Evolution of curves on a surface driven by the geodesic curvature and external force

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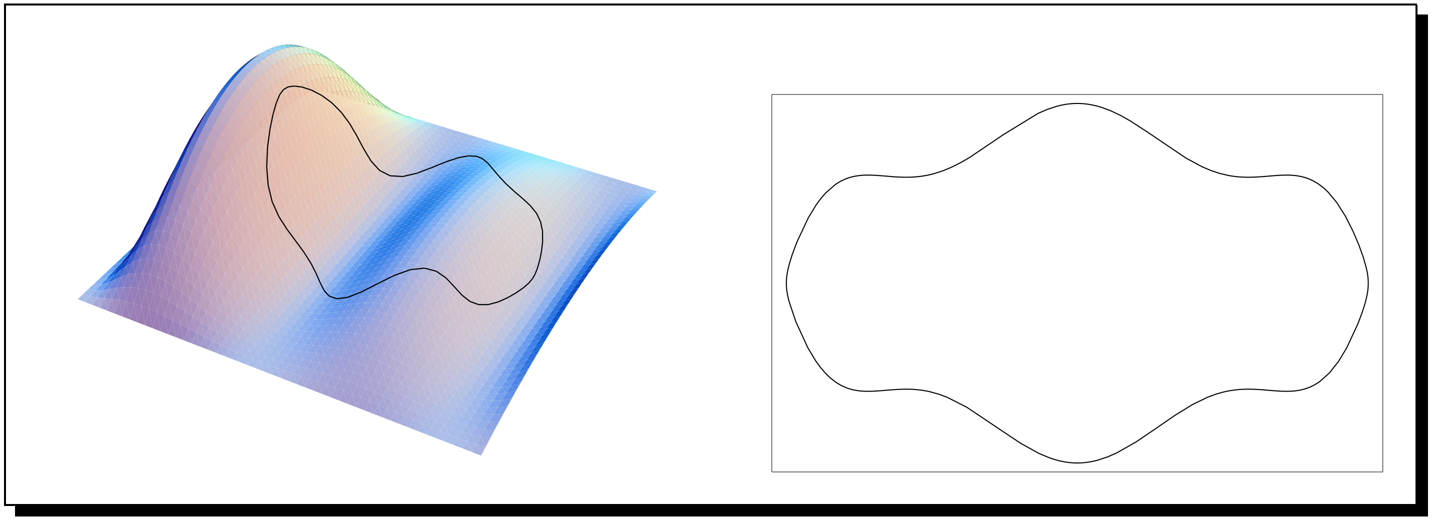
Goals

- study a flow of curves on a given two dimensional surface $\mathcal{M}$ in $\mathbb{R}^3$ where normal velocity $\mathcal{V}$ of a curve $\mathcal{G}$ on $\mathcal{M}$ is a linear function of its geodesic curvature $\mathcal{K}_g$ and external force $\mathcal{F}$:

$$\boxed{\mathcal{V} = \mathcal{K}_g + \mathcal{F}}$$

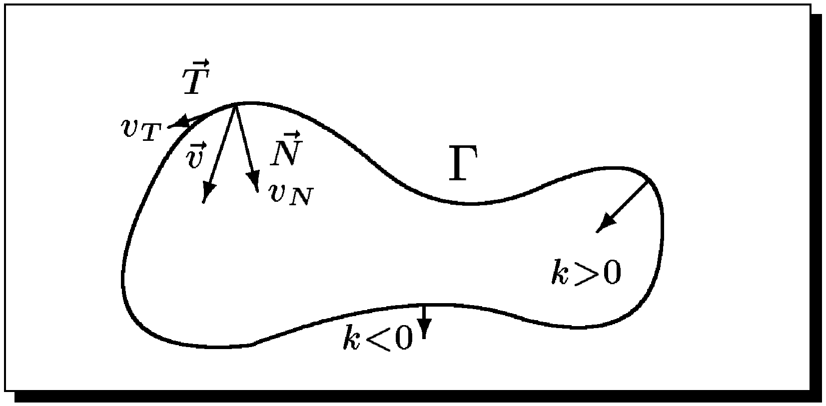


A surface curve $\mathcal{G} \subset \mathcal{M}$ (left). Its vertical projection to a plane curve

- Show how the flow of curves on a given surface driven by the geodesic curvature and external force can be reduced to the flow of curves in the plane driven by the normal velocity

$$\boxed{v = \beta(x, k, \nu)}$$

where k, ν, x are the curvature, tangential angle and position vector of transversally projected planar curve Γ



• Represent the flow of plane curves by a solution to the geometric equation

$$\boxed{\partial_t x = \beta \vec{N} + \alpha \vec{T}}$$

for the position vector $x \in R^2$ representing a curve $\Gamma = \text{Image}\ (x)$.

• Reduce the problem to solution of a system of parabolic PDEs for the curvature, angle and local length of a curve. Analyze qualitative behavior of solutions and stability of closed stationary curves on a surface.

• Suggest a suitable tangential velocity functional α yielding a uniform grid point redistribution along the evolved curve. Compute the flow of curves on various complex surfaces.

Outline

- Transformation of the flow of surface curves to the flow of vertically projected planar curves satisfying $v = \beta(x, k, \nu)$
- Link between the geodesic flow and the edge detection problem in the theory of image segmentation
- Derivation a governing system of PDEs describing the evolution of plane curves satisfying $v = \beta(x, k, \nu)$
- Qualitative aspects of solutions like existence and their limiting behavior. Lyapunov functionals
- Dynamical theory point of view. Closed geodesic curves and their stability.
- Numerical approximation of the geodesic curvature driven flow of surface curves.

Projection of a flow of surface curves to the plane

We consider a flow of surface curves $\mathcal{G}_t \subset \mathcal{M}, t \geq 0$, driven by the geodesic curvature $\mathcal{K}_g$ and external force

$$\boxed{\mathcal{V} = \mathcal{K}_g + \mathcal{F}}$$

where $\mathcal{M} =$ Graph (ϕ), $\phi : \Omega \subset R^2 \to R$ is a surface in R^3, $\mathcal{V}$ is the normal component of the velocity, $\mathcal{K}_g$ is the geodesic curvature of $\mathcal{G}_t$ relative to $\mathcal{M}$ and $\mathcal{F}$ is the normal component of a gravitational like external force

$$\boxed{\vec{G} = -(0, 0, \gamma)}$$

- the geodesic curvature $\mathcal{K}_g$ for a curve $\mathcal{G} = \{(x, \phi(x)) \in \mathbb{R}^3, x \in \Gamma\}$ on a surface $\mathcal{M} = \{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3, \ (x_1, x_2) \in \Omega\}$ can be expressed as a function of the curvature k of its projection to the plane, position vector x and the angle ν.

