FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS
COMENIUS UNIVERSITY

FINANCIAL DERIVATIVES: EXERCISES

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Contents

Preface

These lecture notes provide exercises to an introductory course dealing with analytical and numerical methods for pricing financial derivatives. It takes the partial differential equations approach to the models formulated in terms of stochastic processes.

Exercises collected in these notes start with the concept of options and options strategies. The second chapter deals with random processes, which are the base of the financial models studied later. The first such model is the Black-Scholes model, which is extensive studied in chapter 3 from both the mathematical point of view (analysis of the Black-Scholes partial differential equation) and the financial point of view (financial interpretation of the results, getting intuition about the model, sensitivities, etc.). As an example of possible generalizations of the Black-Scholes model, Leland model for transaction costs is given in chapter 4. The following two chapters contain application of finite difference numerical schemes to options pricing. Firstly, in chapter 5, the European options are considered, when the numerical solution can be compared with the known exact solution. Here, a large space is given to understanding and testing the successive over-relaxation method for solving a system of linear equations. Then, in chapter 6, American options are presented, the free boundary formulation of the pricing problem is transformed to a linear complementarity problem, which is finally numerically solved by projected successive over-relaxation method. The last chapter is an introduction to interest rate modelling and it presents the basics of the Vasicek model - distribution of the short rate, bonds pricing and term structures of interest rates.

Exercises are both theoretical and practical, requiring implementation of the algorithms and their testing. The codes given in the notes and at the website are written for Scilab:

`http://www.scilab.org`

which is a free software with a syntax very similar to Matlab.

The content of these notes is based on the Financial derivatives course, as taught at Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava. The notes were written as a part of the project of transforming the master study program Mathematics of Economy and Finance into English language. There is a website to the English version of the course:

`http://www.iam.fmph.uniba.sk/institute/stehlikova/fd-en.html`

which will be gradually updated.

Bratislava, 31st July 2013

Beáta Stehlíková

Chapter 1

European vanilla options

An **option** gives its holder the right but not the obligation to perform the predetermined contract. So called **European options** are characterized by the fact that it can be exercised only at the specified time, which is called the expiration time. **American options** can be exercised at any time before their expiration. In the first five chapters we are going to deal with European options. Unless stated otherwise, we assume that the stock does not pay dividends.

1.1 Payoff and profit diagrams

1.1.1 Basic definitions. The two basic types of European options, which are referred to as **plain** (or **vanilla**) **options**, are the following:

- The **European call option** is a derivative contract in which the holder of the option has the right but not the obligation to purchase the underlying stock at the specified *expiration time* $t = T$ for the predetermined *exercise price* (also called *strike price*) E
- The **European put option** is a derivative contract in which the holder of the option has the right but not the obligation to sell the underlying stock at the specified *expiration time* $t = T$ for the predetermined *exercise price* (also called *strike price*) E

The value of an option or an option strategy at the time of expiration is called the **payoff**. After subtracting the initial investment (in the case of nonzero interest rates, we subtract its value at expiration) we obtain the **profit**. Graphs of payoff and profit as a function of the price of the underlying asset at the expiration time are referred to as **payoff diagram** and **profit diagram** respectively.

1.1.2 Payoff and profit diagrams for a call option. Let us consider a call option with exercise price E. Its value at the expiration time depends on the price S of the underlying stock at this time:

- If S is lower than or equal to the expiration price E of the option, the option is worthless and hence the payoff is zero in this case.
- If S is greater than the expiration price E, we obtain the profit $S - E$ by exercising the option (i.e., paying the amount E and receiving the asset which has the value S), therefore which makes its value in this case is $S - E$.

Hence, we can write the call option price at expiration in a compact form as $\max(0, S - E)$. Profit diagram of the call option expiring in T years, is then obtained by subtracting Ve^{-rT}, where V is the price of the option and r is the interest rate *per annum*.

Figure 1.1 shows payoff and profit diagrams for a call option. It has been generated by the Scilab code[1] below:

[1]The full code, including formatting of the graph (setting the white background, defining fonts, etc.) can be found on the companion website, see Preface for the link.

```
S=0:20;   // range of stock prices
E=10;     // exercise price
T=1;      // time to expiration
V=3;      // price of the option

Payoff=max(0,S-E);   // payoff of the option

r=0; Profit1=Payoff-V*exp(-r*T);    // profit for r=0
r=0.1; Profit2=Payoff-V*exp(-r*T);  // profit for r=0.1

figure; // new figure window

subplot(1,2,1);
plot(S,Payoff);
xlabel("underlying stock at the expiration time");
ylabel("payoff of the option");

subplot(1,2,2);
plot(S,Profit1);
plot(S,Profit2,"r");
xlabel("underlying stock at expiration time");
ylabel("profit from the option");
legend("r=0","r=0.1 (i.e., 10%)");
```

