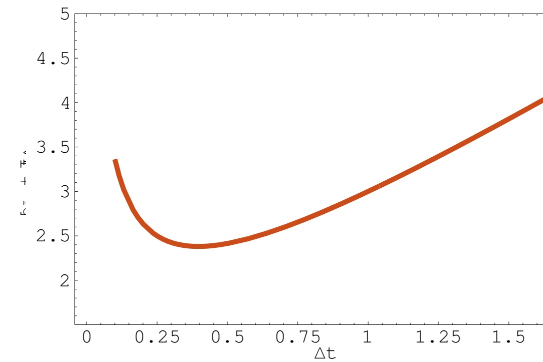
Total risk $r_{TC} + r_{VP}$



Tran. costs risk r_{TC}



Volatile portfolio risk r_{VP}



Total risk $r_{TC} + r_{VP}$

Both r_{TC} and r_{VP} depend on the time lag Δt

$$\Downarrow$$

Minimizing the total risk with respect to the time lag Δt yields

$$\min_{\Delta t} (r_{TC} + r_{VP}) = \frac{3}{2} \left(\frac{C^2 R}{2\pi} \right)^{\frac{1}{3}} \sigma^2 |S \partial_S^2 V|^{\frac{4}{3}}$$

Equation for Risk Adjusted Pricing Methodology model

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \left(1 - \mu (S\partial_S^2 V)^{1/3}\right) \partial_S^2 V + rS\partial_S V - rV = 0$$

$S > 0, t \in (0, T)$ where

$$\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$$

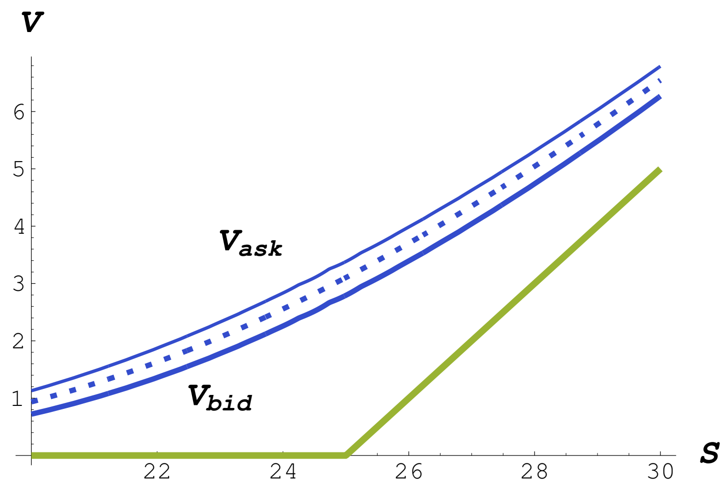
fully nonlinear parabolic equation

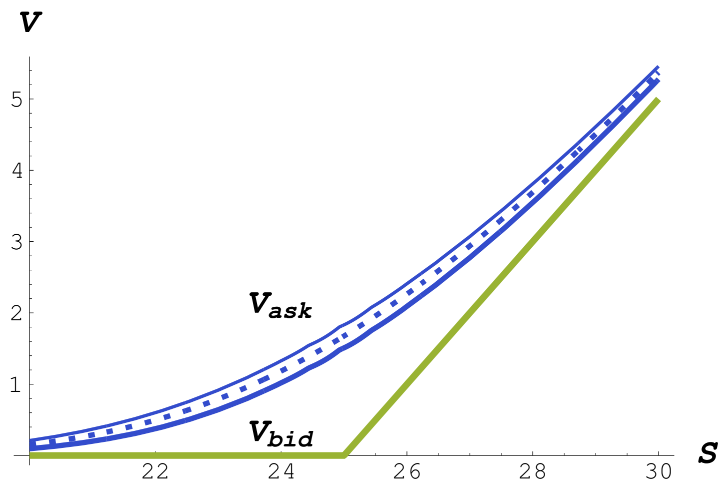
- If $\mu = 0$ (i.e. either $R = 0$ or $C = 0$) the equation reduces to the classical Black-Scholes equation
- minus sign in front of $\mu > 0$ corresponds to Bid option price V_{bid} (price for selling option).

Bid Ask spreads

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \left(1 \pm \mu (S\partial_S^2 V)^{1/3}\right) \partial_S^2 V + rS\partial_S V - rV = 0$$

A comparison of Bid ($-$ sign) and Ask ($+$ sign) option prices computed by means of the RAPM model. The middle dotted line is the option price computed from the Black-Scholes equation.





Volatility smile ;–)

Volatility smile phenomenon is non-constant, convex behavior (near expiration price E) of the implied volatility computed from classical Black-Scholes equation.

By RAPM model we can explain the volatility smile analytically. The Risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

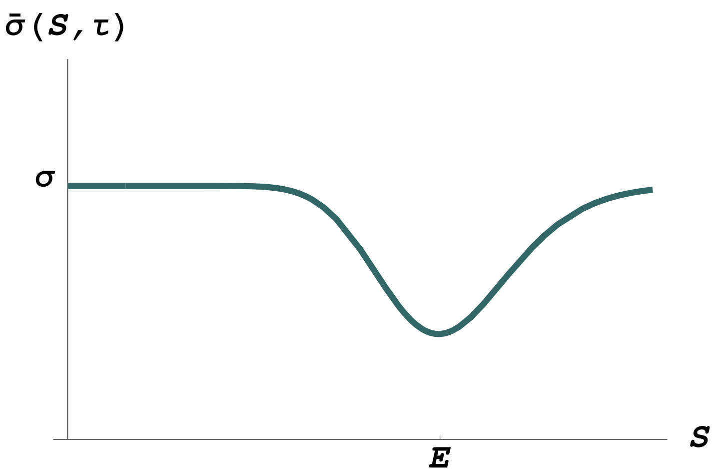
$$\partial_t V + \frac{\bar{\sigma}^2(S,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = 0$$

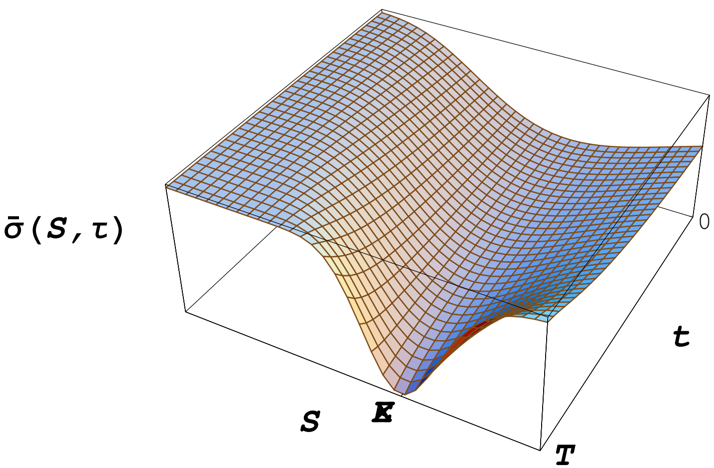
where the volatility $\bar{\sigma}^2(S,t)$ depends itself on a solution $V = V(S,t)$ as follows:

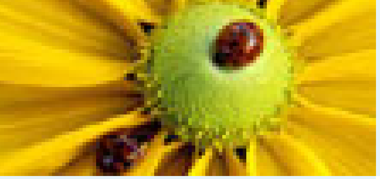
$$\bar{\sigma}^2(S,t) = \sigma^2 \left(1 - \mu (S\partial_S^2 V(S,t))^{1/3}\right) .$$

Explanation of volatility smile

Dependence of $\bar{\sigma}(S,t)$ on S is depicted in the left for t close to T. The mapping $(S,t) \mapsto \bar{\sigma}(S,t)$ is shown in the right.







