Nonlinear curvature driven evolution of plane curves

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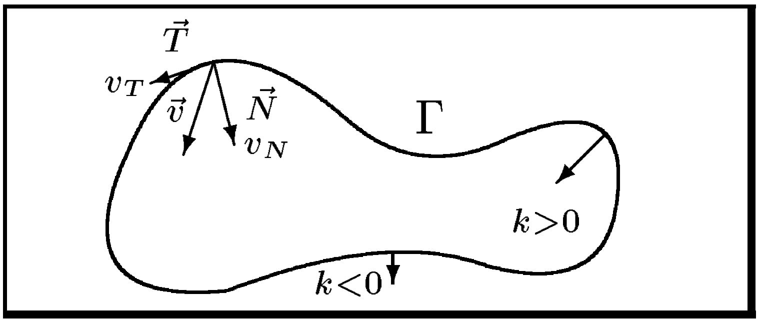
Goals

- To study evolution of plane curves obeying the geometric equation

$$v = \beta(k, \nu)$$

where v is the normal velocity k and ν are the curvature and the tangential angle of a plane curve



$$\beta(k, \nu) = \gamma(\nu)|k|^{m-1}k$$

where $m > 0$ and $\gamma(\nu)$ is a given anisotropy function.

- To represent flow of plane curves by a solution of the intrinsic heat equation

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\frac{\partial x}{\partial s}\right)$$

- To suggest a suitable tangential velocity function α preserving a uniform redistribution of points along the evolution

$$\partial_t x = \beta\, N \ + \alpha\, T$$

- To suggest a computational method for solving geometrical equation and present numerical simulation

Motivation

• Morphological image and shape multiscale analysis of Alvarez, Guichard, Lions & Morel '93 and affine invariant scale space of curves introduced by Sapiro, Tannenbaum & Angenent '94, '98

$$\boxed{\beta(k) = k^{1/3}}$$

• Bifurcation analysis of selfsimilar solutions of the geometric equation by Matano and Ushijima & Yazaki '98.

$$\boxed{\beta(k) = k^{m}}$$

where $m = 1/(n^2 - 1), n = 2, 3, ...$ are bifurcation values.

• Motions of plane phase interfaces and contact conditions studied by Angenent & Gurtin '89 where the normal velocity $v = \beta(k, \nu)$ is given by

$$\boxed{\mu(\nu, v)v = h(\nu)k - g}$$

• Numerical approximation of solutions of the intrinsic heat equation in isotropic and anisotropic case suggested by Dziuk '94, '98

$$\boxed{\beta(k, \nu) = \gamma(\nu)k}$$

Generalized intrinsic heat equation

• The goal is to describe the evolution of plane curves $\Gamma^t = \{x(u,t), u \in S^1\}, \quad t \in [0,T)$ by a solution $x = x(u,t)$ of the intrinsic heat equation

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1} \frac{\partial}{\partial s} \left(\frac{1}{\theta_2} \frac{\partial x}{\partial s} \right)$$

which can be rewritten in terms of a fixed domain time parametrization as ($ds = |x_u| du$) as

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1 |x_u|} \frac{\partial}{\partial u} \left(\frac{1}{\theta_2 |x_u|} \frac{\partial x}{\partial u} \right)$$
$$(u,t) \in S^1 \times [0,T)$$

where θ_1, θ_2 are geometric quantities, i.e they may depend on the signed curvature

$$k = \frac{\partial_u x \wedge \partial_u^2 x}{|x_u|^3}$$

and/or the tangential angle ν.

The fully nonlinear system of PDE's is subject to the initial condition $x(u,0) = x^0(u), u \in S^1$ representing the initial curve Γ^0.

• Notice that the tangential velocity $\alpha = \partial_t x.T$ does not change the shape of a curve and

$$\partial_t x = \beta\, N \; + \alpha\, T$$

$$\theta_1\, \theta_2 = \frac{k}{\beta(k,\nu)}, \qquad \alpha = \frac{1}{\theta_1} \frac{\partial}{\partial s} \left(\frac{1}{\theta_2} \right)$$

• There is still freedom in choice of the geometric quantitities θ_1, θ_2

- Example (Abresch & Langer '86, Dziuk '94)

If $\beta(k) = k$ and $\theta_1 = \theta_2 = 1$ (implying $\alpha = 0$) we obtain

$$\boxed{\frac{\partial x}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial x}{\partial u}\right)}$$

- Example (Deckelnik '97) If $\beta(k) = k$ and $\theta_1 = |x_u|,\ \theta_2 = |x_u|^{-1}$

$$\boxed{\frac{\partial x}{\partial t} = \frac{x_{uu}}{|x_u|^2}}$$

- Example (Skokan '98) For $\beta(k) = k^m, 0 < m \le 1$, choose $\theta_2 = 1 + |k|^3$
- Example (Mikula & Ševčovič '97)