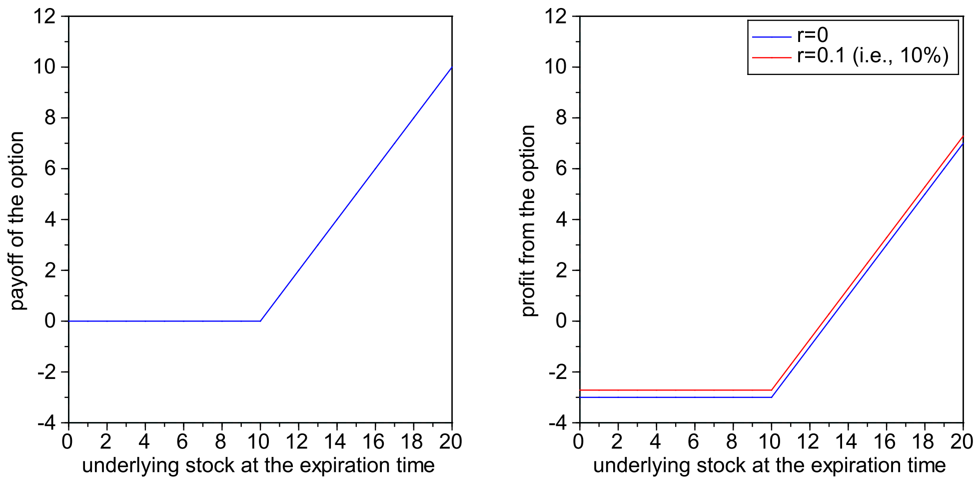


Figure 1.1: Left: Payoff diagram of a call option with the exercise price 10 USD. Right: Profit diagram of such an option which costs 3 USD and expires in one year, for two possible values of the interest rate.

1.1.3 Exercise: Payoff and profit diagrams for a put option. Deduce the payoff of a put option to be equal to $\max(0, E - S)$, where E is the exercise price of the option and S is the price of the underlying stock at the expiration time. Graph the payoff and profit diagram for a put option with selected parameters.

1.1.4 Exercise: Practice with payoff and profit diagrams. Consider buying the call option with the expiration price $E = 60$ USD with expires in one month (i.e., $T = 1/12$), which costs 10 USD. Assume that the interest rate is zero.

- Plot your payoff and profit diagram.

- Is your possible profit bounded? If so, what is the higher possible profit? If no, what behaviour of the underlying stock leads to profit increasing without any bound?
- Suppose that the price of the underlying asset at expiration is 65 USD. Does your position lead to a positive profit or to a loss?
- Determine all the prices of the underlying asset at expiration, for which the positive profit is achieved.

Repeat the same questions for the case when the option considered is a put option. The other parameters remain the same.

1.1.5 Real market data on options prices. Firstly, we should note that in practice the options traded on the individual stocks are American options, which means that can be exercised at any time before the expiration time (not only at the expiration time). Options traded on indices are usually European type options, although some are traded as American options).

Nevertheless, we show the data of options traded on individual stocks as well, to present the notation used. We use the site `http://finance.yahoo.com`, where we start by writing the name of the company or directly the stock symbol, as shown in Figure 1.2.

Unless we deal with pricing American derivatives, we take the option prices given online as the prices of the European style options. Let us remark that later we will show that in the case of a non-dividend paying stock, the prices of a European and an American option are equal.

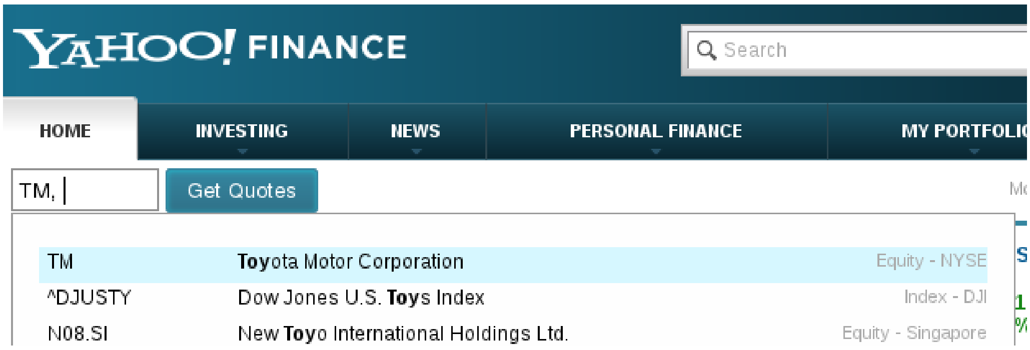


Figure 1.2: Searching options on Toyota Motor Corporation stocks at `finance.yahoo.com`.

Figure 1.3 shows an excerpt from the **option chain** (obtained by clicking on *Options* in the left column), a list of option prices available at the given time. There are options with different exercise prices and such an option chain is available for each expiration time.

