Solution of Nonlinearly Curvature Driven Evolution of Plane Curves

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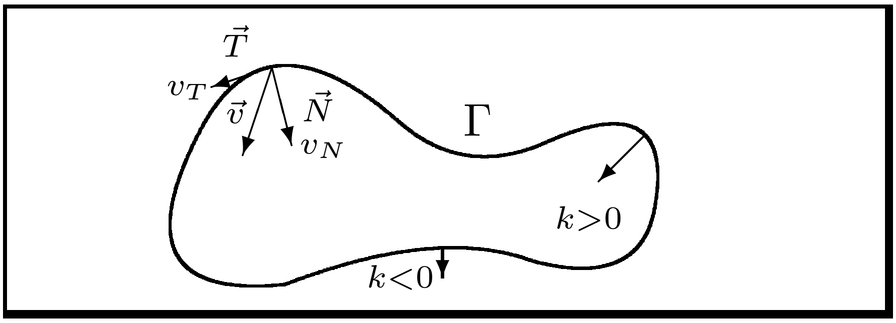
Goals

- To study evolution of plane curves obeying the geom. equation

$$v = \beta(k) \qquad (1)$$

where v is normal velocity and k is the curvature of a plane curve and $\beta : R_0^+ \to R_0^+$ is a smooth function, e.g. $\beta(k) = k^m, m > 0$.



- To suggest a **new computational method** for solving geometrical eq (1)
- To represent eq (1) by a so-called intrinsic heat equation governing the evolution of plane curves with the normal velocity obeying eq (1)
- To present numerical simulations and experiments

Motivation

- Morphological image and shape multiscale analysis of Alvarez, Guichard, Lions and Morel (1993)

$$\beta(k) = k^{1/3}$$

- Affine invariant scale space of curves introduced by Sapiro and Tannenbaum (1994)

$$\beta(k) = k^{1/3}$$

• Isotropic motions of plane phase interfaces studied by Angenent and Gurtin (1989)

$$\beta(k) = k, \quad \beta(k) = k^{1/2}$$

• Numerical approximation of solutions of the intrinsic heat equation (i.e. space mesh size independent form) due to Dziuk (1994)

$$\beta(k) = k$$

Parametrization

• parametrization of a smooth curve Γ the plane R^2, by a C^2 smooth function $x : R/Z \to R^2$ such that

$$\Gamma = \{x(u), u \in [0,1]\}$$

• time evolution $\{\Gamma^t\}, t \in [0, T_{max})$ of a curve Γ^0 we adopt the notation

$$\Gamma^t = \{x(u,t), u \in [0,1]\}, \quad t \in [0, T_{max})$$

where $x : C^2(R/Z \times [0, T_{max}), R^2)$.

• Γ admits various other parametrization! Henceforth, the parameter s will always refer to the arc-length parameter

Intrinsic heat equation

• The goal is to describe the evolution of plane curves $\{\Gamma^t\}$ undergoing the intrinsic heat equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s_*^2} \qquad (2)$$

where s_* is a new parametrization of a curve Γ^t obeying the law

$$ds_* = \vartheta(s) ds$$

• we seek for a function ϑ such that the normal component of the curve-flow velocity v satisfies the equation $v = \beta(k)$.

• using the arc-length parametrization we obtain

$$\frac{\partial x}{\partial t} = \frac{1}{\vartheta(s)} \frac{\partial}{\partial s} \left(\frac{1}{\vartheta(s)} \frac{\partial x}{\partial s} \right) = \frac{1}{\vartheta^2(s)} kN - \frac{\vartheta'(s)}{\vartheta^3(s)} T$$

- Normal velocity $v = (x_t, N)$ fulfills equation $v = \beta(k)$ iff

$$\vartheta = \frac{k^{1/2}}{\beta(k)^{1/2}}$$

- Tangential velocity $v = (x_t, T)$ (Does not change the shape of a curve!)

