Flow of curves on planes and surfaces driven by curvature, external force and anisotropy

Karol Mikula & Daniel Ševčovič

Dept. of Math., Slovak Technical University, Bratislava

Inst. of Applied Math., Comenius University, Bratislava

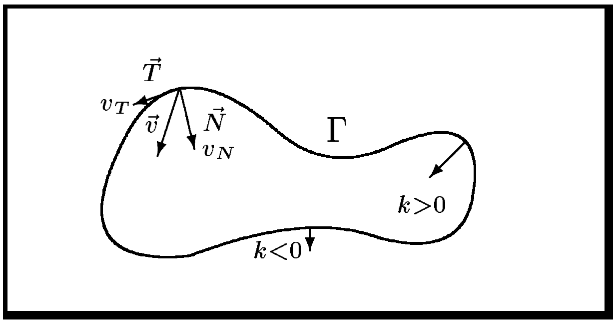
http://www.iam.fmph.uniba.sk/institute/sevcovic

Goals

• The main purpose is to study evolution of curves in the plane satisfying the geometric equation

$$v = \beta(x, k, \nu)$$

where v is the normal velocity k and ν are the curvature and the tangential angle of a plane curve



• To show how the flow of curves on a given surface driven by the geodesic curvature and external force can be reduced to the above geometric equation in the plane

• To represent flow of plane curves by a solution to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

for the position vector $x \in R^2$ representing a curve $\Gamma = \text{Image}\,(x)$. Reduce the problem to solution of a system of parabolic PDEs for the curvature, angle and local length of a curve

• To suggest a suitable tangential velocity function α preserving a uniform redistribution of points along the evolution

• To suggest a computational method for solving geometric equation and present numerical simulation

Outline

- **Motivation**
  - motion of phase interfaces arising from thermo-mechanics
  - geometric flow of curves on surfaces driven by geodesic curvature and external force

  $\updownarrow$ linked to

  - edge detection in the theory of image segmentation

- **Review of numerical aspects**
  - numerical artefacts of naive approximation scheme
  - the role of tangential redistribution in computing
  - local and non-local tangential velocities

- **Governing equations**
  - curve evolution as a solution to a system of nonlinear parabolic equations
  - local and global existence of a smooth solution (curve flow)
  - first intergrals and energy functionals

- **Computational results**
  - planar flow with a nontrivial driving force
  - formation of corner singularities
  - flow on a surface driven by the geodesic curvature
  - the role of an external force and the link to the edge detection problem
  - dynamic visualisation

Motivation

• **Motions of plane phase interfaces and contact conditions** studied by Angenent & Gurtin '89 where the normal velocity $v = \beta(k, \nu)$ is given by

$$\boxed{\mu(\nu, v)v = h(\nu)k - g}$$

where μ is a kinematic (mobility) coefficient, h is a anisotropy of the media, g is an external force

• Morphological image and shape multiscale analysis of Alvarez, Guichard, Lions & Morel '93 and affine invariant scale space of curves introduced by Sapiro, Tannenbaum & Angenent '94, '98

$$\boxed{\beta(k) = k^{1/3}}$$

• Bifurcation analysis of selfsimilar solutions of the geometric equation by Matano and Ushijima & Yazaki '98.

$$\boxed{\beta(k) = k^m}$$

where $m = 1/(n^2 - 1), n = 2, 3, ...$ are bifurcation values.

• Intrinsic heat equation as a model for isotropic and anisotropic motion by mean curvature studied by Dziuk '94, '98

$$\boxed{\beta(k, \nu) = \gamma(\nu)k}$$

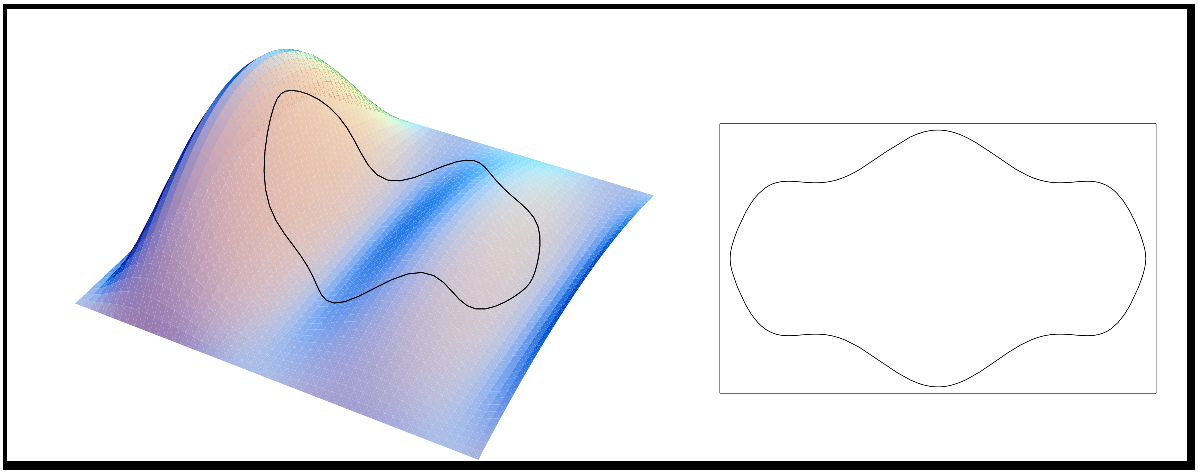
- **Flow of curves on a surface driven by the geodesic curvature**

$\gamma_t \subset M, t \geq 0$, driven by the geodesic curvature K_g and external force

$$V = K_g + C$$

where M = Graph (ϕ), $\phi : \Omega \subset R^2 \to R$ is a surface in R^3, V is the normal component of the velocity, K_g is the geodesic curvature of γ_t relative to M and C is the normal component of a gravitational like external force

$$\vec{G} = (0, 0, g(x))$$



A surface curve $\gamma \subset M$ (left) and its vertical projection into a plane curve

The geodesic surface flow can be reduced to a flow of plane curves $\Gamma_t \subset R^2, t \geq 0$, with the the normal velocity $v = \beta(x, k, \nu)$

$$\beta(x, k, \nu) = a(x, \nu)\, k - b(x, \nu)\, \nabla_x \phi(x).\vec{N}$$

where $\vec{T} = (\cos(\nu), \sin(\nu))$, $\vec{N} = \vec{T}^{\perp}$, and $a, b : \Omega \subset R^2 \to R$