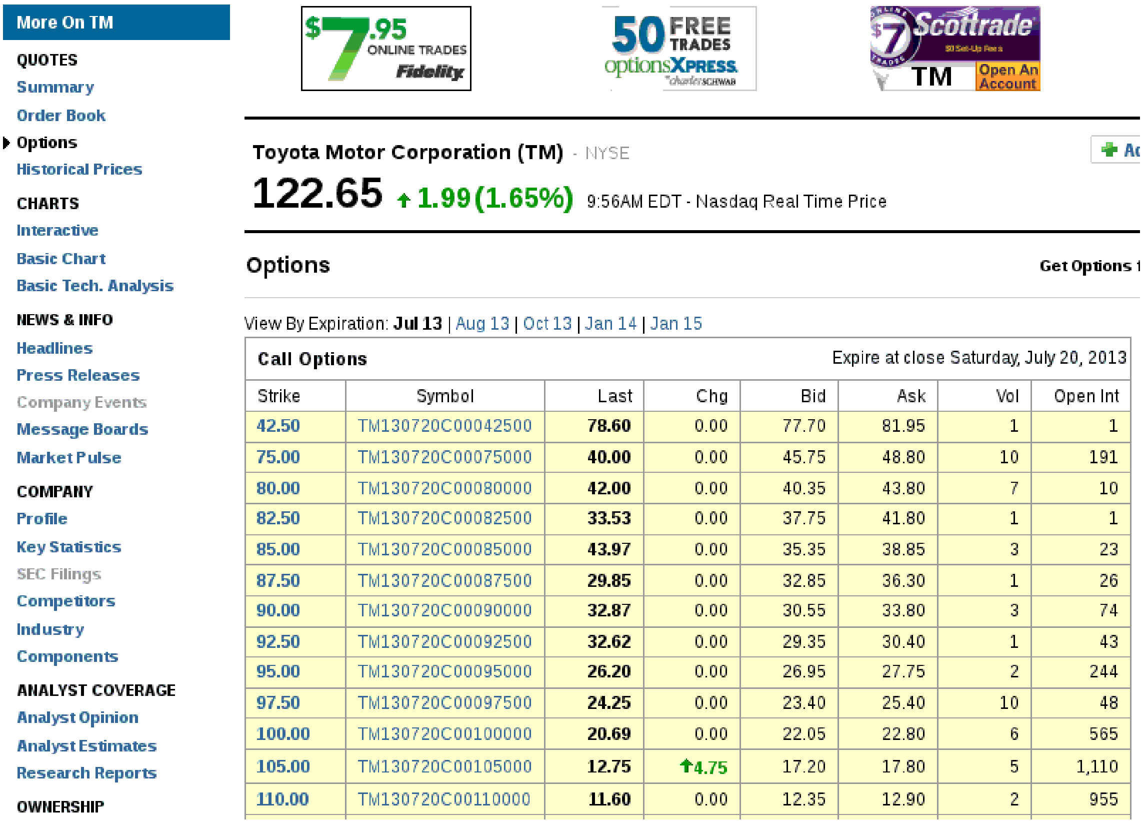
As indicated in the header, these options expire on Saturday, 20th July 2013. American options typically expire the third Saturday of the month and they are closed for trading the preceding Friday[2]. Therefore the options with other expiration dates are indicated only by the month of the expiration. The options not following this format can be distinguished by their symbol which will be explained in the following paragraphs.

The options are characterized by their symbol, see the column **Symbol** in the table. The format of option symbols was created by the Options Clearing Corporation. The option symbol is made up of 17 to 21 characters, depending on the length of the symbol representing the underlying security. Symbols are constructed as follows:

symbol of the underlying + expiration date + call/put + exercise price

The symbol of the underlying has 2 to 6 characters, which results in different lengths of the option symbols. The expiration date is written in the form YYMMDD and the type of the option is given by **C** (call) or **P** (put). The strike price is given in dollars to three decimal places and consists of 8 digits: the first 5 digits give the dollar value (the first digits are set to zero if necessary) and the last 3 digits give the cents.

[2] However, there are also weekly options which are traded for a week and then expire on Friday, quarterly options which expire on the last trading day of the quarter, etc.



More On TM
QUOTES
Summary
Order Book
Options
Historical Prices
CHARTS
Interactive
Basic Chart
Basic Tech. Analysis
NEWS & INFO
Headlines
Press Releases
Company Events
Message Boards
Market Pulse
COMPANY
Profile
Key Statistics
SEC Filings
Competitors
Industry
Components
ANALYST COVERAGE
Analyst Opinion
Analyst Estimates
Research Reports
OWNERSHIP

Toyota Motor Corporation (TM) - NYSE

122.65 ↑ 1.99 (1.65%) 9:56AM EDT - Nasdaq Real Time Price

Options

Get Options f

View By Expiration: **Jul 13** | Aug 13 | Oct 13 | Jan 14 | Jan 15

Call Options Expire at close Saturday, July 20, 2013

Strike	Symbol	Last	Chg	Bid	Ask	Vol	Open Int
42.50	TM130720C00042500	78.60	0.00	77.70	81.95	1	1
75.00	TM130720C00075000	40.00	0.00	45.75	48.80	10	191
80.00	TM130720C00080000	42.00	0.00	40.35	43.80	7	10
82.50	TM130720C00082500	33.53	0.00	37.75	41.80	1	1
85.00	TM130720C00085000	43.97	0.00	35.35	38.85	3	23
87.50	TM130720C00087500	29.85	0.00	32.85	36.30	1	26
90.00	TM130720C00090000	32.87	0.00	30.55	33.80	3	74
92.50	TM130720C00092500	32.62	0.00	29.35	30.40	1	43
95.00	TM130720C00095000	26.20	0.00	26.95	27.75	2	244
97.50	TM130720C00097500	24.25	0.00	23.40	25.40	10	48
100.00	TM130720C00100000	20.69	0.00	22.05	22.80	6	565
105.00	TM130720C00105000	12.75	↑4.75	17.20	17.80	5	1,110
110.00	TM130720C00110000	11.60	0.00	12.35	12.90	2	955

Figure 1.3: Options on Toyota Motor Corporation stocks at `finance.yahoo.com`

As an example, let us take the symbol **TM130720C00100000**. This is an option traded on **TM**, i.e., the Toyota Motor Corporation stocks. The expiration time is given by **130720** which stands for 20th July 2013. The letter **C** indicates a call option. Finally, the expiration price, deciphered as **00100000**, is 00100,000 USD, i.e., 100 USD.

In practice, the options are not traded individually, i.e., one does not get the right to buy one underlying stock. Typically, they are traded in hundreds. There is an exception, so called *mini options* on selected underlyings, which are traded in tens. Their symbol is distinguished by the number 7 following the symbol of the underlying asset, see Figure 1.4

775.00	GOOG130817C00775000	117.20	↑19.80	114.20	117.30	1	1
780.00	GOOG130817C00780000	106.26	0.00	108.90	111.70	1	28
780.00	GOOG7130817C00780000	105.50	0.00	109.10	112.00	19	1
795.00	GOOG130817C00795000	87.40	0.00	95.60	99.00	1	15
800.00	GOOG130817C00800000	84.75	0.00	91.00	94.00	1	33
800.00	GOOG7130817C00800000	87.90	0.00	91.30	93.30	4	6

Figure 1.4: Regular and mini options at `finance.yahoo.com`.