Γ equation - quasilinear PDE

- denote $\beta(H) = \frac{\sigma^2}{2}(1 - \mu H^{\frac{1}{3}})H$
- reverse time $\tau = T - t$ (time to maturity)
- use logarithmic scale $x = \ln(S/E)$ $(x \in R \leftrightarrow S > 0)$
- introduce new variable $H(x,\tau) = S\partial_S^2 V(S,t)$

Then the RAPM equation can be transformed into quasilinear equation

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H \qquad \tau \in (0,T), x \in R$$

- Boundary conditions: $H(-\infty,\tau) = H(\infty,\tau) = 0$
- Initial condition: $H(x,0) = \frac{PDF(d_1)}{\sigma\sqrt{\tau^*}}$ $\quad d_1 = \frac{x + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau^*}}$ where $0 < \tau^* << 1$ is the switching time.

Γ equation - numerical scheme

$$\partial_\tau H = \partial_x^2 \beta(H) + \partial_x \beta(H) + r\partial_x H \qquad \tau \in (0,T), x \in R$$

$$H_i^j \approx H(ih, jk) \qquad \Downarrow \qquad k = \frac{T}{m}, \quad h = \frac{L}{n}$$

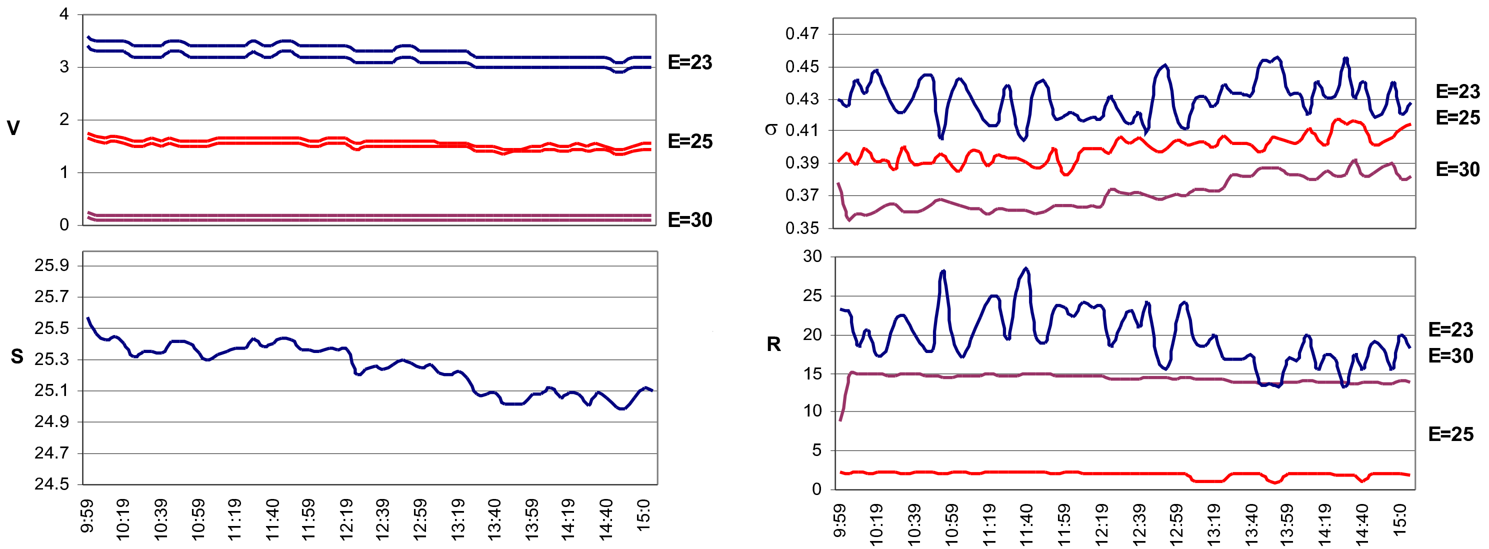
$$a_i^j H_{i-1}^j + b_i^j H_i^j + c_i^j H_{i+1}^j = d_i^j, \quad H_{-n}^j = 0, \quad H_n^j = 0\,,$$

for $i = -n+1, ..., n-1$, and $j = 1, ..., m$, where $H_i^0 = H(x_i, 0)$

$$\begin{aligned} a_i^j &= -\frac{k}{h^2}\beta'(H_{i-1}^{j-1}) + \frac{k}{h} r\,, \qquad b_i^j = 1 - (a_i^j + c_i^j)\,, \\ c_i^j &= -\frac{k}{h^2}\beta'(H_i^{j-1})\,, \quad d_i^j = H_i^{j-1} + \frac{k}{h}\Big(\beta(H_i^{j-1}) - \beta(H_{i-1}^{j-1})\Big)\,. \end{aligned}$$

