A TWO PHASE METHOD FOR CALIBRATION OF INTEREST RATE MODELS AND ITS APPLICATION TO CENTRAL EUROPEAN INTERBANK OFFER RATES

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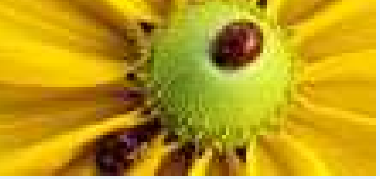
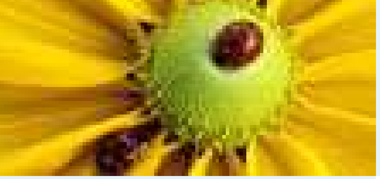
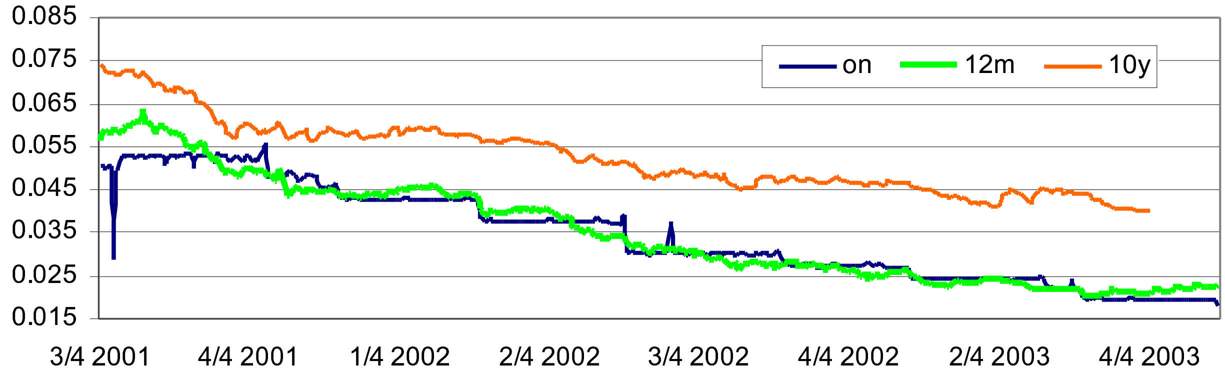


What are Term structure and Interest rate models?

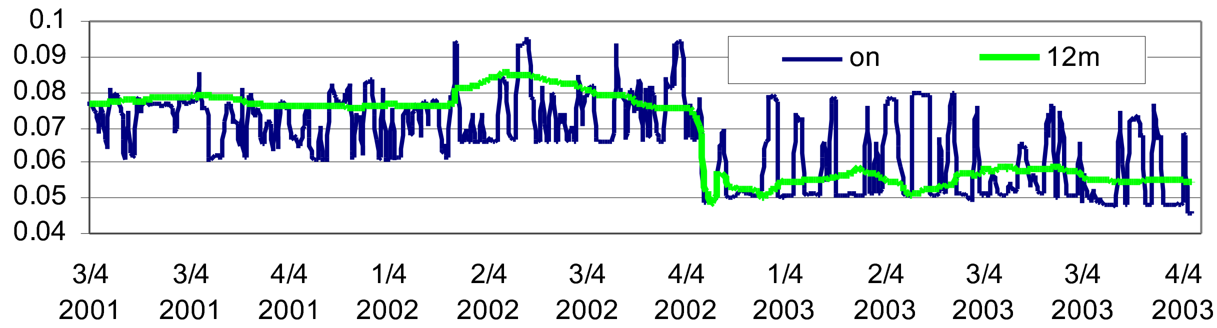
The term structure is a functional dependence between the time to maturity of a discount bond and its present price.



