Mean curvature flow driven by external force and anisotropy

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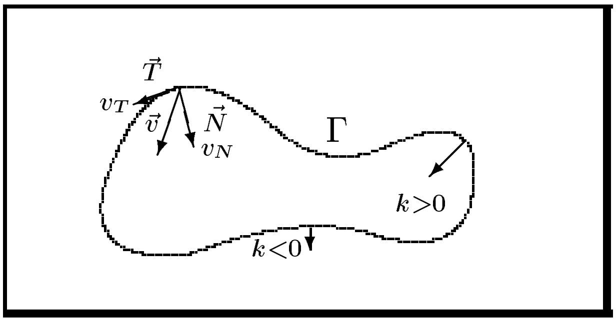
Goals

• The main purpose is to study evolution of curves in the plane satisfying the geometric equation

$$\boxed{v = \beta(x, k, \nu)}$$

where v is the normal velocity k and ν are the curvature and the tangential angle of a plane curve



• To represent flow of plane curves by a solution to the geometric equation

$$\boxed{\partial_t x = \beta\,\vec{N} + \alpha\,\vec{T}}$$

for the position vector $x \in R^2$ representing a curve $\Gamma = \text{Image } (x)$

• To suggest a suitable tangential velocity function α preserving a uniform redistribution of points along the evolution

• To suggest a computational method for solving geometric equation and present numerical simulation

Motivation

• **Motions of plane phase interfaces and contact conditions** studied by Angenent & Gurtin '89 where the normal velocity $v = \beta(k, \nu)$ is given by

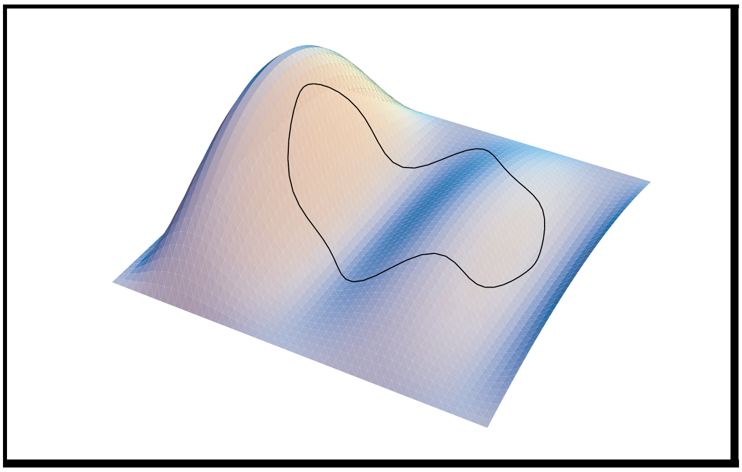
$$\boxed{\mu(\nu, v)v = h(\nu)k - g}$$

where μ is a kinematic (mobility) coefficient, h is a anisotropy of the media, g is an external force

• **Flow of surface curves** $\gamma_t \subset M, t \geq 0$, driven by the geodesic curvature K_g and external force

$$\boxed{V = \varepsilon K_g + C}$$

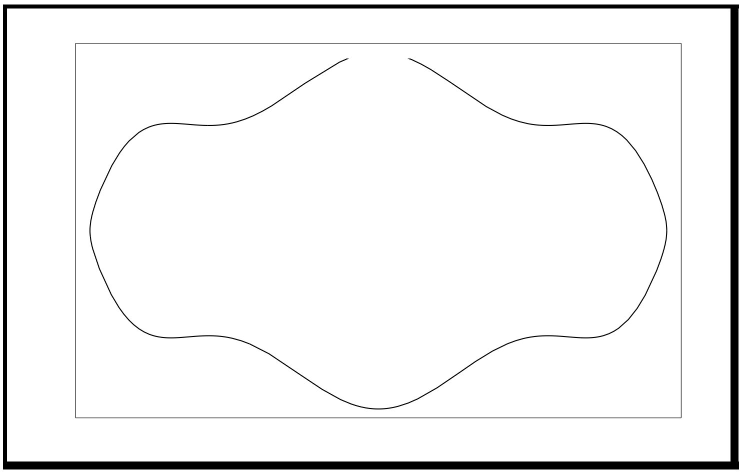
where M = Graph (ϕ), $\phi : \Omega \subset R^2 \to R$ is a surface in R^3, V is the normal component of the velocity, K_g is the geodesic curvature of γ_t relative to M and C is the normal component of an external force $\vec{G}$, $\varepsilon > 0$ is a constant.



The geodesic surface flow can be reduced to a flow of plane curves $\Gamma_t \subset R^2, t \geq 0$, with the the normal velocity $v = \beta(x, k, \nu)$

$$\boxed{\beta(x, k, \nu) = a(x)\, k - b(x)\, (\nabla_x \phi(x).\vec{N})}$$

where $a, b : \Omega \subset R^2 \to R$ are suitable functions, $\vec{N} = (\cos(\nu), \sin(\nu))$