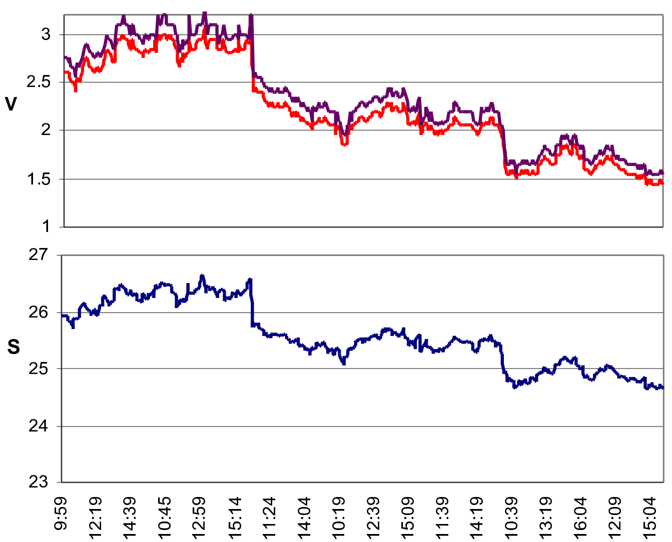
Calibration of RAPM model

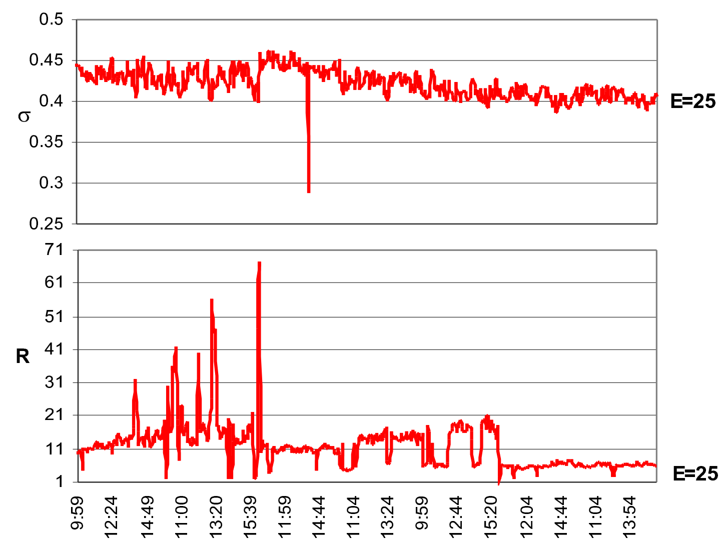
Intra-day behavior of Microsoft stocks (April 4, 2003) and shortly expiring Call options with expiry date April 19, 2003. Computed implied volatilities σ_{RAPM} and risk premium coefficients R.



Calibration of RAPM model

One week behavior of Microsoft stocks (March 20 - 27, 2003) and Call options with expiration date April 19, 2003. Computed implied volatilities σ_{RAPM} and risk premiums R.





Summary of RAPM modeling

- Transaction costs and the risk from volatile portfolio can be incoporated into the Black-Scholes theory.
  - The resulting governing equation is a fully nonlinear parabolic equation.
  - The RAPM model can explain Bid-Ask spreads as well as the volatility smile
  - Implied RAPM volatilty and risk premium coefficient can be calibrated from real market data

II. Linear interest rate models whose calibration leads to nonlinear minimization problems

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Joint work with A. Urbánová Csajková

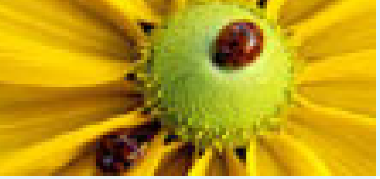
What are Term structure and Interest rate models?

The term structure R is a functional dependence between the time to maturity t of a discount bond and its present price.

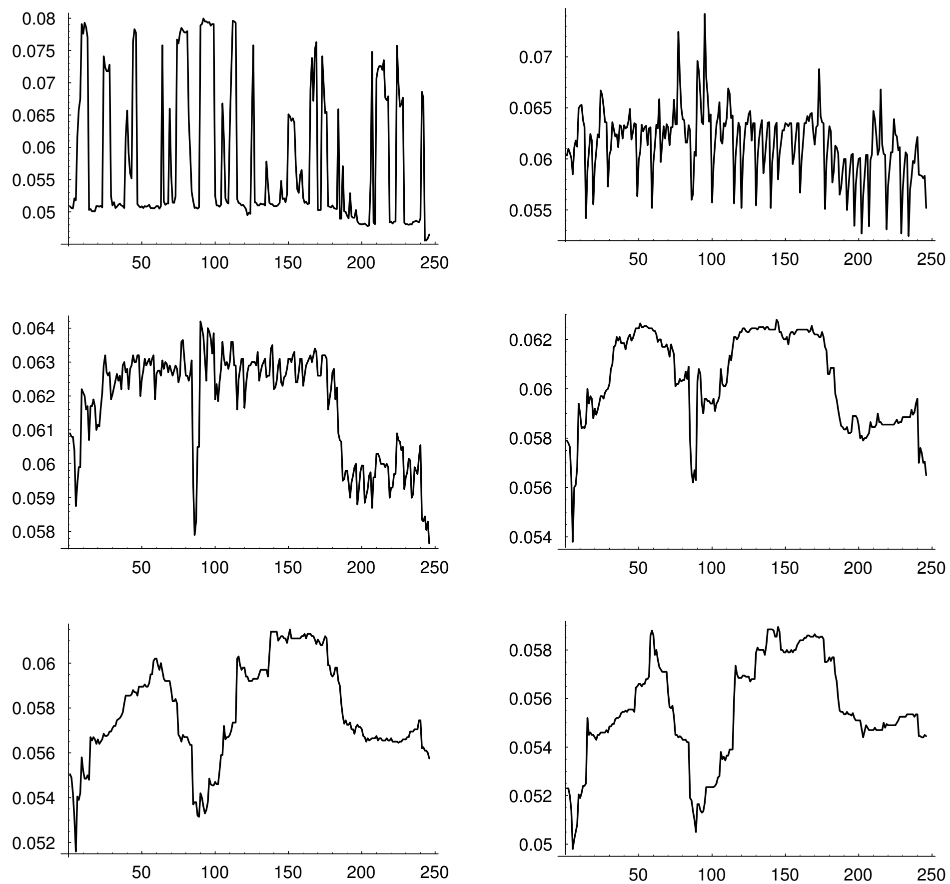
$$P(r,t) = e^{-R(r,t)(T-t)}$$

where

- $P = P(r,t)$ is the price of the zero-coupon bond at the time $t \in [0,T]$ with maturity T. $P(r,T) = 1$ at maturity $t = T$.
- $R = R(r,t)$ is the yield to maturity forming the term structure for $t \in [0,T]$
- r is the short-rate (overnight or instantaneous rate).