Term structures

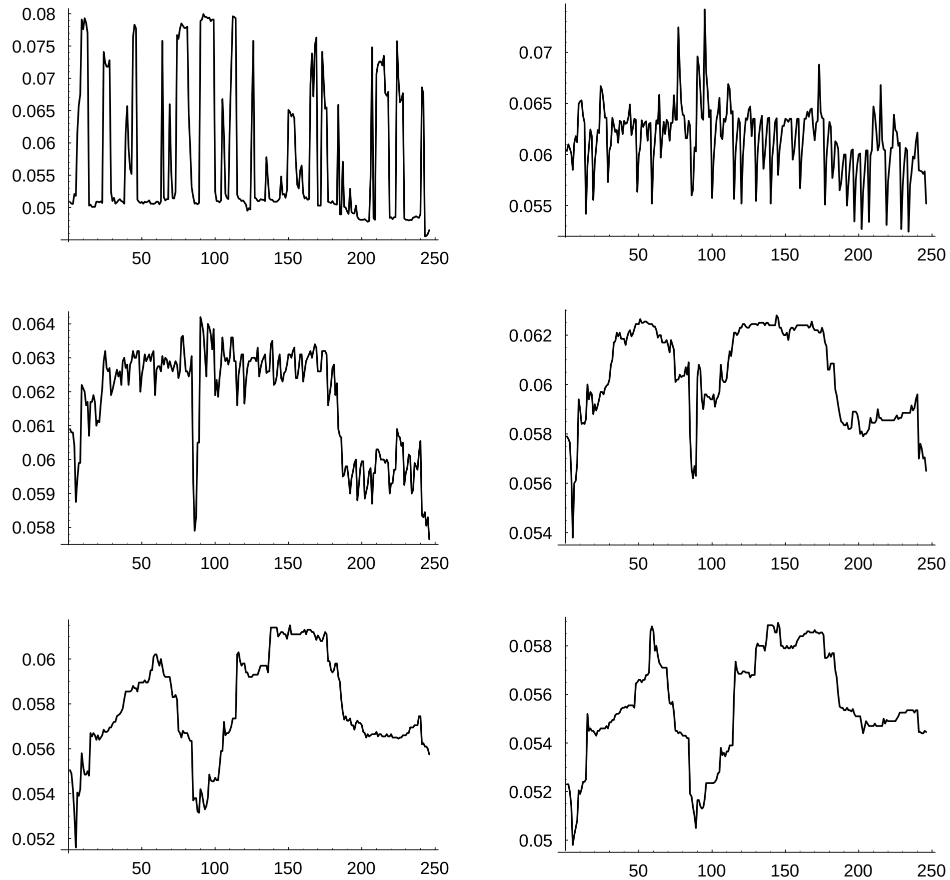


Time evolution of Czech interbank offer rates (PRIBOR). Short rate



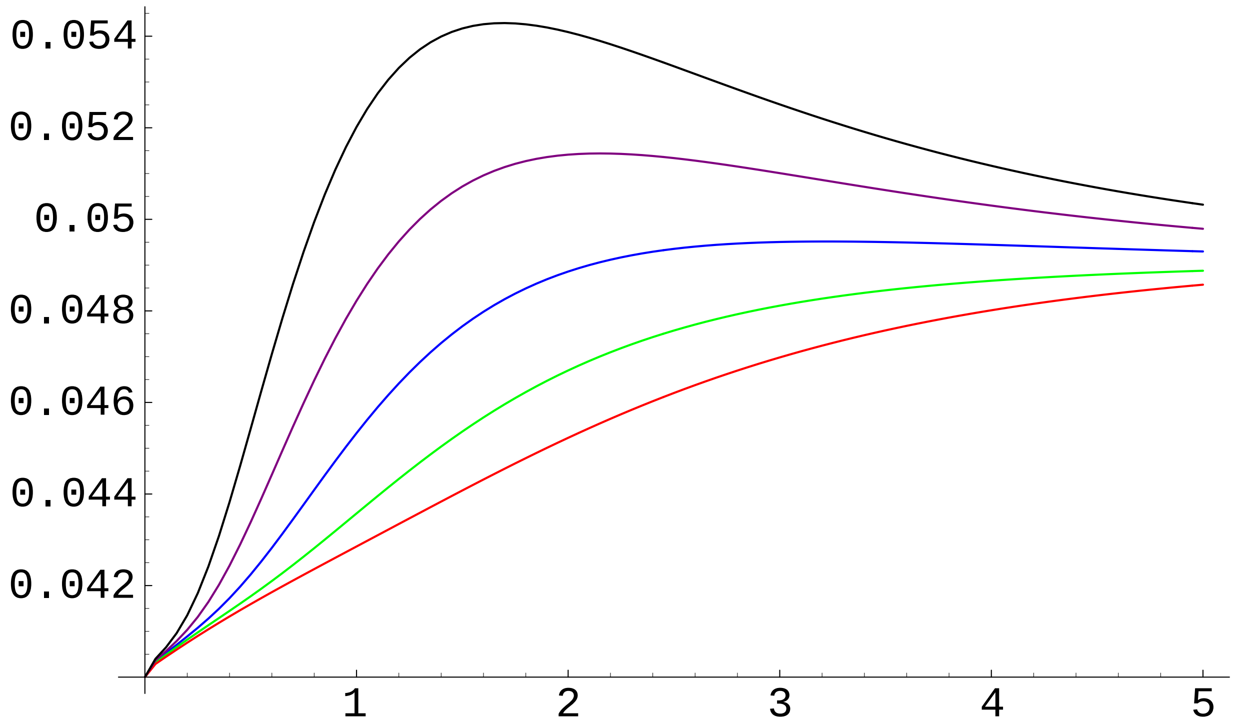
Time evolution of Slovak interbank offer rates (BRIBOR). Highly volatile short rate and rates of bonds with 1-year maturities.

Term structures



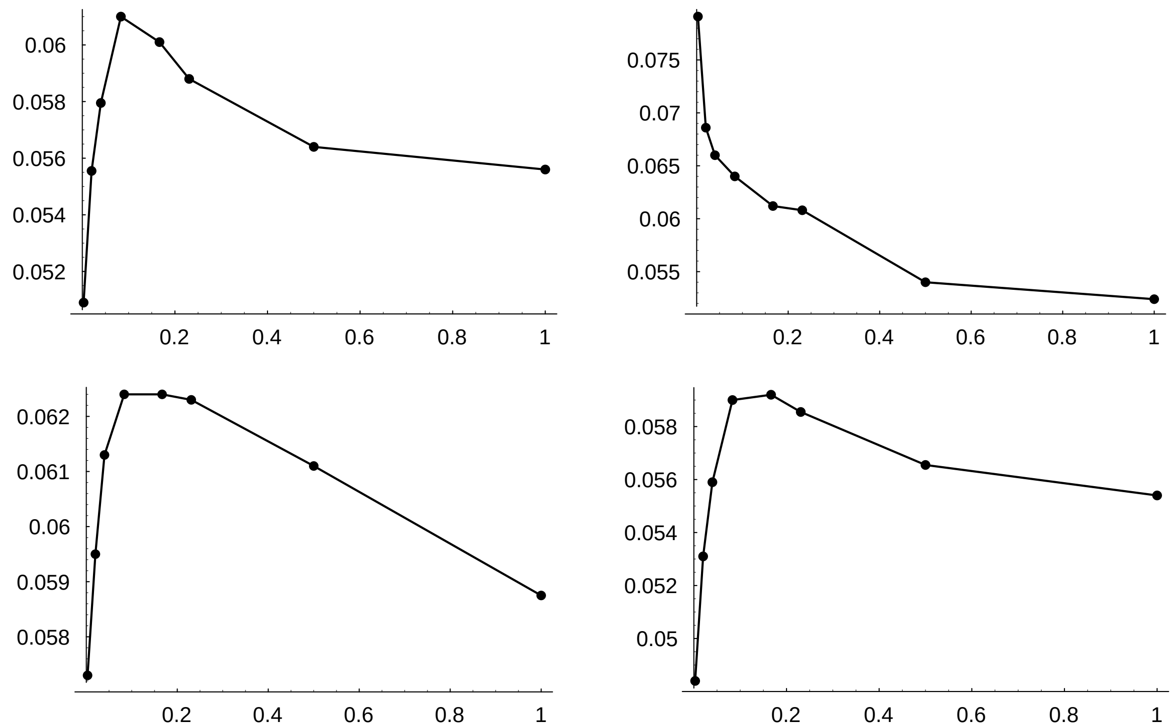
Slovak interbank offer rates. Short rate, 1-week, 1-month, 3-months, 6-months and 1-year interest rates on bonds. Bonds with larger maturities are usually less volatile compared to shorter ones.

