ARMA modely
časť 2: moving average modely (MA)

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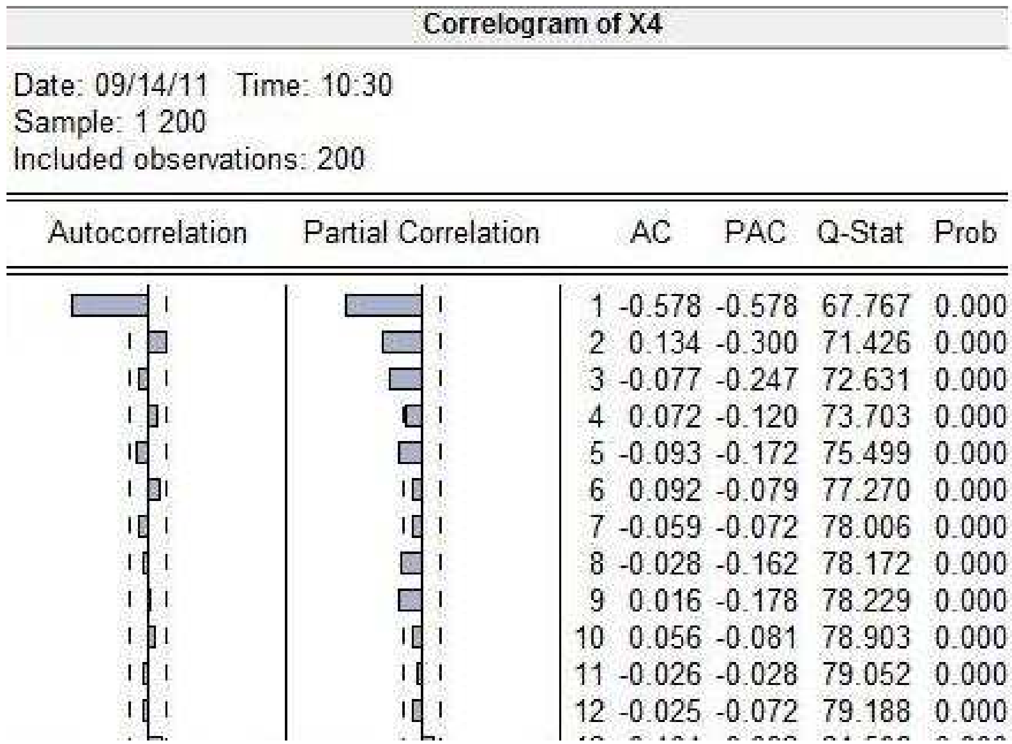
Časové rady, FMFI UK, 2011/2012

V.

Moving average proces prvého rádu - MA(1)

Simulované dáta z minulej prednášky



Correlogram of X4

Date: 09/14/11 Time: 10:30
Sample: 1 200
Included observations: 200

	AC	PAC	Q-Stat	Prob
1	-0.578	-0.578	67.767	0.000
2	0.134	-0.300	71.426	0.000
3	-0.077	-0.247	72.631	0.000
4	0.072	-0.120	73.703	0.000
5	-0.093	-0.172	75.499	0.000
6	0.092	-0.079	77.270	0.000
7	-0.059	-0.072	78.006	0.000
8	-0.028	-0.162	78.172	0.000
9	0.016	-0.178	78.229	0.000
10	0.056	-0.081	78.903	0.000
11	-0.026	-0.028	79.052	0.000
12	-0.025	-0.072	79.188	0.000