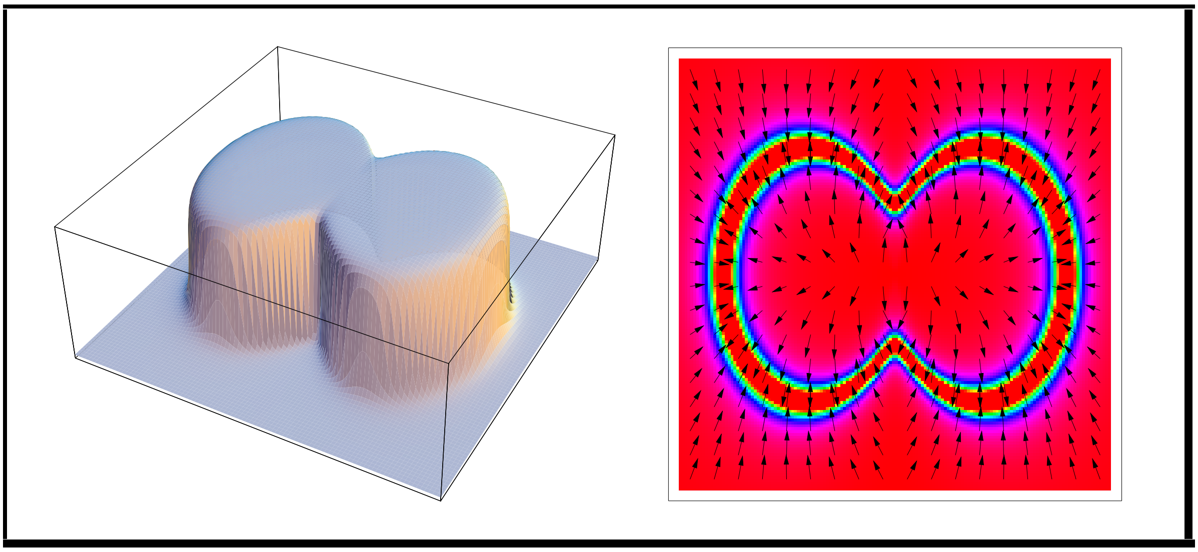


Vertical projection of a surface curve $\gamma \subset M$ onto a plane curve $\Gamma \subset R^2$

• In the **image segmentation**, detection of image silhouettes plays an important role. Suppose that an image is represented by a given intensity function $u_0 : R^2 \to [0, 1]$. The problem is to detect edges of the image, i.e. planar curves on which the gradient ∇u_0 is very large. The idea is to construct an evolving family of planar curves converging to an edge of the image according to the normal velocity

$$\boxed{\beta(x, k, \nu) = \varepsilon\phi(x)k - \nabla\phi(x).\vec{N}}$$

where $\phi(x) = h(|\nabla u_0(x)|)$, h is a suitable image contrast function, e.g. $h(s) = e^{-s}$ (Caselles, Kimmel, Sapiro, Sbert 1997; Kichenassamy, Kumar, Olver, Tannenbaum, Yezzi, 1996)

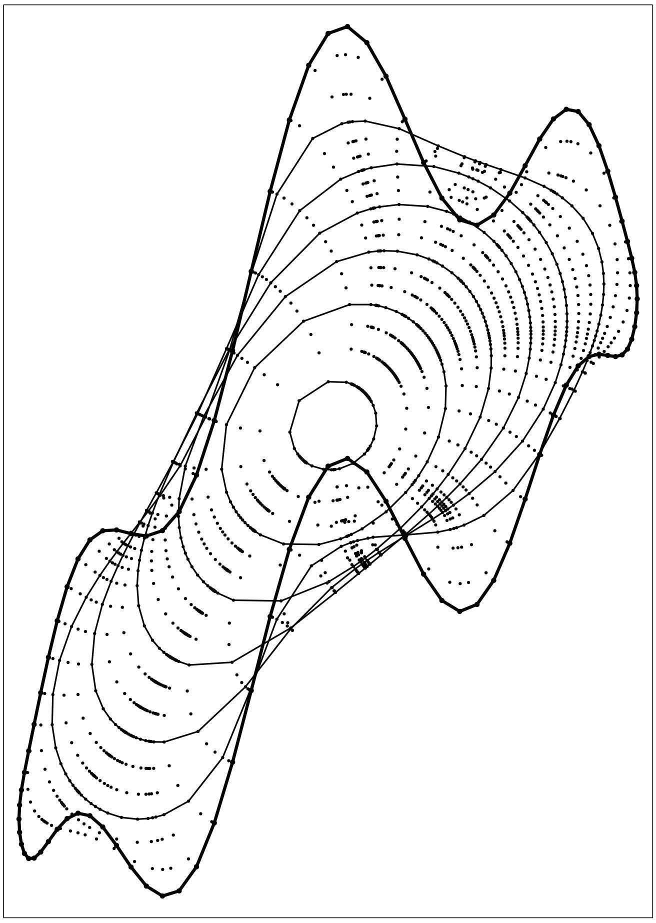


The image intensity function u_0 (left) and the density plot of the function ϕ and corresponding vector field $-\nabla\phi(x)$ (right).

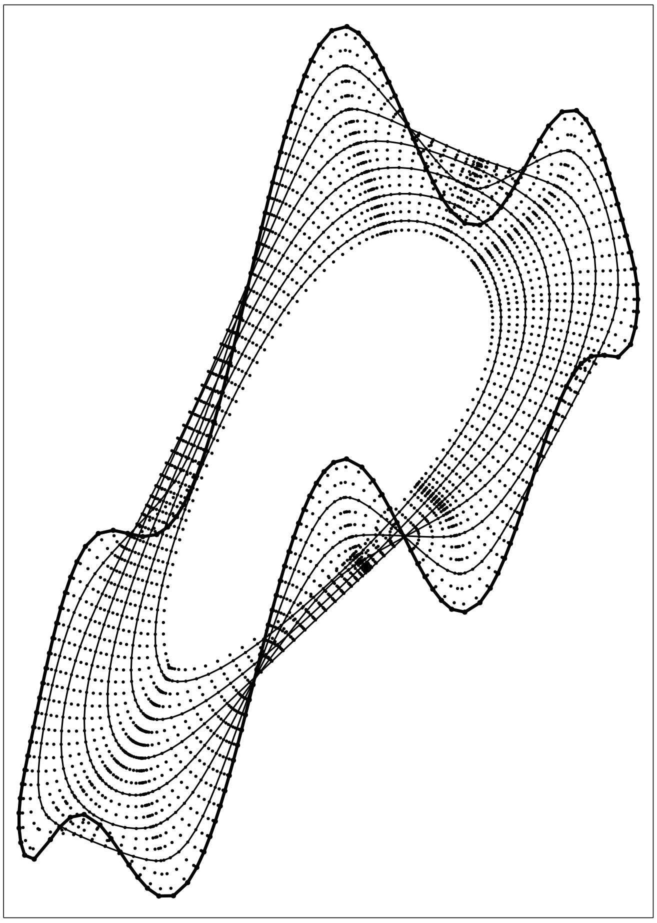
• **Numerical aspects** of approximation approximation of the mean curvature flow. In isotropic and anisotropic case the numerical suggested by Dziuk '94, '98 yields zero tangential velocity, i.e. $\alpha = 0$ and

$$\boxed{\partial_t x = \beta \, \vec{N}}$$

The numerical scheme having no tangential redistribution leads to a formation of various instabilities, like e.g. swallow tails