$$x_t = \beta\, N \;\; + \alpha\, T$$

where, for $\beta(k) = k^m$,

$$\alpha = \frac{m-1}{2} k^{m-1} k_s$$

- Example "Finger"

General intrinsic heat equation

- In terms of time dependent arc-length parametrization

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\frac{\partial x}{\partial s}\right)$$

- In terms of fixed domain time parametrization

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1 |x_u|}\frac{\partial}{\partial u}\left(\frac{1}{\theta_2 |x_u|}\frac{\partial x}{\partial u}\right)$$
$$(u,t) \in [0,1] \times [0, T_{max})$$

where θ_1, θ_2 depend on the curvature

$$k = |x_{ss}| = \frac{x_u \wedge x_{uu}}{|x_u|^3}$$

such that

$$\theta_1(k)\,\theta_2(k) = \frac{k}{\beta(k)}$$

The fully nonlinear system of PDE's is subject to the initial condition $x(u,0) = x^0(u), u \in [0,1]$ and periodic boundary conditions at $u = 0, 1$.

- Example (Dziuk '94) If $\beta(k) = k$ and $\theta_1 = \theta_2 = 1$ we obtain

$$\frac{\partial x}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial x}{\partial u}\right)$$

- Example (Deckelnik '97) If $\beta(k) = k$ and $\theta_1 = |x_u|,\ \theta_2 = |x_u|^{-1}$

$$\frac{\partial x}{\partial t} = \frac{x_{uu}}{|x_u|^2}$$

- Example (Skokan '98) If $\beta(k) = k^m,\ m > 0$ or $\beta(k) = -1 + \varepsilon k$ the appropriate choice of θ_2 is $\theta_2 = 1 + |k|^3$

Eqs. for the curvature and length functions

- Fully nonlinear system of degenerate parabolic PDE's

$$\frac{\partial k}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial}{\partial u}\beta(k)\right) + \alpha(k)\frac{1}{|x_u|}\frac{\partial k}{\partial u} + k^2\beta(k)$$

$$\frac{\partial |x_u|}{\partial t} = -\ |x_u| k\beta(k)\ +\ \frac{\partial \alpha(k)}{\partial u}$$

where

$$\alpha(k) = \frac{1}{\theta_1 |x_u|}\frac{\partial}{\partial u}\left(\frac{1}{\theta_2}\right)$$

- these equations imply curve shortening and area decreasing properties

$$\frac{d}{dt} Length(\Gamma^t) + \int_{\Gamma^t} k\beta(k)\, ds = 0$$

$$\frac{d}{dt} Area(\Gamma^t) + \int_{\Gamma^t} \beta(k)\, ds = 0$$

Self-similar solutions

- we seek for a solution $x(u,t)$ having the form

$$x(u,t) = \phi(t)\tilde{x}(u)$$

Proposition. *Assume that* $\beta(k) = k^{1/\alpha}$, $\alpha > 0$. *A function* $x \in C^{2,1}(R/Z \times [0,T), R^2)$ *of the form* $x(u,t) = \phi(t)\tilde{x}(u)$ *is a solution of (1.10) iff*

- $\Gamma = Image(\tilde{x})$ *is a circle for* $0 < \alpha \neq 3$, *or*
- $\Gamma = Image(\tilde{x})$ *is an ellipse for* $\alpha = 3$.

In other words, the only self-similar solutions of (1.10) are either shrinking circles ($0 < \alpha \neq 3$) *or shrinking ellipses* ($\alpha = 3$).

- Blow-up times.