Term structures



A typical shape of the term structure in some time. Bonds with larger maturities often have higher interest rates.

Term structures



Real interbank offer rates. Bonds with larger maturities need not have higher interest rates. (Previous three plots are courtesy of Stehlikova, 2004).

What are Interest rate models?

Interest rate models describe the bond prices (or yields) as a function of time to maturity, state variables like e.g. instantaneous interest rate as well as several model parameters.

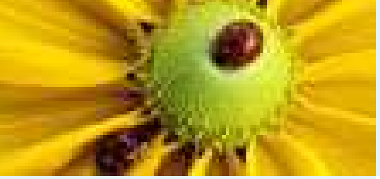
Variety of models available

- Vasicek (1977)
- Cox, Ingersoll, and Ross (1985)
- Ho and Lee (1986)
- Brennan and Schwartz (1982)
- Hull and White (1990)

and many others ...

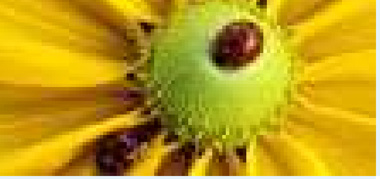
Why to calibrate Interest rate models?

- estimated model parameters provide better insight in analyzed term structures
- enable us to perform qualitative analysis of data and explain some unobserved phenomena like e.g. identification of market price of risk, long term interest rates etc



Historical overview

- Cox, Ingersoll, and Ross (1985) derived the interest rate model from basic assumption made on the form of the instantaneous (short rate) interest rate.
- Brown and Dybvig (1986) applied the CIR model in order to interpret nominal bond prices
- Chan, Karolyi, Longstaff and Sanders (1992) applied the Generalized Method of Moments to identify model parameters
- Pearson and Sun (1994) have shown failure of calibration of a two-factor extension of the CIR model applied to portfolios of US Treasury bills
- Nowman, Saltoglu (2003) applied Gaussian estimation methods of Nowman (1997) to continuous time interest rate models

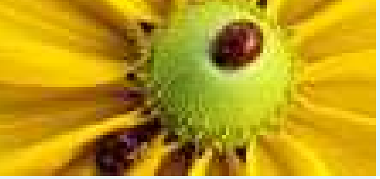


Historical overview

- Takahashi and Sato (2001) developed new methodology for estimation of general class of term structure models based on a Monte Carlo filtering approach. The method was applied to LIBORs and interest rates swaps in the Japanese market
- Modern calibration methods are based on other interest rate derivatives like e.g. prices of caps and floors have been proposed by Rebonato (1999) Such derivatives are still not available in most transitional countries
- Vojtek (2003) applied the Brace, Gatarek, Musiela model (1997) and various types of GARCH models in order to calibrate interest rate models for term structures transitional economies like Central European countries including Czech republic, Slovakia, Poland and Hungary

Content

- Cox, Ingersoll, and Ross interest model (CIR)
- New two phase minmax calibration method based on CIR model
- Evolution strategies used as optimization methods
- Calibration of Central European Interbank offer rates and comparison to stable Western Europe term structures
- Conclusions

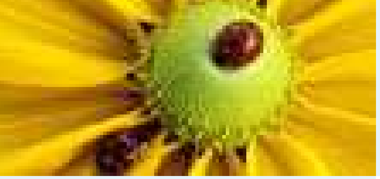


CIR interest rate model

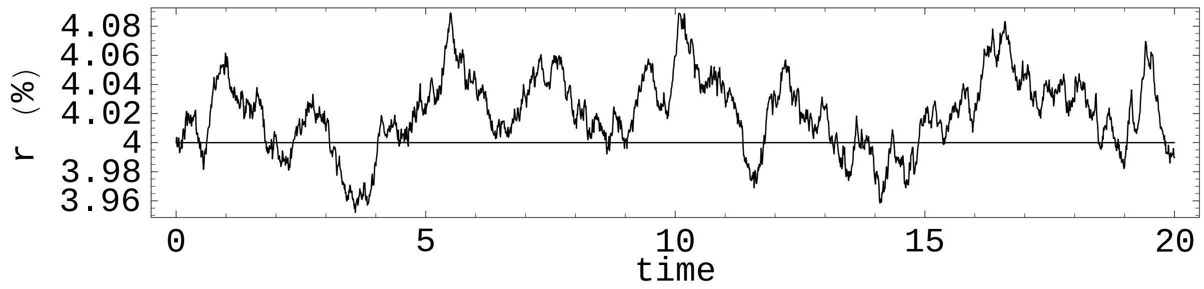
The Cox-Ingersoll-Ross interest rate model is derived from a basic assumption made on the form of the stochastic process driving the instantaneous interest rate $r_t, t \in [0,T]$.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dw_t$$

where $\{w_t, t \geq 0\}$, denotes the standard Wiener process. Positive constants κ, θ and σ denote the adjustment speed of reversion, the long term interest rate and volatility factor of the process, respectively.



CIR interest rate model

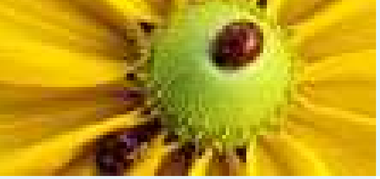


Simulation of a mean reverting process driven by the Orstein-Uhlenbeck mean-reverting process with parameters: $\kappa = 1, \sigma = 0.2, \theta = 0.04$ within the time interval $(0, 20)$.

Time discretized version of the mean reverting equation has the form:

$$r_{t+1} - r_t = \kappa(\theta - r_t) + \sigma\sqrt{r_t}\varepsilon_t$$

with $\varepsilon_t \approx N(0,1)$ is normally distributed random variable.



CIR interest rate model

In the CIR theory the price $P = P(t, T, r)$ of a zero coupon bond is assumed to be a function of the present time $t \in [0, T]$, expiration time $T > 0$ and the present value of the short rate interest rate $r = r_t$.

$$P = e^{-R(t,T)(T-t)}$$

where $R(t, T)$ is the yield on the bond at time t with maturity T, $r_t = R(t, t)$.

