An analytic approximation formula for pricing zero-coupon bonds

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Abstract

This paper presents an analytic approximation formula for pricing zero-coupon bonds, when the dynamics of the short-term interest rate are driven by a one-factor mean-reverting process in which changes in the volatility of the interest rate are a function of the level of the interest rate.

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1. Introduction

Risk-free short-term interest rate (or short rate) is used in financial economics to determine term structure of interest rates, bond prices and derivative security prices. Short rates also serve as an important element in the development of tools for risk management and in empirical work on term premiums and yield curves, where short rates are treated as reference rates for other interest rates.

For analytical tractability, this paper deals with a one-factor model of the short rate. It can be viewed as a building block for more complicated models of the term structure in which the short rate is only one of the factors. Existing empirical studies on one-factor short-rate models show that the most successful models in terms of capturing the dynamics of the short rate are those...
which allow for changes in the volatility of interest rate to be highly sensitive to the level of the interest rate.\(^1\) Such dynamics for the short rate, denoted as \(r_t\), with a single source of uncertainty can be written as a stochastic process of the form
\[
\frac{dr_t}{r_t} = (\alpha + \beta r_t) dt + \sigma r_t^\gamma dW_t, \tag{1}
\]
where \(\alpha\), \(\beta\), \(\sigma\), and \(\gamma\) are time-homogeneous parameters and \(dW_t\) is the increment of a standard Brownian motion. The drift term in (1) is specified to be linear in the short rate, \(\alpha + \beta r_t\). It captures the tendency for the short rate to revert to its long-run equilibrium (or unconditional mean) level. In particular the magnitude of the parameter \(\beta\) measures the speed with which the short rate reverts to its long-run mean value, \(-\alpha/\beta\). The volatility term is specified to be nonlinear in the level of the short rate, \(\sigma r_t^\gamma\). In particular the parameter \(\gamma\) \(>0\) in the volatility term parameterizes the extent to which changes in the short rate depend on the level of the short rate.

Chan et al. (1992) are the first to estimate the general one-factor model in (1) and a variety of its nested versions using one-month U.S. Treasury bill yields. They show that the value of \(\gamma\) distinguishes a variety of one-factor models of the short rate that have been studied in finance. In addition, they report that models favored by statistical tests typically have an unconstrained estimate of \(\gamma\) that is greater than 1. However this value of \(\gamma\) exceeds the values set in most theoretical one-factor models; for example, the Vasicek (1977) model sets \(\gamma = 0\) and the Cox et al. (1985)—CIR—model sets \(\gamma = 1/2\). Other studies, such as Tse (1995) and Dahlquist (1996), which extend the analysis of Chan et al. (1992) to other countries short-term interest rates, generally found that the effect of the level of short rates on the volatility of interest rates are also positive, but of smaller magnitudes than the one reported for the U.S. data. In a related study, McLeish and Kolkiewicz (1995) propose a parameter estimation based on higher order Itô–Taylor expansions, which alleviate much of the approximation errors that arise from estimation of a continuous-time process with discrete-time data. Using 30 year Treasury bond yields, they report an unconstrained estimate of \(\gamma\) to be 2.5.

Given a wide range of \(\gamma\) estimates in the empirical literature, it is useful to obtain a closed-form solution of zero-coupon bonds in a general one-factor model in which \(\gamma\) is treated as a free parameter. To this end, we provide in this paper an analytic approximation formula for zero-coupon bonds in the one-factor model of the short-rate in (1). For comparison purposes, we also plot the price difference between the exact formula and the approximation formula for the CIR model where \(\gamma = 1/2\).

The remainder of the paper is organized as follows. Section 2 derives an approximation formula for the general one-factor model of zero-coupon bonds and Section 3 presents the performance level for the case of the CIR model. Section 4 contains concluding remarks.