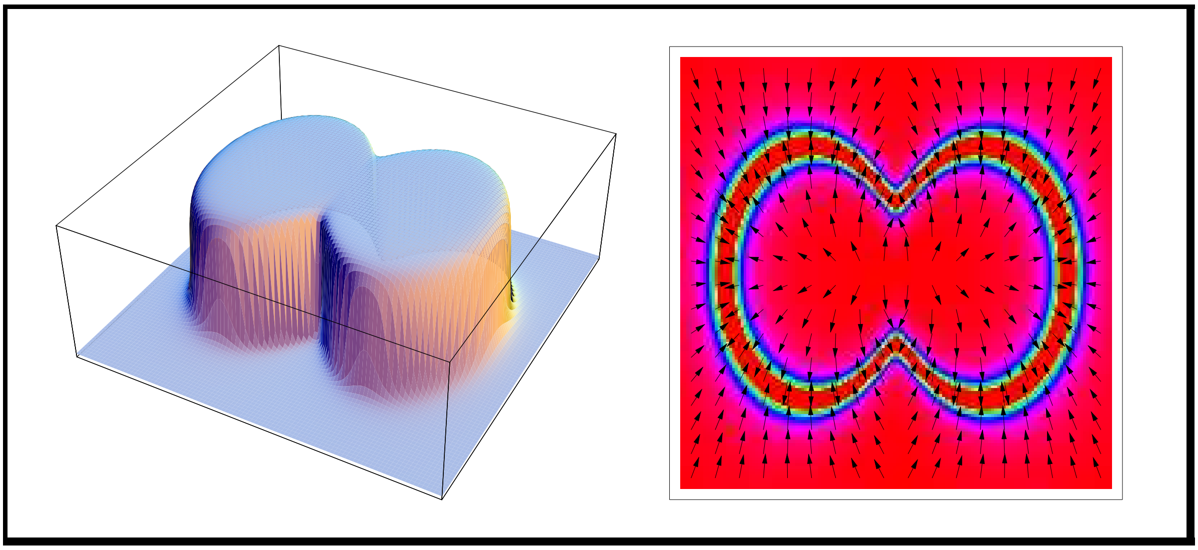
$$a(x, \nu) = \frac{1}{1 + (\nabla \phi.\vec{T})^2}$$

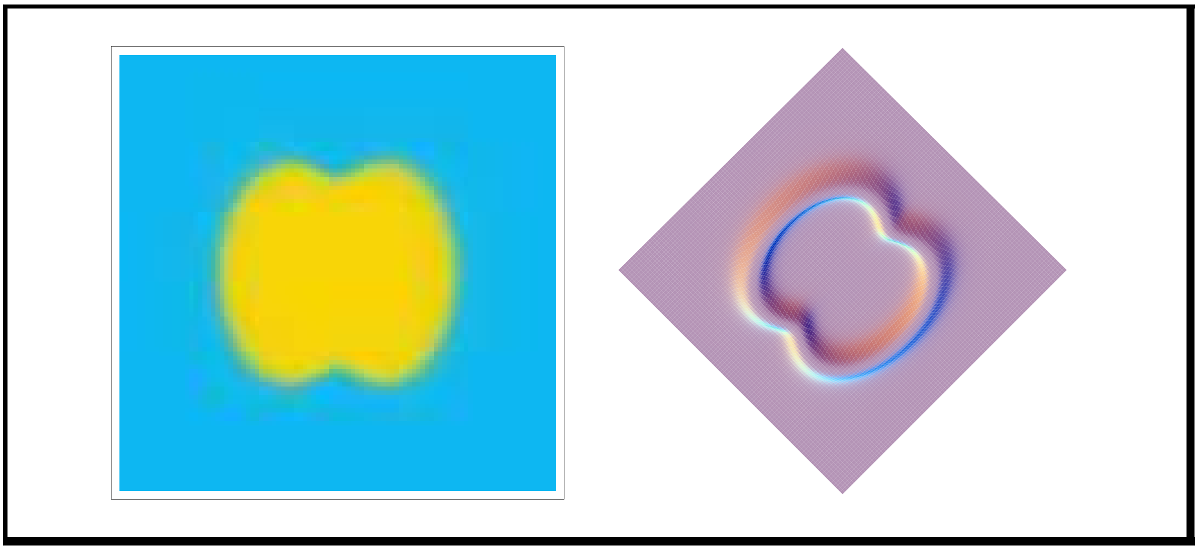
$$b(x, \nu) = \frac{1}{1 + |\nabla \phi|^2} \left(g(x) - \frac{\vec{T}^T \nabla^2 \phi\, \vec{T}}{1 + (\nabla \phi.\vec{T})^2} \right)$$

• **Image segmentation - link to the flow on a surface**
detection of image silhouettes plays an important role. Suppose that an image is represented by a given intensity function $u_0 : R^2 \to [0,1]$. The problem is to detect edges of the image, i.e. planar curves on which the gradient ∇u_0 is very large. The idea is to construct an evolving family of planar curves converging to an edge of the image according to the normal velocity

$$\boxed{\beta(x,k,\nu) = \varepsilon\phi(x)k - \nabla\phi(x).\vec{N}}$$

where $\phi(x) = h(|\nabla u_0(x)|)$, h is a suitable image contrast function, e.g. $h(s) = e^{-s}$ (Caselles et al 1997; Kichenassamy et al 1996)





The image intensity function u_0 (top left) and its density plot (bottom left). 3D plot of Casseles' functional ϕ (bottom-right) and the corresponding vector field $-\nabla\phi(x)$ (top-right)