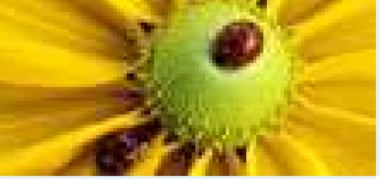
CIR interest rate model

Constructing a risk-less portfolio containing two bonds with different maturities and applying stochastic differential calculus (Itô lemma) we can conclude that the price of the zero coupon bond $P = P(t, T, r)$ satisfies a backward parabolic partial differential equation

$$\frac{\partial P}{\partial t} + (\kappa(\theta - r) - \lambda r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} - rP = 0\,, \quad t \in (0, T)\,,\ r > 0\,.$$

subject to the terminal condition $P(T, T, r) = 1$ for any $r > 0$.

The parameter $\lambda \in \mathbb{R}$ represents the so-called market price of risk.



CIR model has an explicit solution!

$$P(T-\tau, T, r) = A(\tau)e^{-B(\tau)r}, \quad \tau = T - t \in [0, T],$$

where

$$B(\tau) = \frac{2(e^{\eta\tau}-1)}{(\kappa+\lambda+\eta)(e^{\eta\tau}-1)+2\eta}, \quad A(\tau) = \left(\frac{\eta e^{(\kappa+\lambda+\eta)\tau/2}}{e^{\eta\tau}-1}B(\tau)\right)^{\frac{2\kappa\theta}{\sigma^2}},$$

and $\eta = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$.

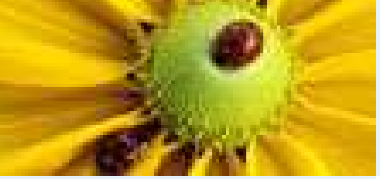
Four parameters $\kappa, \sigma, \theta, \lambda$ enter analytic formula

CIR interest rate model

As it has been already pointed by Pearson and Sun (1994) the adjustment speed κ and the risk premium λ enter the analytic formula only in summation

$$\kappa + \lambda$$

$\Rightarrow$ four CIR parameters can be reduced to three essential parameters fully describing the behavior of the functions A, B and thus the whole term structure



Three parameters reduction

by introducing the following new parameters:

1. $\beta = e^{-\eta} \quad \in [0,1]$
2. $\xi = \frac{\kappa+\lambda+\eta}{2\eta} \quad \in [0,1]$
3. $\varrho = \frac{2\kappa\theta}{\sigma^2} \quad \in \mathbb{R}$

where $\eta = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$.

Summarizing, in the CIR model the price of bonds and, consequently, the corresponding yield curve depend only on three transformed parameters β, ξ and ϱ.

Two-phase minmax method

1. we identify one dimensional curve of the CIR parameters by minimizing the cost functional that mimics least squares approach in linear regression methods. The global minimum is attained at a unique point $\check{\beta}, \check{\xi}, \check{\varrho}$ of transformed vars $\Rightarrow$ on a one dimensional curve of genuine CIR parameters $\kappa, \sigma, \theta, \lambda$
2. maximization of the likelihood function restricted to that one dimensional curve of global minimizers

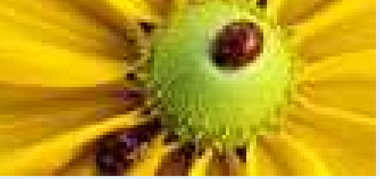
The restricted maximum likelihood is attained in a unique point - desired estimation of the CIR model parameters.

Nonlinear least square minimization

We minimize the weighted least square sum of differences between real market yield curve interest rates and those predicted by CIR model.

$$U(\beta, \xi, \varrho) = \frac{1}{m} \sum_{j=1}^{m} \frac{1}{n} \sum_{i=1}^{n} (R_j^i - \bar{R}_j^i)^2 \tau_j^2$$

- $\{R_j^i, j = 1, ..., m\}$ is the yield curve of the length m at time $i = 1, ..., n$, with time τ_j to maturity.
- $\{\bar{R}_j^i, j = 1, ..., m\}$ is the yield curve computed by using the CIR model with parameters β, ξ, ϱ.
- The over-night (short-rate) interest rate at time $i = 1, ..., n$, is denoted by R_0^i.



Nonlinear least square minimization

values $\bar{R}^i_j$ can be calculated from the bond price - yield curve relationship:

$$A_j e^{-B_j R^i_0} = P = e^{-\bar{R}^i_j \tau_j} .$$

$\Rightarrow \bar{R}^i_j \tau_j = B_j R^i_0 - \ln A_j$ and the cost functional U can be rewritten as

$$U(\beta, \xi, \varrho) = \frac{1}{m} \sum_{j=1}^{m} \left((\tau_j E(R_j) - B_j E(R_0) + \ln A_j)^2 + D(\tau_j R_j - B_j R_0) \right.$$

Nonlinear least square minimization

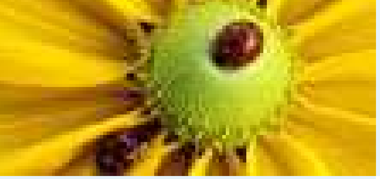
the cost function U attains a unique global minimum

$$U(\check\beta, \check\xi, \check\varrho) = \min_{(\beta,\xi,\varrho)\in\Omega} U(\beta, \xi, \varrho)\,.$$

In terms of genuine CIR parameters:

$$(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda) \quad \leftrightarrow \quad (\check\beta, \check\xi, \check\varrho)$$

the global minimum is attained on a λ-parameterized curve of global minimizers $(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda),\ \lambda \in \mathbb{R}$.



Nonlinear regression problem!

- the function $U = U(\beta, \xi, \varrho)$ is nonlinear and it is not know whether is convex or not
- standard steepest descent gradient minimization method of Newton-Kantorovich type fail to converge to a global minimum of U
- we need a robust and efficient numerical method unconditionally converging to a unique global minimum of U

$\Rightarrow$ we had to apple a variant of the evolution strategy (ES) globally converging algorithm

Evolution strategy (ES) is the one of the most successful stochastic algorithm which was invented to solve technical optimization problems (see e.g. Schwefel (1977, 1995, 1998).