For our exercises in this and the following chapters, we are going to use the current stock price (in the boldface in the top, here 120.66 USD), the last traded option price (in the **Last** column), the price for which we are able to buy the option immediately (so called *ask price*, column **Ask** in the table) and the price for which we are able to dell the option immediately (so called *bid price*, column **Bid** in the table).

1.1.6 Exercise: Using real market options prices. Find he current prices of the call and put options for a selected stock. Based on your expected evolution of the stock decide on buying a call or put option with a suitable exercise price. Alternatively, consider a protection against a certain movement of the stock price (for example, if you have stocks in your portfolio, you might want to protect yourself against the possibility that the stock price declines).

Plot the payoff and profit diagrams for the chosen strategy, taking the interest rate equal to the Treasure Bills yield with a maturity close to the exercise time of the option. For what values of the underlying stock at expiration time is your strategy profitable?

Sample solution. We provide a sample solution using the options shown earlier in this chapter in Table 1.6.

Consider buying the call option with exercise price of 100 USD, expiring in July 2013, i.e., on 20th July. It can be bought for its ask price which is equal to 22.80 USD. The data shown in Figure 1.3 come from the beginning of the trading on July 1st which leaves 14 trading days[3] until the expiration of the option. Considering the convention of 252 trading days a year, we have $T = 14/252$. As the interest rate we take the four-week Treasury Bills rate[4] from the previous trading day which equals 0.02%, i.e., $r = 0.0002$. This enables us to plot the diagrams shown in Figure 1.5

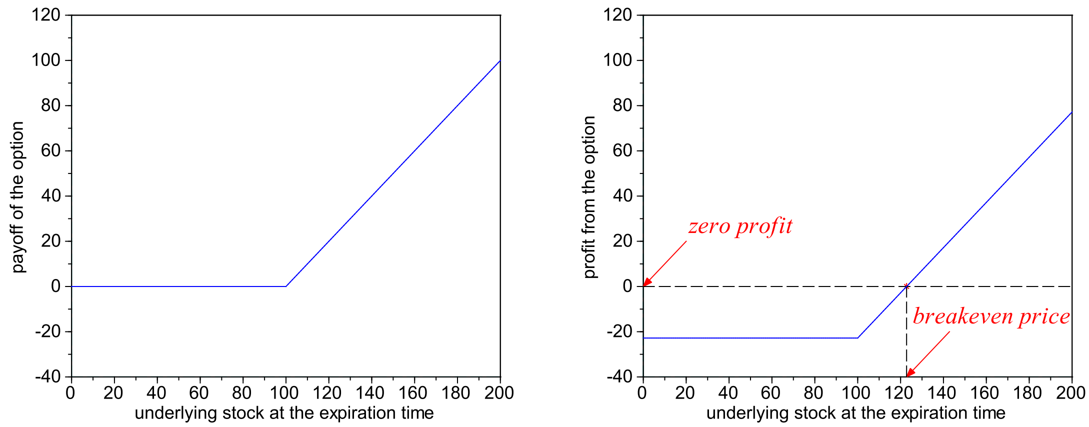


Figure 1.5: Payoff diagram (left) and profit diagram (right) of the option.

Since the profit is a piecewise linear function, it is easy to explicitly compute its intersection with the horizontal axis. This value of the stock is called a **breakeven price**; at this price the profit is exactly zero. Having computed this price, we are able to answer the question, when our profit would be positive.

The profit equals

$$\text{profit}(S) = \begin{cases} -22.80 \times e^{-0.0002 \times 14/252} & \text{for } S \leq 100, \\ S - 100 - 22.80 \times e^{-0.0002 \times 14/252} & \text{for } S > 100. \end{cases}$$

Hence the breakeven price equals

$$S^* = 100 + 22.80\, e^{-0.0002 \times 14/252}$$

and the strategy results in a profit if the stock price at expiration is higher than S^*.

1.1.7 Exercise. Take a look on a selected options chain. Typically, the options are ordered so that they start with the smallest exercise price and continue with options with increasing exercise price. What pattern do you notice for call and put options? How does the prices of call and put options depend on the exercise price? Why?

1.1.8 Exercise: Selling (or writing) options. In the same way we can analyze the situation when we write an option. This means we have an obligation to perform the determined contract if the holder

[3] Note that this time interval includes the Independence day (4th July), when the stock market is closed in the USA. A useful site for determining public holidays in different countries is, for example, `http://calendar.retira.eu`.

[4] Taken from `http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=billrates`

of the option chooses to exercise his right. Simple strategies of writing a call or put option are known as **naked call** and **naked put** respectively. The term *naked* refers to the fact that the risk resulting from writing the options is not covered by buying/selling another financial instrument. We will return to this when discusing combined strategies.

For now, plot payoff and profit diagrams for these strategies. Is the highest possible profit bounded? Is the highest possible loss bounded?

1.1.9 Exercise. In the preface to the book *Options Math for Traders: How To Pick the Best Option Strategies for Your Market Outlook*[5], the author writes:

> One strategy that you won't see discussed at all, even though logically extending some of the phenomena might make one think it would be a profitable strategy, is naked call selling. Please don't do it, even if your broker will let you. Little good and much regret can come of it. I'll explain naked put selling, which some might say is being hypocritical, but even when we discuss selling naked puts they're not really naked. ... On the other hand, it's impossible to set aside enough cash to cover the risk from selling naked calls.

Can you explain this reasoning? What is the difference between a naked call and a naked put? Why is "a naked put not really naked" and why is it "impossible to set aside enough cash to cover the risk from selling naked calls"?