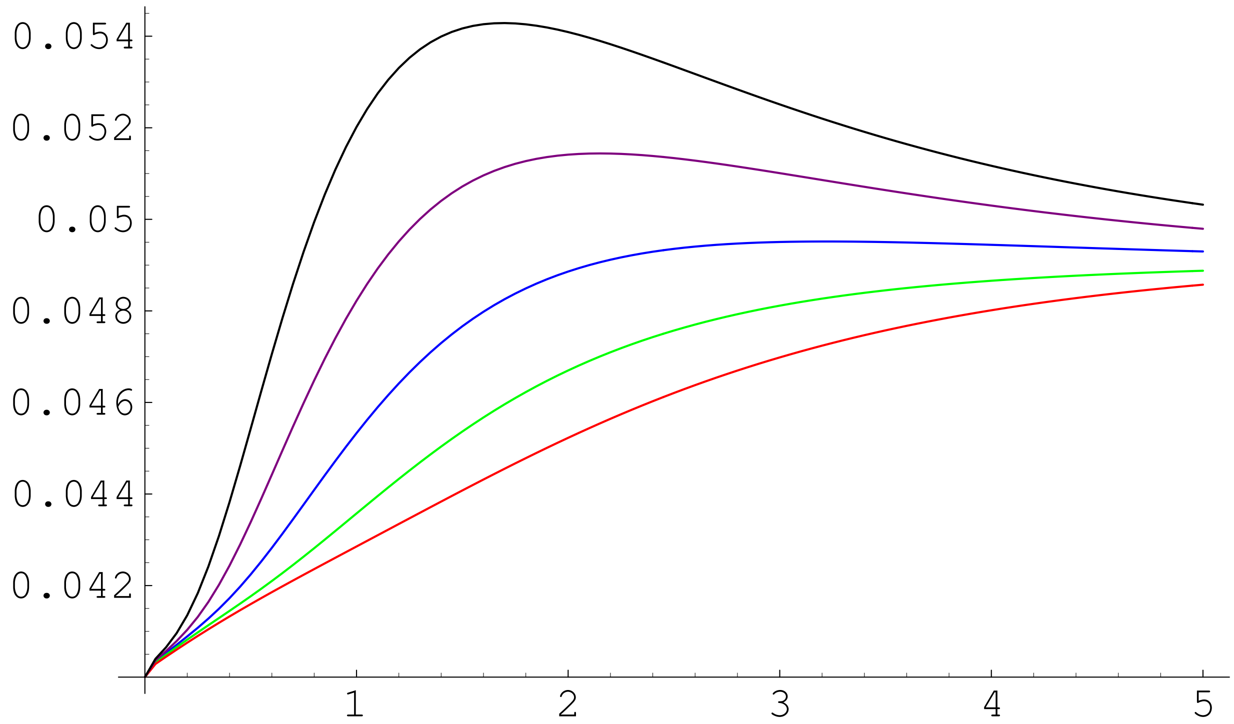


Term structures



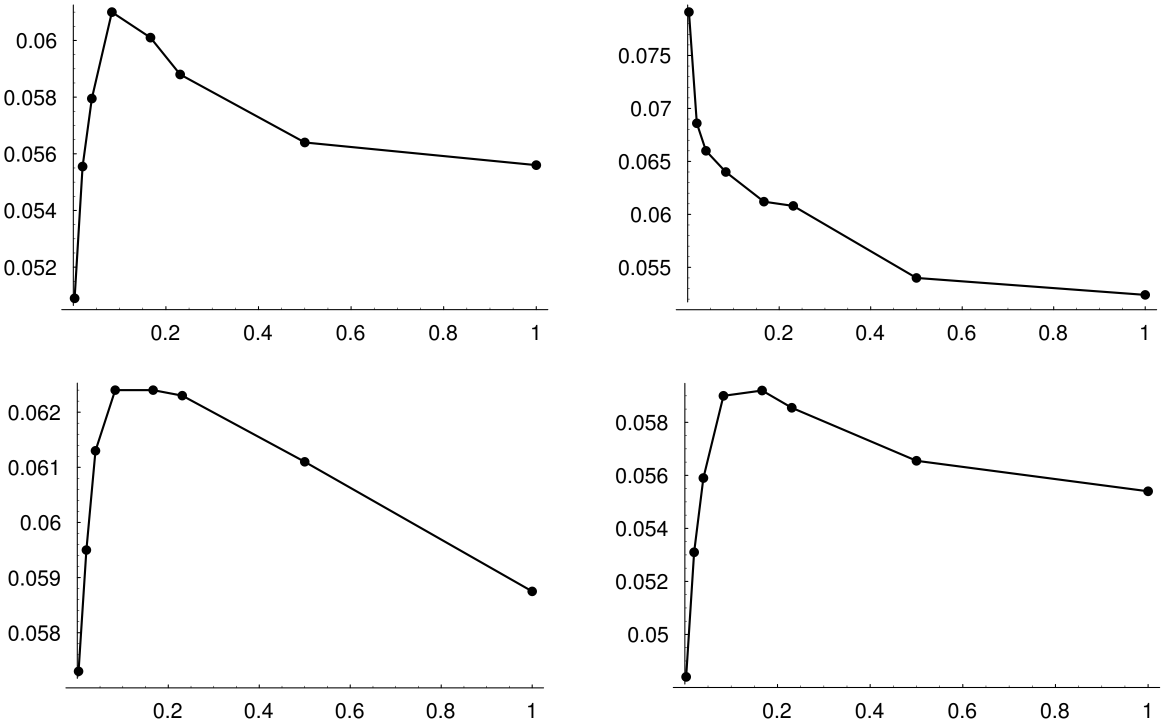
Slovak inter-bank offer rates. Short rate, 1-week, 1-month, 3-months, 6-months and 1-year interest rates on bonds. Bonds with larger maturities are usually less volatile compared to shorter ones.

Term structures



A typical shape of the term structure in some time. Bonds with larger maturities often have higher interest rates.

Term structures



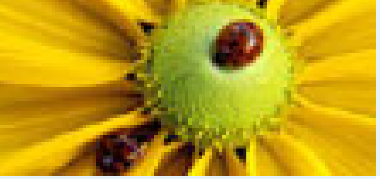
Real inter-bank offer rates. Bonds with larger maturities need not have higher interest rates.

What are Interest rate models?

Interest rate models describe the bond prices (or yields) as a function of time to maturity, state variables like e.g. instantaneous interest rate as well as several model parameters.

- Vašíček (1977)
- Cox, Ingersoll, and Ross (1985)
- Ho and Lee (1986)
- Brennan and Schwartz (1982)
- Hull and White (1990)

and many others ...



Why to calibrate Interest rate models?

- estimated model parameters provide better insight in analyzed term structures
- enable us to perform qualitative analysis of data and explain some unobserved phenomena like e.g. identification of market price of risk, long term interest rates etc

Historical overview

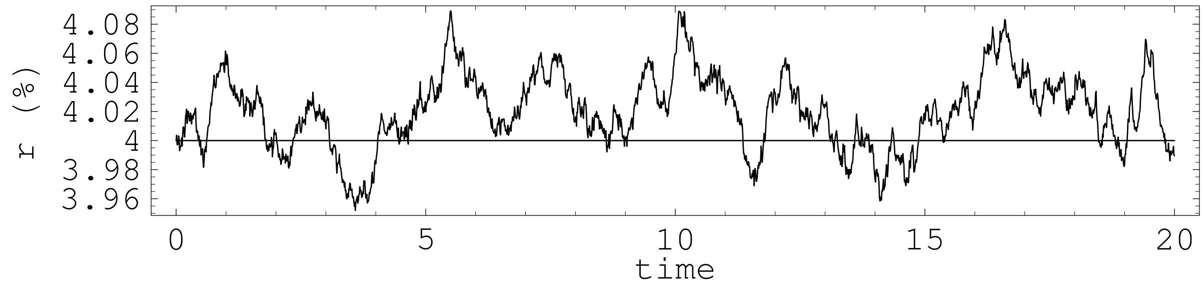
- Chan, Karolyi, Longstaff and Sanders (1992) applied the Generalized Method of Moments to identify model parameters
- Pearson and Sun (1994) have shown failure of calibration of a two-factor extension of the CIR model applied to portfolios of US Treasury bills
- Nowman, Saltoglu (2003) applied Gaussian estimation methods of Nowman (1997) to continuous time interest rate models
- Frühwirth-Schnatter and Geier applied Markov-chain-Monte-Carlo method (Bayesian framework, Gibbs sampling method, Metropolis-Hastings algorithm).
- Vojtek calibrated term structures for post-transitional countries by GARCH autoregressive model.