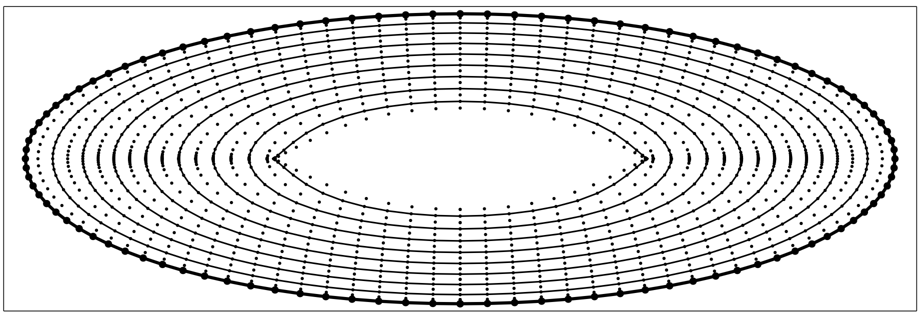


$\beta(k) = k$

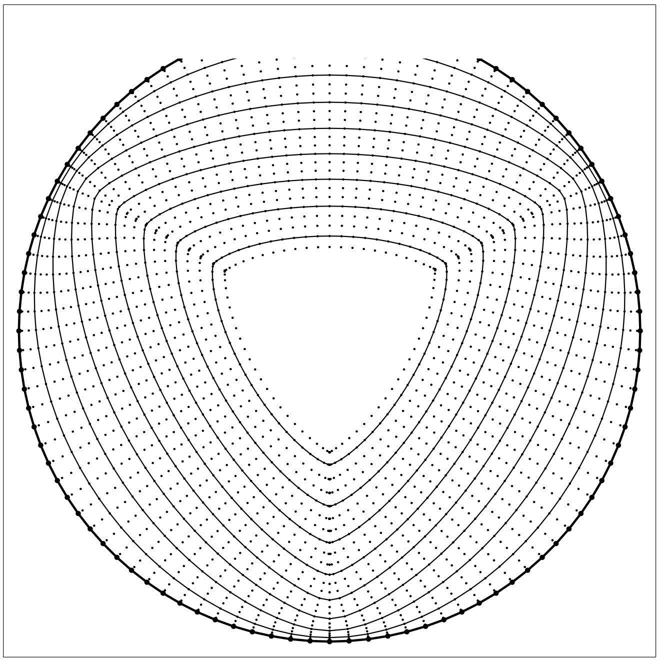


$\beta(k) = k^{1/3}$

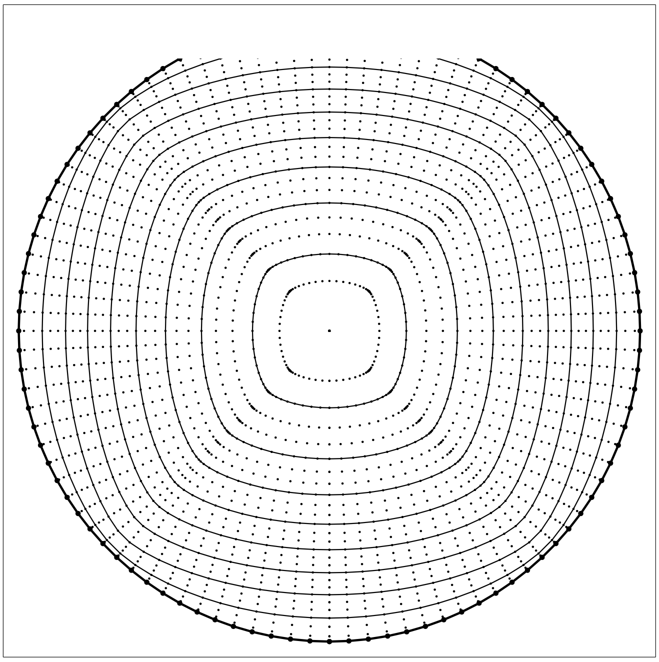
Level set equation approaches: Osher, Sethian 1988;
Allen-Cahn equation approach: Beneš, Mikula, Chalupecký (2000)



Affine invariant motion of ellipse, $\beta(k) = k^{1/3}$



$\beta(k,\nu) = (1 - \frac{7}{9}\cos(3\nu))k^{1/2}$



$\beta(k,\nu) = (1 - 0.8\cos(4\nu - \pi))k$

A method how overcome this difficulty has been suggested by K.Mikula and D.Ševčovič in 1999, 2001. It consists in considering a tangential velocity functional α satisfying some non-local equation and having the property

$$\boxed{\frac{|\partial_u x(u,t)|}{L_t} = \frac{|\partial_u x(u,0)|}{L_0}}$$

It means that the ratio of the local length and total length of a curve is preserved along the time.

K.Mikula and D.Ševčovič: Solution of nonlinearly curvature driven evolution of plane curves, Applied Numerical Mathematics Vol 31, No.2 (1999) pp. 191-207

K.Mikula and D.Ševčovič: Evolution of plane curves driven by a nonlinear function of curvature and anisotropy, SIAM J. Appl. Math., 61, (2001) 1473–1501.

Governing equations

- Fully nonlinear system of parabolic PDEs

$$\boxed{\begin{aligned} &\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta\,, \\ &\partial_t \nu = \partial_s \beta + \alpha k\,, \\ &\partial_t g = -g\, k\beta \,+\, \partial_u \alpha\,, \\ &\partial_t x = \beta \vec{N} + \alpha \vec{T} \end{aligned}}$$

where $\beta = \beta(x, k, \nu)$ and α are the normal and tangential velocities, $\vec{N} = (\cos(\nu), \sin(\nu)), \quad \vec{T} \perp \vec{N}$,
$k = k(u, t)$ is the curvature
$\nu = \nu(u, t)$ is the tangential angle
$g = g(u, t) = |\partial_u x(u, t)|$ is the local length element,
$x = x(u, t)$ is the position vector of a curve Γ_t
$ds = g\, du$ is the arc-length parameterization, $(u, t) \in Q_T = S^1 \times [0, T)$.