Review of numerical aspects

- **Simple example**

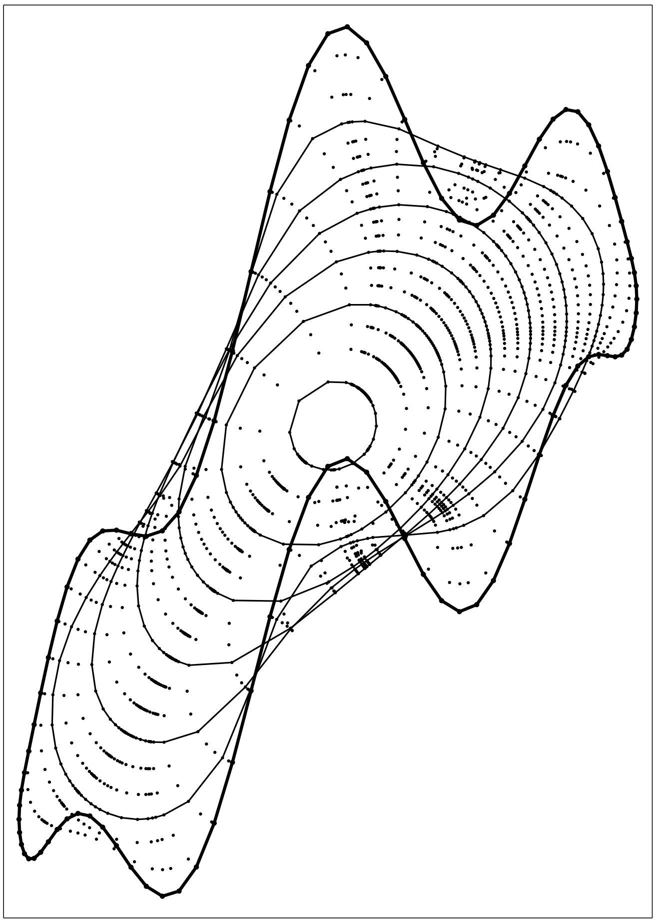
In isotropic and anisotropic case the numerical scheme for solving the geometric equation $v = k$ follows from the intrinsic heat equation suggested by Abresch & Langer '86 and numerically approximated by Dziuk '94,

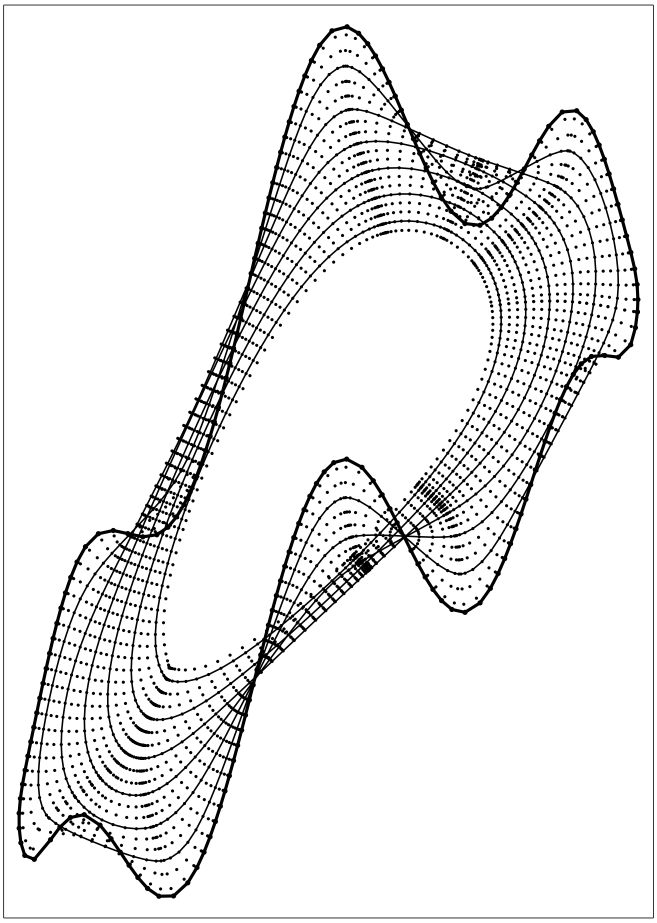
$$\frac{\partial x}{\partial t} = \frac{1}{|x_u|}\frac{\partial}{\partial u}\left(\frac{1}{|x_u|}\frac{\partial x}{\partial u}\right)$$

This approach yields zero tangential velocity, i.e. $\alpha = 0$, i.e.

$$\partial_t x = k\,\vec{N}$$

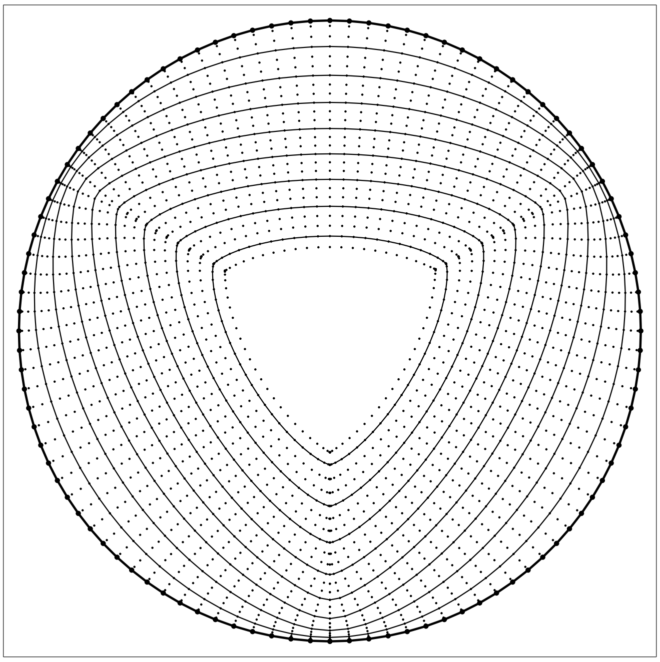
The numerical scheme having no tangential redistribution however leads to formation of various instabilities, like e.g. swallow tails



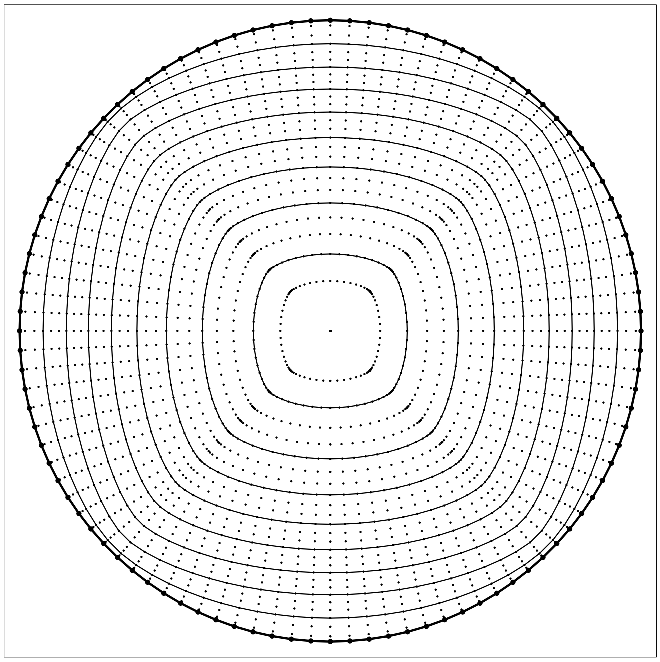


$v = k$ $\qquad$ $v = k^{1/3}$

- Level set equation approaches: Osher, Sethian 1988;
- Allen-Cahn equation approach: Beneš, Mikula, Chalupecký (2000)