$\beta(k) = k^m$ and $\theta_1 = \theta_2 = \theta = |k|^{\frac{m-1}{2}}$

$$\boxed{\frac{\partial x}{\partial t} = \frac{1}{\theta}\frac{\partial}{\partial s}\left(\frac{1}{\theta}\frac{\partial x}{\partial s}\right)}$$

$$\boxed{\alpha = \frac{m-1}{2}|k|^{m-3}k\partial_s k}$$

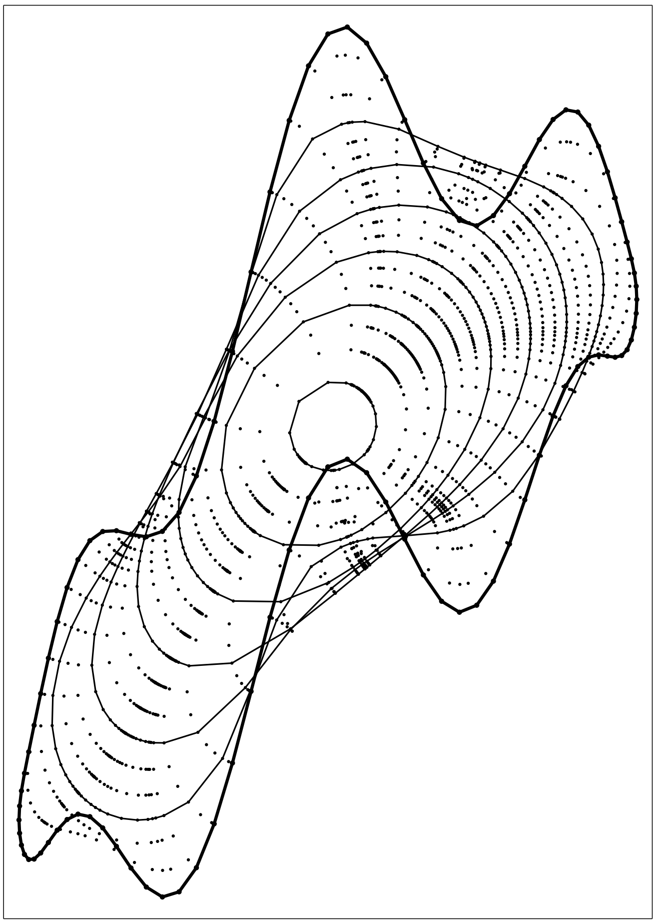
if $0 < m \le 1$ (fast diffusion) maintains redistribution whereas
if $m > 1$ the tangential velocity yields bad redistribution of grid points

- Example (zero tangential velocity)

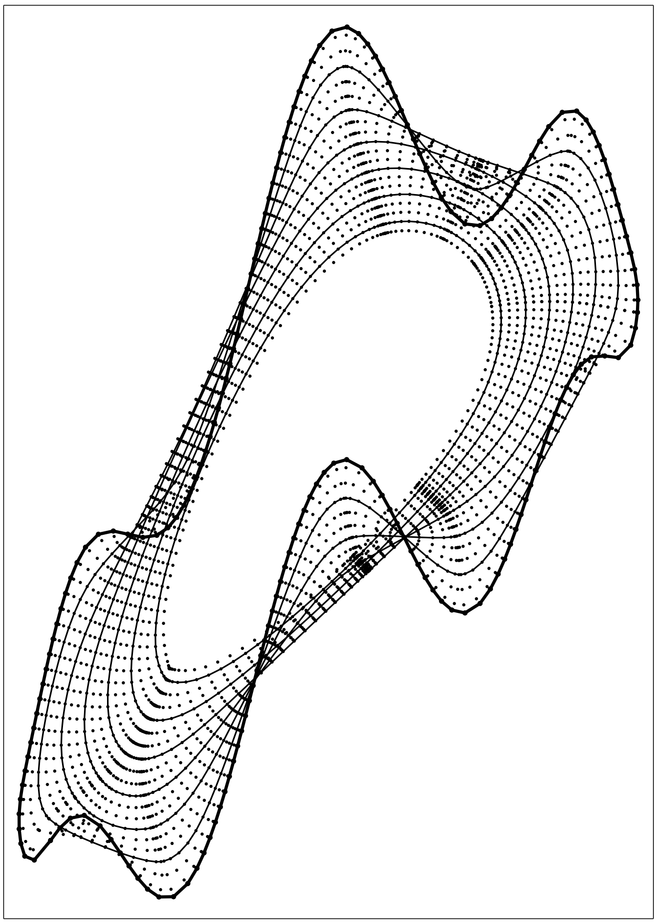
$\beta(k) = k^m$ and $\theta_1 = \frac{k}{\beta(k)},\ \theta_2 = 1$ (implying $\alpha = 0$)

$$\boxed{\frac{\partial x}{\partial t} = \frac{\beta(k)}{k}\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial s}\right)}$$