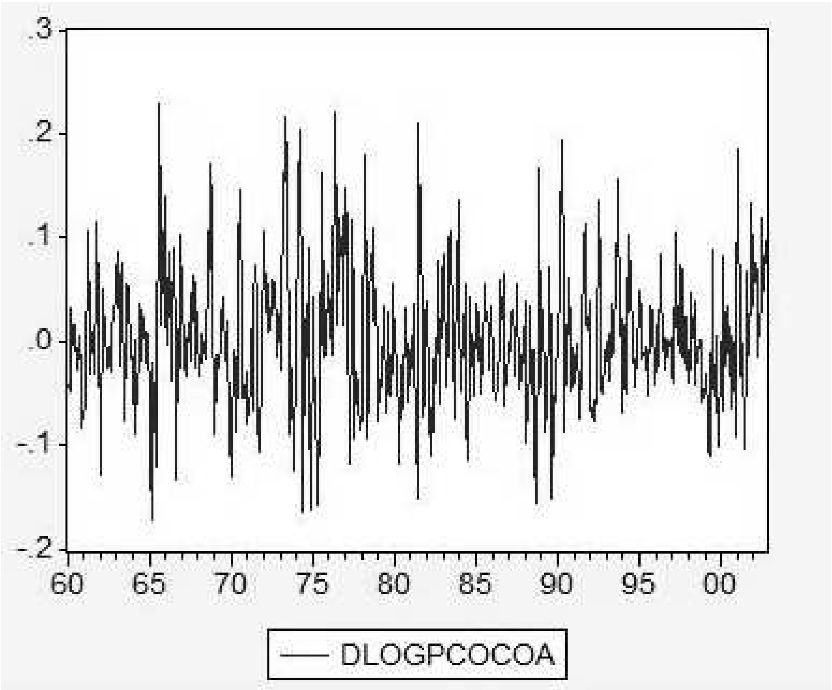
- AR(p) proces - ACF rýchlo klesá (monotónne alebo oscilujúco), PACF sa po p hodnotách rovná nule
- tu je to v podstate naopak → nebude to AR proces

Reálne dáta s podobnou ACF

Ben Vogelvang: **Econometrics. Theory and Applications with EViews.** Pearson Education Limited, 2005.

Chapter 14.7. - The Box-Jenkins Approach in Practice

- Mesačné dáta, január 1960 - september 2002
- $pcocoa_t$ - cena kakaa, zlogaritmujeme a kvôli stacionarite budeme pracovat' s diferenciami



Reálne dáta - pokračovanie

- Odhadnutá ACF:

Correlogram of LOGPCOCOA

Sample: 1960M01 2002M09
Included observations: 513

	AC
1	0.990
2	0.978
3	0.966
4	0.955
5	0.944
6	0.934
7	0.924
8	0.915
9	0.905
10	0.894
11	0.882

Correlogram of D(LOGPCOCOA)

Sample: 1960M01 2002M09
Included observations: 512

	AC	PAC	Q-Stat	Prob
1	0.313	0.313	50.303	0.000
2	-0.012	-0.122	50.382	0.000
3	-0.007	0.040	50.405	0.000
4	0.010	-0.003	50.452	0.000
5	-0.035	-0.043	51.101	0.000
6	0.000	0.032	51.101	0.000
7	-0.043	-0.067	52.087	0.000
8	0.017	0.062	52.235	0.000
9	0.071	0.045	54.862	0.000
10	0.097	0.066	59.751	0.000
11	0.018	-0.026	59.930	0.000
12	-0.012	-0.007	60.012	0.000
13	-0.021	-0.013	60.247	0.000
14	-0.033	-0.030	60.840	0.000

- Logaritmy - ACF klesá vel'mi pomaly → typické pre dáta, ktoré treba zdiferencovat'
- Diferencie logaritmov - jedna výrazne nenulová autokorelácia, ostatné skoro nulové

Príklad z prvej prednášky

- Nech u_t je biely šum, definujeme

$$x_t = u_t + u_{t-1}$$

- Vypočítali sme:

$$E[x_t] = 0, \quad Var[x_t] = 2\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = \begin{cases} \sigma^2 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} 1/2 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

- ACF je nulová pre $\tau = 2, 3, \ldots$ - presne tá vlastnosť, ktorú potrebujeme

Zovšeobecnenie - MA(1) proces

- Nech u_t je biely šum, potom

$$x_t = \mu + u_t - \beta u_{t-1}$$

  sa nazýva moving average proces prvého rádu - MA(1)

- Woldova reprezentácia: $x_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}$
  MA(1) proces: $\psi_0 = 1, \psi_1 = -\beta, \psi_j = 0$ pre $j = 2, 3, \ldots$

- Momenty a ACF:

$$E[x_t] = \mu, \quad Var[x_t] = (1 + \beta^2)\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = \begin{cases} -\beta\sigma^2 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} -\frac{\beta}{1+\beta^2} & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \ldots \end{cases}$$