$\beta(k,\nu) = (1 - \frac{7}{9}\cos(3\nu))k^{1/2}$



$\beta(k,\nu) = (1 - 0.8\cos(4\nu - \pi))k$

A method how to overcome this difficulty has been suggested by Hou et al in '94 and by K.Mikula and D.Ševčovič in 1999, 2001. It consists in introducing a tangential velocity functional α satisfying some non-local equation. The presence of a non-locally defined tangential redistribution functional prevents the constructed curve from forming numerical singularities

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T.Y. Hou, J. Lowengrub, M. Shelley: Removing the stiffness from interfacial flows and surface tension, J. Comput. Phys., 114 (1994), pp. 312–338.

K.Mikula and D.Ševčovič: Solution of nonlinearly curvature driven evolution of plane curves, Applied Numerical Mathematics Vol 31, No.2 (1999) pp. 191-207

K.Mikula and D.Ševčovič: Evolution of plane curves driven by a nonlinear function of curvature and anisotropy, SIAM J. Appl. Math., 61, (2001) 1473–1501.

Controlling tangential velocity - local dependence

Methods based on solution to the intrinsic heat equation

- Example (Deckelnik '97) $\beta(k) = k$

$$\boxed{\frac{\partial x}{\partial t} = \frac{x_{uu}}{|x_u|^2}}$$

$$\Downarrow$$

$$\boxed{\alpha = \partial_s \ln |x_u|}$$

- Example (Mikula & Ševčovič '99) $\beta(k) = k^m$ and $\theta = |k|^{\frac{m-1}{2}}$

$$\boxed{\frac{\partial x}{\partial t} = \frac{1}{\theta}\frac{\partial}{\partial s}\left(\frac{1}{\theta}\frac{\partial x}{\partial s}\right)}$$

$$\Downarrow$$

$$\boxed{\alpha = \frac{1}{2}\partial_s |k|^{m-1}}$$

works fine for the case of fast diffusion in which $0 < m \leq 1$

Controlling tangential velocity - non-local dependence

- Hou et al '94, '97 showed that if α is a unique (up to an additive constant) solution to the nonlocal geometric equation

$$\boxed{\frac{\partial \alpha}{\partial s} = k\beta - \frac{1}{L}\int_\Gamma k\beta \, ds}$$

then, combining the total and local length equations, we obtain that α is the tangential velocity preserving relative local length, i.e.

$$\boxed{\frac{|\partial_u x(u,t)|}{L_t} = \frac{|\partial_u x(u,0)|}{L_0}}$$

for any $t \in [0,T)$ and $u \in S^1$. It means that the redistribution is preserved along the evolution of curves.

- Mikula and Ševčovič '01 showed that if α is a unique solution to

$$\boxed{\frac{\partial \alpha}{\partial s} = k\beta - \frac{1+\kappa}{L}\int_\Gamma k\beta \, ds + \frac{\kappa}{g}\int_\Gamma k\beta \, ds}$$

then

$$\boxed{\lim_{t\to T} \frac{|\partial_u x(u,t)|}{L_t} = 1}$$

for any $u \in S^1$. Here $\beta = \beta(x,k,\nu)$. It means that the redistribution becomes uniform as time approaches maximal time of existence.

Governing equations

- Fully nonlinear system of parabolic PDEs

$$
\begin{aligned}
\partial_t k &= \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta\,, \\
\partial_t \nu &= \partial_s \beta + \alpha k\,, \\
\partial_t g &= -g\, k\beta \,+\, \partial_u \alpha\,, \\
\partial_t x &= \beta \vec{N} + \alpha \vec{T}
\end{aligned}
$$

where $\beta = \beta(x, k, \nu)$ and α are the normal and tangential velocities, $\vec{T} = (\cos(\nu), \sin(\nu)), \quad \vec{T} \perp \vec{N}$,
$k = k(u,t)$ is the curvature, $\quad \nu = \nu(u,t)$ is the tangential angle
$g = g(u,t) = |\partial_u x(u,t)|$ is the local length element,
$x = x(u,t)$ is the position vector of a curve Γ_t
$ds = g\, du$ is the arc-length parameterization, $(u,t) \in Q_T = S^1 \times [0,T)$.

A solution is subject to initial and periodic boundary conditions corresponding to an initial curve and is searched in the functional space

$$E_k = c^{2k+\delta}(S^1) \times c^{2k+\delta}(S^1) \times c^{1+\delta}(S^1) \times (c^{2+\delta}(S^1))^2$$

Theorem Assume that $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector and $g_0 = |\partial_u x_0| > 0$ is the local length element of the initial regular curve $\Gamma_0 =$ Image (x_0). If $\beta = \beta(x,k,\nu)$ is a C^4 smooth function such that

$$\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$$

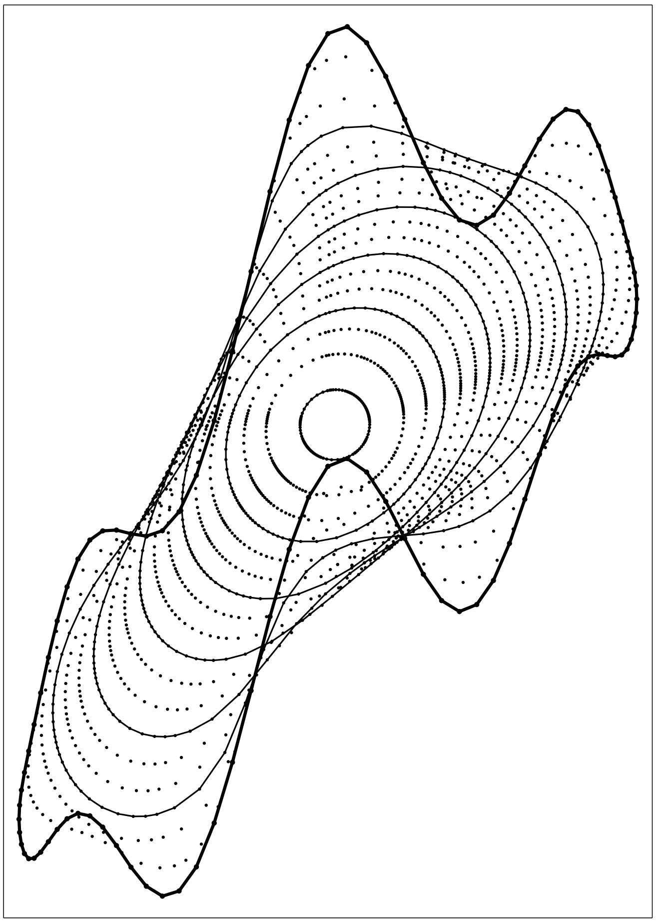
then there exists a unique classical solution