- The external vector field $\vec{G}$ is assumed to be perpendicular to the plane $\mathbb{R}^2$ and it depends on the vertical coordinate $z = \phi(x)$ only, i.e.

$$\boxed{\vec{G}(x) = -(0, 0, \gamma)}$$

where $\gamma = \gamma(z) = \gamma(\phi(x))$ is a given scalar "gravity" functional

• taking the normal component of such an external force we obtain expression for the driving term $\mathcal{F} = \vec{G}.\vec{\mathcal{N}}$

$$\boxed{\mathcal{F} = -\frac{\gamma(\phi(x))}{\left((1+|\nabla\phi|^2)(1+(\nabla\phi.\vec{T})^2)\right)^{\frac{1}{2}}}\nabla\phi.\vec{N}}$$

• $\vec{\mathcal{N}} \subset T_x(\mathcal{M})$ is the unit inward normal vector to the surface curve $\mathcal{G}_t$ relative to the surface $\mathcal{M}$

• $\vec{N}, \vec{T}$ are unit inward normal and tangent vectors to the projected planar curve Γ_t.

• the flow of surface curves $\mathcal{G}_t \subset \mathcal{M}$ fulfills $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ iff the normal velocity v of the flow of planar curves $\Gamma_t, t \geq 0$, satisfies the geometric equation

$$\boxed{v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla_x \phi(x).\vec{N}}$$

where $\vec{T} = (\cos(\nu), \sin(\nu))$, $\vec{N} = \vec{T}^\perp$, and $a, b : \Omega \subset \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

$$a(x, \nu) = \frac{1}{1 + (\nabla\phi.\vec{T})^2}$$

$$b(x, \nu) = \frac{1}{1 + |\nabla\phi|^2}\left(\gamma - \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{1 + (\nabla\phi.\vec{T})^2}\right)$$

Image segmentation - link to the flow on a surface

geometric flow of curves on surfaces driven by geodesic curvature and external force

$\updownarrow$ linked to

edge detection in the theory of image segmentation

• In image segmentation, detection of image silhouettes plays an important role. An image is represented by a given intensity function $u_0 : R^2 \to [0, 1]$.

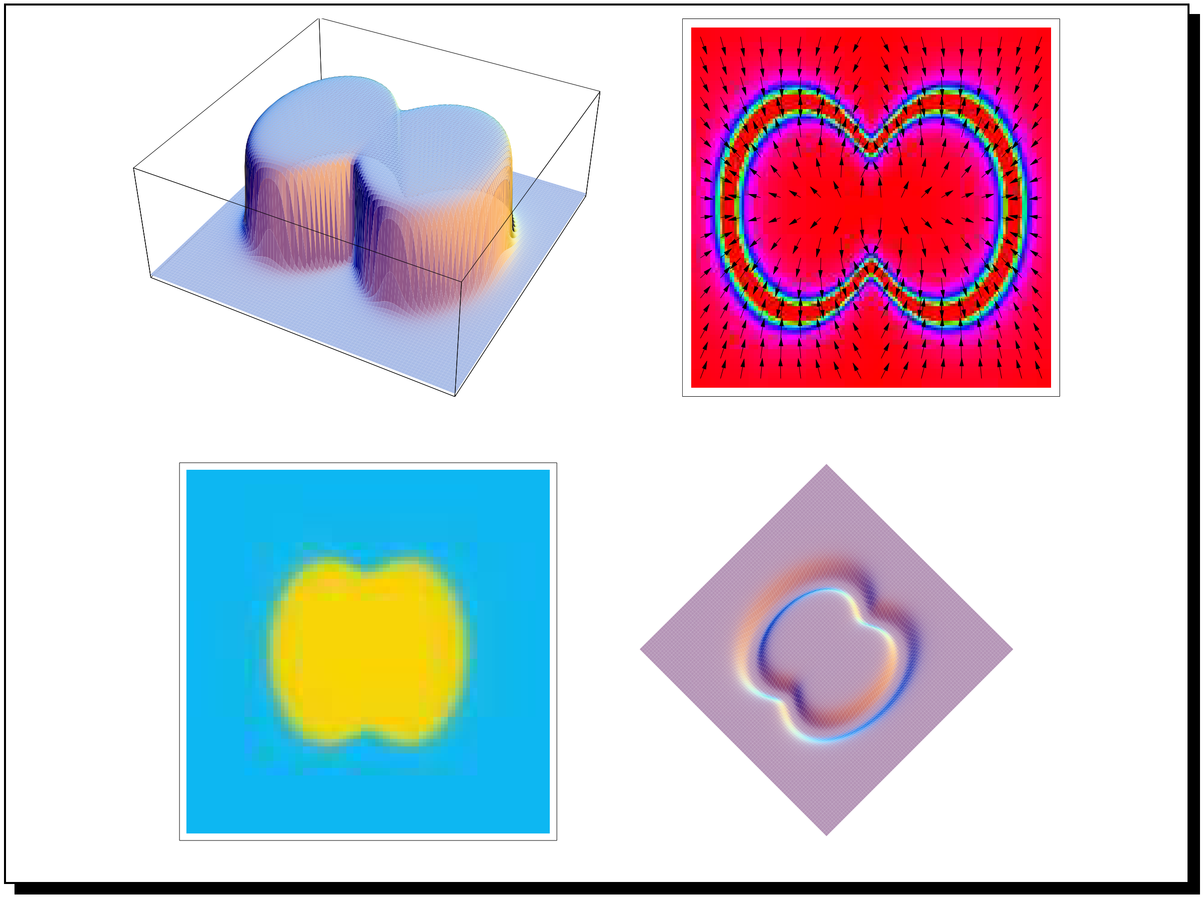
• The problem is to detect edges of the image, i.e. planar curves on which the gradient ∇u_0 is very large.