MA(1) proces - príklady

1. Nech u_t je biely šum s rozdelením $N(0,4)$, definujme

$$x_t = u_t + \frac{1}{2}u_{t-1}$$

Potom: $E[x_t] = 0, \quad Var[x_t] = (1 + (1/2)^2) \times 4 = 5$

$$Cor[x_t, x_{t+\tau}] = \begin{cases} \frac{1/2}{1+1/4} = 2/5 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \dots \end{cases}$$

2. Nech u_t je biely šum s rozdelením $N(0,1)$, definujme

$$y_t = u_t + 2u_{t-1}$$

Potom: $E[y_t] = 0, \quad Var[y_t] = (1 + 4) \times 1 = 5$

$$Cor[y_t, y_{t+\tau}] = \begin{cases} \frac{2}{1+4} = 2/5 & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \dots \end{cases}$$

Procesy x_t a y_t majú rovnakú ACF $\rightarrow$ nedajú sa rozlíšiť

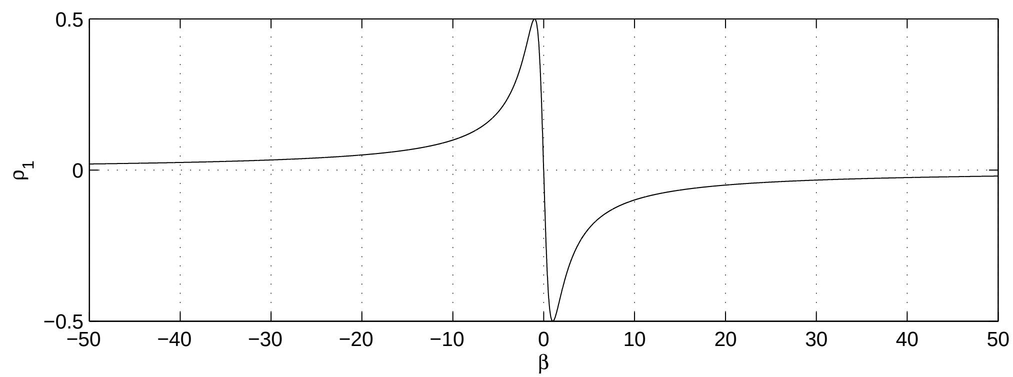
MA(1) proces - zovšeobecnenie príkladu

- Majme MA(1) proces, t.j. ACF tvaru

$$Cor[x_t, x_{t+\tau}] = \begin{cases} -\frac{\beta}{1+\beta^2} & \text{pre } \tau = 1 \\ 0 & \text{pre } \tau = 2, 3, \dots \end{cases}$$

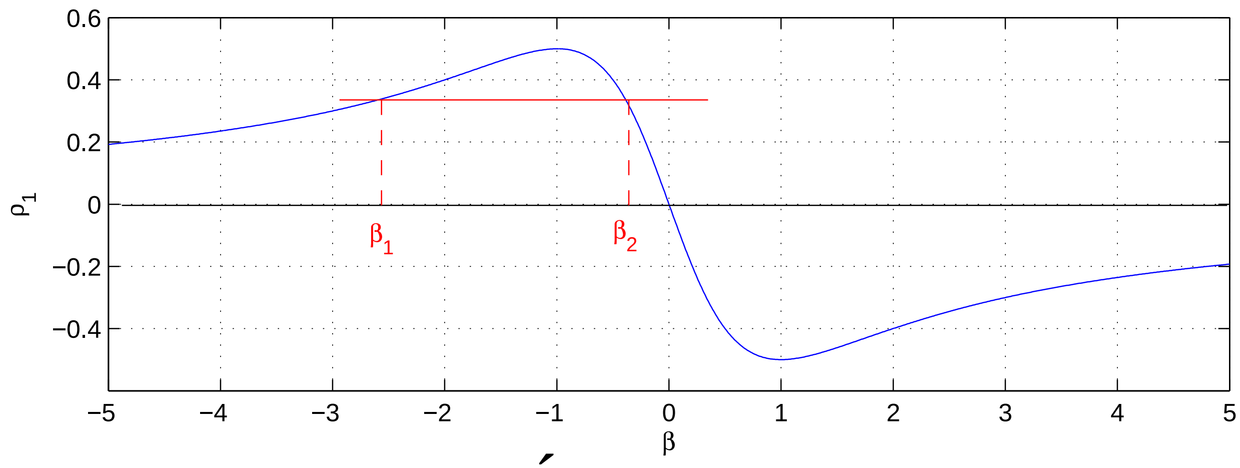
- Predpokladajme teraz, že máme danú hodnotu $\rho_1 = \rho(1)$ a chceme z nej spätne určiť koeficient β, t.j.