Shrinking circles The case $0 < \alpha \neq 3$. If the initial curve Γ^0 is a circle with radius a then the solution blows up at

$$T_{max} = \frac{\alpha}{\alpha+1} a^{\frac{\alpha+1}{\alpha}}$$

Shrinking ellipses The case $\alpha = 3$, i.e. $\beta(k) = k^{1/3}$. If the initial curve Γ^0 is an ellipse with halfaxes a, b then the solution blows up at

$$T_{max} = \frac{3(ab)^{\frac{2}{3}}}{4}$$

A-priori estimates

- Curve shortening property

Proposition. *Let* x *be a nondegenerate classical solution of equation (3.1),* $\varepsilon \geq 0$. *Then, for each* $t \in (0, T_{max})$,

$$\frac{d}{dt}\int_0^1 |x_u(.,t)| + \int_0^1 k\beta(k)|x_u(.,t)| = 0$$

where $k = k(x_u, x_{uu})$

$$\frac{d}{dt}Length(\Gamma^t) + \int_{\Gamma^t} k\beta(k)\, ds = 0$$

• the length $|\Gamma^t|$ of the curve Γ^t decreases along the time, i.e. $\{\Gamma^t\}, t \in [0, T_{max})$, is a curve shortening flow.

Numerical scheme

• Let $[0, T]$ be an interval and let $\tau = T/n, n \in N$, denote the time discretization step. By $x^i, i = 0, 1, ..., n$, we denote the approximation of a true solution at time $t = i\tau$, i.e. $x^i(.) = x(., i\tau)$.

• time discretization scheme is based on approximation of the intrinsic heat equation by the backward Euler method

$$\frac{x^i - x^{i-1}}{\tau} = \frac{\partial^2 x^i}{ds_*^2}$$

where the parameterization s_* is computed from the previous time step x^{i-1}. The 'Eulerian form' of the above scheme reads as follows

$$x^i - \frac{\tau}{g^{i-1}} \frac{\partial}{\partial u} \left(\frac{1}{g^{i-1}} \frac{\partial x^i}{\partial u} \right) = x^{i-1}$$

$i = 1, 2, ..., n$, where $g^{i-1} = |x_u^{i-1}| \theta_\beta(x_u^{i-1}, x_{uu}^{i-1})$ and x^0 is the initial condition.

• full space-time discretization scheme. We use the uniform spatial grid $u_j = jh$ $(j = 0, ..., m)$ with $h = 1/m$. The smooth solution x is then approximated by the discrete values x_j^i corresponding to $x(jh, i\tau)$. Using quite natural finite difference approximations of spatial differential terms we end up with semi-implicit difference scheme

$$\frac{1}{2}(g_j^{i-1} + g_{j+1}^{i-1}) \frac{x_j^i - x_j^{i-1}}{\tau} = \frac{x_{j+1}^i - x_j^i}{g_{j+1}^{i-1}} - \frac{x_j^i - x_{j-1}^i}{g_j^{i-1}}$$

$i = 1, ..., n, \qquad j = 1, ..., m$, where

$$g_j^{i-1} = h_j^{i-1} \left(\frac{k_j^{i-1}}{\beta(k_j^{i-1})} \right)^{1/2}, \quad h_j^{i-1} = |x_j^{i-1} - x_{j-1}^{i-1}|$$

and the curvature can be approximated as

$$k_j^{i-1} = \frac{1}{h_j^{i-1}} \left| \text{arccos} \left(\frac{x_{j+1}^{i-1} - x_{j-1}^{i-1}}{|x_{j+1}^{i-1} - x_{j-1}^{i-1}|}, \frac{x_j^{i-1} - x_{j-2}^{i-1}}{|x_j^{i-1} - x_{j-2}^{i-1}|} \right) \right|$$

- Intrinsic property.

The 'intrinsic property' of the governing equation causes that the spatial parametrization step is not involved in the approximation scheme and therefore only the spatial position of points of a curve Γ and the curvature of Γ play the role in the discretization scheme suggested

Conclusions

In this paper we have

- proposed a governing equations capable of describing evolution of plane curves having the normal component of the velocity equal to $\beta(k)$
- studied special self-similar solutions of the governing equations and their blow-up times
- shown some a-priori estimates yielding, in particular, curve shortening property
- designed semi-implicit full time-space discretization scheme for numerical approximating of solutions
- obtained a scheme which is intrinsic, only the spatial position of points of a curve Γ its the curvature of Γ occur in the scheme

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This document and the paper are available at

http://www.iam.fmph.uniba.sk/institute/sevcovic