- The idea is to construct an evolving family of planar curves converging to an edge of the image according to the normal velocity

$$\boxed{\beta(x,k,\nu) = \varepsilon\phi(x)k - \nabla\phi(x).\vec{N}}$$

where $\phi(x) = h(|\nabla u_0(x)|)$, h is a suitable image contrast function, e.g. $h(s) = e^{-s}$

- (Caselles et al 1997; Kichenassamy *et al.* 1996)



The image intensity function u_0 (top left) and its density plot (bottom left). 3D plot of Casseles' functional ϕ (bottom-right) and the corresponding vector field $-\nabla\phi(x)$ (top-right)

Governing equations and tangential velocity functional

- An embedded regular plane curve Γ can be parameterized by a smooth function $x : S^1 \to \mathbb{R}^2$, i.e. $\Gamma = \text{Image}(x) := \{x(u), u \in S^1\}$

- We represent the flow of plane curves by a solution x to the geometric equation

$$\boxed{\partial_t x = \beta \vec{N} + \alpha \vec{T}}$$

for the position vector $x \in R^2$ representing a curve $\Gamma = \text{Image}\ (x)$.

- The curvature k, tangential angle ν and position vector x satisfies

$$\boxed{\begin{array}{rcl} g &=& |\partial_u x| \\ \vec{T} &=& (\cos\nu, \sin\nu) = \partial_s x = g^{-1}\partial_u x \\ k &=& \partial_s x \wedge \partial_s^2 x = g^{-3}\partial_u x \wedge \partial_u^2 x \end{array}}$$

- Straightforward calculations using Frenet's formulae yield the fully nonlinear system of governing parabolic PDEs

$$
\boxed{
\begin{aligned}
\partial_t k &= \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta \\
\partial_t \nu &= \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu) \partial_s \nu + \nabla_x \beta . \vec{T} \\
\partial_t g &= -g k \beta + \partial_u \alpha \\
\partial_t x &= \beta \vec{N} + \alpha \vec{T}
\end{aligned}
}
$$

where $\beta = \beta(x, k, \nu)$ and α are the normal and tangential velocities, $\vec{T} = (\cos(\nu), \sin(\nu)), \quad \vec{T} \perp \vec{N}$,

$g = g(u, t) = |\partial_u x(u, t)|$ is the local length element,

$ds = g\, du$ is the arc-length parameterization, $(u, t) \in Q_T = S^1 \times [0, T)$.

- A solution is subject to initial and periodic boundary conditions corresponding to an initial curve and is searched in the functional space

$$\boxed{\begin{array}{c} E_k = \underbrace{c^{2k+\delta}(S^1)}_{k} \times \underbrace{c^{2k+\delta}(S^1)}_{\nu} \times \underbrace{c^{1+\delta}(S^1)}_{g} \times \underbrace{(c^{2k+\delta}(S^1))^2}_{x} \end{array}}$$

where $k = 0, \frac{1}{2}, 1$, and $c^{2k+\varrho} = c^{2k+\varrho}(S^1)$ is the "little" Hölder space, i.e. the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{2k+\varrho}(S^1)$

Theorem. Assume that $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector and $g_0 = |\partial_u x_0| > 0$ is the local length element of the initial regular curve $\Gamma_0 = \text{Image } (x_0)$. If $\beta = \beta(x, k, \nu)$ is a C^4 smooth function such that

$$\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$$

and α is an admissible tangential velocity functional. Then there exists a unique classical solution

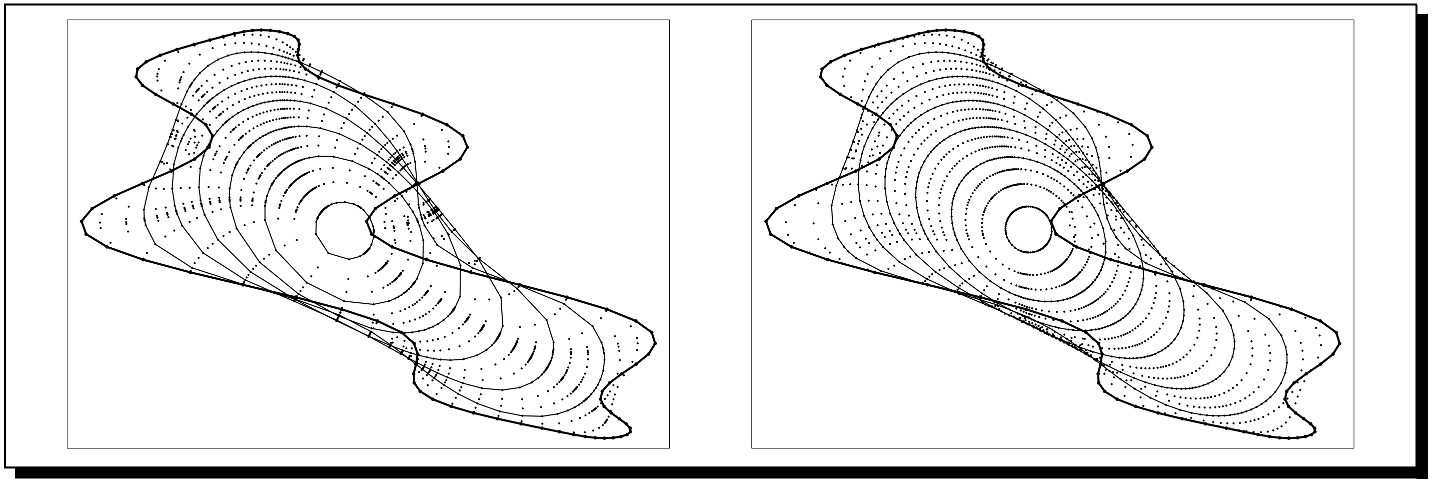
$$\Phi = (k, \nu, g, x) \in C([0, T], E_1) \cap C^1([0, T], E_0)$$

of the governing system of equations defined on some small time interval $[0, T]$, $T > 0$. If Φ is a maximal solution defined on $[0, T_{max})$ then either $T_{max} = +\infty$ or $\liminf_{t \to T_{max}^-} \min_{\Gamma_t} \beta'_k(x, k, \nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma_t} |k| \to \infty$ as $t \to T_{max}$.

- Consequence of the abstract theory of fully nonlinear parabolic equations due to Angenent (1990) and Lunardi.

Review of numerical aspects of tangential velocity functional

- usual choice of the tangential velocity $\alpha = 0$ fails and may lead to serious numerical instabilities



Merging of numerically computed grid points in the case $\alpha = 0$ (left). Impact of a suitable tangential velocity functional α defined as:

$$\boxed{\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma}$$

on enhancement of the spatial grids redistribution (right). (Hou et al in '94, K.Mikula and D.Ševčovič in 1999, 2001).