A solution is subject to initial and periodic boundary conditions corresponding to an initial curve.

$$\boxed{E_k = c^{2k+\delta}(S^1) \times c^{2k+\delta}(S^1) \times c^{1+\delta}(S^1) \times (c^{2+\delta}(S^1))^2}$$

Theorem Assume that $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector and $g_0 = |\partial_u x_0| > 0$ is the local length element of the initial regular curve $\Gamma_0 =$ Image (x_0). If $\beta = \beta(x, k, \nu)$ is a C^4 smooth function such that

$$\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$$

then there exists a unique classical solution

$$\Phi = (k, \nu, g, x) \in C([0, T], E_1) \cap C^1([0, T], E_0)$$

of the governing system of equations defined on some small time interval $[0, T]$, $T > 0$. Moreover, if Φ is a maximal solution defined on $[0, T_{max})$ then either $T_{max} = +\infty$ or $\liminf_{t \to T_{max}^-} \min_{\Gamma_t} \beta'_k(x, k, \nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma_t} |k| \to \infty$ as $t \to T_{max}$.

Consequence of the abstract theory of fully nonlinear parabolic equations due to Angenent (1990) and Lunardi.

Controlling tangential velocity

- In 2001, the authors showed that if α is a unique (up to an additive constant) solution to the nonlocal geometric equation

$$\boxed{\frac{\partial \alpha}{\partial s} = k\beta - \frac{1}{L}\int_\Gamma k\beta\, ds}$$

then, combining the total and local length equations, we obtain that α is the tangential velocity preserving relative local length, i.e.

$$\boxed{\frac{|\partial_u x(u,t)|}{L_t} = \frac{|\partial_u x(u,0)|}{L_0}}$$

for any $t \in [0, T)$ and $u \in S^1$. It means that the redistribution is preserved along the evolution of curves

- More generally, if α is a unique solution to

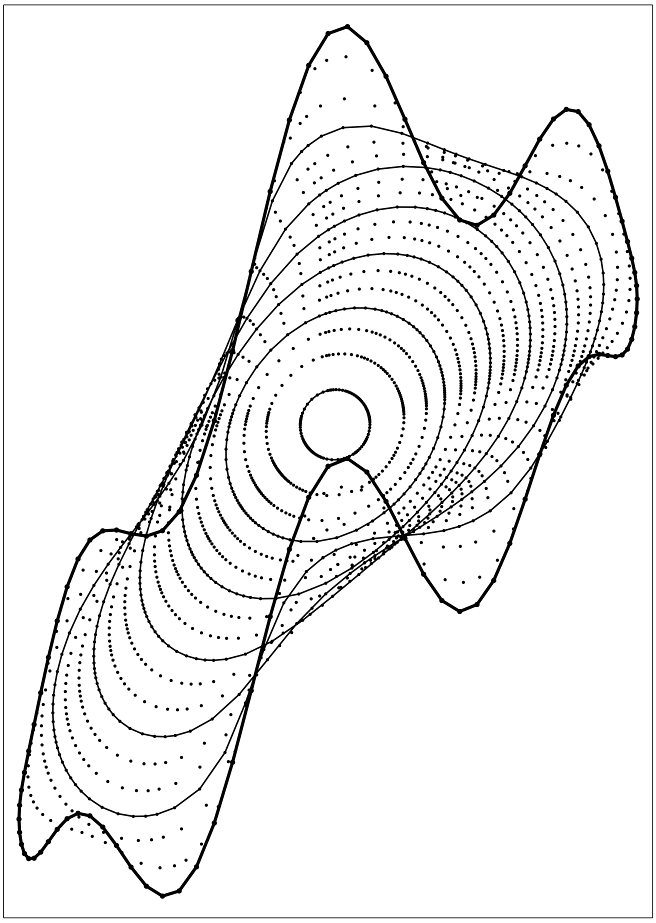
$$\boxed{\frac{\partial \alpha}{\partial s} = k\beta - \frac{1+\kappa}{L}\int_\Gamma k\beta\, ds + \frac{\kappa}{g}\int_\Gamma k\beta\, ds}$$

then

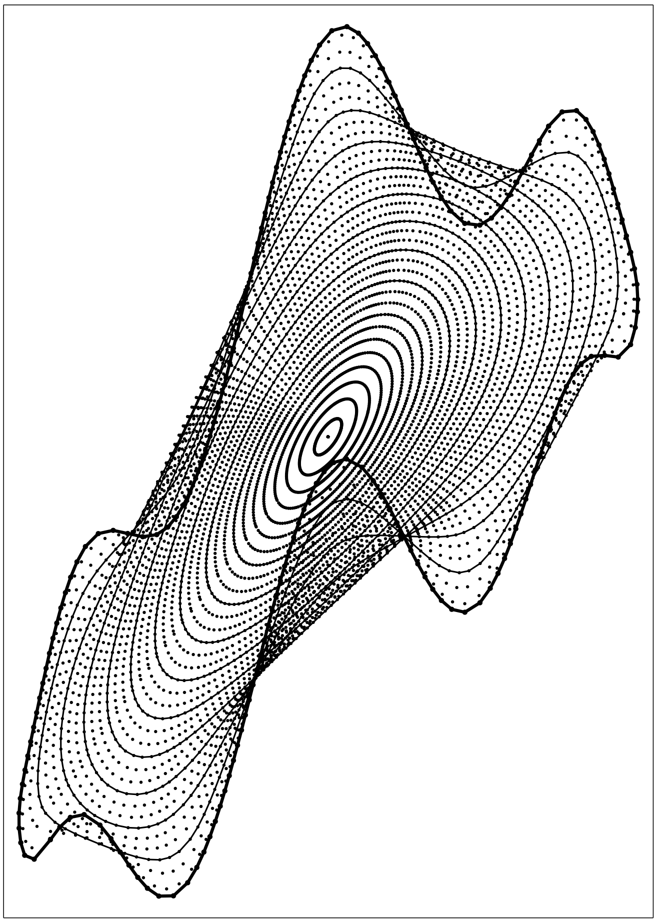
$$\boxed{\lim_{t\to T}\frac{|\partial_u x(u,t)|}{L_t} = 1}$$

for any $u \in S^1$. Here $\beta = \beta(x, k, \nu))$. It means that the redistribution becomes uniform as time approaches maximal time of existence.