2. Zero-coupon bond pricing

The absence of arbitrage opportunities is characterized by the existence of risk-neutral measure \(\mathbb{Q}\), so that the time-\(t\) price of a zero-coupon bond maturing at time \(T\), \(P_{t,T-t}\), is given by
\[
P_{t,T-t} = E^Q_t \left[ \exp \left( - \int_t^T r_s \, ds \right) \right]. \tag{2}
\]

\(^1\) A partial list of these studies includes Brennan and Schwartz (1982), Grossman et al. (1987), Longstaff (1989, 1990), and Chan et al. (1992).
For simplicity and without loss of generality, we assume that the current time is given by $t = 0$. The trapezoidal rule is employed to approximate the price $P_{0,T}$, so that the integral part can be written as

$$\int_0^T r_s \, ds \approx h \left[ \frac{1}{2} r_0 + r_1 + \cdots + r_{n-1} + \frac{1}{2} r_n \right],$$

where the interval $[0, T]$ is partitioned into $t_0 = 0 < t_1 < \cdots < t_n = T$ with $t_k = kh$, $h = \frac{T}{n}$, and $r_k = r_{tk}$. For simplicity, we write $E^Q_{tk}$ as $E_k$ for $k = 0, 1, \ldots, n - 1$. It follows as an application of the tree property of conditional expectation that

$$P_{0,T} = E_0 E_1 \cdots E_{n-1} \left[ \exp \left( - \int_0^T r_s \, ds \right) \right].$$

To obtain a general expression from the sequential application of conditional expectations, we need a lemma about the following sequence.\(^2\)

**Lemma 1.** Suppose that the sequence $\{a_n\}$ is given by $a_1 = 1$, and

$$a_n = 2 + (1 + \beta h)a_{n-1} \quad \text{for} \ n \geq 2.$$

Then $a_n$ can be explicitly rewritten as

$$a_n = \frac{2}{\beta h} \left[ (1 + \beta h)^n - 1 \right].$$

To compute a conditional expectation on the lognormal distribution, we need another lemma specified below.

**Lemma 2.** Suppose that $\phi$ is a standard normal distribution, i.e., $\phi \sim N(0, 1)$. Then we have

$$E\left[ e^{b\sqrt{T} \phi} \right] = e^{\frac{1}{2} b^2 T}.$$

To approximate $r_{k+1}$ by the information of $r_k$, we first recall that the discrete-time model based on the Euler–Maruyama approximation scheme is given by

$$r_{k+1} - r_k \approx (\alpha + \beta r_k)h + \sigma r^\gamma_k \Delta W_k(h),$$

where $\Delta W_k(h) = W_{tk+1} - W_{tk}$. Then, approximating $r_n$ by the Euler–Maruyama scheme and applying Lemma 2 to $E_{n-1}$, we obtain the result that

$$E_{n-1} \left[ \exp \left( - \int_0^T r_s \, ds \right) \right] = \exp \left( -h \left[ \frac{r_0}{2} + r_1 + \cdots + r_{n-2} \right] - hr_{n-1} \right) \times E_{n-1} \left[ \exp \left( -\frac{h}{2} \left[ r_{n-1} + (\alpha + \beta r_{n-1})h + \sigma r^\gamma_{n-1} \Delta W_{n-1}(h) \right] \right) \right].$$

\(^2\) Substituting $a_{n-1}$ by a function of $a_{n-2}$ and $\beta h$ and then repeating this process yield the intended result.
\[
\begin{align*}
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-2}\right] - \frac{\alpha h^2}{2} - \frac{h}{2}(3 + \beta h)r_{n-1}\right) \exp\left(h \frac{3}{8} \sigma^2 r_{n-1}^2\right).
\end{align*}
\]