1.1.10 Intrinsic value and time value of an option. The price of an option can be decomposed into so called **intrinsic value** and **time value**. The intrinsic value of an option is the value of the option if exercised immediately. The rest of the value is called the time value.

For example, suppose that the stock price is 100 USD. Let us consider the put option with the exercise price of 105 USD which costs 9.50 USD. Then the intrinsic value of the option is 5 USD and the time value is 4.50 USD. If we consider another put option with exercise price of 90 USD which costs 2.60 USD, then its intrinsic value is zero and the time value is 2.60 USD.

1.1.11 Moneyness. The term moneyness refers to the relation of the exercise price of the option to the current stock price. If an option would make a profit if exercised now, it is said to be **in-the-money** (abbreviated as ITM). If exercising an option immediately would lead to a loss, the option is said to be **out-of-the-money** (OTM). If the current price of the stock and exercise price of an option are equal, the option is said to be **at-the-money** (ATM).

In the tables presenting the options chains, the ITM and OTM options are often distinguished graphically, for example on both `finance.yahoo.com` and `finance.google.com` the ITM options are highlighted, see Figure 1.6.

23.00	YHOO130817C00023000	2.80	↑0.30	2.75	2.78	19	2,560
24.00	YHOO130817C00024000	1.94	↑0.13	2.01	2.05	4	922
25.00	YHOO130817C00025000	1.41	↑0.18	1.40	1.42	32	7,294
26.00	YHOO130817C00026000	0.89	↑0.08	0.91	0.92	10	2,881
27.00	YHOO130817C00027000	0.50	0.00	0.55	0.57	62	5,042
28.00	YHOO130817C00028000	0.32	↑0.05	0.31	0.34	13	1,347

Figure 1.6: Highlighted ITM options at `finance.yahoo.com`; the current stock price is 25.47 USD.

1.1.12 Exercise. What is the time value and the intrinsic value for each of the alternatives - an OTM, an ITM and an ATM option?

1.1.13 Exercise. Select a stock and consider the option chain for a selected expiration time. You are asked to buy an in the money call option. List some alternatives which you have. For each of them plot the profit diagram and compute the breakeven price.

[5] S. Nations,s *Options Math for Traders. How To Pick the Best Option Strategies for Your Market Outlook*. John Wiley & Sons, 2012, p. xii

1.2 Combined strategies

1.2.1 Example. Consider the part of the option chain presented in Figure 1.6 and suppose the investor buys the call option with the strike price of 23 USD, anticipating the higher value of the stock at the exercise time. On the other hand, he does not expect the price to rise so sharply that, for example, the call option with the strike price 28 USD would be useful. However, he sees that there is a demand for these options, so he sells such an option. In this way, he receives a certain cash at the beginning and expects no payments at the expiration time.

Let us compare these two strategies (to simplify the computations we consider zero interest rate, note that the expiration time is close and the interest rates are low):

- Buying the call option with the strike price of 23 USD has the profit

$$\text{profit}_1(S) = \max(0, S-23) - 2.78 = \begin{cases} -2.78 & \text{for } S \le 23, \\ S - 25.78 & \text{for } S > 23. \end{cases}$$

- Buying the call option with the strike price of 23 USD and selling the call option with the strike price of 28 USD has the profit

$$\text{profit}_2(S) = \max(0, S-23) - \max(0, S-28) - 2.78 + 0.31 = \begin{cases} -2.47 & \text{for } S \le 23, \\ S - 25.47 & \text{for } 23 < S \le 28, \\ 2.53 & \text{for } S > 28. \end{cases}$$

At this point it is useful to define functions in Scilab for a payoff of call and put options as functions of the stock price S and the exercise price E:

```
function [cp]=CallPayoff(S,E)
        cp=max(0,S-E);
endfunction

function [pp]=PutPayoff(S,E)
        pp=max(0,E-S);
endfunction
```

Their use makes the subsequent code more readable:

```
function [p1]=profit1(S)
        p1=CallPayoff(S,23)-2.78;
endfunction

function [p2]=profit2(S)
        p2=CallPayoff(S,23)-CallPayoff(S,28)-2.78+0.31;
endfunction

S=10:35;
figure;
plot(S,profit1(S));
plot(S,profit2(S));
```

The profit diagrams are shown in Figure 1.7. Based on them, we can make several observations:

- The first strategy is profitable if the stock price at the time of the expiration exceeds 25.78 USD. The second strategy is profitable it exceeds 25.47. Hence a smaller value of the stock price is required for the profitability of the second strategy. This is caused by selling the option which brought certain cash at the beginning.
- The first strategy brings higher profit than the second one if the stock price exceeds 28.31 USD, i.e., for a high value of the stock. Again, this is caused by selling the option with the strike price 28

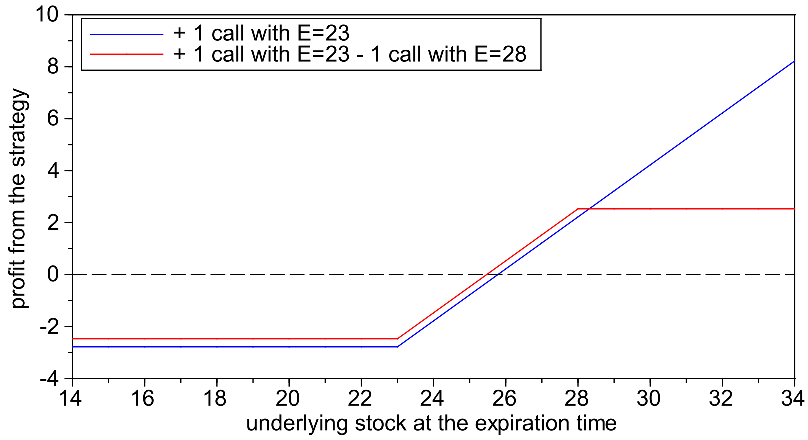


Figure 1.7: Comparison of the option strategies.