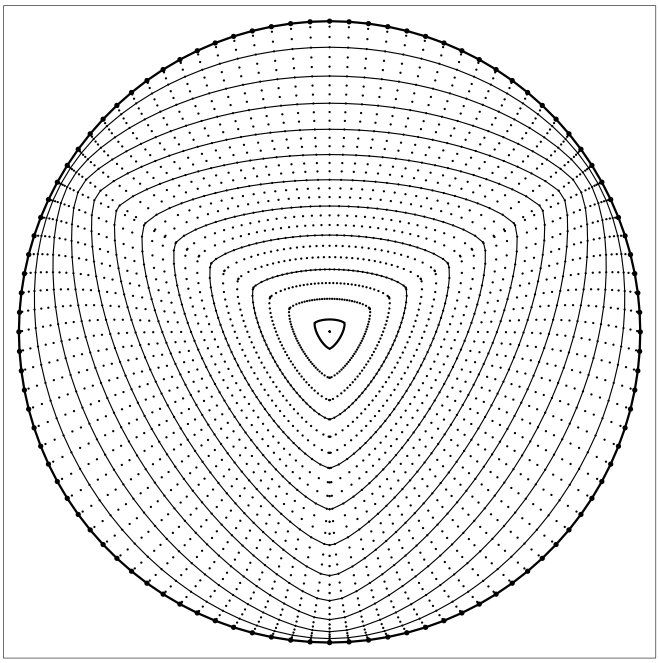
- Evolution o plane curves with uniform tangential redistribution plane curves with uniform tangential redistribution



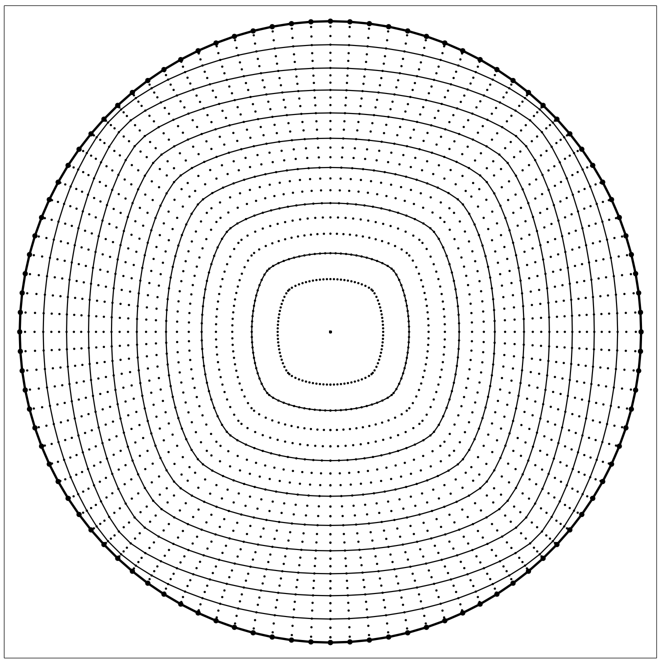
$\beta(k) = k$



$\beta(k) = k^{1/3}$



$\beta(k, \nu) = (1 - \frac{7}{9}\cos(3\nu))k^{1/2}$

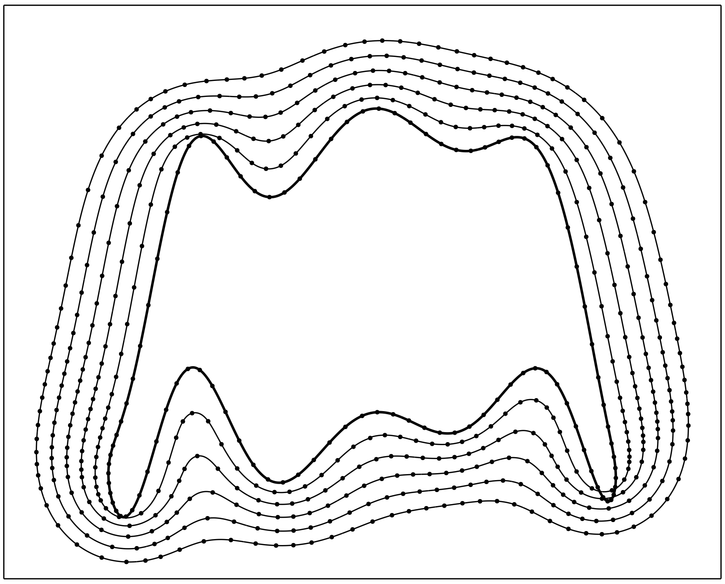


$\beta(k, \nu) = (1 - 0.8\cos(4\nu - \pi))k$

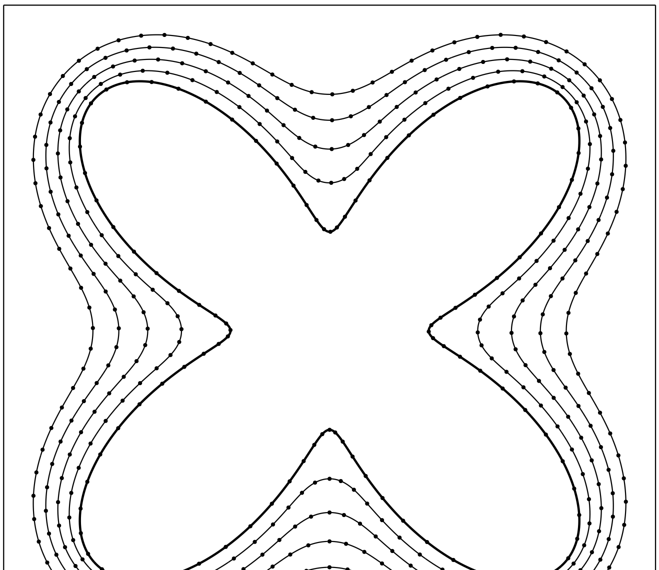
Mean curvature flow with a nontrivial driving force