With the notation \(AF_k = \exp\left(h \frac{3}{8} \sigma^2 a_k^2 r_{n-k}^2\right)\), where \(a_k\) is provided in the form of recursive relation and explicit representation in Lemma 1, the conditional expectation based on the information until \(t_{n-2}\) is similarly obtained as follows

\[
\begin{align*}
E_{n-2}\left[\exp\left(-\int_{t}^{T} r_s ds\right) / AF_1\right]
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-3}\right] - hr_{n-2} - \frac{\alpha h^2}{2}\right) \\
&\quad \times E_{n-2}\left[\exp\left(-\frac{h}{2}(3 + \beta h)\left[r_{n-2} + (\alpha + \beta r_{n-2})h + \sigma r_{n-2}^\gamma \Delta W_{n-2}(h)\right]\right] \\
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-3}\right] - \frac{\alpha h^2}{2}\left[1 + (3 + \beta h)\right]\right) \\
&\quad \times \exp\left(-\frac{h}{2}r_{n-2}[2 + (1 + \beta h)(3 + \beta h)] + \frac{h^3}{8}(3 + \beta h)^2 r_{n-2}^2\right) \\
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-3}\right] - \frac{\alpha h^2}{2}(a_1 + a_2) - \frac{h}{2}a_3 r_{n-2} + AF_2\right).
\end{align*}
\]

Repeating this procedure for the sequence \(\{a_k\}\) in Lemma 1, we arrive at the following general formula:

\[
\begin{align*}
E_{n-k}\left[\exp\left(-\int_{t}^{T} r_s ds\right) / \prod_{i=1}^{k-1} AF_i\right]
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-k-1}\right] - hr_{n-k} - \frac{\alpha h^2}{2}\sum_{i=1}^{k-1} a_i\right) \\
&\quad \times E_{n-k}\left[\exp\left(-\frac{h}{2}a_k\left[r_{n-k} + (\alpha + \beta r_{n-k})h + \sigma r_{n-k}^\gamma \Delta W_{n-k}(h)\right]\right] \\
&= \exp\left(-h\left[r_0 + r_1 + \cdots + r_{n-k-1}\right] - \frac{\alpha h^2}{2}\sum_{i=1}^{k} a_i - \frac{h}{2}r_{n-k} a_{k+1} + AF_k\right).
\end{align*}
\]

Finally, we have the result that

\[
\begin{align*}
E_0\left[\exp\left(-\int_{t}^{T} r_s ds\right) / \prod_{i=1}^{n-1} AF_i\right] &= \exp\left(-h\frac{r_0}{2} - \frac{\alpha h^2}{2}\sum_{i=1}^{n-1} a_i\right) \\
&\quad \times E_0\left[\exp\left(-\frac{h}{2}a_{n}\left[r_0 + (\alpha + \beta r_0)h + \sigma r_0^\gamma \Delta W_0(h)\right]\right] \\
&= \exp\left(-\frac{\alpha h^2}{2}\sum_{i=1}^{n} a_i - \frac{h}{2}r_0\left[1 + a_n(1 + \beta h)\right] + AF_n\right).
\end{align*}
\]
To obtain a convergence result as \( n \) approaches \(+\infty\), we need several simple computational formulas. Recall that the number \( e \) is defined as the limit of the sequence

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.
\]

Applying this above result to Lemma 1, we have the following three lemmas.

**Lemma 3.** The term \( \frac{h^2}{2} [1 + a_n (1 + \beta h)] \) converges to

\[
B(T) = B(0, T) = \frac{1}{\beta} \left( e^{\beta T} - 1 \right),
\]

as the number of sub-intervals in \([0, T]\), which is \( n \), approaches to \( \infty \).