USD, which gets exercised. For high value of the stock, the cash received at the beginning is not sufficient to make up for the lost caused by having to sell the option for the predetermined price of 28 USD.

- The profit of the second strategy is bounded. If the stock price exceeds 28 USD, both options are exercised. This means that the investor buys the stock for 23 USD (because he uses his option) and at the same time sells it for 28 USD (the holder of the option exercises it too). This brings him the payoff of 5 USD, but we need to subtract the initial investment to receive the profit (he payed 2.78 USD and received 0.31 at that time).

The strategy of buying an ITM call option and selling an OTM call option is known as **bullish call spread**. The term *bullish* refers to the expectation of a rise of the stock price, contrary to so called *bearish* strategies which are based on the expected fall of the price.

1.2.2 Exercise: bullish put spread. Show that a similar strategy can be constructed using put options. Construct a **bullish put spread** which is constructed by buying an OTM put option and selling an ITM put option. Why is it a bullish strategy?

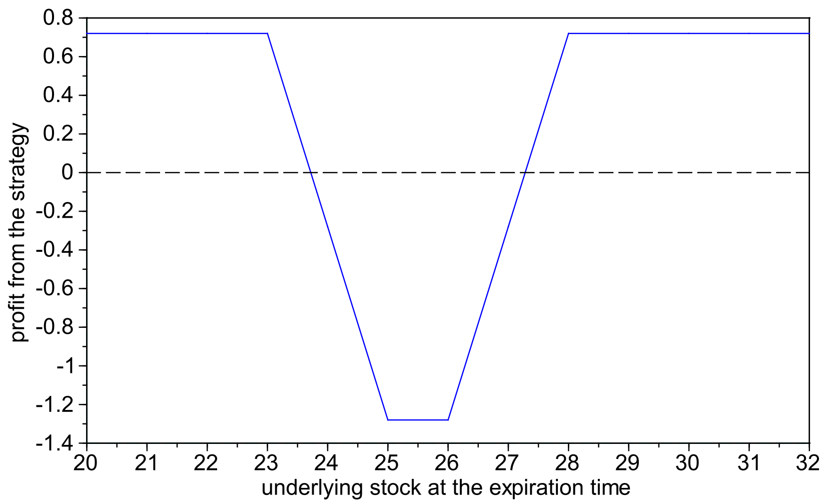


Figure 1.8: Profit from the options strategy.

1.2.3 Exercise. Consider a strategy which consists of buying one call option with exercise price of 25 USD and one with exercise price of 26 USD, while selling one call option with with exercise price of 23 USD and one with exercise price of 28 USD. The profit diagram of this strategy, taking option prices from Figure 1.3 and assuming zero interest rate, is presented in Figure 1.8. What does it say about the investor's expectation about the stock price? When is it reasonable to have such an expectation? Compute the prices for which the strategy brings profit.

Remark on numerical computation. Instead of an exact breakeven price, a numerical solution suffices and its computation might be more convenient in the cases like this one, where the strategy involves more options.

We define `profit` as a function of the stock price at expiration and compute the value for which it equals zero using the Scilab function `fsolve`[6], which finds, in general, a zero of a system of nonlinear equations. Our profit function is "problematic" in some ways - there are points where it is not differentiable and intervals where it has constant values - but when the starting value for the optimization is chosen to be close to the solution we can avoid these problems and use `fsolve` also in this case. The initial point can be chosen from the graph, see Figure 1.8.

Firstly, we define the function

```
function [p]=profit(S)
        init_inv=-2.75+1.42+0.92-0.31    // initial investment
        p=-CallPayoff(S,23)+CallPayoff(S,25)+CallPayoff(S,26)-...
                CallPayoff(S,28)-init_inv;
endfunction
```

and then we can use for example the following initial values

```
b1=fsolve(23.5,profit);
b2=fsolve(27.5,profit);
```

to compute the two breakeven prices.

1.2.4 A "herbarium" of options strategies. For a variety of options strategies, we refer to the website `http://www.theoptionsguide.com/`. It lists option strategies according to the expectation about the stock price: bullish, bearish, non-directional. In the last case we distinguish whether we expect the price to stay at about the same level as it is now, or we anticipate a big change in the price but we are unsure about its direction (for example depending on the results of the company which are to be announced). Note that when constructing some of the strategies, there may not be an exactly ATM option available; in such a case, an option with the strike price close to the stock price should be used.

The website provides profit diagrams of the strategies, method of their construction, maximum profit and loss, examples. You should get an overview about different possible strategies, understand their analysis and possible use.

1.2.5 Exercise. Find a combination of options which has a payoff diagram equal to each of those shown in Figure 1.9. Furthemore, answer the following questions:

- Why would anyone consider the strategies from the bottom row, which never have positive payoff?
- Why are the strategies in the first row, which always have nonnegative payoff (especially the strategy in the right, which has a positive payoff, regardless of the stock price at expiration time), not arbitrage[7]?

1.2.6 Exercise. Give an example of an option strategy which has a limited loss and is profitable if there is a small change in stock price in either direction.

1.2.7 Exercise. Consider the strategy of selling an ATM call and an ATM put option.

- Sketch the payoff diagram and show that the possible loss is not bounded.

[6] See `http://help.scilab.org/docs/5.4.1/en_US/fsolve.html`

[7] An *arbitrage* is a strategy which leads to a riskless profit.