$$\Phi = (k, \nu, g, x) \in C([0,T], E_1) \cap C^1([0,T], E_0)$$

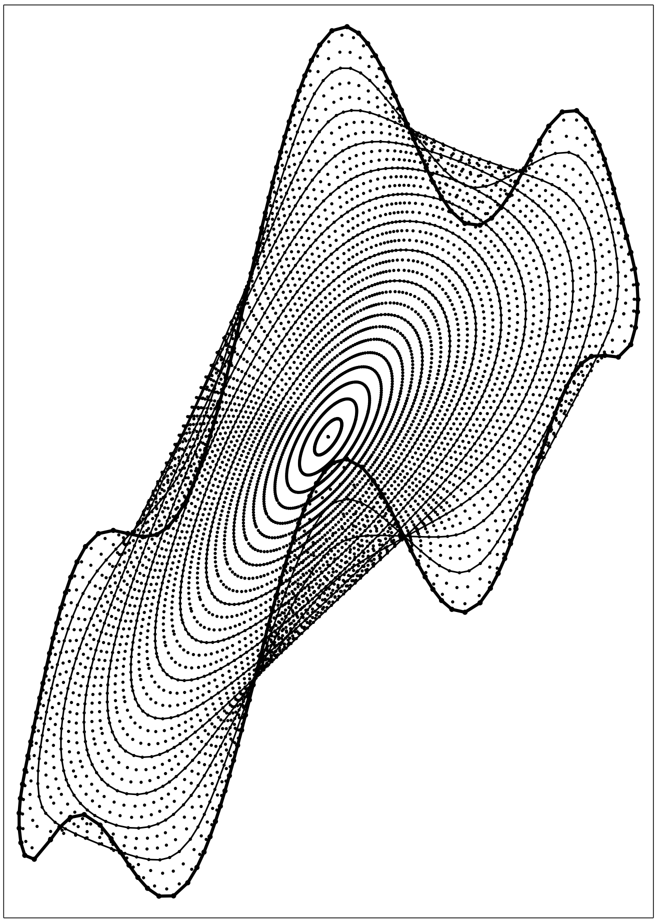
of the governing system of equations defined on some small time interval $[0,T]$, $T > 0$. If Φ is a maximal solution defined on $[0, T_{max})$ then either $T_{max} = +\infty$ or $\liminf_{t \to T_{max}^-} \min_{\Gamma_t} \beta'_k(x,k,\nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma_t} |k| \to \infty$ as $t \to T_{max}$.

Consequence of the abstract theory of fully nonlinear parabolic equations due to Angenent (1990) and Lunardi.

- Evolution o plane curves with uniform tangential redistribution plane curves with uniform tangential redistribution



$v = k$



$v = k^{1/3}$

Inserting non-local tangential velocity functional

$$\frac{\partial \alpha}{\partial s} = k\beta - \frac{1}{L}\int_\Gamma k\beta \, ds \quad =: k\beta - \langle k\beta \rangle_\Gamma$$

into governing equations yields:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta$$

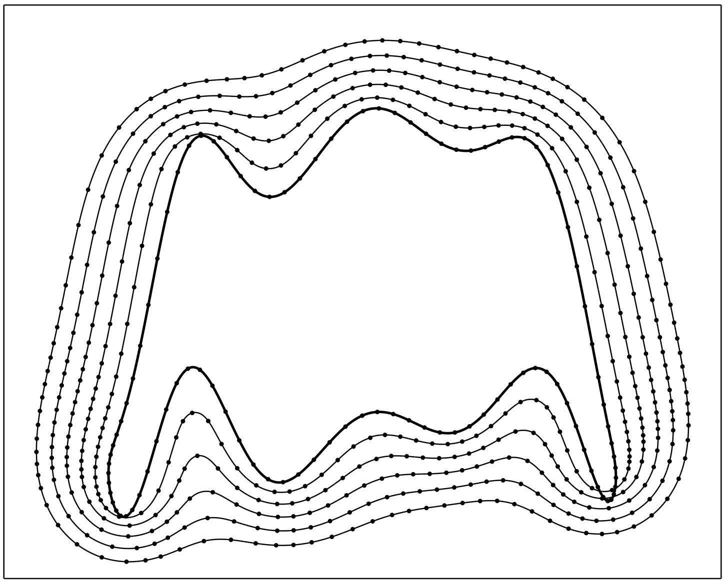
$$\Downarrow$$

$$\partial_t k = \partial_s^2 \beta + \partial_s(\alpha k) + k \langle k\beta \rangle_\Gamma$$

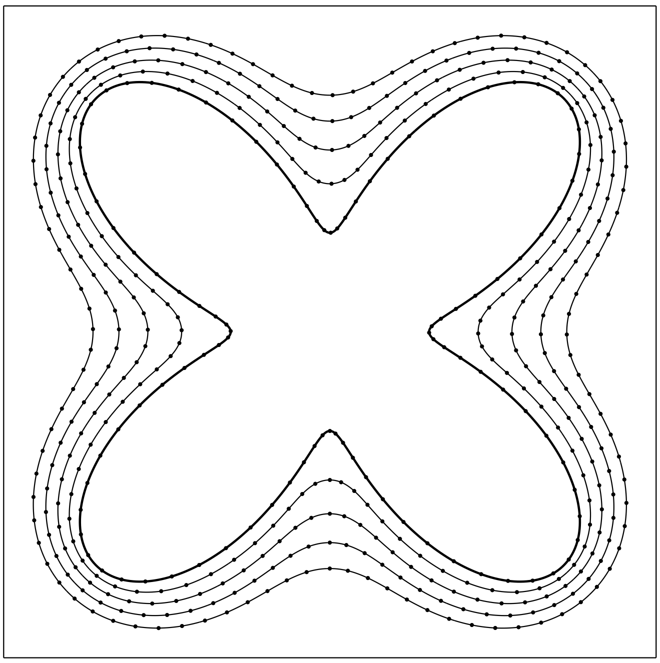
Mean curvature flow with a nontrivial driving force