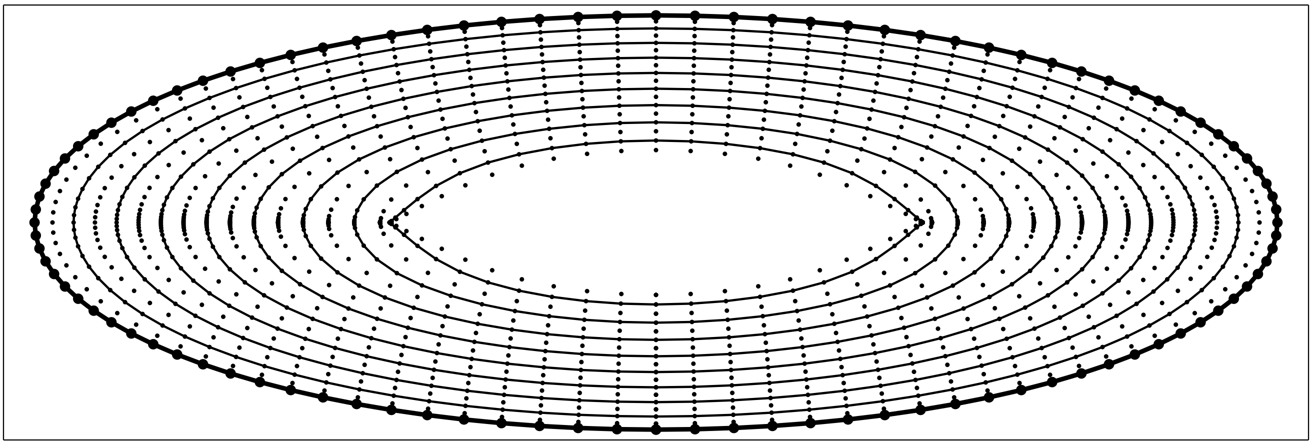
Evolution o plane curves with zero tangential velocity