Restricted likelihood function

Gaussian estimation of parameters of the mean reverting process driving the short rate r_t

$$\ln L(\kappa,\sigma,\theta) = -\frac{1}{2}\sum_{t=2}^{n}\left(\ln v_t^2 + \frac{\varepsilon_t^2}{v_t^2}\right)$$

$$v_t^2 = \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa}\right) r_{t-1}, \qquad \varepsilon_t = r_t - e^{-\kappa} r_{t-1} - \theta\left(1-e^{-\kappa}\right).$$

Combining information gained from minimization of the cost functional U and likelihood parameter estimation $\Rightarrow$ restricted maximum likelihood function $\Rightarrow$ maximize $\ln L$ over a one dimensional curve $(\kappa_\lambda, \sigma_\lambda, \theta_\lambda, \lambda), \ \lambda \in \mathbb{R}$.

Restricted likelihood function

The argument $(\kappa_{\bar{\lambda}}, \sigma_{\bar{\lambda}}, \theta_{\bar{\lambda}})$ of the restricted likelihood function

$$\ln L^r = \ln L(\kappa_{\bar{\lambda}}, \sigma_{\bar{\lambda}}, \theta_{\bar{\lambda}}) = \max_{\lambda} \ln L(\kappa_\lambda, \sigma_\lambda, \theta_\lambda)\,.$$

is now identified the resulting optimal values as estimated CIR

parameters obtained by the two-phase minmax optimization method.

Restricted maximum likelihood ratio

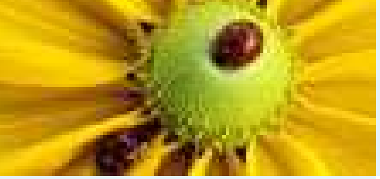
The ratio between the unrestricted maximum likelihood

$$\ln L^u = \ln L(\kappa^u, \sigma^u, \theta^u) = \max_{\kappa,\sigma,\theta>0} \ln L(\kappa, \sigma, \theta)\,.$$

and the restricted maximum likelihood $\ln L^r$ is called the maximum likelihood ratio (MLR)

$$MLR = \frac{\ln L^r}{\ln L^u}\,.$$

- $MLR \leq 1$
- value of MLR close to 1 indicates that the restricted maximum likelihood value $\ln L^r$ is close to unrestricted one $\Rightarrow$ simple estimation of parameters of mean reversion equation is also suitable for calibration of the whole term structure



Nonlinear R^2 ratio

We introduce the nonlinear R^2 ratio as follows:

$$R^2 = 1 - \frac{U(\check{\beta}, \check{\xi}, \check{\rho})}{U(1,1,1)}$$

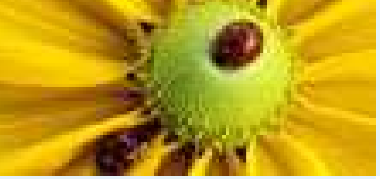
where $(\check{\beta}, \check{\xi}, \check{\rho})$ is the argument of the unique global minimum of the cost functional U

- $0 \le R^2 \le 1$
- a value of R^2 close to one indicate that $U(\check{\beta}, \check{\xi}, \check{\rho}) \ll U(1,1,1)$
  $\Rightarrow$ the cost functional U is close to zero
  $\Rightarrow$ almost perfect matching of the CIR yield curve computed for parameters $(\check{\beta}, \check{\xi}, \check{\rho})$ and that of the given real market data set.

Calibration results

We present results of the two-phase calibration for terms structures for various Central European countries and comparison to stable western European financial markets.

- EURIBOR - Euro-zone
- London inter-bank offer rates EURO-LIBOR and USD-LIBOR
- Slovakia (BRIBOR),
- Hungary (BUBOR)
- Czech republic (PRIBOR)
- Poland (WIBOR)



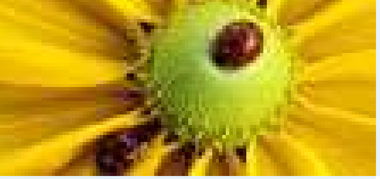
Term structures description

- The data for overnight interest rate are available for BRIBOR, PRIBOR, WIBOR, BUBOR as well as for London Inter-bank Offer Rates (LIBOR).
- For the Euro-zone term structure EURIBOR is concerned we recall that there exists a substitute for the overnight rate which is called EONIA.

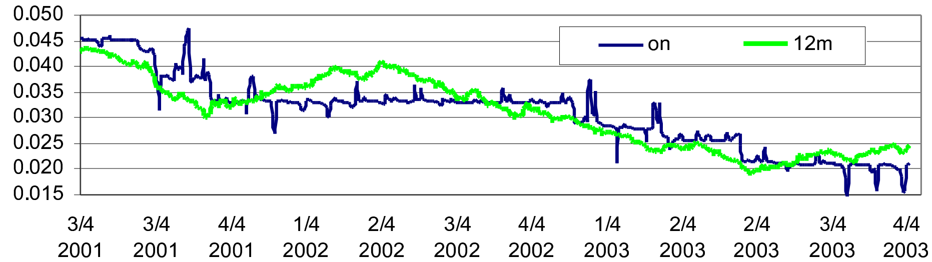
Calibration covers the period from the year 2001 up to 2003 and is done on quarterly basis