Qualitative behavior of solutions

- General function $\beta(x, k, \nu)$

$$\boxed{\frac{d}{dt}L(\Gamma^t) + \int_{\Gamma^t} k\beta\, ds = 0, \quad \frac{d}{dt}A(\Gamma^t) + \int_{\Gamma^t} \beta\, ds = 0}$$

where $L(\Gamma)$ and $A(\Gamma)$ denote the length and enclosed area of Γ

- Casseles' functional in the image segmentation

$\beta(x, k, \nu) = a(\phi)k - b(\phi)\nabla_x\phi(x).\vec{N}$

$$\boxed{\frac{d}{dt}\int_{\Gamma_t} H(\phi)\, ds + \int_{\Gamma_t} \frac{H(\phi)}{a(\phi)}\beta^2\, ds = 0}$$

where $H = H(\phi)$ is a solution to: $H' = \frac{b}{a}H$.

• General first integral for the flow driven by the geodesic curvature on a surface

$$\boxed{\frac{d}{dt}\mathcal{L}_t + \int_{\mathcal{G}_t} \mathcal{K}_g \mathcal{V} \, dS = 0}$$

It is a Lyapunov functional if $\mathcal{F} = 0$. Then $\mathcal{V}\mathcal{K}_g = \mathcal{K}_g^2$

• Flow driven by the geodesic curvature $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ on a surface $\mathcal{M} = Graph(\phi)$. Then vertically projected planar curves have the normal velocity $\beta(x, k, \nu) = a(x, \nu)k - b(x, \nu)\nabla_x \phi(x).\vec{N}$ and

$$\boxed{\frac{d}{dt}\int_{\mathcal{G}_t} H(\phi(x)) \, dS + \int_{\mathcal{G}_t} H(\phi(x))\mathcal{V}^2 \, dS = 0}$$

where $H' = \gamma H$ (γ is the vertical component of the gravitational like external force $\vec{G} = -(0, 0, \gamma)$)

Stationary solution. Closed geodesic curves on a surface

- Goal is to analyze stationary surface curves with respect to the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$, i.e. surface curves satisfying $\mathcal{K}_g + \mathcal{F} = 0$.

- Motivation

  - Construction of closed geodesic curves ($\mathcal{K}_g = 0$)

  - Analysis of stability of resolved edges in the image

- $\mathcal{V} = 0$ on $\mathcal{M}$ iff $v \equiv \beta(x, k, \nu) = 0$ in the plane

Definition A closed C^2 smooth planar curve $\bar{\Gamma} = \text{Image}(\bar{x})$ is called a stationary curve with respect to the normal velocity β iff $\beta(\bar{x}, \bar{k}, \bar{\nu}) = 0$ on $\bar{\Gamma}$ where $\bar{x}, \bar{k}$ and $\bar{\nu}$ are the position vector, curvature and tangential angle of the curve $\bar{\Gamma}$.

Principle of linearized stability

- Tangential velocity functional in the system of governing equations has no impact on the shape of evolving curves. For $\alpha = 0$ the governing system of equations reduces to:

$$\boxed{\begin{array}{rcl} \partial_t k &=& g^{-1}\partial_u(g^{-1}\partial_u\beta) + k^2\beta \\ \partial_t \nu &=& g^{-1}\partial_u\beta \\ \partial_t g &=& -gk\beta \\ \partial_t x &=& \beta\vec{N} \end{array}}$$

- In order to analyze stability of $\bar{\Gamma}$ we have to investigate the behavior of infinitesimal variations of k, ν, g and x

- $\delta k, \delta\nu, \delta g$ and δx satisfy the linearized system

$$\boxed{\begin{array}{rcl} \partial_t \delta k &=& \bar{g}^{-1}\partial_u(\bar{g}^{-1}\partial_u \delta\beta) + \bar{k}^2 \delta\beta \\ \partial_t \delta\nu &=& \bar{g}^{-1}\partial_u \delta\beta \\ \partial_t \delta g &=& -\bar{g}\bar{k}\delta\beta \\ \partial_t \delta x &=& \delta\beta \vec{\bar{N}} \end{array}}$$

where

$$\boxed{\begin{aligned} \delta\beta = \beta(\bar{x}+\delta x, \bar{k}+\delta k, \bar{\nu}+\delta\nu) - \beta(\bar{x}, \bar{k}, \bar{\nu}) + h.o.t. \\ = \nabla_x \bar{\beta}.\delta x + \bar{\beta}'_k \delta k + \bar{\beta}'_\nu \delta\nu \end{aligned}}$$

- the total variation $\delta = \delta\beta$ satisfies the scalar parabolic equation

$$\boxed{\partial_t \delta = P \partial_u^2 \delta + R \partial_u \delta + Q \delta}$$

subject to periodic boundary conditions at $u = 0, 1$ where

$$\boxed{P = \bar{g}^{-2} \bar{\beta}'_k \qquad R = \bar{g}^{-1} \bar{\beta}'_\nu + \bar{g}^{-1} \bar{\beta}'_k \partial_u \bar{g}^{-1} \qquad Q = \bar{\beta}'_k \bar{k}^2 + \nabla_x \bar{\beta} . \vec{\bar{N}}}$$

Functions P, Q and R are 1-periodic in u and depend on the $\bar{\Gamma}$ only.

Definition. A stationary curve $\bar{\Gamma} = \mathrm{Image}(\bar{x})$ is called linearly stable if the zero solution is exponentially asymptotically stable in the space $L^2(S^1)$, i.e. there exist constants $M, \omega > 0$ such that $\|\delta(., t)\|_{L^2(S^1)} \leq M e^{-\omega t} \|\delta(., 0)\|_{L^2(S^1)}$ for any initial condition $\delta(., 0) \in L^2(S^1)$.

Lemma. Suppose $P, R, Q \in C^1(S^1)$, $P > 0$. If $\int_0^1 \frac{R(u)}{P(u)} du = 0$ then the linear operator $A : D(A) \subset L^2(S^1, w) \to L^2(S^1, w), D(A) = W^{2,2}(S^1)$, is selfadjoint operator in the weighted Lebesgue space $L^2(S^1, w)$ with the weight defined as: $w(u) = P(u)^{-1} \exp(\int_0^u \frac{R(v)}{P(v)} dv)$.