- The following examples show computational results of mean curvature flow with a nontrivial driving force

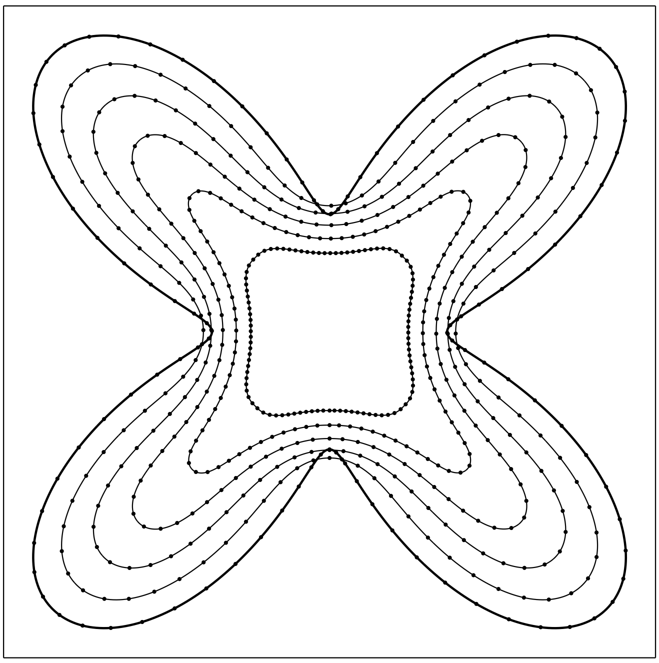
$$\beta(k) = \varepsilon k + c$$



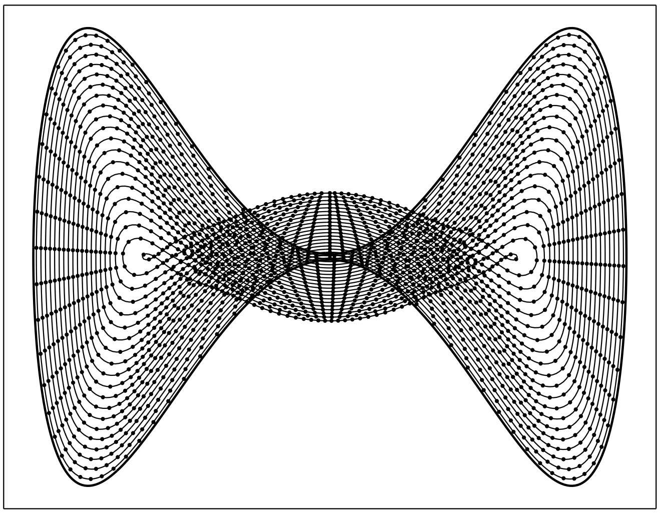
$\varepsilon = 1,\ c = -10$



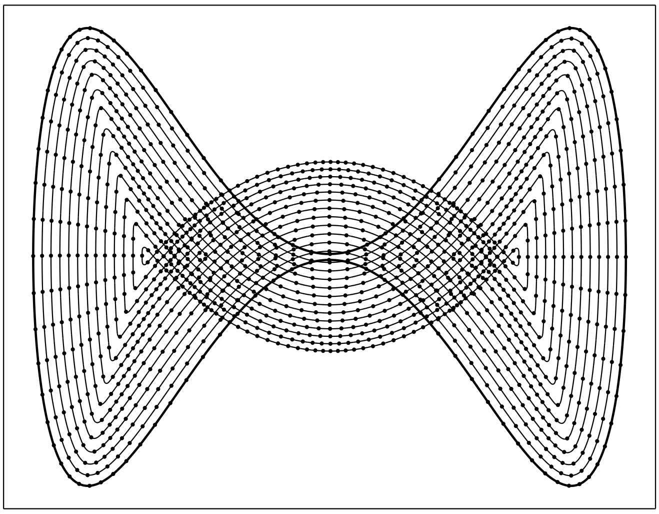
$\varepsilon = 1,\ c = -10$



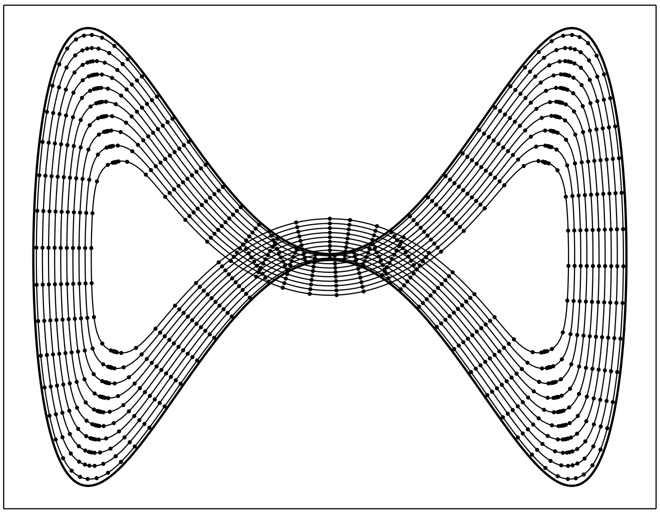
$\varepsilon = 1,\ c = 10$



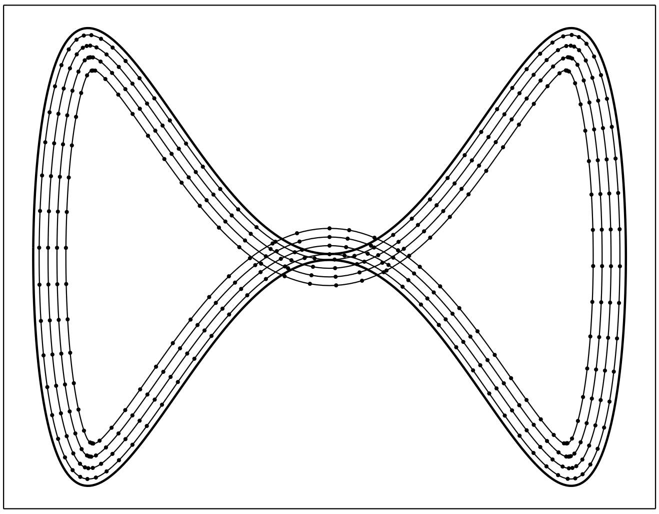
$\varepsilon = 1, \;\; c = 10$



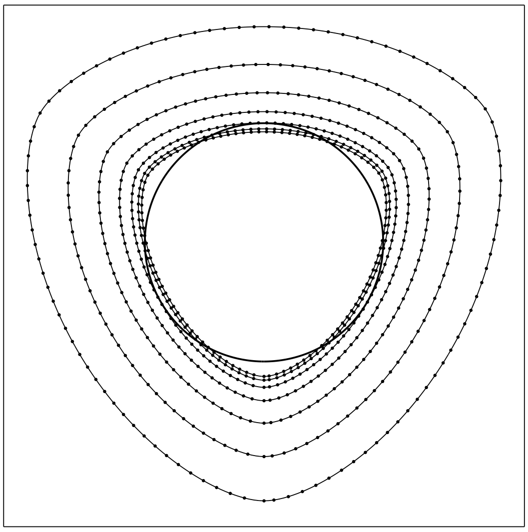
$\varepsilon = 1, \;\; c = 100$



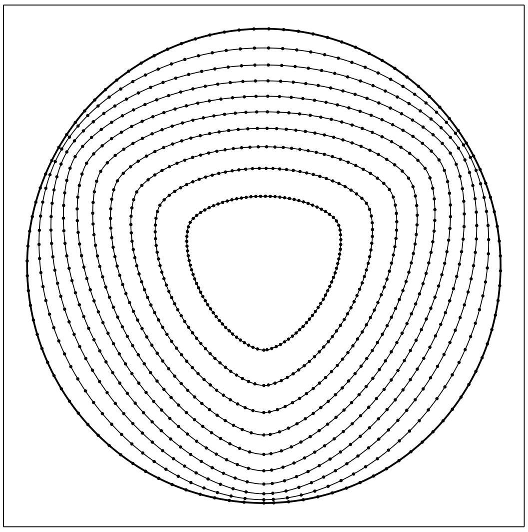
$\varepsilon = 1, \;\; c = 10$
(no tangential redistribution)



$\varepsilon = 1, \;\; c = 100$
(no tangential redistribution)



$\beta = \gamma(\nu)k - 1$



$\beta = \gamma(\nu)k$

First integrals and conserved quantities

- General function $\beta(x, k, \nu)$

$$\boxed{\frac{d}{dt}L(\Gamma^t) + \int_{\Gamma^t} k\beta \, ds = 0, \quad \frac{d}{dt}A(\Gamma^t) + \int_{\Gamma^t} \beta \, ds = 0}$$

where $L(\Gamma)$ and $A(\Gamma)$ denote the length and enclosed area of a curve Γ

- **Casseles' functional in the image segmentation**