**Lemma 4.** The partial sum \( \sum_{i=1}^{n} a_i \) is

\[
-\frac{2n}{\beta h} + 2 \left(1 + \frac{\beta h}{2}\right) \left(1 + \beta h\right)^n - \frac{1}{\beta^2 h^2}.
\]

And the limit of \( \frac{h^2}{2} \sum_{i=1}^{n} a_i \) is

\[
-\frac{1}{\beta} \left(T - B(T)\right).
\]

**Lemma 5.** If the parameter \( \gamma \) is 0, then the adjusting factor \( \ln(\prod_{i=1}^{n} A_{F_i}) \) is reduced to

\[
\frac{h^3 \sigma^2}{8} \sum_{i=1}^{n} a_i^2 = \frac{\sigma^2}{2 \beta^2} \left[ \left(1 + \frac{\beta h}{2}\right) \left(1 + \beta h\right)^{2n} - \frac{1}{\beta} \left(1 + \beta h\right)^n - 1 + nh \right],
\]

and its limit is given by

\[
\frac{\sigma^2}{4 \beta^3} \left( (e^{\beta T} - 1)^2 - 2(e^{\beta T} - 1) \right) + \frac{\sigma^2}{2 \beta^2} T = \frac{\sigma^2}{4 \beta} B^2(T) + \frac{\sigma^2}{2 \beta^2} (T - B(T)).
\]

Applying Lemmas 3–5 to the equation in (3) immediately yields the result for the Vasicek (1977) model where \( \gamma = 0 \). This is stated in the following theorem.

**Theorem 1.** Provided that \( \gamma = 0 \), the time-\( t \) price of a zero-coupon bond maturing at time \( T \), \( P_{t, T-t} \), is given by

\[
P_{0, T} = E_0^Q \left[ \exp \left( -\int_0^T r_s \, ds \right) \right]
\]

\[
= \exp \left( -r_0 B(T) + \frac{\sigma^2}{4 \beta} B^2(T) + \frac{\alpha \beta + \sigma^2 / 2}{\beta^2} (T - B(T)) \right). \tag{4}
\]

Scrutinizing the process of obtaining (3), we see that it would be difficult to compute the expression \( E_{n-k-1}[A_{F_k}] \) explicitly. However it turns out to be much less difficult to simply approximate the expression \( A_{F_k} \) at time \( t_{n-k} \) by \( E_0[A_{F_k}] \). First, recall that \( A_{F_k} = \exp \left( \frac{h^3 \sigma^2}{8} a_k^2 r_{n-k}^{2\gamma} \right) \). To
compute the unconditional expectation of $A F_k$ at time $t = 0$, the distribution function of $r_{n-k}$ at zero is needed and obtained through the application of Itô’s formula to the function $f(x) = x^{2\gamma}$ as follows

$$d(r_t^{2\gamma}) = q_t \, dt + 2\gamma \sigma r_t^{3\gamma-1} \, dW_t$$

$$\Rightarrow r_{n-k}^{2\gamma} \approx r_0^{2\gamma} + q_0(n-k)h + 2\gamma \sigma r_0^{3\gamma-1} \sqrt{(n-k)}h \phi,$$ \hspace{1cm} (5)

where $q_t = \gamma ((2\gamma - 1)\sigma^2 r_t^{2\gamma(2\gamma-1)} + 2\gamma r_t^{2\gamma-1}(\alpha + \beta r_t)$ and $\phi$ is a standard normal distribution, i.e., $\phi \sim N(0, 1)$.\(^3\) Since the only random term of the right-hand side of $r_{n-k}^{2\gamma}$ in Eq. (5) is related to $\phi$, applying Lemma 2 to $\exp(h^3\sigma^2 a_k^2 c_k \phi)$ with the notation $c_k = 2\gamma \sigma r_0^{3\gamma-1} \sqrt{(n-k)}h$, gives us

$$\ln(AF_k)_{n-k} \approx \frac{h^3\sigma^2}{8} a_k^2 \left[r_0^{2\gamma} + q_0(n-k)h\right] + \frac{h^7\sigma^6}{32} \gamma^2 r_0^{2(3\gamma-1)} a_k^4 (n-k).$$

which is known at any time $t \in [0, T]$ and therefore, can be treated as a constant under the conditional expectation. Thus, \(^4\)

$$\ln\left(\prod_{k=1}^{n} AF_k\right) \approx \left[r_0^{2\gamma} + q_0 T\right] \frac{h^3\sigma^2}{8} \sum_{k=1}^{n} a_k^2 - q_0 \frac{h^4\sigma^2}{8} \sum_{k=1}^{n} k a_k^2$$

$$+ \frac{\sigma^6}{32} \gamma^2 r_0^{2(3\gamma-1)} h^6 \left[T \sum_{k=1}^{n} a_k^4 - h \sum_{k=1}^{n} k a_k^4\right].$$ \hspace{1cm} (6)

To obtain the approximation solution, the following lemma is needed.