CIR interest rate model

The Cox-Ingersoll-Ross interest rate model is derived from a basic assumption made on the form of the stochastic process driving the instantaneous interest rate $r_t, t \in [0,T]$.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dw_t$$

where $\{w_t, t \geq 0\}$, denotes the standard Wiener process. Positive constants κ, θ and σ denote the adjustment speed of reversion, the long term interest rate and volatility factor of the process, respectively.



Simulation of a stochastic process driven by mean-reverting process with $\kappa = 1, \sigma = 0.2, \theta = 4\%$.

CIR interest rate model

In the CIR theory the price $P = P(t, T, r)$ of a zero coupon bond is assumed to be a function of the present time $t \in [0, T]$, expiration time $T > 0$ and the present value of the short rate interest rate $r = r_t$.

$$P = e^{-R(t,T)(T-t)}$$

where $R(t, T)$ is the yield on the bond at time t with maturity T, $r_t = R(t, t)$.

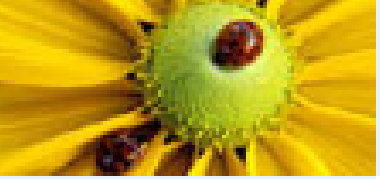
CIR interest rate model

Constructing a risk-less portfolio containing two bonds with different maturities and applying stochastic differential calculus (Itô lemma) we can conclude that the price of the zero coupon bond $P = P(t, T, r)$ satisfies a backward parabolic partial differential equation

$$\frac{\partial P}{\partial t} + (\kappa(\theta - r) - \lambda r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} - rP = 0\,, \quad t \in (0, T)\,,\ r > 0\,.$$

subject to the terminal condition $P(T, T, r) = 1$ for any $r > 0$.

The parameter $\lambda \in \mathbb{R}$ represents the so-called market price of risk.



CIR model has an explicit solution!

$$P(T-\tau, T, r) = A(\tau)e^{-B(\tau)r}, \quad \tau = T - t \in [0, T],$$

where

$$B(\tau) = \frac{2(e^{\eta\tau} - 1)}{(\kappa + \lambda + \eta)(e^{\eta\tau} - 1) + 2\eta}, \quad A(\tau) = \left(\frac{\eta e^{(\kappa+\lambda+\eta)\tau/2}}{e^{\eta\tau} - 1} B(\tau)\right)^{\frac{2\kappa\theta}{\sigma^2}},$$

and $\eta = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}$.

Four parameters $\kappa, \sigma, \theta, \lambda$ enter analytic formula

Three parameters reduction

by introducing the following new parameters:

1. $\beta = e^{-\eta} \quad \in [0,1]$
2. $\xi = \frac{\kappa+\lambda+\eta}{2\eta} \quad \in [0,1]$
3. $\varrho = \frac{2\kappa\theta}{\sigma^2} \quad \in \mathbb{R}$

where $\eta = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$.

Summarizing, in the CIR model the price of bonds and, consequently, the corresponding yield curve depend only on three transformed parameters β, ξ and ϱ.

Two-phase minmax method

1. we identify one dimensional curve of the CIR parameters by minimizing the cost functional that mimics least squares approach in linear regression methods. The global minimum is attained at a unique point $\check{\beta}, \check{\xi}, \check{\varrho}$ of transformed vars $\Rightarrow$ on a one dimensional curve of genuine CIR parameters $\kappa, \sigma, \theta, \lambda$
2. maximization of the likelihood function restricted to that one dimensional curve of global minimizers

The restricted maximum likelihood is attained in a unique point - desired estimation of the CIR model parameters.

Nonlinear least square minimization

We minimize the weighted least square sum of differences between real market yield curve interest rates and those predicted by CIR model.

$$U(\beta,\xi,\varrho) = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{n}\sum_{i=1}^{n}(R_j^i - \bar{R}_j^i)^2\tau_j^2$$

- $\{R_j^i, j = 1, \dots, m\}$ is the yield curve of the length m at time $i = 1, \dots, n$, with time τ_j to maturity.
- $\{\bar{R}_j^i, j = 1, \dots, m\}$ is the yield curve computed by using the CIR model with parameters β, ξ, ϱ.
- The over-night (short-rate) interest rate at time $i = 1, \dots, n$, is denoted by R_0^i.

Nonlinear least square minimization

the cost function U attains a unique global minimum

$$U(\check\beta, \check\xi, \check\varrho) = \min_{(\beta,\xi,\varrho)\in\Omega} U(\beta, \xi, \varrho)\,.$$

In terms of genuine CIR parameters:

$$(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda) \quad \leftrightarrow \quad (\check\beta, \check\xi, \check\varrho)$$

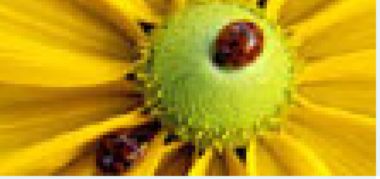
the global minimum is attained on a λ-parameterized curve of global minimizers $(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda),\ \lambda \in \mathbb{R}$.

Nonlinear regression problem!

- the function $U = U(\beta, \xi, \varrho)$ is nonlinear and it is not know whether is convex or not
- standard steepest descent gradient minimization method of Newton-Kantorovich type fail to converge to a global minimum of U
- we need a robust and efficient numerical method unconditionally converging to a unique global minimum of U

$\Rightarrow$ we had to apple a variant of the evolution strategy (ES) globally converging algorithm

Evolution strategy (ES) is the one of the most successful stochastic algorithm which was invented to solve technical optimization problems (see e.g. Schwefel (1977, 1995, 1998).



Restricted likelihood function

Gaussian estimation of parameters of the mean reverting process driving the short rate r_t

$$\ln L(\kappa, \sigma, \theta) = -\frac{1}{2} \sum_{t=2}^{n} \left(\ln v_t^2 + \frac{\varepsilon_t^2}{v_t^2} \right)$$

$$v_t^2 = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa}\right) r_{t-1}, \qquad \varepsilon_t = r_t - e^{-\kappa} r_{t-1} - \theta\left(1 - e^{-\kappa}\right).$$

Combining information gained from minimization of the cost functional U and likelihood parameter estimation $\Rightarrow$ restricted maximum likelihood function $\Rightarrow$ maximize $\ln L$ over a one dimensional curve $(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda),\ \lambda \in \mathbb{R}$.

Restricted likelihood function

The argument $(\kappa_{\bar\lambda}, \sigma_{\bar\lambda}, \theta_{\bar\lambda})$ of the restricted likelihood function

$$\ln L^r = \ln L(\kappa_{\bar\lambda}, \sigma_{\bar\lambda}, \theta_{\bar\lambda}) = \max_\lambda \ln L(\kappa_\lambda, \sigma_\lambda, \theta_\lambda)\,.$$

is now identified the resulting optimal values as estimated CIR

parameters obtained by the two-phase minmax optimization method.