If a flow of planar curves evolves according to the normal velocity

$$\boxed{\beta(x, k, \nu) = a(\phi)k - b(\phi)\nabla_x \phi(x).\vec{N}}$$

where $a = a(\phi), b = b(\phi), \phi = \phi(x)$ are smooth functions, $a(\phi) > 0$, $\phi = \phi(x) \in [0, 1]$ then

$$\boxed{\frac{d}{dt}\int_{\Gamma_t} H(\phi) \, ds + \int_{\Gamma_t} \frac{H(\phi)}{a(\phi)}\beta^2 \, ds = 0}$$

where $H = H(\phi)$ is a solution to: $H' = \frac{b}{a}H$.

- **Flow driven by the geodesic curvature**

$V = K_g + C$ on a surface $M = Graph(\phi)$

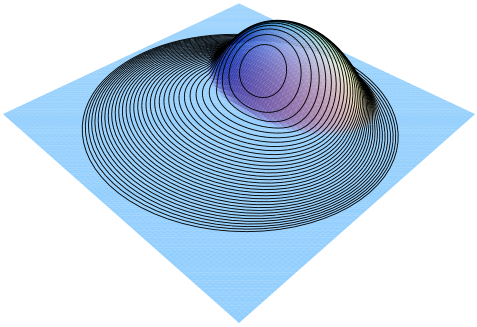
$$\boxed{\beta(x, k, \nu) = a(x, \nu)k - b(x, \nu)\nabla_x \phi(x).\vec{N}}$$

then

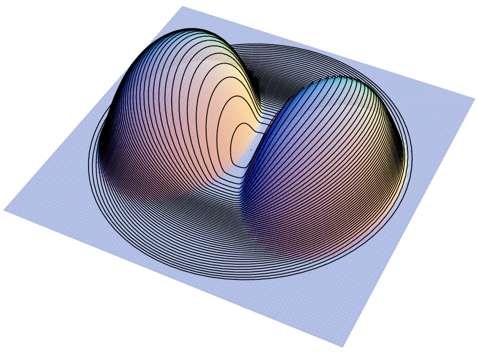
$$\boxed{\frac{d}{dt}\int_{\gamma_t} H(\phi(x)) \, dS + \int_{\gamma_t} H(\phi(x))\beta^2 \, dS = 0}$$

where $H' = gH$ (g is the vertical component of the gravitational like external force $\vec{G} = (0, 0, g)$

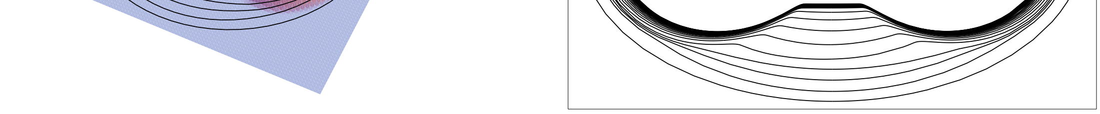
Flow on a surface driven by the geodesic curvature



Evolved curve passes through the hill and then selfsimilarly shrinks to a point in finite time.