**Lemma 6.** The limit of the partial sum $h^4 \sum_{k=1}^{n} k a_k^2$ is

$$\frac{1}{\beta^2} B^2(T)(2\beta T - 1) - \frac{2}{\beta^3} B(T)(2\beta T - 3).$$

Also, the limits of the partial sum $h^5 \sum_{k=1}^{n} a_k^4$ and $h^6 \sum_{k=1}^{n} k a_k^4$ are bounded by some positive number $M < \infty$.

Applying Lemmas 3–6 to the last equation of (3) and the left-hand side of Eq. (6), we have an analytical approximation solution for the time-0 price of a zero-coupon bond maturing at time $T$ for the general $\gamma$ case. This is given by the next theorem.

**Theorem 2.** For a general case of $\gamma$, the time-0 price of a zero-coupon bond maturing at time $T$, $P_{0,T}$, is given by

$$\ln(P_{0,T}) = -r_0 B(T) + \frac{\alpha}{\beta} (T - B(T))$$

$$+ \left[r_0^{2\gamma} + q_0 T\right] \frac{\sigma^2}{4\beta} \left[B^2(T) + \frac{2}{\beta} [T - B(T)]\right]$$

---

\(^3\) The corresponding Milstein scheme is given by $r_{n-k}^{2\gamma} \approx r_0^{2\gamma} + q_0(n-k)h + 2\gamma \sigma r_0^{3\gamma-1} \sqrt{(n-k)}h \phi + \sigma^2 \gamma (3\gamma - 1) \times r_0^{4\gamma-2}(n-k)h(\phi^2 - 1).$
Fig. 1. Price or relative mispricing as a function of time to maturity. The parameter values are set at $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$, $\gamma = 1/2$ (the CIR model), $r_0 = 8\%$, and $T = [0, 20]$ years. The exact price (EP) is obtained from the formula for the CIR model and the approximating price (AP) is obtained from our analytic formula in (5). Relative mispricing is defined as $(AP - EP)/EP$.

$$- q_0 \frac{\sigma^2}{8\beta^2} \left[ B^2(T)[2\beta T - 1] - 2B(T)[2T - 3/\beta] + 2T^2 - 6T/\beta \right],$$

(7)

where $q_0 = \gamma (2\gamma - 1)\sigma^2 r_0^{2(2\gamma - 1)} + 2\gamma r_0^{2\gamma - 1}(\alpha + \beta r_0)$ and $B(T) = \frac{1}{\beta}(e^{\beta T} - 1)$.

3. The performance of an approximation pricing formula

In this section we demonstrate the performance of the approximation pricing formula derived in the previous section. For the case of Vasicek (1977) model where $\gamma = 0$, we obtain an exact formula with different proof steps. For the case where $\gamma \neq 0$ or $\gamma \neq 1/2$, the exact pricing formula is not available. For this reason, we concentrate on the case of the CIR model where $\gamma = 1/2$.

To analyze the mispricing of our formula relative to the exact formula, the parameter values are set at $\alpha = 0.00315$, $\beta = -0.0555$, and $\sigma = 0.0894$. Figure 1 shows that the price pattern of our formula is similar to that of the exact solution and the relative mispricing is increasing with the time to maturity. But it is less than 1% as long as the time to maturity is less than 10 years. In addition Fig. 2 shows that the relative mispricing is increasing with the time to maturity and the initial short-term interest rate. However the figure suggests that the relative mispricing is very small provided that the time to maturity is less than 5 years.