- The following examples show computational results of mean curvature flow with a nontrivial driving force

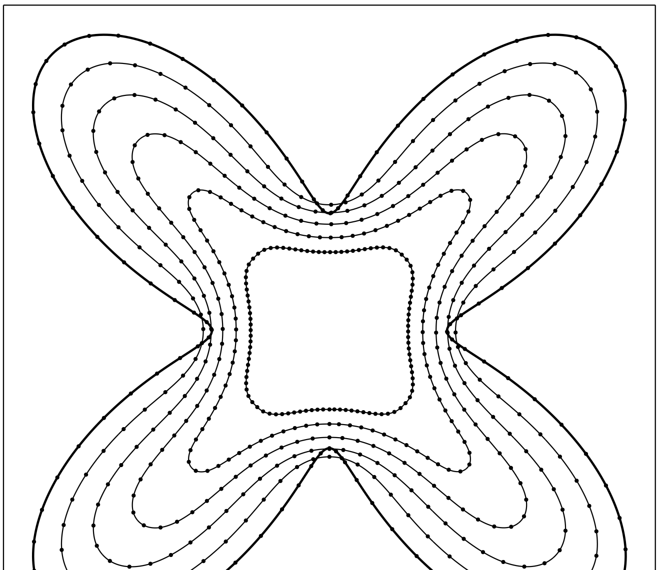
$$\boxed{\beta(k) = \varepsilon k + c}$$