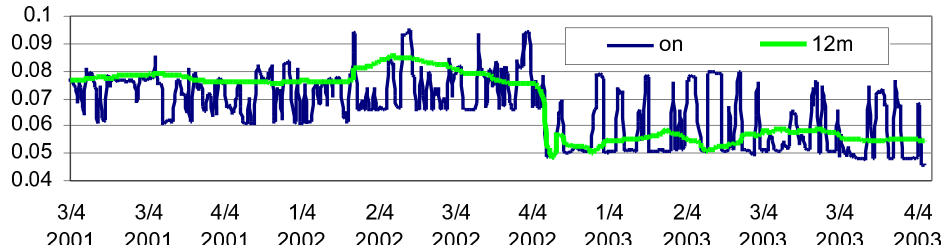
Term structures description

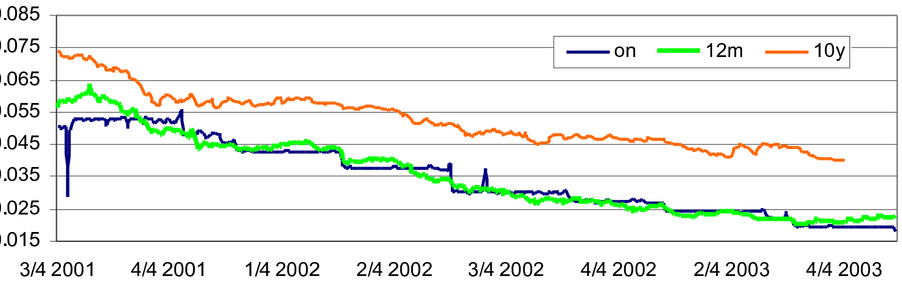
- EURO-LIBOR and USD-LIBOR contains bonds with maturities: 1 week, 1 up to 12 month, i.e. its length is $m = 13$.
- EURIBOR, in addition, contains 2 and 3 weeks maturities, $m = 15$.
- BRIBOR contains 1, 2 weeks and 1, 2, 3, 6, 9, 12 months maturities
- BUBOR, PRIBOR, and WIBOR contain maturities: 1 and 2 weeks, 1, 2, 3, 6, 9, 12 months, and then each quarter up to 10 years maturity



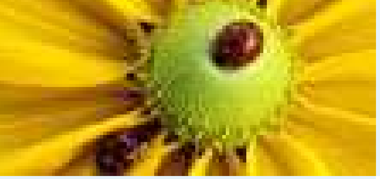
Term structures description



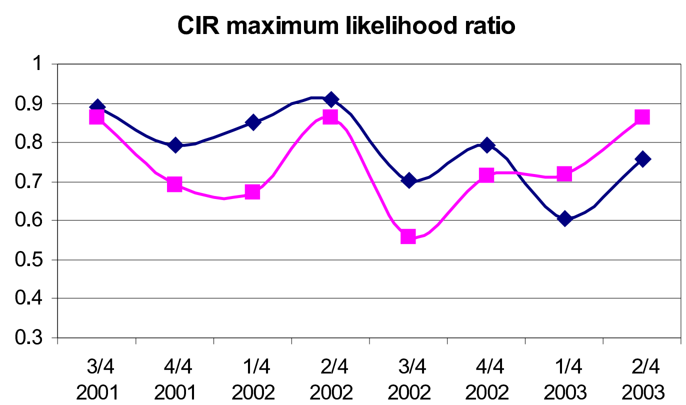


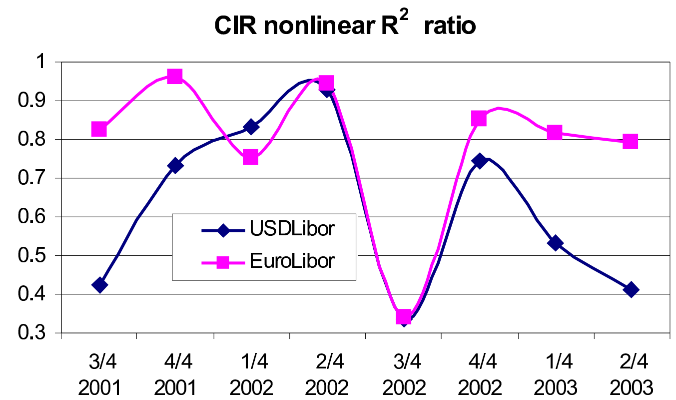


Graphical description of overnight (short-rate) interest rates and those of bond with longer maturity. Daily data are plotted for EURO-LIBOR, BRIBOR and PRIBOR.

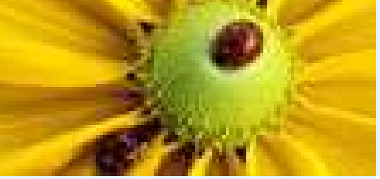


Calibration results





Results of calibration for term structures. Maximum likelihood ratio (a) and R^2 ratio (b) for EURO-LIBOR and USD-LIBOR.