Footnote 4: These parameters are estimated from the Korea Certificate of Deposit 90 days rate of the data period 1998-06-01–2002-10-01. The long-run equilibrium value $-\alpha/\beta$ is 0.0568.
Fig. 2. Relative mispricing as a function of $T$ and $r_0$. The parameter values are chosen to be $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$, $\gamma = 1/2$, $r_0 = [1, 20]$% and $T = [0, 20]$ years. The exact price (EP) is obtained from the formula for the CIR model and an approximating price (AP) is obtained from our analytic formula in (5). Relative mispricing is defined as $(AP - EP)/EP$.

4. Conclusion

In this paper we derived an analytic approximation formula for pricing zero-coupon bonds when the dynamics of the short-term interest rate were driven by a one-factor mean-reverting process in which changes in the volatility of interest rate is specified to be a function of the level of the interest rate. For the CIR model, the mispricing of our formula was shown to be very small relative to the exact solution, provided that the time to maturity was not longer than 5 years.

As a final caveat, we mention that since the domain of interest in this literature appears to be for values of $\gamma > 1$, it would be useful to investigate how changes in the parameter $\gamma$ would affect the pricing error. Even when no closed-form pricing formula is available for this case, comparison can be made between the analytic approximation and the results that can be obtained from Monte Carlo simulation. This is left for future research.

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Appendix A

Proof of Lemma 1. If the sequence $\{a_n\}$ is written in the summation form, we have

$$a_n = 2 + (1 + \beta h)a_{n-1}$$
\[
= 2 + (1 + \beta h)[2 + (1 + \beta h)a_{n-2}]
\]
\[= 2 + 2(1 + \beta h) + 2(1 + \beta h)^2 + \cdots + 2(1 + \beta h)^{n-2} + (1 + \beta h)^{n-1}a_1
\]
\[= \frac{2}{\beta h}[{(1 + \beta h)^{n-1}} (1 + \frac{\beta h}{2}) - 1]. \quad \Box
\]

**Proof of Lemma 3.** It is clear that
\[
\frac{h}{2}[1 + a_n(1 + \beta h)] = \frac{h}{2} + \frac{1}{\beta} \left[ (1 + \beta h)^n \left( 1 + \frac{\beta h}{2} \right) - (1 + \beta h) \right]
\]
\[\xrightarrow{n \to \infty} \frac{1}{\beta} \left[ e^{\beta(T-t)} - 1 \right]. \quad \Box
\]

**Proof of Lemma 4.** Since \( T = nh \), it is clear that
\[
\sum_{i=1}^{n} a_i = \frac{2}{\beta h} \sum_{i=1}^{n} \left[ (1 + \beta h)^{i-1} (1 + \frac{\beta h}{2}) - 1 \right]
\]
\[= -\frac{2(T-t)}{\beta h^2} + 2 \left( 1 + \frac{\beta h}{2} \right) \frac{(1 + \beta h)^n - 1}{\beta^2 h^2},
\]
which implies the desired limit of \( \frac{h^2}{2} \sum_{i=1}^{n} a_i. \) \( \Box \)

**Proof of Lemma 5.** It is clear that
\[
\frac{h^2}{2} \sum_{i=1}^{n} a_i^2 = \frac{4h}{\beta^2} \sum_{i=1}^{n} \left[ (1 + \beta h)^{i-1} \left( 1 + \frac{\beta h}{2} \right) - 1 \right]^2
\]
\[= \frac{4h}{\beta^2} \left[ \left( 1 + \frac{\beta h}{2} \right)^2 \frac{(1 + \beta h)^{2n} - 1}{2\beta h} - 2 \left( 1 + \frac{\beta h}{2} \right) \frac{(1 + \beta h)^n - 1}{\beta h} + n \right]
\]
\[\xrightarrow{n \to \infty} \frac{2}{\beta^3} [e^{2\beta T} - 1] - \frac{8}{\beta^3} [e^{\beta T} - 1] + \frac{4}{\beta^2} T,
\]
which gives us the intended result. \( \Box \)