$$\rho_1 = -\frac{\beta}{1+\beta^2} \Rightarrow \beta = ?$$



MA(1) proces - zovšeobecnenie príkladu

- Máme teda rovnicu: $\rho_1 = -\frac{\beta}{1+\beta^2} \Rightarrow \beta^2 + \frac{1}{\rho_1}\beta + 1 = 0$



$\longrightarrow$ dve riešenia β_1, β_2, spĺňajú $\beta_1\beta_2 = 1$.

- Procesy

$$x_t = \mu + u_t - \beta u_{t-1}, \quad x_t = \mu + u_t - \frac{1}{\beta}u_{t-1}$$

majú rovnakú ACF

- Ak chceme jednoznačnú parametrizáciu, potrebujeme dodať ďalšiu podmienku.

Invertovateľnosť procesu

- Budeme sa snažiť zapísať proces v tvare AR(∞):

$$x_t = \hat{\mu} + u_t + \psi_1 x_{t-1} + \psi_2 x_{t-2} + \psi_3 x_{t-3} + \dots$$

  - ak sa to dá spraviť, proces sa nazýva invertovateľný
- Pre MA(1) proces:

$$\begin{aligned} x_t &= \mu + (1 - \beta L) u_t \\ (1 - \beta L)^{-1} x_t &= (1 - \beta L)^{-1} \mu + u_t \end{aligned}$$

  $(1 - \beta L)^{-1}$ existuje pre $|\beta| < 1$, vtedy

$$(1 + \beta L + \beta^2 L^2 + \dots) x_t = \mu / (1 - \beta) + u_t$$

$$x_t + \beta x_{t-1} + \beta^2 x_{t-2} + \dots = \mu / (1 - \beta) + u_t$$

MA(1) - invertovateľnosť procesu

- Dostali sme teda podmienku invertovateľnosti MA(1) procesu: $|\beta| < 1$
- Iný zápis tejto podmienky:
  - máme proces $x_t = \mu + (1 - \beta L)u_t$
  - koreň polynómu $1 - \beta L$ je $1/\beta$
  - podmienka invertovateľnosti teda hovorí, že koreň $1 - \beta L = 0$ musí byť v absolútnej hodnote väčší ako 1, teda mimo jednotkového kruhu

MA(1) - výpočet PACF

- Pripomeňme si všeobecný vzorec;

$$\Phi_{kk} = \frac{\det \begin{pmatrix} 1 & \rho(1) & \dots & \rho(1) \\ \rho(1) & 1 & \dots & \rho(2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & \rho(k) \end{pmatrix}}{\det \begin{pmatrix} 1 & \rho(1) & \dots & \rho(k-1) \\ \rho(1) & 1 & \dots & \rho(k-2) \\ & \dots & \dots & \\ \rho(k-1) & \rho(k-2) & \dots & 1 \end{pmatrix}} \qquad (1)$$

- Pre MA(1) proces je $\rho(k) = 0$ pre $k = 2, 3, \dots$

MA(1) - výpočet PACF

- PACF sa (na rozdiel od AR procesu) nevynuluje:

$$\Phi_{11} = \rho(1)$$

$$\Phi_{22} = \frac{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}} = \frac{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}} = \frac{-\rho(1)^2}{1-\rho(1)^2}$$

$$\Phi_{33} = \frac{\det\begin{pmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix}} = \frac{\det\begin{pmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{pmatrix}} = \frac{\rho(1)^3}{1-2\rho(1)}$$

$$\Phi_4 = \frac{-\rho(1)^4}{(1-\rho(1)^2)^2-\rho(1)^2}$$

$$\ldots$$

Reálne dáta - ceny kakaa

- Dáta zo začiatku prednášky
- MA(1) model pre diferencie logaritmov cien:

Dependent Variable: D(LOG(PCOCOA))
Method: Least Squares
Date: 08/14/09 Time: 14:51
Sample (adjusted): 1960M02 2002M09
Included observations: 512 after adjustments
Convergence achieved after 6 iterations
MA Backcast: 1960M01