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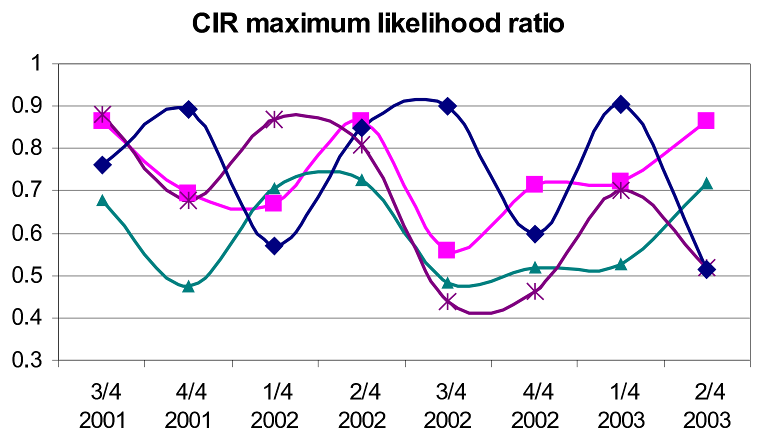
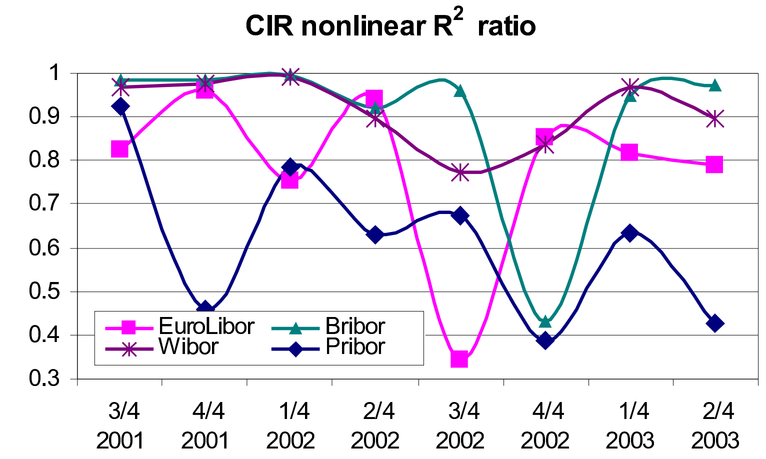
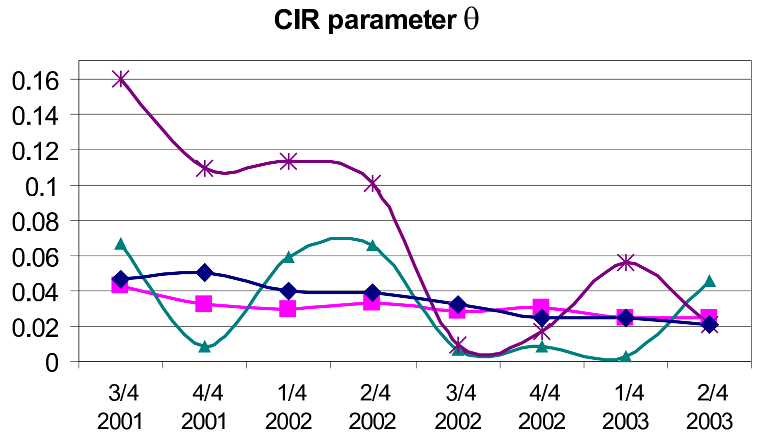
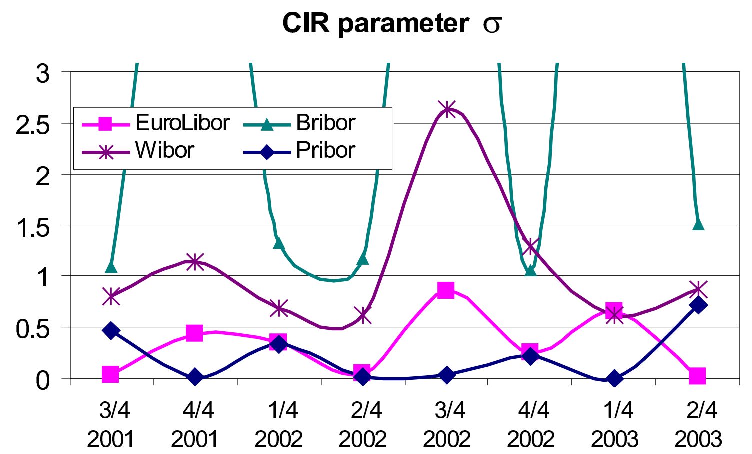
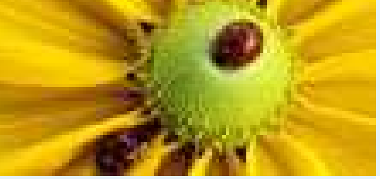








Maximum likelihood ratio and R^2 ratio for EURO-LIBOR and comparison to BRIBOR, PRIBOR and WIBOR. Estimated parameters θ and σ (below).



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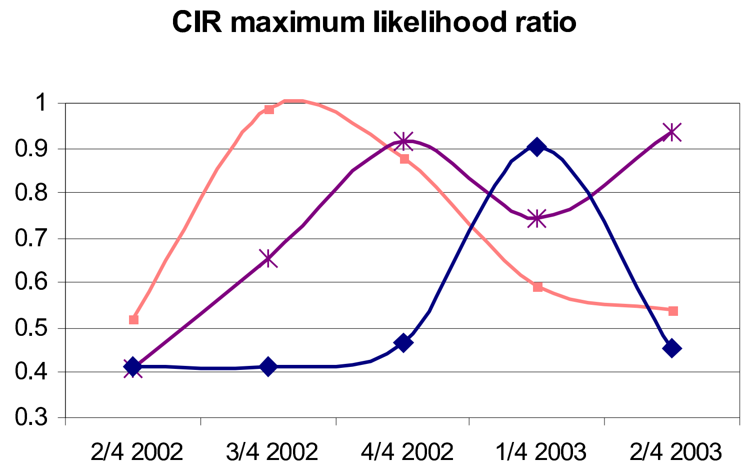
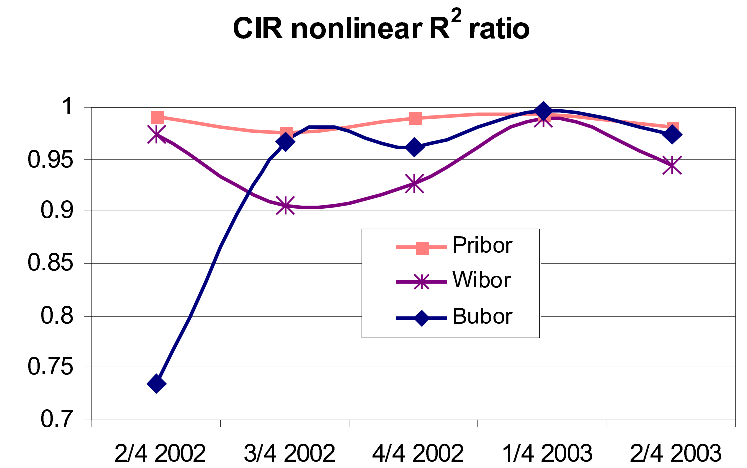
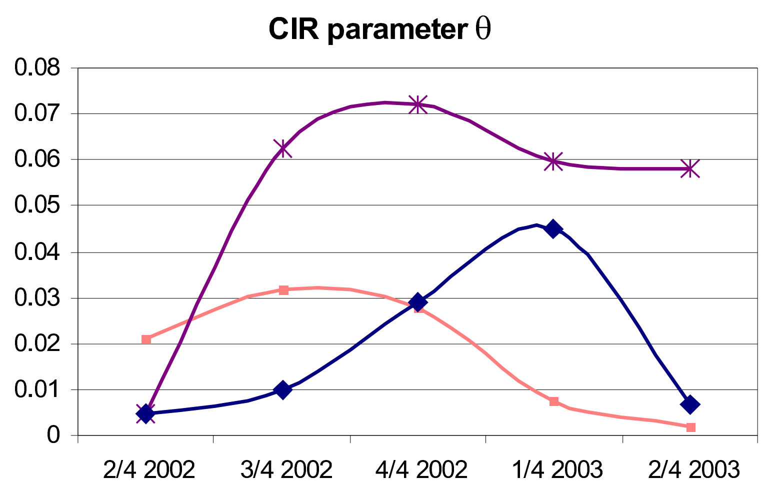
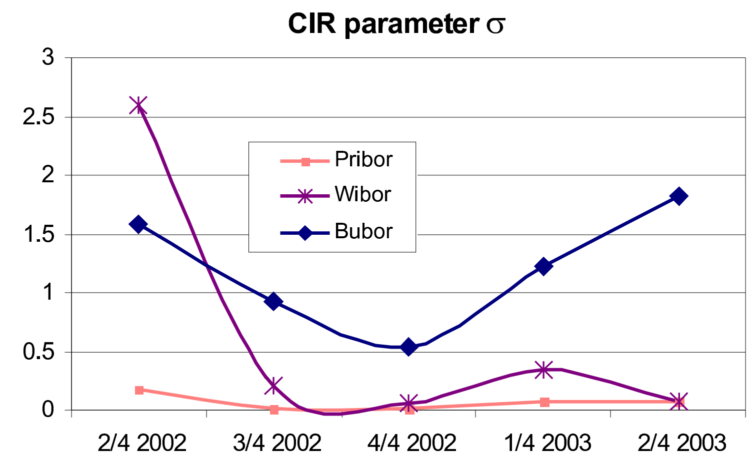