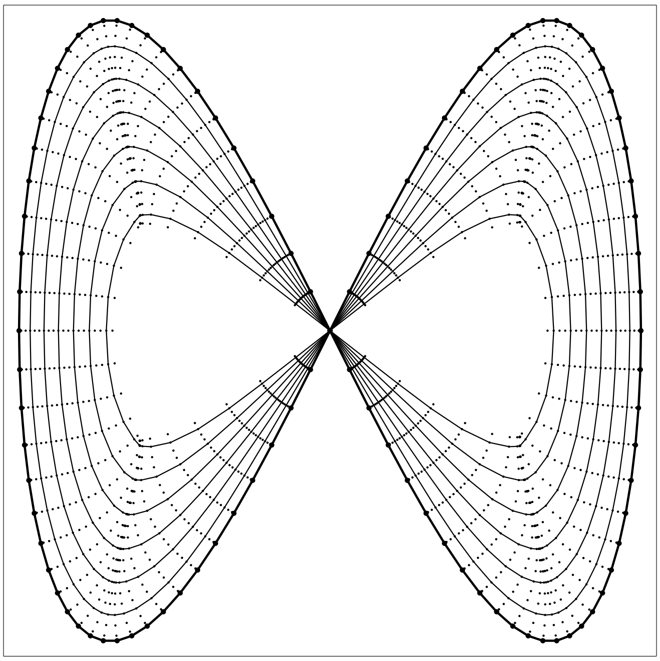
$\beta(k) = k$



$\beta(k) = k^{1/3}$

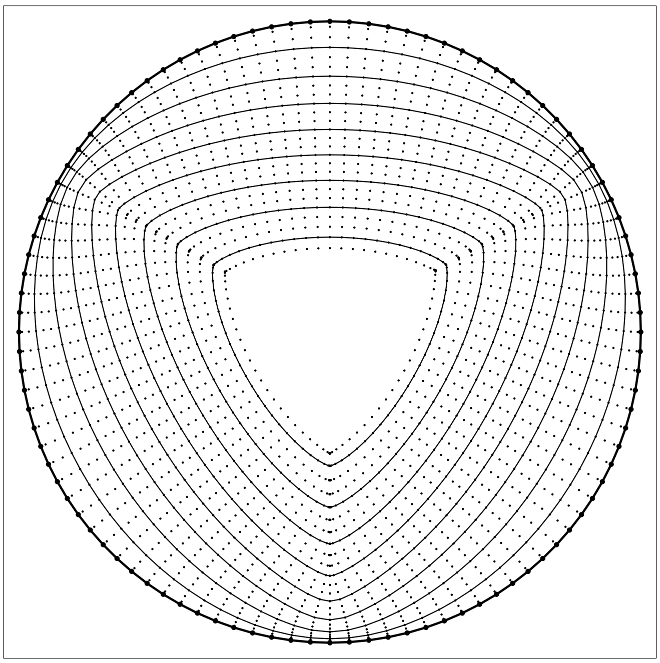


Affine invariant motion of ellipse

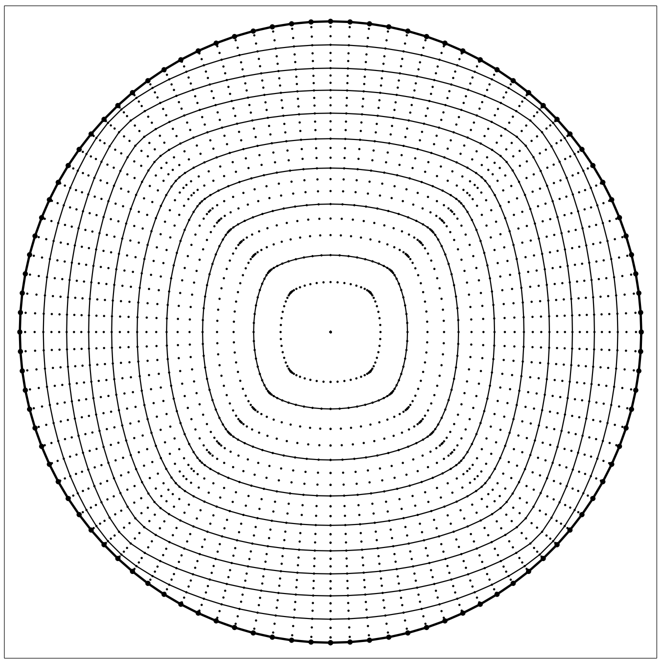


$\beta(k) = k^{1/4}$

Anisotropic evolution o plane curves with zero tangential velocity



$\beta(k,\nu) = (1 - \frac{7}{9}\cos(3\nu))k^{1/2}$



$\beta(k,\nu) = (1 - 0.8\cos(4\nu - \pi))k$

Eqs. for the curvature and the local length

- Fully nonlinear system of degenerate parabolic PDE's

$$\frac{\partial k}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial}{\partial u}\beta(k)\right) + \alpha(k)\frac{1}{|x_u|}\frac{\partial k}{\partial u} + k^2\beta(k)$$

$$\frac{\partial |x_u|}{\partial t} \; = \; - \; |x_u| k\beta(k) \; + \; \frac{\partial \alpha(k)}{\partial u}$$

where

$$\alpha(k) = \frac{1}{\theta_1 |x_u|}\frac{\partial}{\partial u}\left(\frac{1}{\theta_2}\right), \qquad \theta_1\,\theta_2 = \frac{k}{\beta(k)}$$

- these equations imply curve shortening and area decreasing properties

$$\frac{d}{dt} Length(\Gamma^t) + \int_{\Gamma^t} k\beta(k)\, ds = 0$$

$$\frac{d}{dt} Area(\Gamma^t) + \int_{\Gamma^t} \beta(k)\, ds = 0$$

Tangential velocity preserving uniform redistribution

- The idea is to keep the relative local length constant, i.e.

$$\frac{|x_u(u,t)|}{Length(\Gamma^t)} = \frac{|x_u(u,0)|}{Length(\Gamma^0)}$$

for all $(u,t) \in Q_T = S^1 \times [0,T)$

- Combining the total and local length equations we obtain α is the tangential velocity preserving relative local length iff it is a solution of

$$\frac{\partial \alpha}{\partial s} = k\beta(k) - \frac{1}{L}\int_\Gamma k\beta(k)\, ds$$

α can be determined uniquely assuming normalization of θ_2.

Closed system of governing equations

$$\frac{\partial k}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial}{\partial u}\beta(k)\right) + \alpha\frac{1}{|x_u|}\frac{\partial k}{\partial u} + k^2\beta(k)$$

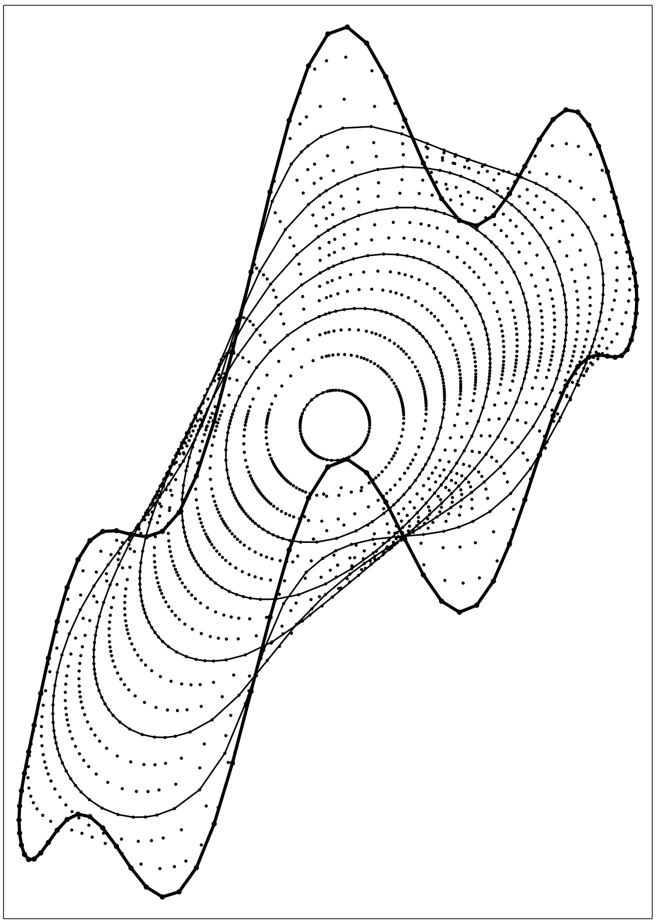
$$\frac{\partial |x_u|}{\partial t} = -|x_u|k\beta(k) + \frac{\partial \alpha}{\partial u}$$