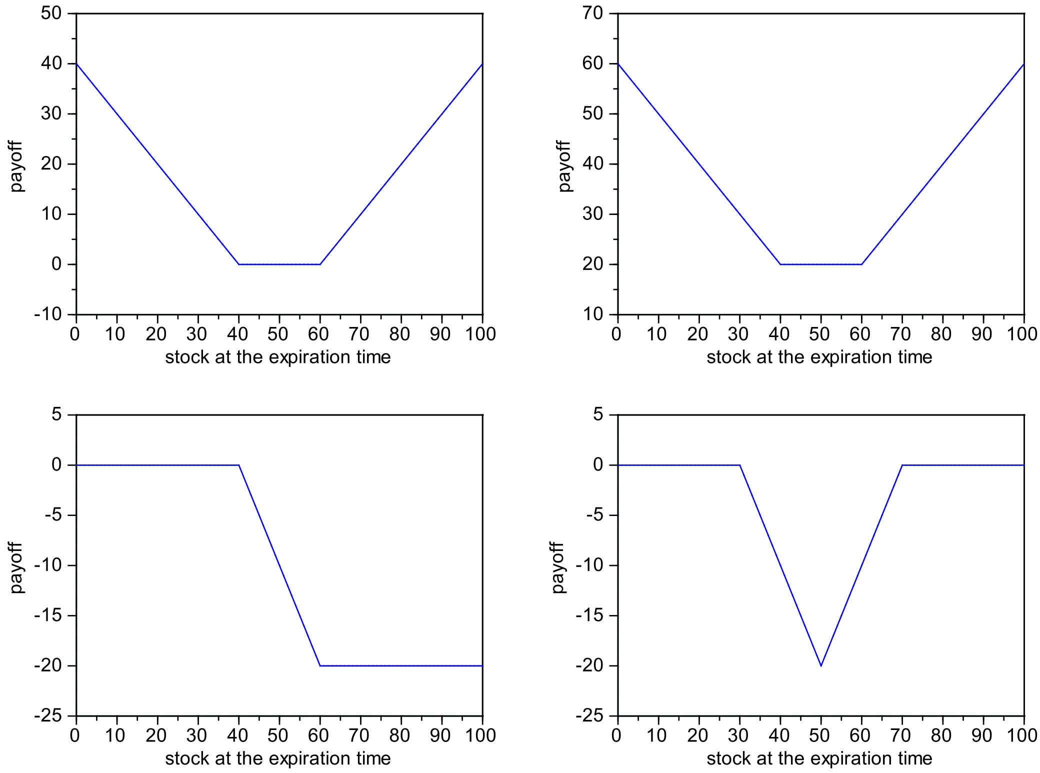


Figure 1.9: Payoffs of the strategies.

- Add an option to the portfolio, so that the possible loss is bounded, while the character of the strategy (bearish, bullish, a given type of an non-directional strategy) is preserved.

1.2.8 Exercise: covered call. So far we considered strategies which involved selling and buying options. However, we can include also stocks into out portfolio.

In the previous section we have shown a warning against the *naked call* strategy - selling a call option alone. Now we consider the **covered call strategy**, which consists of selling a call option, while holding the underlying asset. Sketch the payoff diagram of this strategy and explain how the unlimited risk (which was present when selling the call) is covered.

1.3 Put-call parity

1.3.1 Put-call parity. From a construction of a certain portfolio we can deduce an important property of the options. It relates the price of a call and a put option with the same expiration price, the same expiration time, written on the same underlying asset; if it was not satisfied, it would create an arbitrage opportunity - a certain strategy leading to a riskless profit.

In what follows, the interest rate is denoted by r, the common expiration price of the options by E, the current time by t and the expiration time by T. We assume that the bid and ask prices coincide. A further assumption is, that the *short selling*[8] is permitted.

Consider a portfolio consisting of one put option, minus one call option (i.e., selling one call option) and one stock. The payoff of such a portfolio is

$$\max(0, E - S) - \max(0, S - E) + S = E,$$

[8]*Short selling* a stock means borrowing a stock to sell it immediately, thus creating "debt" of one stock which needs to be returned at a later time to clear the short position; mathematically it means having "minus one stock" in the portfolio.

regardless of the price of the stock at the expiration time. If we receive E USD with certainty, the price of this portfolio today (at time t) has to be equal to $Ee^{-r(T-t)}$, otherwise there would be an arbitrage. It means that

$$\text{price(put)} - \text{price(call)} + S = Ee^{-r(T-t)}. \tag{1.1}$$

1.3.2 Exercise. Assume the following prices on the market, for which the put-call parity (1.1) does not hold: The interest rate is zero. The current price of the stock is 10 USD and there are options with exercise price 12 USD which expire in one month. The call option costs 3 USD, while the put option costs 4 USD. Show that this situation would lead to an arbitrage.

Hint. Let us consider the portfolio from the derivation of the put-call parity, i.e., one put option, minus one call option and one stock. Its price today should be equal to $Ee^{-r(T-t)} = 10$. However, in reality, its price is $4 - 3 + 10 = 11$. The portfolio is more expensive than it should be, hence it is natural that the arbitrage strategy includes selling this portfolio. Finish this idea by explicitly stating the strategy and showing that it indeed produces a riskless profit.

1.3.3 Exercise. Make your own example violating the put-call parity, where the arbitrage is present only for certain positive interest rates (choose a particular value) and show how the arbitrage strategy looks like.

1.3.4 Exercise. Consider a call and a put option on a stock, which both have the exercise price of 55 USD and both expire in one year. The current price of the stock is 53 USD and the price of the call is 0.1 USD higher than the price of the put option. What is the interest rate?

1.3.5 Exercise. Suppose that the interest rate is in fact higher than computed in the previous question. What arbitrage would it cause?