Restricted maximum likelihood ratio

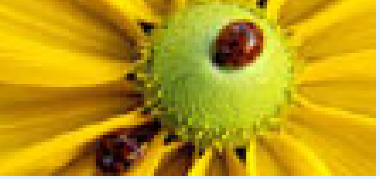
The ratio between the unrestricted maximum likelihood

$$\ln L^u = \ln L(\kappa^u, \sigma^u, \theta^u) = \max_{\kappa,\sigma,\theta>0} \ln L(\kappa, \sigma, \theta) .$$

and the restricted maximum likelihood $\ln L^r$ is called the maximum likelihood ratio (MLR)

$$MLR = \frac{\ln L^r}{\ln L^u} .$$

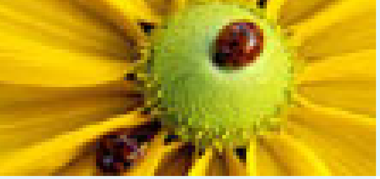
- $MLR \leq 1$
- value of MLR close to 1 indicates that the restricted maximum likelihood value $\ln L^r$ is close to unrestricted one $\Rightarrow$ simple estimation of parameters of mean reversion equation is also suitable for calibration of the whole term structure



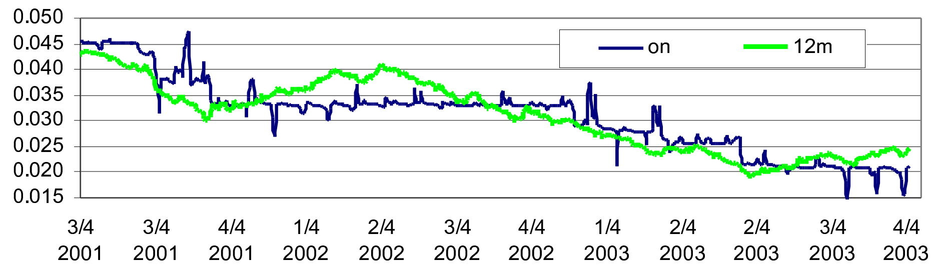
Term structures description

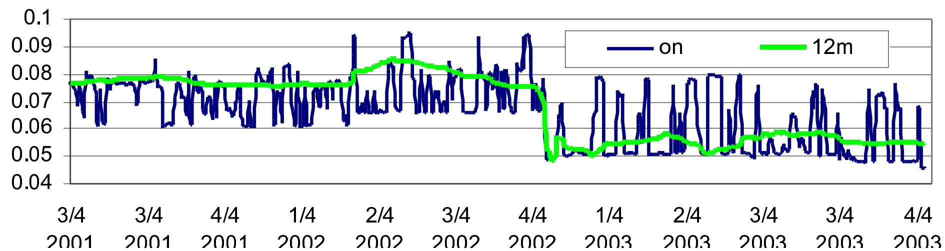
- The data for overnight interest rate are available for BRIBOR, PRIBOR, WIBOR, BUBOR as well as for London Inter-bank Offer Rates (LIBOR).
- For the Euro-zone term structure EURIBOR is concerned we recall that there exists a substitute for the overnight rate which is called EONIA.

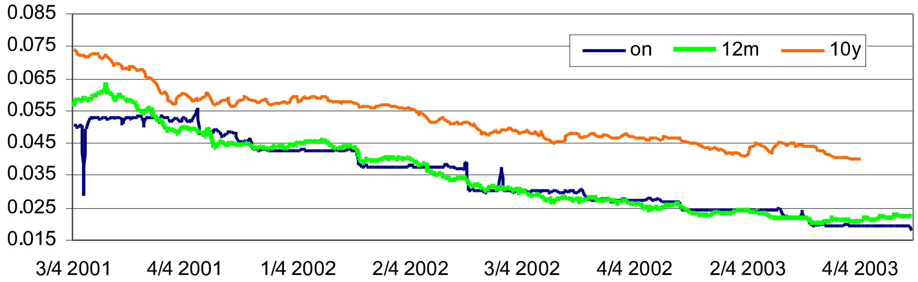
Calibration covers the period from the year 2001 up to 2003 and is done on quarterly basis



Term structures description

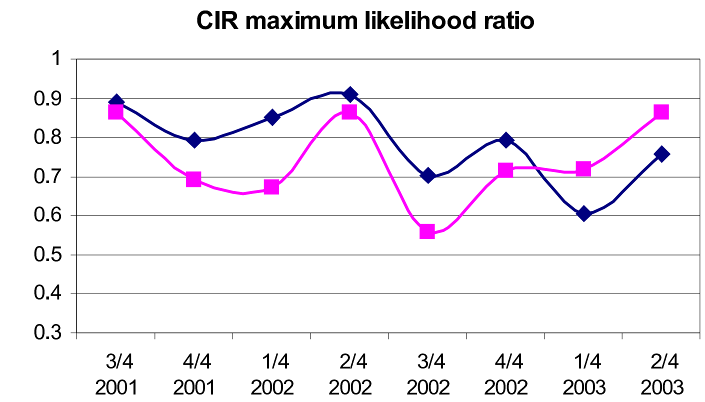


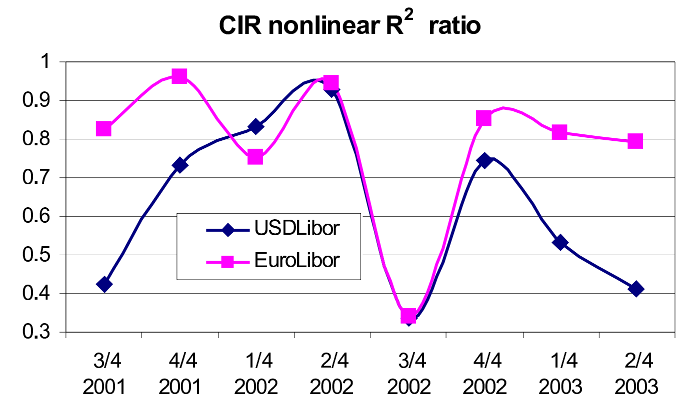




Graphical description of overnight (short-rate) interest rates and those of bond with longer maturity. Daily data are plotted for EURO-LIBOR, BRIBOR and PRIBOR.