where α is a nonlocal geometric functional satisfying

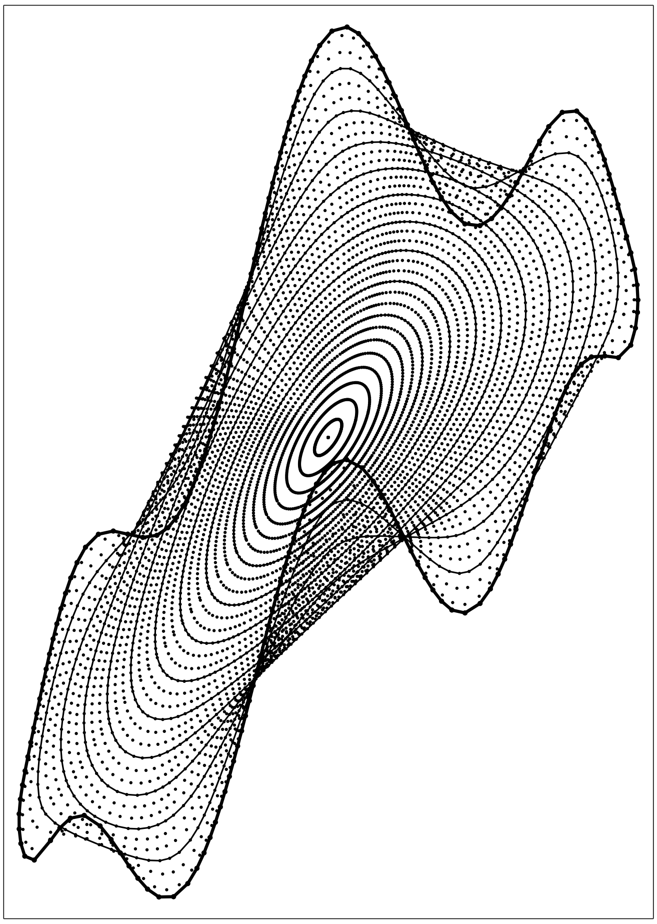
$$\frac{\partial \alpha}{\partial s} = k\beta(k) - \frac{1}{L}\int_\Gamma k\beta(k)\, ds$$

- Initial conditions for $k(.,0)$ and $|x_u(.,0)|$ correspond to the initial curve
- If $\beta = \beta(k,\nu)$ the third parabolic equation for ν must be added into the system of governing equations

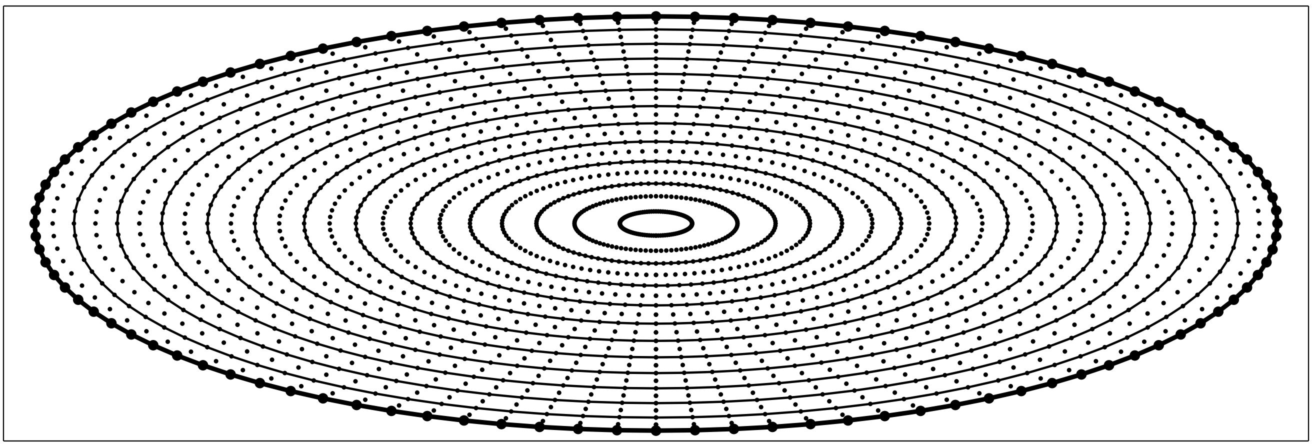
Evolution o plane curves with uniform tangential redistribution