- Notice $\int_0^1 \frac{R(u)}{P(u)} du = 0$ if and only if $\int_{\bar\Gamma} \frac{\bar\beta'_\nu}{\bar\beta'_k} ds = 0$

Proposition. Let $\beta(x, k, \nu) = a(x, \nu)k - b(x, \nu)\nabla\phi.\vec{N}$ where a, b correspond to projected flow of surface curves and $\phi(x)$ is C^2 smooth function. Then

$$\int_{\bar\Gamma} \frac{\bar\beta'_\nu}{\bar\beta'_k} ds = 0$$

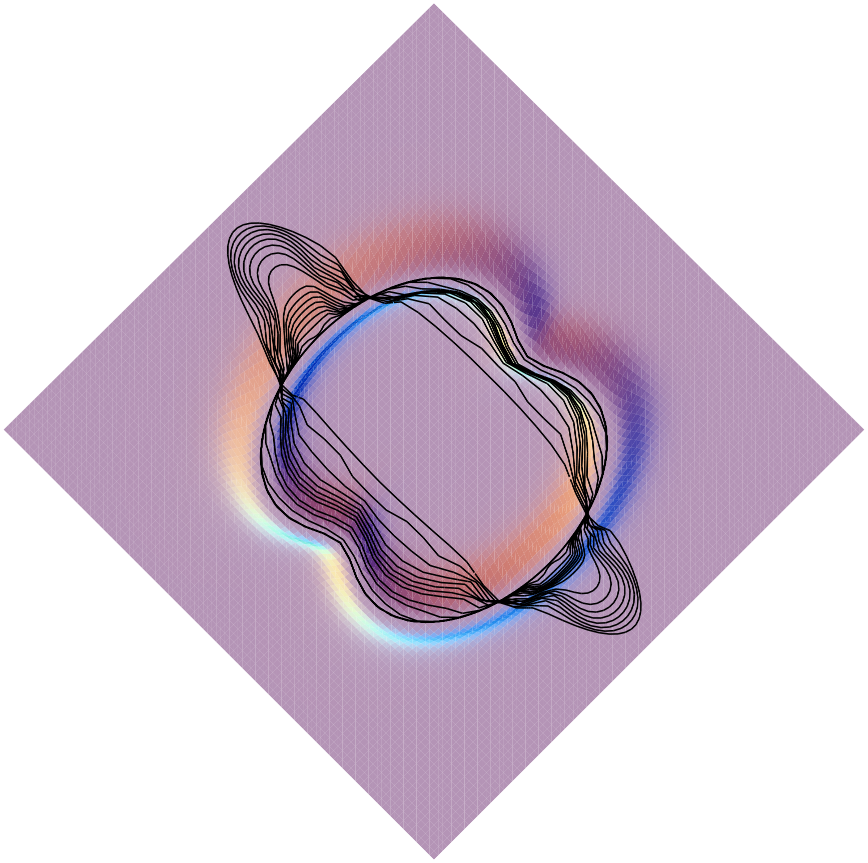
for any closed stationary curve $\bar\Gamma = \mathrm{Image}(\bar x)$ where $\bar\beta = \beta(\bar x, \bar k, \bar\nu)$.

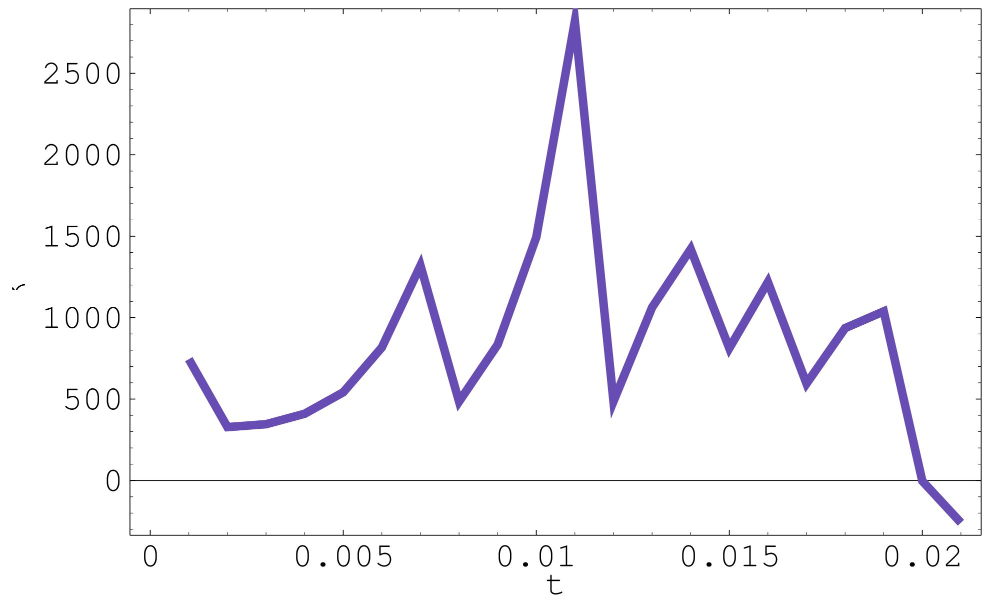
Theorem. Suppose that $\bar{\Gamma}$ is a stationary curve with respect to the normal velocity corresponding to the projected flow of surface curves, i.e. $\bar{\Gamma}$ is a vertical projection of a stationary surface curve $\mathcal{G}$. Then

- $\bar{\Gamma}$ is linearly stable if $\sup_{\bar{\Gamma}} Q < 0$;
- $\bar{\Gamma}$ is unstable if $\int_0^1 Q w \, du > 0$

where $Q = \bar{\beta}'_k \bar{k}^2 + \nabla_x \bar{\beta}.\vec{\bar{N}}$ and w is the weight in $L^2_w(S^1)$

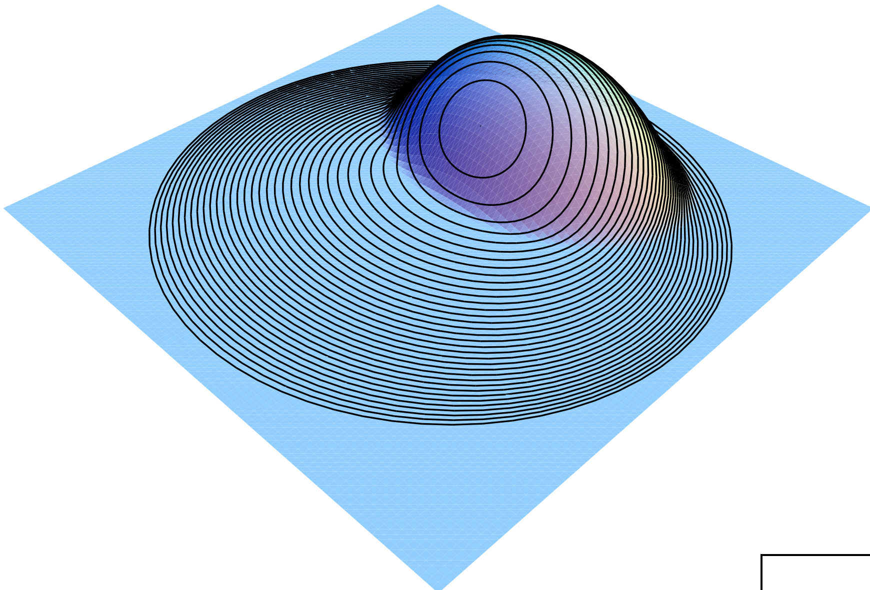
- Proof follows from selfadjoint property of the parabolic equation for the variation $\delta\beta$ and analysis of the Rayleigh quotient.