	Coefficient	Std. Error	t-Statistic	Prob.
C	0.002454	0.003738	0.656614	0.5117
MA(1)	0.352651	0.041550	8.487292	0.0000

R-squared	0.112273	Mean dependent var	0.002398
Adjusted R-squared	0.110532	S.D. dependent var	0.066320
S.E. of regression	0.062548	Akaike info criterion	-2.701880
Sum squared resid	1.995221	Schwarz criterion	-2.685324
Log likelihood	693.6813	Hannan-Quinn criter.	-2.695390
F-statistic	64.50096	Durbin-Watson stat	1.987519
Prob(F-statistic)	0.000000		

Inverted MA Roots	-.35

Reálne dáta - ceny kakaa

- Invertovatel'nost':

MA Root(s)	Modulus	Cycle
-0.352651	0.352651	

No root lies outside the unit circle.
ARMA model is invertible.

- ACF rezíduí:

Correlogram of Residuals

Sample: 1960M02 2002M09
Included observations: 512
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.003	0.003	0.0055	
		2	-0.008	-0.008	0.0371	0.847
		3	-0.014	-0.014	0.1382	0.933
		4	0.032	0.032	0.6730	0.880
		5	-0.057	-0.057	2.3587	0.670
		6	0.037	0.038	3.0731	0.689
		7	-0.062	-0.063	5.0529	0.537
		8	0.026	0.025	5.3941	0.612
		9	0.034	0.037	6.0032	0.647
		10	0.089	0.083	10.181	0.336
		11	-0.004	0.004	10.190	0.424
		12	-0.008	-0.015	10.224	0.510
		13	-0.015	-0.008	10.336	0.586
		14	-0.016	-0.024	10.471	0.655

VI.

Moving average proces q-teho rádu - MA(q)

MA(q) proces - definícia a vlastnosti

- Nech u_t je biely šum, potom

$$x_t = \mu + u_t - \beta_1 u_{t-1} - \beta_2 u_{t-2} - \ldots - \beta_q u_{t-q}$$

sa nazýva moving average proces q-teho rádu - MA(q)

- Woldova reprezentácia: $x_t = \mu + \sum_{j=0}^{\infty} \psi_j u_{t-j}$
MA(q) proces: $\psi_0 = 1, \psi_1 = -\beta_1, \ldots \psi_q = -\beta_q, \psi_j = 0$ pre $j > q$ $\rightarrow$ MA(q) proces je vždy stacionárny
- Momenty, ACF, PACF:

$$E[x_t] = \mu, \;\; Var[x_t] = (1 + \beta_1^2 + \ldots \beta_q^2)\sigma^2$$

$$Cov[x_t, x_{t+\tau}] = 0 \text{ pre } \tau = q+1, q+2, \ldots$$

$$\Rightarrow Cor[x_t, x_{t+\tau}] = 0 \text{ pre } \tau = q+1, q+2, \ldots$$

MA(q) proces - definícia a vlastnosti

- Výpočet prvých q autokorelácií (môžeme uvažovať $\mu = 0$):

$$
\begin{aligned}
Cov[x_t, x_{t+\tau}] &= E[(u_t - \beta_1 u_{t-1} - \ldots - \beta_q u_{t-q}) \times \\
&\quad (u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})]
\end{aligned}
$$

$$
\begin{aligned}
= \; & E[u_t(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})] \\
& -\beta_1 E[u_{t-1}(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})] \\
& \ldots \\
& -\beta_q E[u_{t-q}(u_{t+\tau} - \beta_1 u_{t+\tau-1} - \ldots - \beta_q u_{t+\tau-q})]
\end{aligned}
$$

MA(q) proces - definícia a vlastnosti

- Výpočet prvých q autokorelácií - pokračovanie:

$$\begin{aligned}
\tau = 1 \quad &\Rightarrow \quad \gamma(1) = (-\beta_1 + \beta_1\beta_2 + \ldots + \beta_{q-1}\beta_q)\sigma^2 \\
\tau = 2 \quad &\Rightarrow \quad \gamma(2) = (-\beta_2 + \beta_1\beta_3 + \ldots + \beta_{q-2}\beta_q)\sigma^2 \\
& \quad\quad \ldots \\
\tau = q \quad &\Rightarrow \quad \gamma(q) = (-\beta_q)\sigma^2
\end{aligned}$$