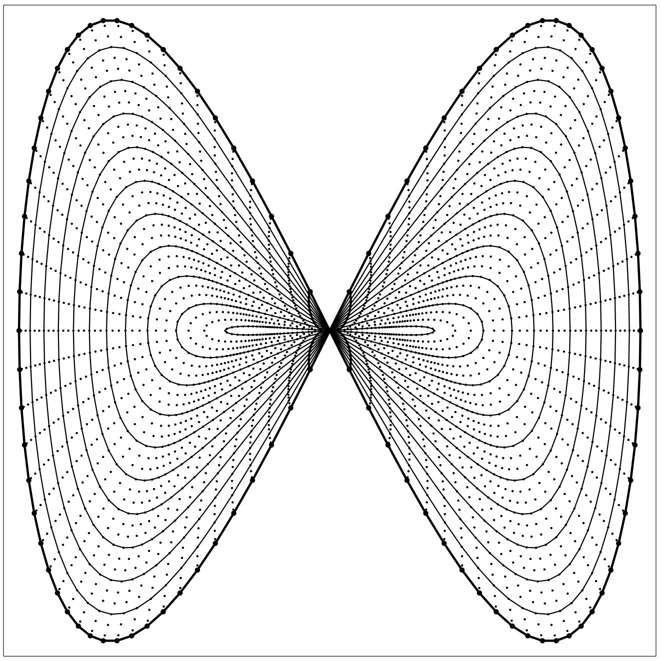
$\beta(k) = k$



$\beta(k) = k^{1/3}$

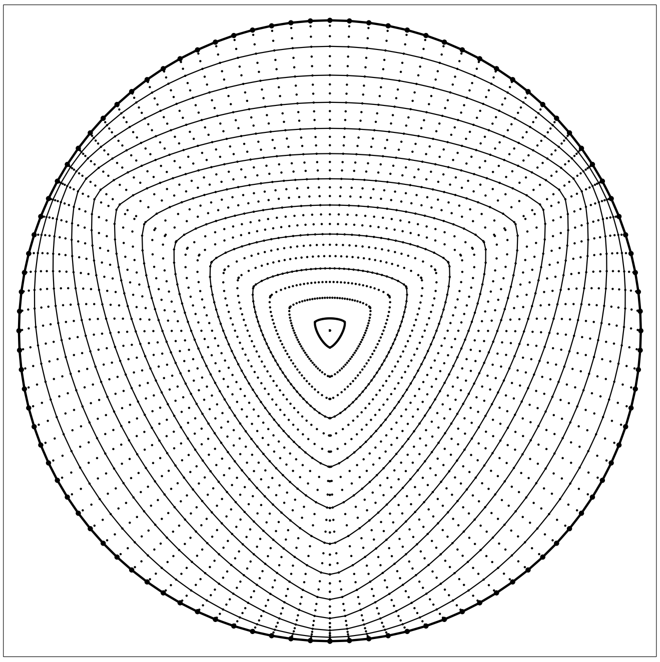


Affine invariant motion of ellipse

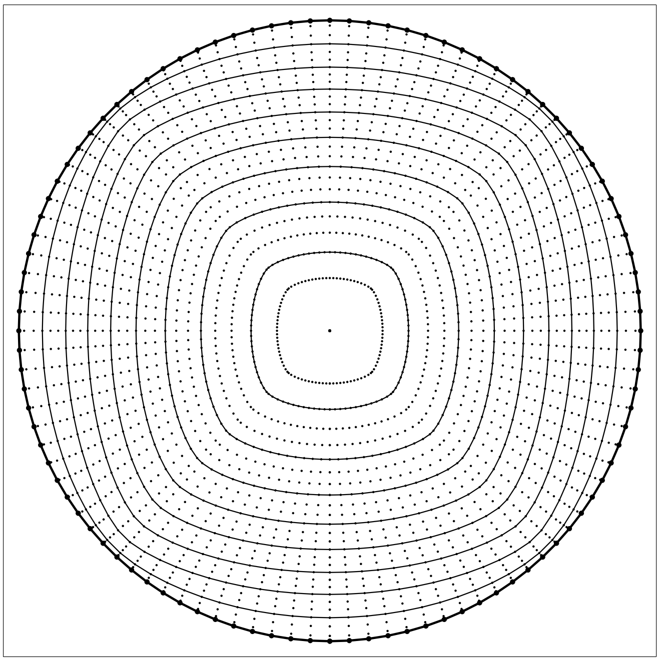


$\beta(k) = k^{1/4}$

Anisotropic evolution o plane curves with tangential redistribution

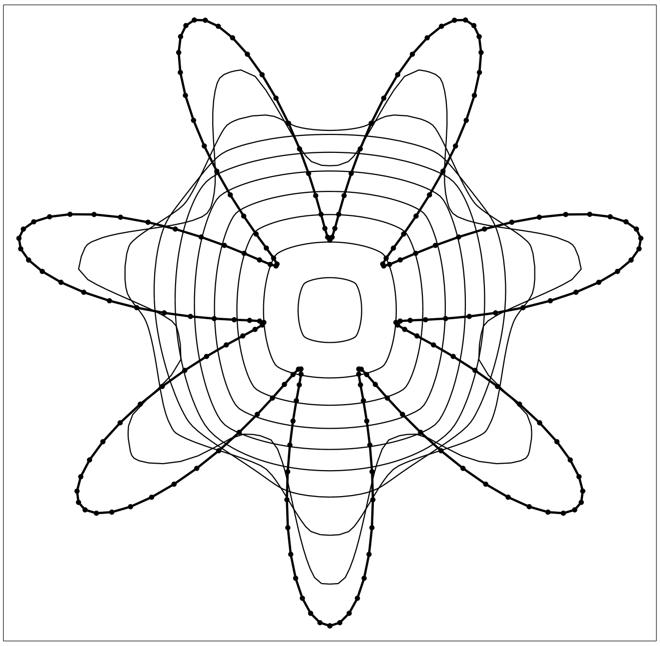


$\beta(k,\nu) = (1 - \frac{7}{9}\cos(3\nu))k^{1/2}$

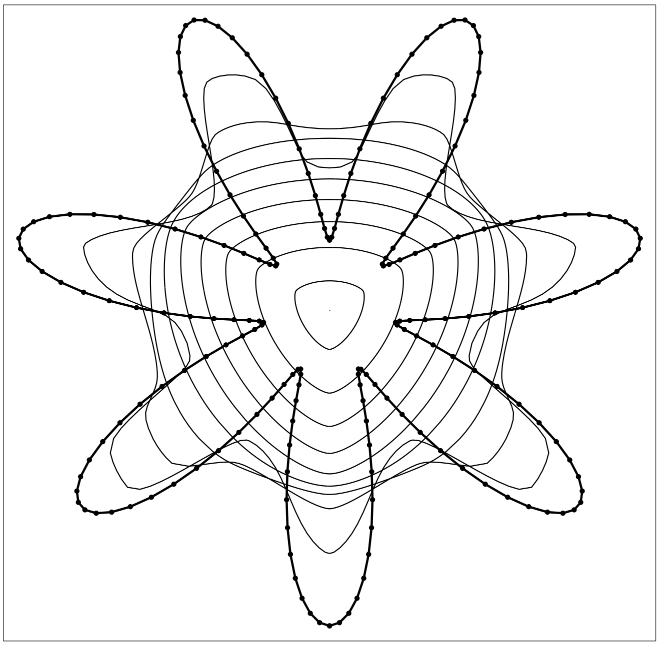


$\beta(k,\nu) = (1 - 0.8\cos(4\nu - \pi))k$

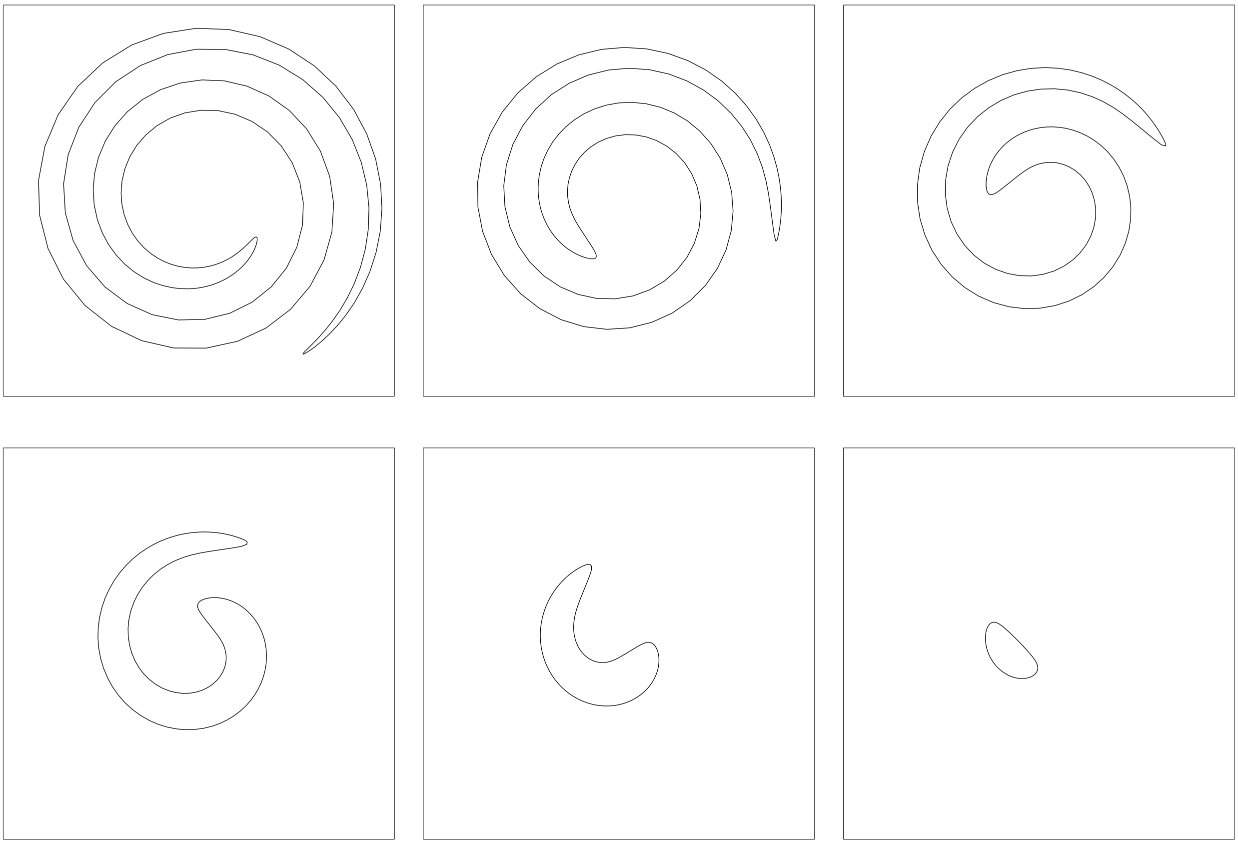
Isotropic and anisotropic motion with tangential redistribution



$\beta(k,\nu) = (1 - \frac{7}{9}\cos(3\nu))k^{3/4}$



$\beta(k,\nu) = (1 - 0.8\cos(4\nu - \pi))k^{3/4}$



The sequence of evolving spirals for $\beta(k,\nu) = k^{1/3}$ *using redistribution. The limiting curve is an ellipse rounded point.*

Local existence of classical solutions

$$\frac{\partial k}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial}{\partial u}\beta(k)\right) + \alpha\frac{1}{|x_u|}\frac{\partial k}{\partial u} + k^2\beta(k)$$

$$\frac{\partial |x_u|}{\partial t} \;=\; -\,|x_u|k\beta(k) \;+\; \frac{\partial \alpha}{\partial u}$$

where α is a nonlocal geometric functional satisfying

$$\frac{\partial \alpha}{\partial s} = k\beta(k) - \frac{1}{L}\int_\Gamma k\beta(k)\,ds$$

Regular case

$$0 < \lambda_- \leq \beta_k'(k,\nu) \leq \lambda_+ < \infty$$

• Using the general theory due to Angenent '90 we can prove the existence of a classical solution

$$(k, |x_u|) \in C([0,T], E_1) \cap C^1([0,T], E_0)$$

where $E_0 = c^\sigma(S^1) \times c^{1+\sigma}(S^1), E_1 = c^{2+\sigma}(S^1) \times c^{1+\sigma}(S^1)$, $0 < \sigma < 1$.