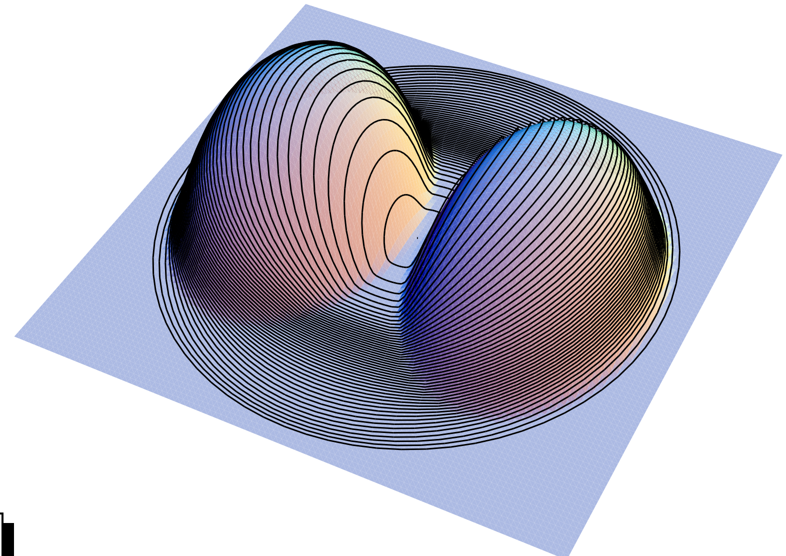




Geodesic flow on Casseles' functional surface drives evolved curves to the boundary of the image (left). Time evolution of the quantity $Q = \bar{\beta}'_k \bar{k}^2 + \nabla_x \bar{\beta}.\vec{\bar{N}}$ (right). It eventually becomes negative when $t \to \infty$

Flow on a surface driven by the geodesic curvature

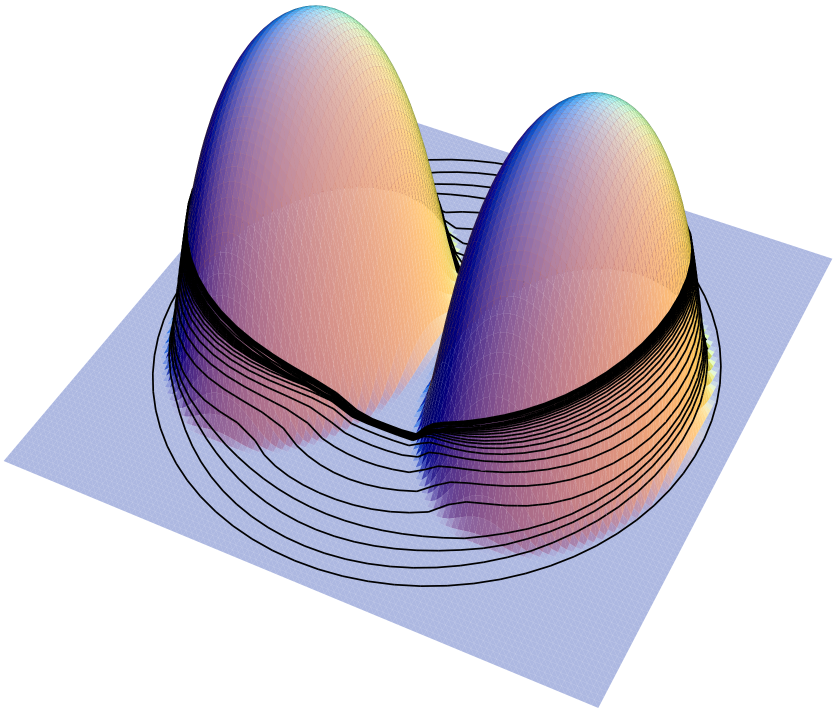


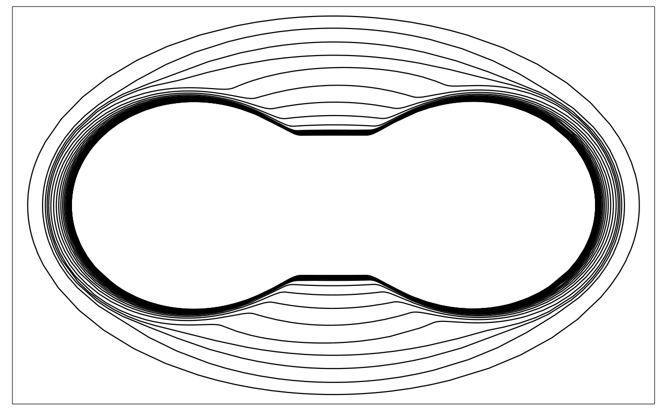


$$\boxed{\mathcal{V} = \mathcal{K}_g}$$

- Left: Evolved curve passes through the hill and then selfsimilarly shrinks to a point in finite time.