- PACF - dosadením vypočítaných autokorelácií do všeobecného vzorca (1)

MA(q) proces - definícia a vlastnosti

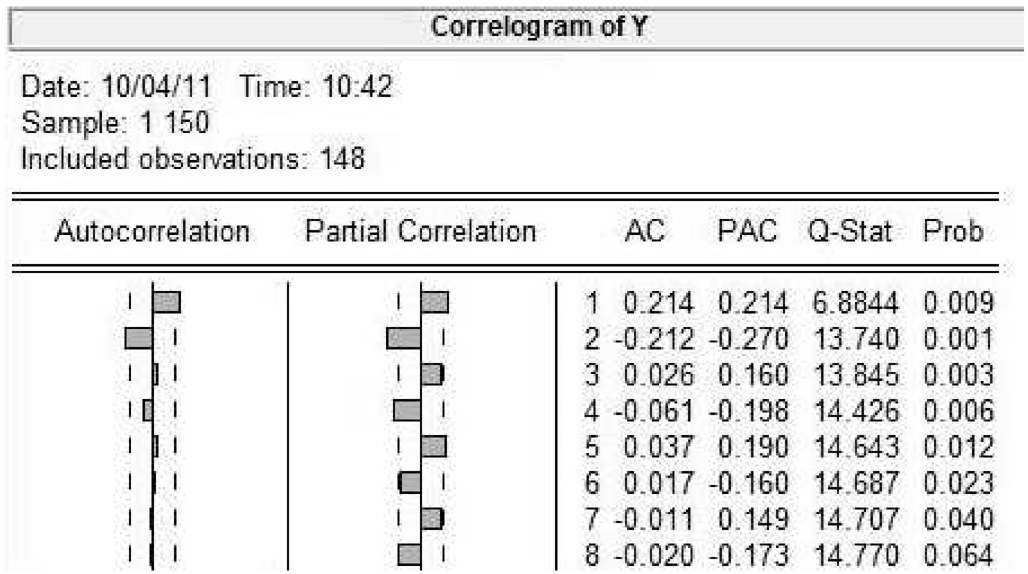
- Invertovateľnosť:

$$
\begin{aligned}
x_t &= \mu + u_t - \beta_1 u_{t-1} - \beta_2 u_{t-2} - \ldots - \beta_q u_{t-q} \\
x_t &= \mu + (1 - \beta_1 L - \ldots - \beta_q L^q) u_t
\end{aligned}
$$

- Existencia inverzného operátora $(1 - \beta_1 L - \ldots - \beta_q L^q)^{-1}$ - korene polynómu $1 - \beta_1 L - \ldots - \beta_q L^q = 0$ musia byť mimo jednotkového kruhu

Cvičenie : simulované dáta

- Výberová ACF a PACF z dát:



Correlogram of Y

Date: 10/04/11 Time: 10:42
Sample: 1 150
Included observations: 148

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.214	0.214	6.8844	0.009
		2	-0.212	-0.270	13.740	0.001
		3	0.026	0.160	13.845	0.003
		4	-0.061	-0.198	14.426	0.006
		5	0.037	0.190	14.643	0.012
		6	0.017	-0.160	14.687	0.023
		7	-0.011	0.149	14.707	0.040
		8	-0.020	-0.173	14.770	0.064

- PACF nie je po nejakom počte členov nulová $\rightarrow$ nebude to AR proces
- ACF má prvé dve hodnoty výraznejšie, ostatné skoro nulové $\rightarrow$ skúsime odhadnúť MA(2) proces

Cvičenie : simulované dáta

- Odhadnutý MA(2) proces:

Dependent Variable: Y
Method: Least Squares
Date: 10/04/11 Time: 10:42
Sample (adjusted): 3 150
Included observations: 148 after adjustments
Convergence achieved after 7 iterations
Backcast: 1 2

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.538822	0.025448	21.17305	0.0000
MA(1)	0.505575	0.078037	6.478657	0.0000
MA(2)	-0.373660	0.078641	-4.751449	0.0000

R-squared	0.262481	Mean dependent var	0.538148
Adjusted R-squared	0.252309	S.D. dependent var	0.315777
S.E. of regression	0.273049	Akaike info criterion	0.261734
Sum squared resid	10.81062	Schwarz criterion	0.322489
Log likelihood	-16.36834	F-statistic	25.80258
Durbin-Watson stat	1.998332	Prob(F-statistic)	0.000000