**Proof of Lemma 6.** It is well known that
\[
\sum_{i=1}^{n} i(1 + r)^{i-1} = \frac{(1 + r)^n - 1}{r} - n(1 + r)^n.
\]
Applying the above result, we obtain
\[
\frac{h^4}{\beta^2} \sum_{i=1}^{n} a_i^2 = \frac{4h^2}{\beta^2} \sum_{i=1}^{n} i \left[ (1 + \beta h)^{i-1} \left( 1 + \frac{\beta h}{2} \right) - 1 \right]^2
\]
\[= \frac{4h^2}{\beta^2} \sum_{i=1}^{n} i \left[ \left( 1 + \frac{\beta h}{2} \right)^2 (1 + \beta h)^{2(i-1)} - 2 \left( 1 + \frac{\beta h}{2} \right) (1 + \beta h)^{i-1} + 1 \right]
\]
\[
\frac{h^5 \beta^4}{16} \sum_{i=1}^{n} a_i^4 = \left(1 + \frac{\beta h}{2}\right)^4 \left(1 + \beta h\right)^{4(i-1)} - 4 \left(1 + \frac{\beta h}{2}\right)^3 \left(1 + \beta h\right)^{3(i-1)} \\
+ 6 \left(1 + \frac{\beta h}{2}\right) \left(1 + \beta h\right)^{2(i-1)} - 4 \left(1 + \frac{\beta h}{2}\right) \left(1 + \beta h\right)^{i-1} + 1
\]

Thus,
\[
\frac{h^5 \beta^4}{16} \sum_{i=1}^{n} a_i^4 = \left(1 + \frac{\beta h}{2}\right)^4 \frac{1 + \beta h)^4n - 1}{\beta(4 + 6\beta h + \cdots)} - 4 \left(1 + \frac{\beta h}{2}\right)^3 \frac{(1 + \beta h)^3n - 1}{\beta(3 + 3\beta h + \cdots)} \\
+ 6 \left(1 + \frac{\beta h}{2}\right)^2 \frac{(1 + \beta h)^2n - 1}{2\beta(1 + \beta h/2)} - 4 \left(1 + \frac{\beta h}{2}\right) \frac{(1 + \beta h)^n - 1}{\beta} + T,
\]

which is bounded by positive number \(M = M(T)\). Similarly,
\[
\frac{h^6 \beta^4}{16} \sum_{i=1}^{n} ia_i^4 = \left(1 + \frac{\beta h}{2}\right)^4 \left[ \frac{T(1 + \beta h)^4n}{\beta(4 + 6\beta h + \cdots)} + \frac{1 - (1 + \beta h)^4n}{\beta^2(4 + 6\beta h + \cdots)^2} \right] \\
- 4 \left(1 + \frac{\beta h}{2}\right)^3 \left[ \frac{T(1 + \beta h)^3n}{\beta(3 + 3\beta h + \cdots)} + \frac{1 - (1 + \beta h)^3n}{\beta^2(3 + 3\beta h + \cdots)^2} \right] \\
+ 6 \left(1 + \frac{\beta h}{2}\right)^2 \left[ \frac{T(1 + \beta h)^2n}{2\beta(1 + \beta h/2)} + \frac{1 - (1 + \beta h)^2n}{4\beta^2(1 + \beta h/2)^2} \right] \\
- 4 \left(1 + \frac{\beta h}{2}\right) \left[ \frac{T(1 + \beta h)^n}{\beta} + \frac{1 - (1 + \beta h)^n}{\beta^2} \right] + \frac{T(T + h)}{2},
\]

which is bounded by positive number \(M = M(T)\). 

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\]

**References**