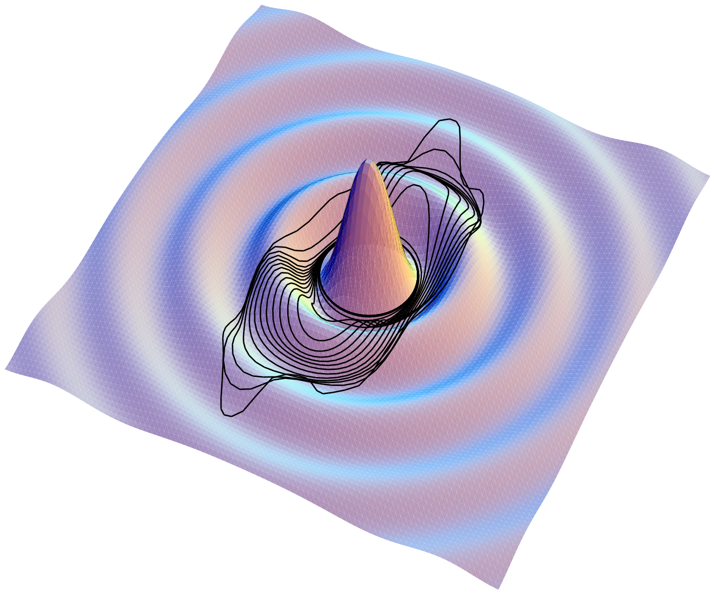
- Right: Evolved curve passes through both hills and then selfsimilarly shrinks to a point in finite time

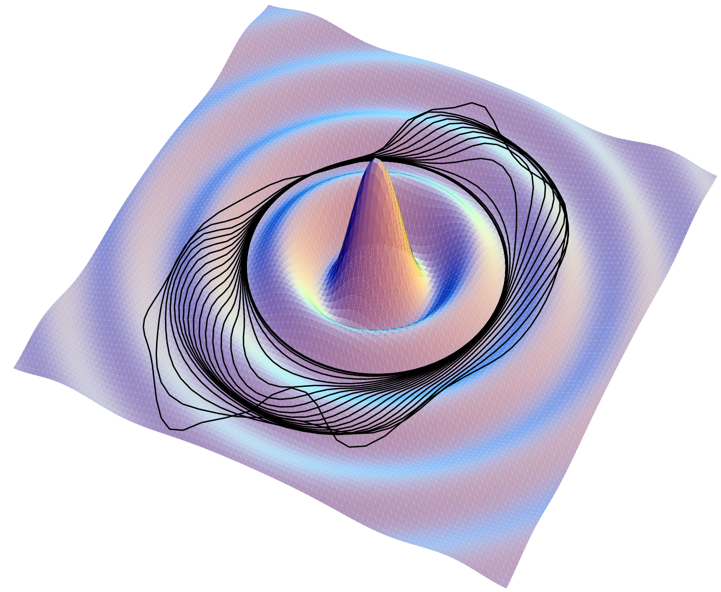




$$\boxed{\mathcal{V} = \mathcal{K}_g}$$

Evolved curve tries to pass through equally high humps. They constitute an obstacle for the evolution. The curve approaches closed geodesic curve on the surface in infinite time

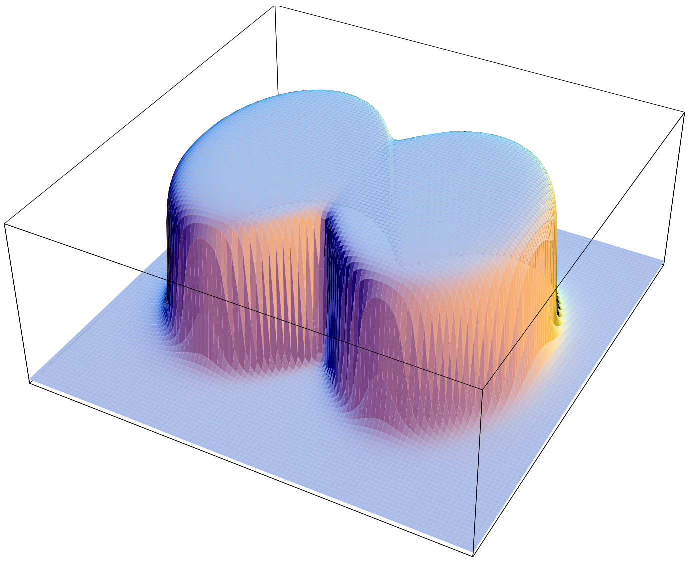


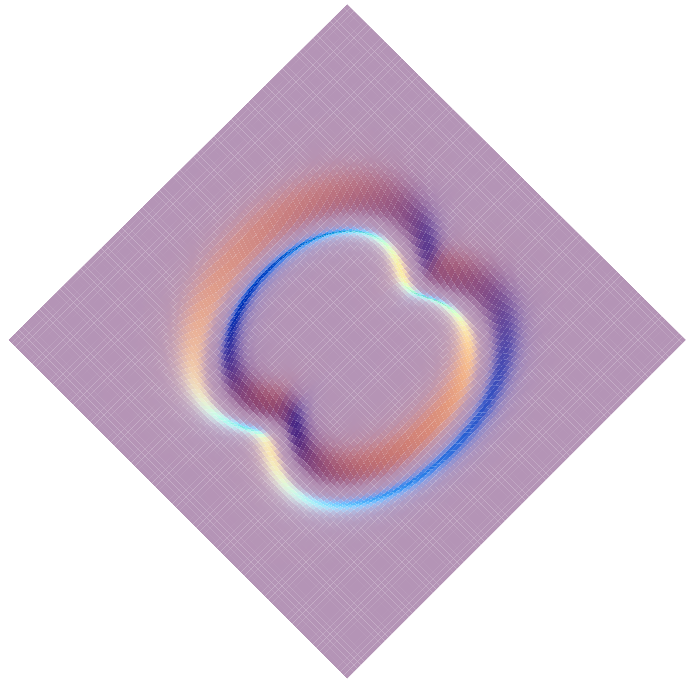


$$\boxed{\mathcal{V} = \mathcal{K}_g + \mathcal{F}}$$

A surface flow on a wave-let like surface driven by the geodesic curvature and strong external force $\mathcal{F}$. Surface curves converge to the stable stationary circular curve with smallest radius (left) and second smallest radius (right).