Inverted MA Roots	.41	-.91

Cvičenie : simulované dáta

- Stacionarita - MA proces je vždy stacionárny
- Invertovatel'nosť - je invertovatel'ný:

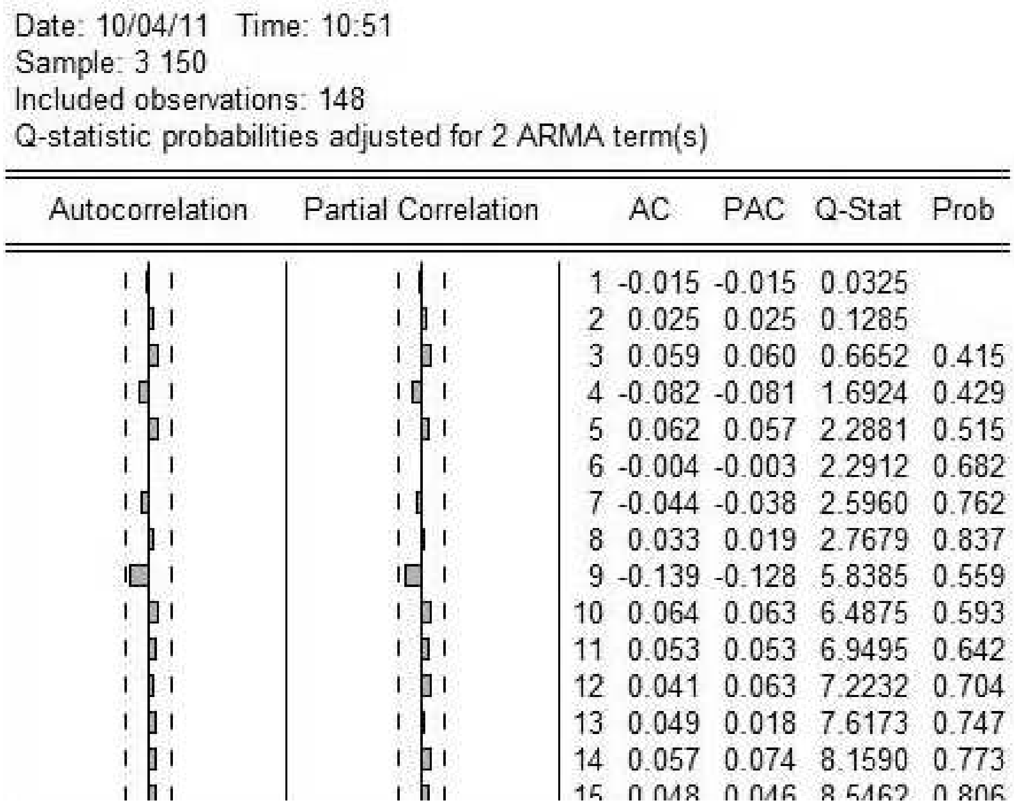
Inverse Roots of AR/MA Polynomial(s)
Specification: Y C MA(1) MA(2)
Date: 10/04/11 Time: 10:44
Sample: 1 150
Included observations: 148

MA Root(s)	Modulus	Cycle
-0.914272	0.914272	
0.408697	0.408697	

No root lies outside the unit circle.
ARMA model is invertible.

Cvičenie : simulované dáta

- ACF rezíduí, Q štatistika - v poriadku:



Date: 10/04/11 Time: 10:51
Sample: 3 150
Included observations: 148
Q-statistic probabilities adjusted for 2 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.015	-0.015	0.0325	
		2	0.025	0.025	0.1285	
		3	0.059	0.060	0.6652	0.415
		4	-0.082	-0.081	1.6924	0.429
		5	0.062	0.057	2.2881	0.515
		6	-0.004	-0.003	2.2912	0.682
		7	-0.044	-0.038	2.5960	0.762
		8	0.033	0.019	2.7679	0.837
		9	-0.139	-0.128	5.8385	0.559
		10	0.064	0.063	6.4875	0.593
		11	0.053	0.053	6.9495	0.642
		12	0.041	0.063	7.2232	0.704
		13	0.049	0.018	7.6173	0.747
		14	0.057	0.074	8.1590	0.773
		15	0.048	0.046	8.5462	0.806