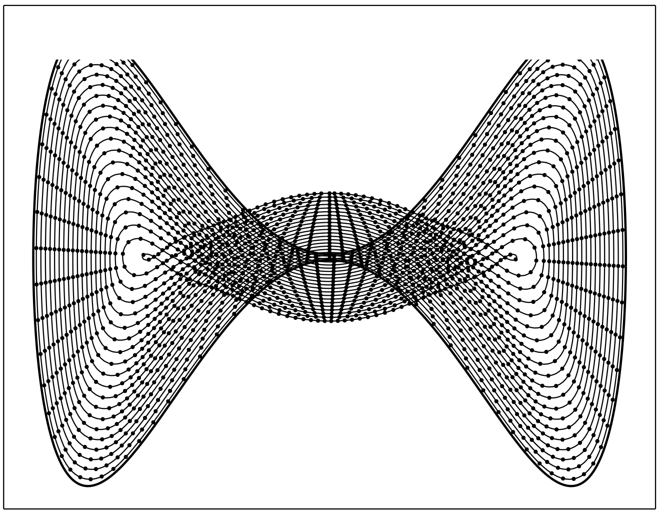
$\varepsilon = 1, \; c = -10$



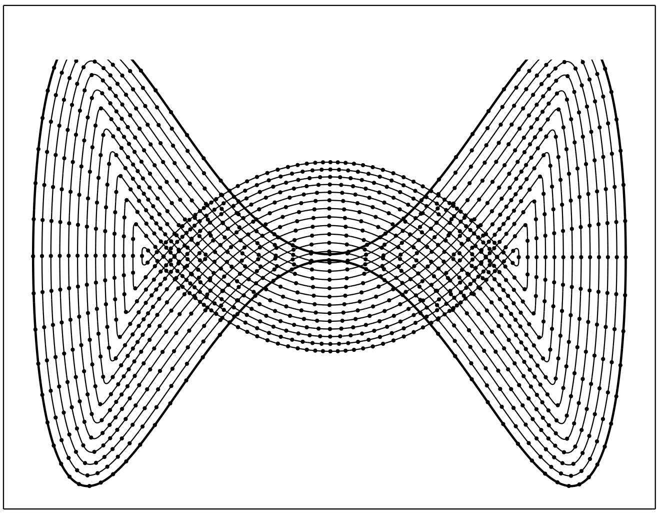
$\varepsilon = 1, \; c = -10$



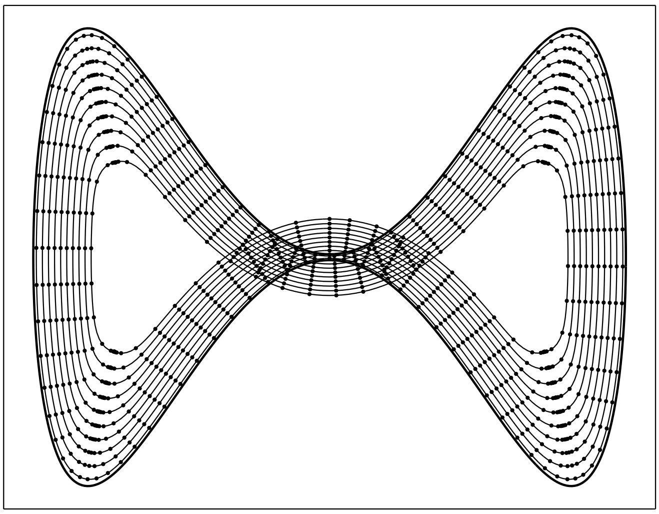
$\varepsilon = 1, \; c = 10$



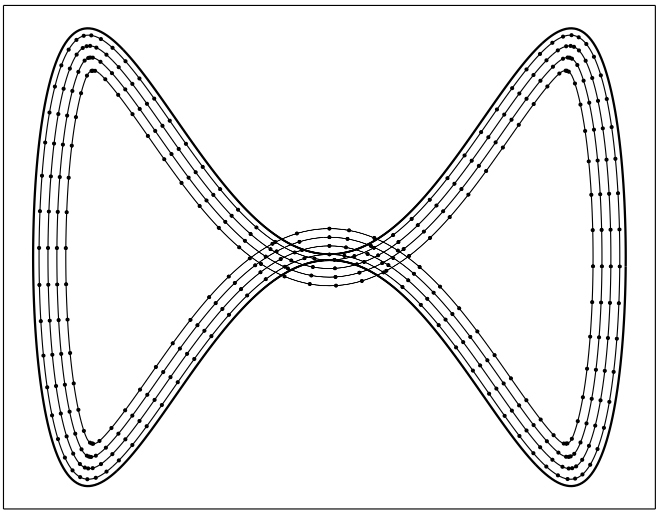
$\varepsilon = 1, \;\; c = 10$



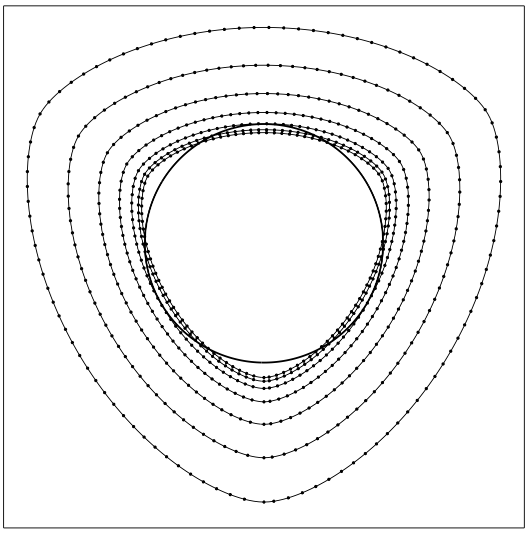
$\varepsilon = 1, \;\; c = 100$



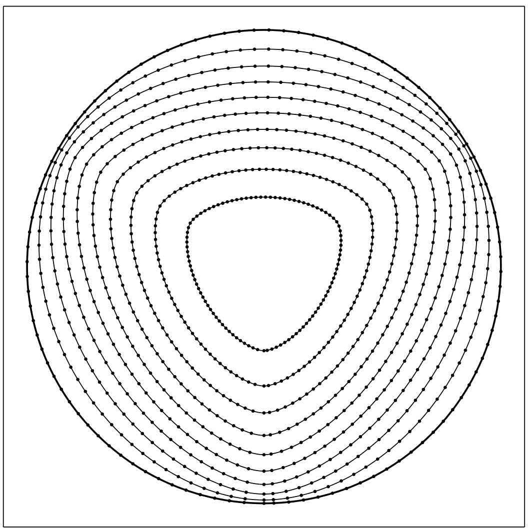
$\varepsilon = 1, \;\; c = 10$
(no tangential redistribution)



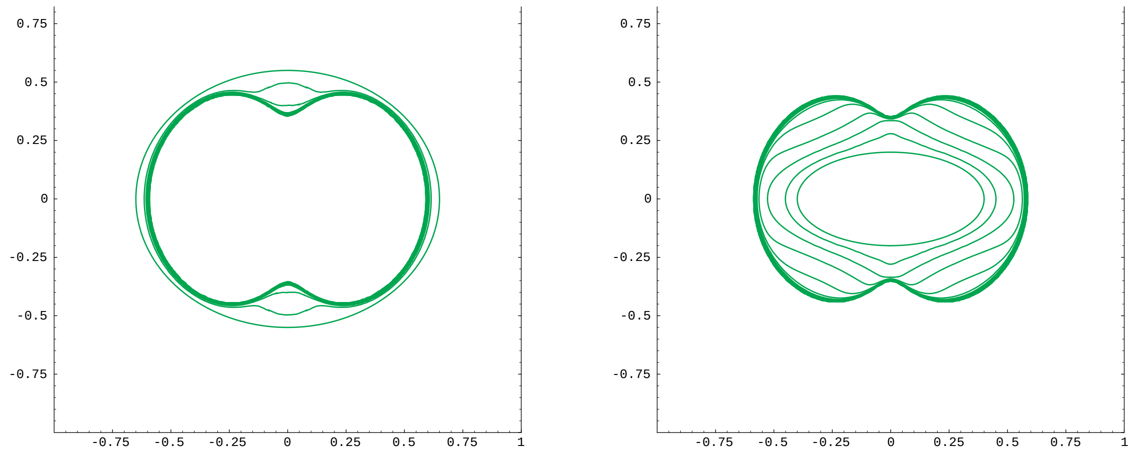
$\varepsilon = 1, \;\; c = 100$
(no tangential redistribution)