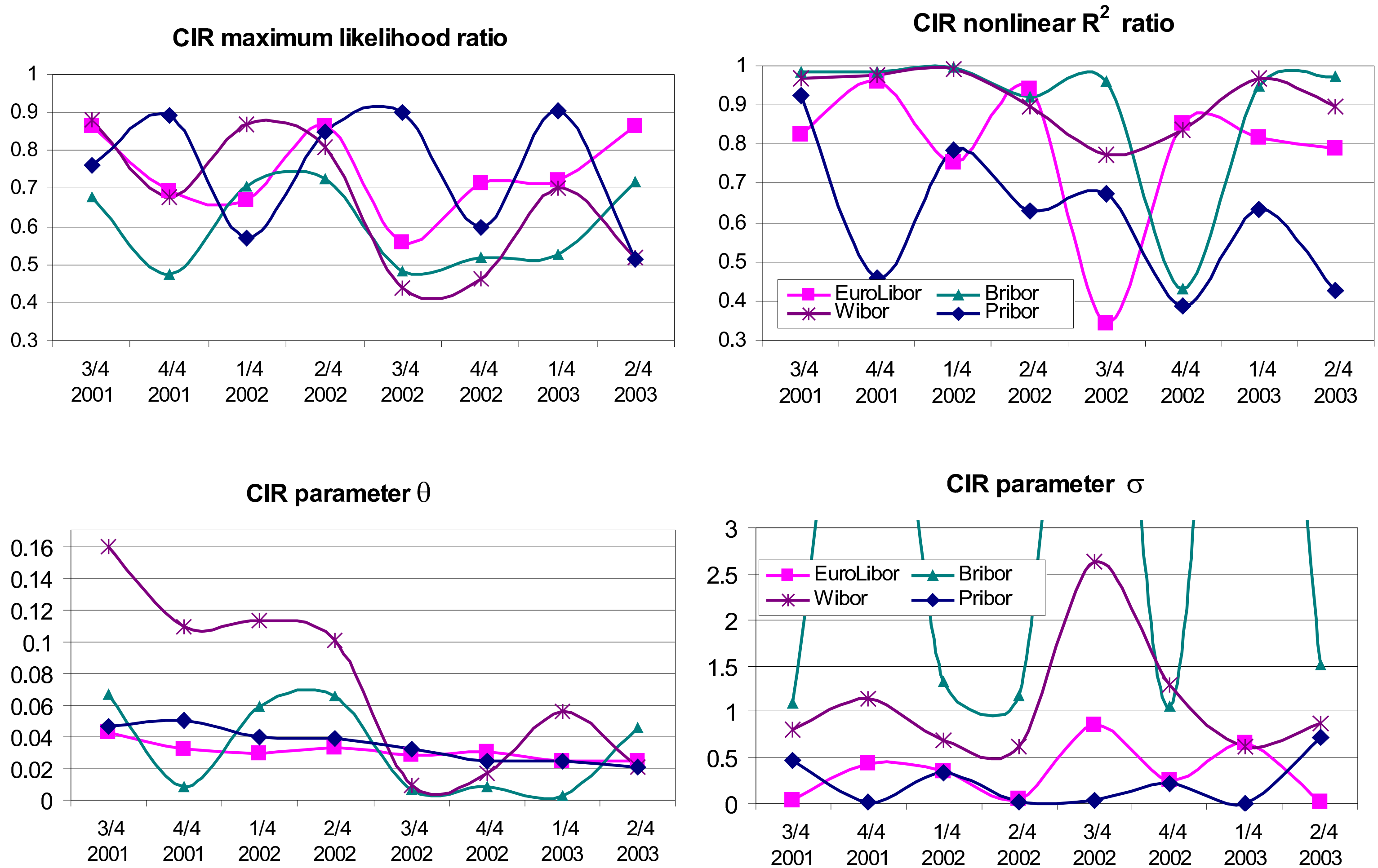
Calibration results





Results of calibration for term structures. Maximum likelihood ratio (a) and R^2 ratio (b) for EURO-LIBOR and USD-LIBOR.

Calibration results



Maximum likelihood ratio and R^2 ratio for EURO-LIBOR and comparison to BRIBOR, PRIBOR and WIBOR. Estimated parameters θ and σ (below).

Summary of CIR model calibration

- a two-phase minmax optimization method for calibration of the CIR one factor interest rate model. It is based on minimization of the cost functional together with maximization of the likelihood function restricted to the set of minimizers.
- We have tested the calibration method various term structures including both stable western inter-bank offer rates as well as those of emerging economies.
- Based on our results of calibration of the CIR one-factor model we can state that the western European term structure data are better described with CIR model compared to transitional economies represented by Central European countries

III. Stochastically nonlinear linear interest rate models

(Linear interest rate models that are nonlinear with respect to stochastic averaging)

Joint work with B. Stehlíková

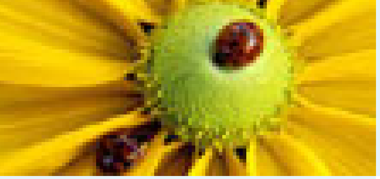
Averaging of linear partial differential equations

Motivation from thermodynamical physical problem. Passive scalar (for temperature θ) linear transport equation with diffusion

$$\frac{\partial \theta}{\partial t} + (\vec{v}.\nabla)\theta = \Delta\theta, \qquad \theta(x,0) = \theta_0(x)$$

where $\vec{v}$ is a given random vector field with zero mean $\langle \vec{v} \rangle = 0$. The formally averaged linear PDE is:

$$\frac{\partial \langle\theta\rangle}{\partial t} = \Delta\langle\theta\rangle, \qquad \theta(x,0) = \theta_0(x)$$

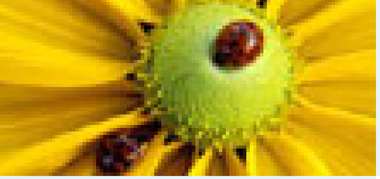


Stochastically nonlinear models

What is the correct equation the ensemble average $\langle\theta\rangle$? The answer (Majda's example) is not the formally averaged equation but the averaged PDE with enhanced diffusion coefficient $\kappa > 1$

$$\frac{\partial\langle\theta\rangle}{\partial t} = \kappa\Delta\langle\theta\rangle, \qquad \langle\theta(x,0)\rangle = \theta_0(x)$$

Although $\theta \mapsto (\vec{v}.\nabla)\theta$ is linear in θ we have $\langle(\vec{v}.\nabla)\theta\rangle \neq \langle(\vec{v}.\nabla)\rangle.\langle\theta\rangle = 0\langle\theta\rangle = 0$, in general (unless $\vec{v}$ and θ are independent).



Fong-Vašíček model

The Fong-Vašíček two-factor interest rate model for valuing term structures. The volatility of the short rate process is assumed to be stochastic and it satisfies a stochastic differential equation of the mean reversion type.

$$
\begin{aligned}
dr &= \kappa_1(\theta_1 - r)dt + \sqrt{y}dw_1 \\
dy &= \kappa_2(\theta_2 - y)dt + v\sqrt{y}dw_2
\end{aligned}
$$

- $\theta_1 > 0, \theta_2 > 0$ are given constant characterizing the long term average interest rate and average volatility, resp.
- $v > 0$ is the constant volatility of the volatility
- $\kappa_1, \kappa_2 > 0$ are rates of reversion for the short rate and volatility, resp.
- w_1, w_2 are two Wiener processes with correlation $\rho \in [-1, 1]$ of increments, i.e. $\rho dt = E(dw_1(t)dw_2(t))$.