Singular case

$$\beta(k) = |k|^{m-1}k$$

- We must go through a regularization argument $\beta \leftrightarrow \beta^\varepsilon$, $0 < \varepsilon \ll 1$
- We have to establish a uniform in ε bound for the gradient of β, i.e.

$$|\partial_u \beta^\varepsilon| \leq M t^{-\frac{3}{4}}$$

This can be done similarly as in Angenent et al '98 by using the Nash-Moser iterative technique for estimating

$$X_p(t) = \int_{\Gamma^t} |\partial_s \beta^\varepsilon|^p ds$$

for $p = 2, 2^2, ..., 2^k \to \infty$.

- The case $0 < m < 1$ (fast diffusion) is similar to the particular case $\beta(k) = k^{\frac{1}{3}}$ studied by Angenent.
- The case $1 < m$ (slow diffusion) is more involved. To prove the required a-priori bound we have to assume the initial curve Γ^0 satisfies the structural condition

$$\int_{\Gamma^0} \frac{k}{\beta(k)} ds < \infty \quad \text{and} \quad 1 < m \leq 2$$

The condition is satisfied for any nonconvex smooth curve whose inflection points have at most $2 + \frac{1}{m-1}$ order contact with their tangents. (Bernoulli lemniscate $(x_1^2 + x_2^2)^2 = 4x_1x_2$ and $m < 2$)

Theorem There exists $T > 0$ and a family of regular plane curves $\Gamma^t = Image(x(.,t)), t \in [0,T]$ satisfying $\partial_t x = \beta N + \alpha T$

- α is the tangential velocity preserving the relative local length.
- $x, \partial_u x \in (C(\overline{Q_T}))^2, \;\; \partial_u^2 x, \; \partial_t x, \; \partial_u \partial_t x \in (L^\infty(Q_T))^2$;

Numerical scheme

- Let $[0,T]$ be an interval and let $\tau = T/n, n \in N$, denote the time discretization step. By $x^i, i = 0,1,...,n$, we denote the approximation of a true solution at time $t = i\tau$, i.e. $x^i(.) = x(., i\tau)$.
- full space-time discretization scheme. We use the uniform spatial grid $u_j = jh$ $(j = 0,...,m)$ with $h = 1/m$. The smooth solution x is then approximated by the discrete values x^i_j corresponding to $x(jh, i\tau)$. Using quite natural finite difference approximations of spatial differential terms we end up with semi-implicit difference scheme

$$\frac{1}{2}(g_j^{i-1} + g_{j+1}^{i-1})\frac{x_j^i - x_j^{i-1}}{\tau} = \frac{x_{j+1}^i - x_j^i}{h_{j+1}^{i-1}} - \frac{x_j^i - x_{j-1}^i}{h_j^{i-1}}$$

$i = 1,...,n, \quad j = 1,...,m$, where

$$g_j = |r_j|\theta_{1,j}, \quad h_j = |r_j|\theta_{2,j}, \quad r_j = x_j - x_{j-1}, \quad \theta_{1,j} = \frac{k_j}{\beta_j \theta_{2,j}}$$

the curvature is approximated by

$$k_j = \frac{1}{2|r_j|}\mathrm{sgn}(r_{j-1} \wedge r_{j+1}) \arccos\left(\frac{r_{j+1}.r_{j-1}}{|r_{j+1}||r_{j-1}|}\right)$$

and $\theta_{2,j} = \exp(\vartheta_j)$ where ϑ is a solution of

$$-\frac{\frac{\beta_j}{k_j} + \frac{\beta_{j+1}}{k_{j+1}}}{|r_j| + |r_{j+1}|}(\vartheta_{j+1} - \vartheta_j) + \frac{\frac{\beta_j}{k_j} + \frac{\beta_{j-1}}{k_{j-1}}}{|r_j| + |r_{j-1}|}(\vartheta_j - \vartheta_{j-1})$$

$$= |r_j|\left(k_j\beta_j - \left(\sum_{l=1}^{n}|r_l|k_l\beta_l\right)\left(\sum_{l=1}^{n}|r_l|\right)^{-1}\right)$$

- <u>Intrinsic property.</u>

> *The 'intrinsic property' of the governing equation causes that the spatial parametrization step is not involved in the approximation scheme and therefore only the spatial position of points of a curve Γ and the curvature of Γ play the role in the discretization scheme suggested*

Conclusions

- We proposed a governing equations for the flow of plane curves capable of describing with the prescribed normal velocity and the tangential velocity preserving the initial redistribution of flowing points
- We have shown a-priori estimates yielding local existence of a flow of plane curves in the case when $\beta(k) = k^m$, $0 < m \le 2$
- We have designed semi-implicit full time-space discretization scheme for numerical approximating of solutions
- We obtained a scheme which is intrinsic, only the spatial position of points of a curve Γ its the curvature of Γ occur in the scheme

This document and the paper are available at

http://www.iam.fmph.uniba.sk/institute/sevcovic