Evolved curve passes through both hills and then selfsimilarly shrinks to a point in finite time



Evolved curve tries to pass through both high hills. They constitute an obstacle for the evolution. The curve approaches closed geodesic curve on the surface in infinite time

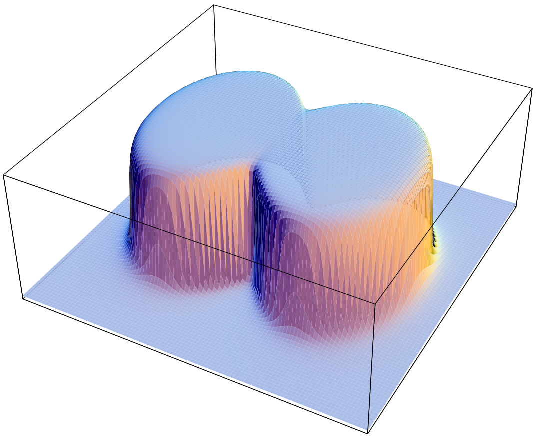
Numerical hard tests - experimental order of convergence

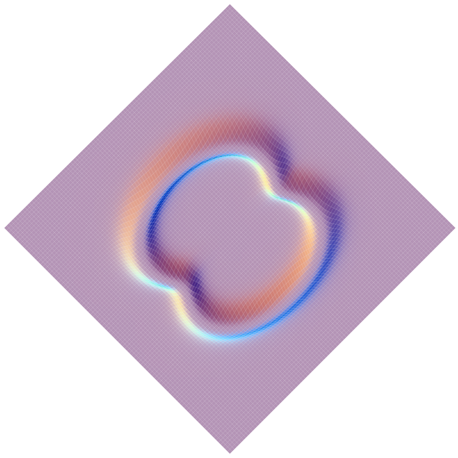
An important tool for testing numerical algorithms is the so-called experimental order of convegence (EOC). If the error $err(h)$ (calculated in a certain norm) of an approximate numerical solution and a true solution satisfies $err(h) \approx h^{\alpha}$ then we halve the discretization step h and determine EOC as follows:

$$\alpha = \log_2 \left(\frac{err(h)}{err(h/2)} \right)$$

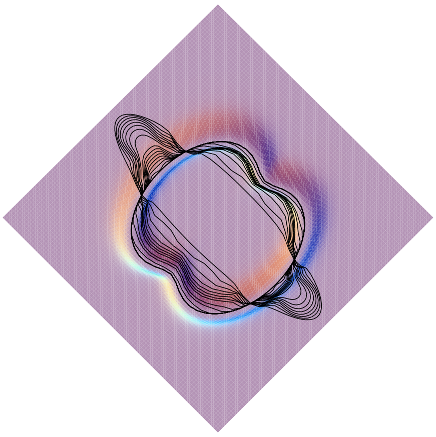
We have tested the experimental order of convergence for the explicit example of shrinking circles on a transversal plane. Since the solved system of governing equations has the parabolic nature we adopted the natural contraint between time stepping and space disctretization: $\tau = h^2/2$.

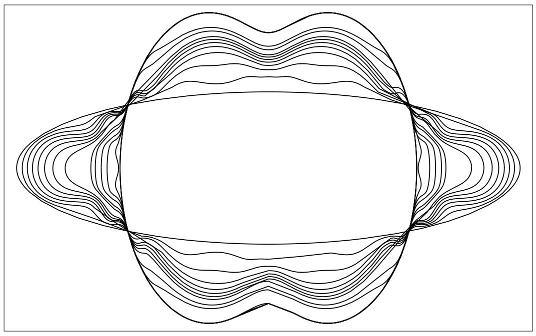
h	x	eoc	k	eoc	ν	eoc	g	eoc
0.1	0.009398		0.07258		0.1286		0.03289	
	0.01421		0.1162		0.1371		0.05433	
0.05	0.004015	1.227	0.03388	1.099	0.06842	0.9109	0.008914	1.884
	0.005955	1.255	0.06471	0.8445	0.07381	0.8939	0.01483	1.873
0.025	0.001877	1.097	0.0165	1.038	0.03473	0.978	0.002296	1.957
	0.002761	1.109	0.03347	0.9508	0.03761	0.9727	0.003808	1.961
0.0125	0.0009225	1.025	0.00827	0.9961	0.01743	0.9944	0.0005814	1.982
	0.001352	1.031	0.01705	0.9734	0.01889	0.9931	0.0009625	1.984
0.00625	0.0004588	1.008	0.004105	1.011	0.008723	0.9991	0.0001454	1.999
	0.0006711	1.01	0.008483	1.007	0.009458	0.9983	0.0002404	2.001



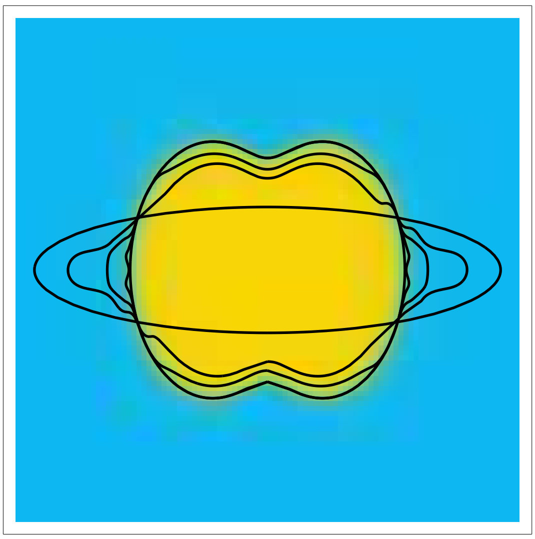


3D plot of a given intensity function I_0 and the surface of the Casseles' functional $\phi(x) = h(|\nabla I_0(x)|)$





Gravitational-like external force drives the evolved curve towards a narrow valley. Geodesic curvature smoothes evolution in the infinite life-span



Geodesic flow on Casseles' functional surface drives evolved curves to the boundary of the image.

Conclusions

• We have studied mean curvature flow of planar curves with the normal velocity depending on the curvature, tangential angle and a position of a curve. Local in time existence of smooth solutions has been show. Various first integrals decreasing along trajectories have been derived and analysed.

• We showed how the flow of curves on a given surface can be reduced to a planar one with the normal velocity described as above.

• Governing system of PDEs equations includes a nontrivial tangential velocity functional. It has no impact on the shape of evolving curves but it can prevent numerically computed solutions from forming instabilities like swallow tails. Redistribution of grid points is asymtotically constant.

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