Using multidimensional Itô's lemma the bond price $P = P(\tau, r, y)$ is a classical solution to the partial differential equation

$$-\frac{\partial P}{\partial \tau} + (\kappa_1(\theta_1 - r) - \tilde{\lambda}_1\sqrt{y})\frac{\partial P}{\partial r} + (\kappa_2(\theta_2 - y) - \tilde{\lambda}_2 v\sqrt{y})\frac{\partial P}{\partial y}$$

$$+\frac{1}{2}y\frac{\partial^2 P}{\partial r^2} + \frac{v^2}{2}y\frac{\partial^2 P}{\partial y^2} + vy\rho\frac{\partial^2 P}{\partial r \partial y} - rP = 0$$

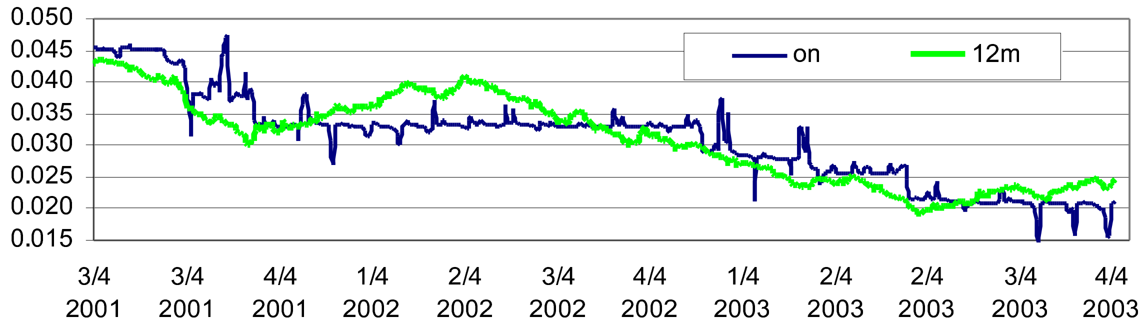
with the initial condition $P(0, r, y) = 1$.

A solution (the bond price) $P(\tau, r, y)$ depends on

- the time $\tau = T - t$ to maturity,
- the short rate r
- the present value y (dispersion)

Observable and hidden variables

- the short rate r at any time is an observable variable (from real inter-bank offer rates data market data)



- But the volatility $\sqrt{y}$ is a hidden variable and can hardly estimated from historical data !!!

Volatility averaging in Fong-Vašíček model

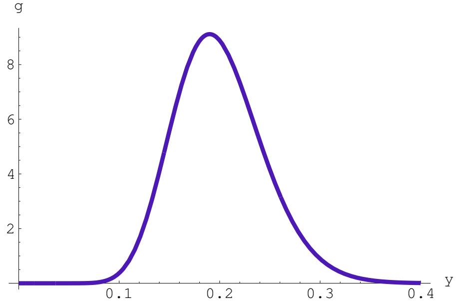
The dispersion y ($\sqrt{y}$ is volatility) satisfies the Bessel square root process SDE

$$dy = \kappa_2(\theta_2 - y)dt + v\sqrt{y}dw_2$$

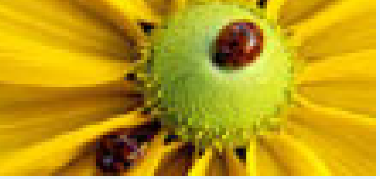
The corresponding Fokker-Planck equation for the conditional density distribution $f(t, y|y(0) = y_0)$ of the random variable $y(t)$

$$-\frac{\partial f}{\partial t} - \frac{\partial}{\partial y}(\kappa_2(\theta_2 - y)f) + \frac{\partial^2}{\partial y^2}\left(\frac{v^2 y}{2} f\right) = 0, \;\; \tau > 0\,,$$

with the initial condition $f(0, y) = \delta_0(y - y_0)$



The limiting density distribution $g(y) = \lim_{t\to\infty} f(t, y)$



Ways out

Average the bond price with respect to the volatility are then given by

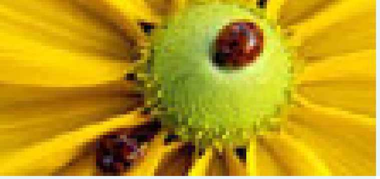
$$\langle P(\tau, r, y)\rangle_y = \int_{-\infty}^{\infty} P(\tau, r, y) g(y) dy,$$

The PDE for $P = P(\tau, r, y)$ is linear in P but it is stochastically nonlinear with respect to averaging

$$\langle P(\tau, r, y)\rangle_y \neq P(\tau, r, \langle y\rangle) = P(\tau, r, \theta_2)$$

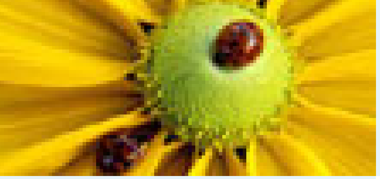
(remember Majda's example from thermomechanics)

What is the averaged price $\langle P(\tau, r, y)\rangle_y$?



Averaging with respect to a hidden variable

- Majda
  - averaging of stochastic PDE arising from thermomechanics
- Cotton, Foque, Papanicolaou, Sircar (2001)
  - Stochastic volatility corrections for interest rate derivatives
- Fouque, Papanicolaou, Sircar (1999)
  - Asymptotic of a two-scale stochastic volatility model. Application to two factor option pricing model.



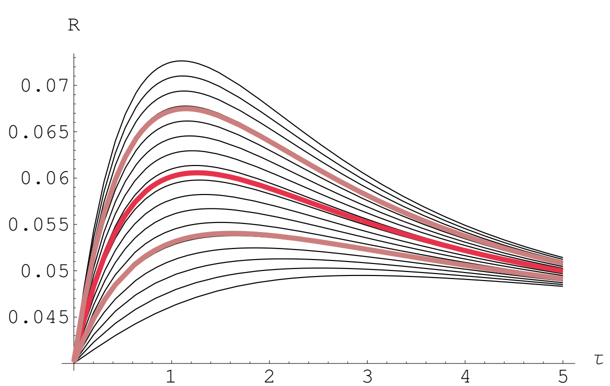
Fong-Vašíček model has an explicit solution

$$P(\tau, r, y) = A(\tau) e^{-B(\tau)r - C(\tau)y}$$

where the functions $A = A(\tau),\ B = B(\tau), C = C(\tau), \tau \in (0, T]$, satisfy the following system of nonlinear ordinary differential equations.

$$\Downarrow$$

$$\langle P(\tau, r, y)\rangle_y = A(\tau) e^{-B(\tau)r} \langle e^{-C(\tau)y}\rangle_y > A(\tau) e^{-B(\tau)r - C(\tau)\langle y\rangle} = P(\tau, r,$$



- The distribution of yield curves for various y
- 95% interval of confidence

Summary of averaging methods

- a two-factor Fong-Vašíček can be averaged with respect to the stochastic volatility
- the averaged bond price is higher then the bond price calculated from averaged value of the volatilty