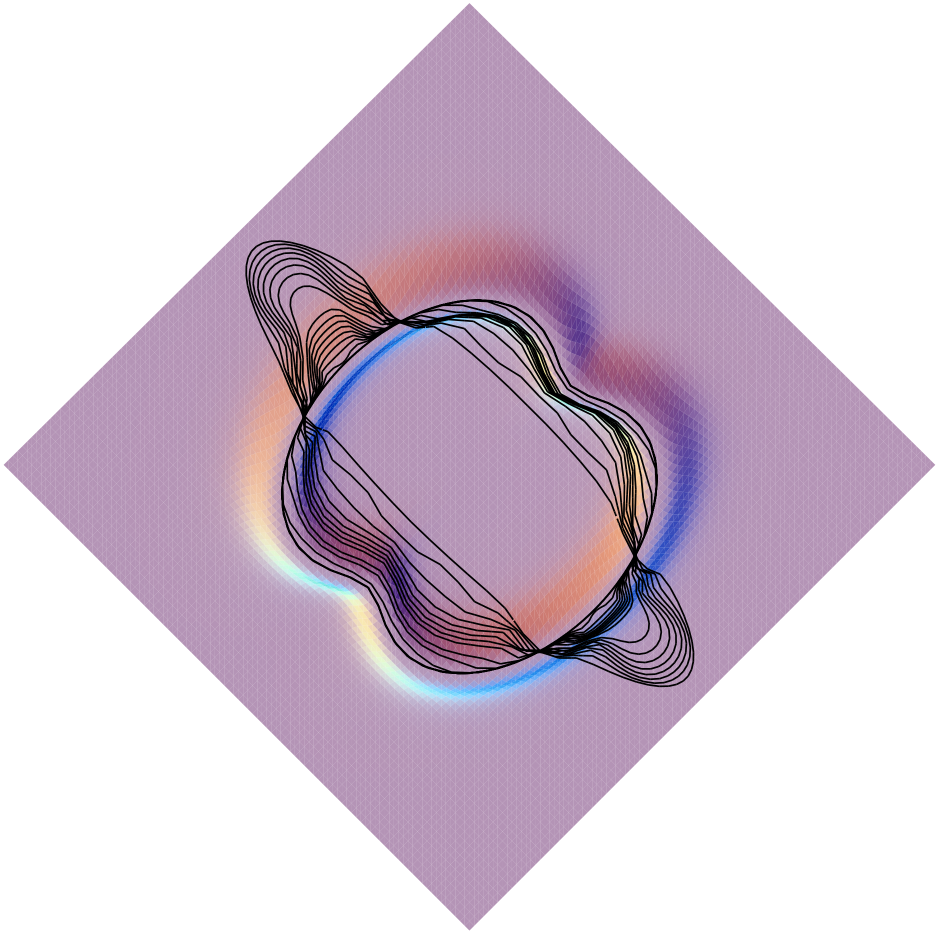
Geodesic flow with external force. Edge detection

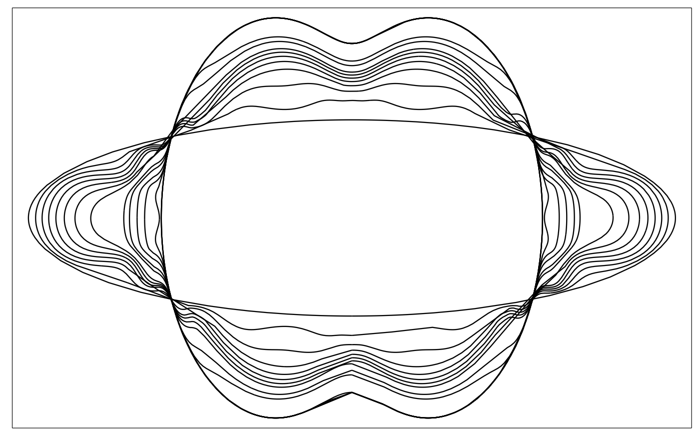




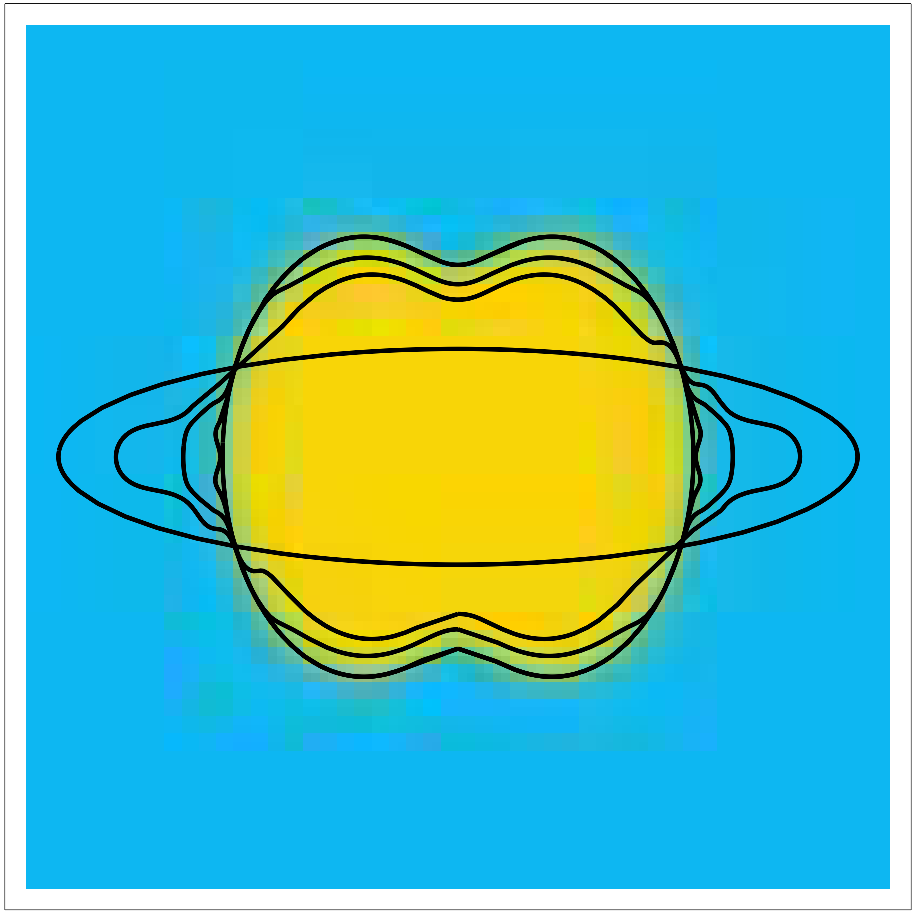
Intensity function I_0 (left) and the Casseles' functional surface,

$\phi(x) = h(|\nabla I_0(x)|)$ (right)





Gravitational-like external force drives the evolved curve towards a narrow valley. Geodesic curvature smoothes evolution in the infinite life-span



Geodesic flow on Casseles' functional surface drives evolved curves to the boundary of the image.

Conclusions

- The flow of curves on a given surface can be reduced to a planar flow with the normal velocity depending on the curvature, position and orientation.

- The geometric problem can be transformed to a fully nonlinear parabolic system of equations for the curvature, position, orientation and local length. Local in time existence of smooth solutions.

- Impact of a nontrivial tangential velocity functional on grid points redistribution has been emphasized.

- Various first integrals decreasing along trajectories have been constructed and analyzed. Closed stationary curves have been identified. Criterion for their linearized stability has been derived.

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