Results of calibration for term structures with maturities up to 10 years for BRIBOR, WIBOR and BUBOR.

Risk premium analysis

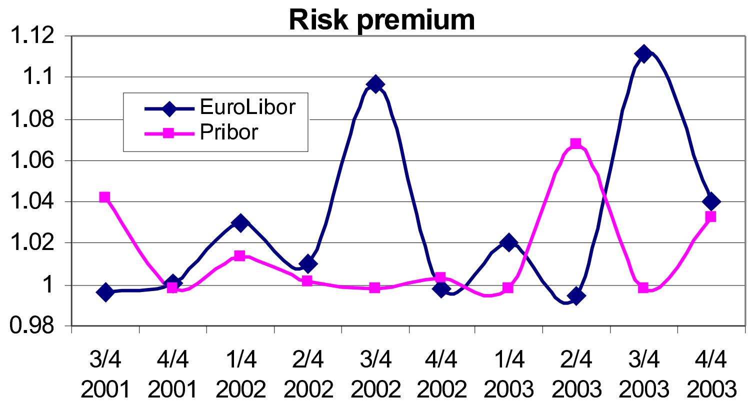
Recall that the parameter λ represents the market price of risk in the CIR model

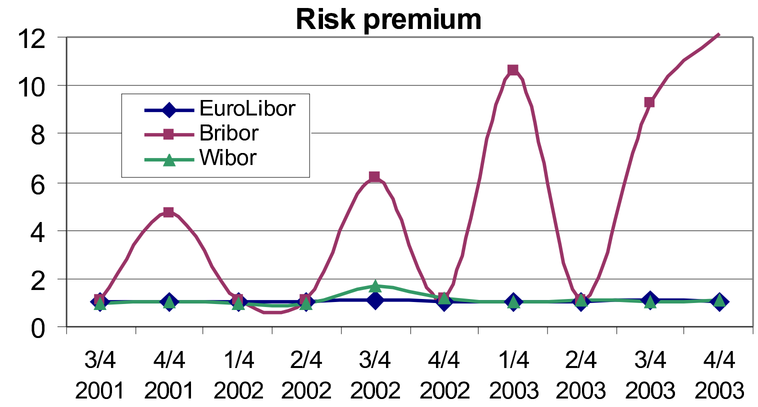
- The bond price $P = P(t, T, r)$ satisfying PDE is given by formula $P(t, T, r) = A(T-t)e^{-B(T-t)r}$. Thus $\partial_r P = -BP$.
- Partial differential equation can be rewritten as

$$\frac{\partial P}{\partial t} + \kappa(\theta - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} - r^* P = 0\,, \quad t \in (0,T)\,,\ r > 0$$

  where $r^* = (1 - \lambda B)r$.
- Multiplier $1 - \lambda B$ can interpreted as the risk premium factor and r^* as the expected rate of return on the bond.
- We have $r^* > r$ iff $\lambda < 0$

Calibration results

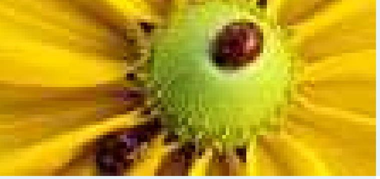




Comparison of risk premium factors $1 - \lambda B_1$ for EURO-LIBOR and BRIBOR (left) and EURO-LIBOR, WIBOR and BRIBOR (right).

Conclusions

- We have proposed a two-phase minmax optimization method for calibration of the CIR one factor interest rate model. It is based on minimization of the cost functional together with maximization of the likelihood function restricted to the set of minimizers.
- We have tested the calibration method various term structures including both stable western inter-bank offer rates as well as those of emerging economies.
- Based on our results of calibration of the CIR one-factor model we can state that the western European term structure data are better described with CIR model compared to transitional economies represented by Central European countries

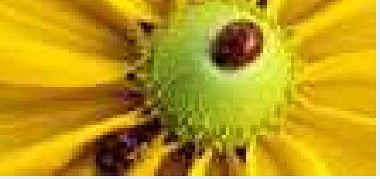


Conclusions

- The CIR model can applied to EURO-LIBOR, USD-LIBOR and EURIBOR. Interestingly, to some extent, it could be also used for Czech PRIBOR term structure
- On the other hand, we can observe, at least partial, failure of the CIR model for other Central European term structures.

The paper as well as presentation and other material can be downloaded from:

http://www.iam.fmph.uniba.sk/institute/sevcovic



Appendix

Term structure descriptive statistics

Tabeled calibration results