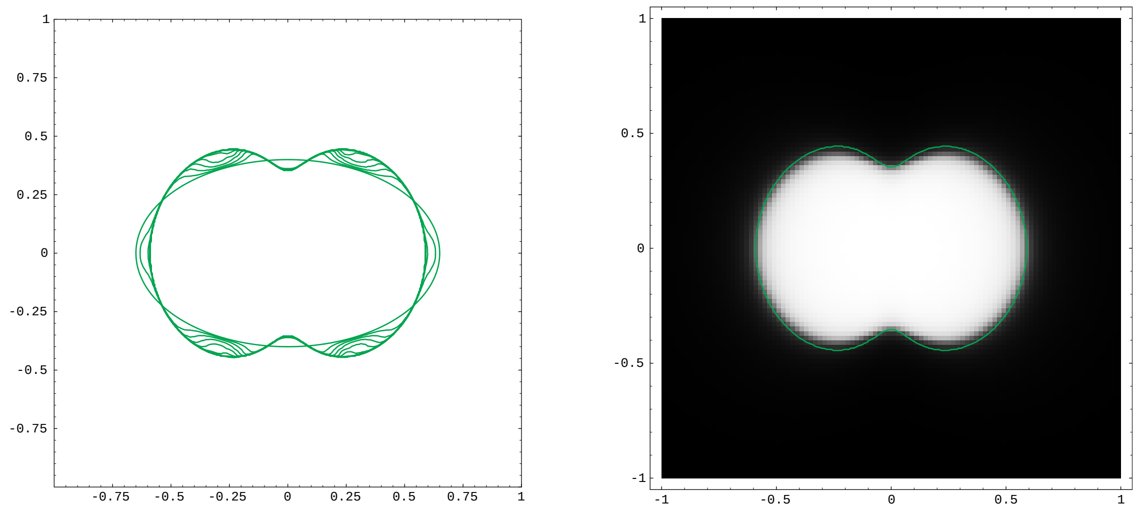
$\beta = \gamma(\nu)k - 1$



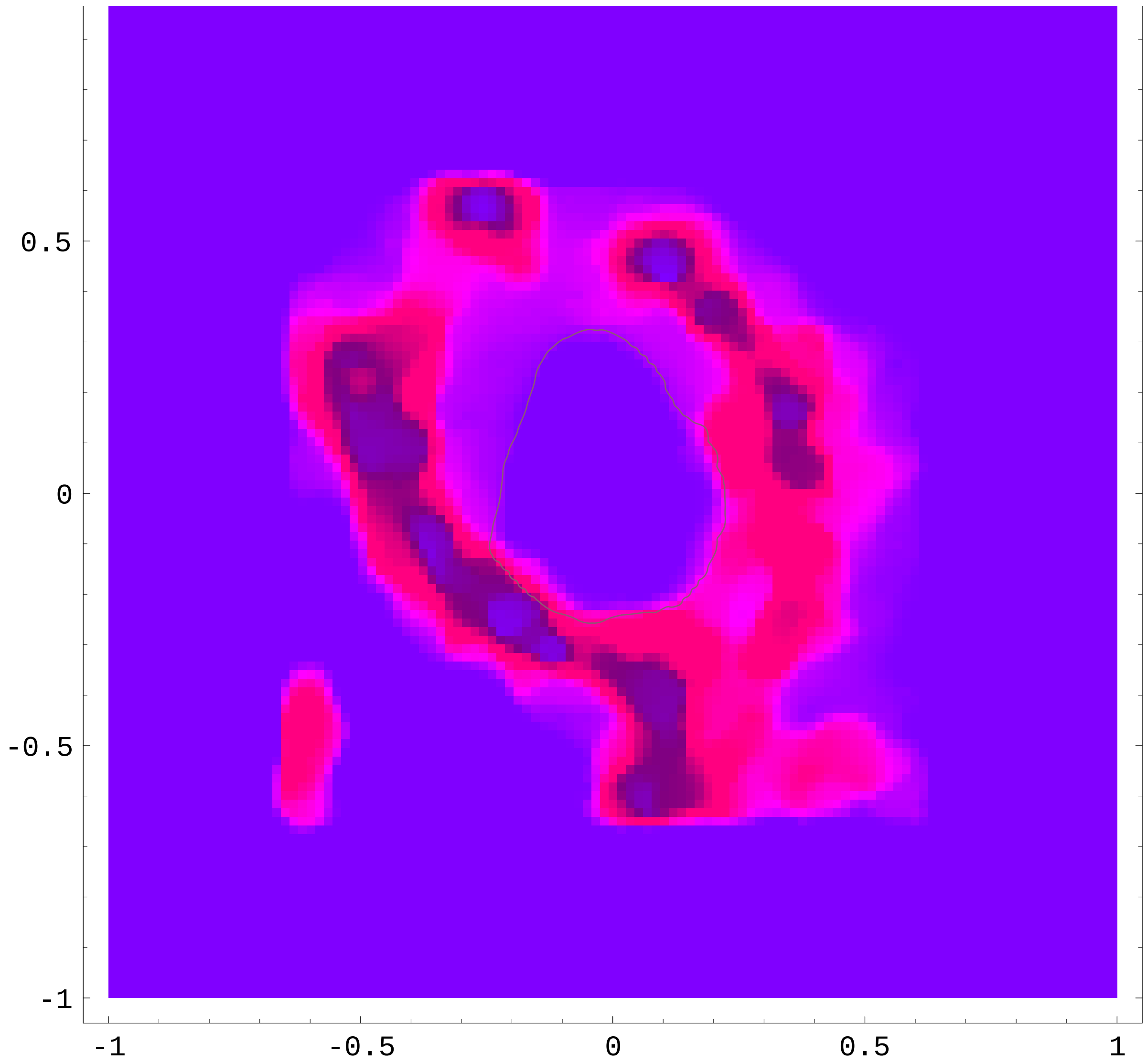
$\beta = \gamma(\nu)k$



Convergence of an initial curve to the edge, from outside (left) and from inside (right).



Convergence to the edge of an initial curve crossing the edge (left); the density plot of the image intensity function together with the limiting curve representing the edge position (right).



2D slice of a prefiltered 3D echocardiography together with the limiting curve representing the edge. The initial curve was an ellipse inserted into the slice.

First integrals and conserved quantities

- General function $\beta(x,k,\nu)$

$$\boxed{\frac{d}{dt}L(\Gamma^t)+\int_{\Gamma^t} k\beta\, ds = 0, \quad \frac{d}{dt}A(\Gamma^t)+\int_{\Gamma^t} \beta\, ds = 0}$$

where $L(\Gamma)$ and $A(\Gamma)$ denote the length and enclosed area of a curve Γ

- If a flow of planar curves evolves according to the normal velocity

$$\boxed{\beta(x,k,\nu) = a(\phi)k - b(\phi)\nabla_x\phi(x).\vec{N}}$$

where $a = a(\phi), b = b(\phi), \phi = \phi(x)$ are smooth functions, $a(\phi) > 0$, $\phi = \phi(x) \in [0,1]$ then

$$\boxed{\frac{d}{dt}\int_{\Gamma_t} H(\phi)\, ds + \int_{\Gamma_t} \frac{H(\phi)}{a(\phi)}\beta^2\, ds = 0}$$

$$\boxed{H(\phi) = e^{\int^{\phi} \frac{b(\xi)}{a(\xi)}\, d\xi}}$$

In the context of the image segmentation theory one can consider either $a(\phi) = \varepsilon\phi$ and $b(\phi) = \frac{1}{1-\phi}$ and then

$$\boxed{H(\phi) = \phi^{\varepsilon}(1-\phi)^{-\varepsilon}}$$

or $a(\phi) = \varepsilon\phi$ and $b(\phi) = 1$, and, then

$$\boxed{H(\phi) = \phi^{\varepsilon}}$$

Conclusions

• We have studied a mean curvature flow of planar curves with the normal velocity depending on the curvature, tangential angle and a position of a curve. Local in time existence of smooth solutions has benn show. Various first integrals decreasing along trajectories have been derived and analysed.

• Governing system of equations includes a nontrivial tangential velocity functional. It has no impact on the shape of evolving curves but it can prevent numerically computed solutions from forming instabilities like swallow tails. Redistribution of grid points is asymtotically constant.

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