1.3.6 Exercise. Consider a call and a put option on a stock, which have the same expiration price and both expire at the same time. Suppose that the stock price is higher than the present value of the exercise price of the option. Prove that the price of the call option is higher than the price of the put option.

1.3.7 Exercise. [1] Consider the following *chooser option*: At time $t = 1$, the holder will choose whether the option becomes a call option or a put option with a strike price of 100 USD and expiration at time $t = 3$. Today, at time $t = 0$, the price of the stock is 95 USD and the price of the chooser option is 20 USD. The interest rate is zero.

Denote by $C(T)$ the price of a call option with exercise price 100 USD at time $t = 0$ which expires at time $t = T$. We are given that $C(1) = 4$. Determine $C(3)$.

Hint. Let $C(S, t, T)$ be the price at time t of a call option with exercise price of 100 USD which expires at time T, if the stock price at time t is S; define $P(S, t, T)$ analogously for a put option. The value of the chooser option at time $t = 1$ is $\max(C(S(1), 1, 3), P(S(1), 1, 3))$, which can be expressed as $C(S(1), 1, 3) + \max(0, P(S(1), 1, 3) - C(S(1), 1, 3))$. Explain why and use the put-call parity.

1.3.8 Exercise: Stock paying dividends. How does the put-call parity change if the stock pays continuous dividends with a dividend rate q?

Hint. Use the same portfolio as in 1.3.1. What is the effect of dividends on its valuation?

1.3.9 Put-call parity and the real data. Use the real data of option prices and use the last traded price for the price of stock and options. Verify the put-call parity using these data - for example, compute the theoretical price of a put option from the remaining data and compare it with its real price.

Why are there differences and why it does not mean that there are many arbitrage opportunities on the market? One reason is, that we took the last traded prices, and these last trades did not necessarily happen at the same moment. Hence at any moment, the real bid and ask prices might have values which did not permit arbitrage. Also, the stock might be paying dividends, but then they are not the continuous dividends considered in the previous question (it is only an approximation). Moreover, in practice there are transaction costs associated with trading.

1.4 Bounds on options prices

1.4.1 Example from the lectures. The price of a call option is a nonincreasing function of the expiration price (when the expiration time is kept fixed). If this was not satisfied, there would be an arbitrage.

The main idea of deriving this property was a construction of two portfolios. If it holds that

$$\text{price(portfolio I.)} \leq \text{price(portfolio II.)}$$

at the expiration time of the options, the same inequality has to be satisfied also today when trading the options. Otherwise there would be an arbitrage.

1.4.2 Notation and assumptions. In what follows, we denote $c(E)$ and $p(E)$ the prices of call and put options with expiration price E respectively. We do not distinguish bid and ask prices, we assume that there is only one option price and one stock price for which any number of options and stocks (also fractional) can be both sold and purchased. In particular, short selling of stocks is permitted. The options considered expire in τ years. The interest rate *per annum* is r and we can borrow any amount of cash for this interest, as well as give any amount of cash to a bank account and receive the interest.

1.4.3 Example: convexity of the call price as a function of the expiration price. We need to show that for $E_1 < E_2$ and $\lambda \in (0, 1)$ the following inequality holds:

$$c(\lambda E_1 + (1 - \lambda)E_2) \leq \lambda c(E_1) + (1 - \lambda)c(E_2). \tag{1.2}$$

Hence we consider the two portfolios:

1. call option with expiration price $\lambda E_1 + (1 - \lambda)E_2$ which we denote by E,
2. λ call options with expiration price E_1 and $1 - \lambda$ call options with expiration price E_2.

We write their values at the time of the expiration into Table 1.1.

stock at expiration	portfolio I.	portfolio II.	comparison
$S \in [0, E_1)$	0	0	$0 = 0$
$S \in [E_1, E)$	0	$\lambda(S - E_1)$	$0 \leq \lambda(S - E_1)$
$S \in [E, E_2)$	$S - E$	$\lambda(S - E_1)$	$S - E \leq \lambda(S - E_1)$
$S \in [E_2, \infty)$	$S - E$	$S - E$	$S - E = S - E$

Table 1.1: Values of the portfolios at the time of expiration.

Most of the entries in the table are straightforward; some comments might be useful in the following cases:

- The payoff of the portfolio I. in the case $S \in [E_2, \infty)$ is $\lambda(S - E_1) + (1 - \lambda(S - E_2) = S - (\lambda E_1 + (1 - \lambda)E_2 = S - E$, as stated in the table.
- The comparison $S - E \leq \lambda(S - E_1)$ for $S \in [E, E_2)$ follows from writing its left hand side as $S - E = \lambda(S - E_1) + (1 - \lambda)(S - E_2)$ and taking into account the interval in which S lies.

We see that at the time of expiration, the value of the portfolio I. is always less than or equal to the value of portfolio II. Hence the same has to be satisfied for their current prices, which is exactly the inequality (1.2).

1.4.4 Example: Finding an arbitrage. Consider the prices of the call options given in Table 1.2. Visualizing them in Figure 1.10 we see that they form a decreasing function (as they are supposed to) but the exercise prices $E_1 = 105$, $E_2 = 115$ and $E = \lambda E_1 + (1 - \lambda)E_2 = 110$ with $\lambda = 1/2$ violate the inequality (1.2). Therefore we sell $c(E)$ and buy λ of $c(E_1)$ and $1 - \lambda$ of $c(E_2)$, again following the principle of buying what is cheaper that it is supposed to be and selling what is more expensive than it is supposed to be.

expiration price	price of the call option
100	19
105	14
110	11
115	4

Table 1.2: Option prices for the arbitrage finding example.

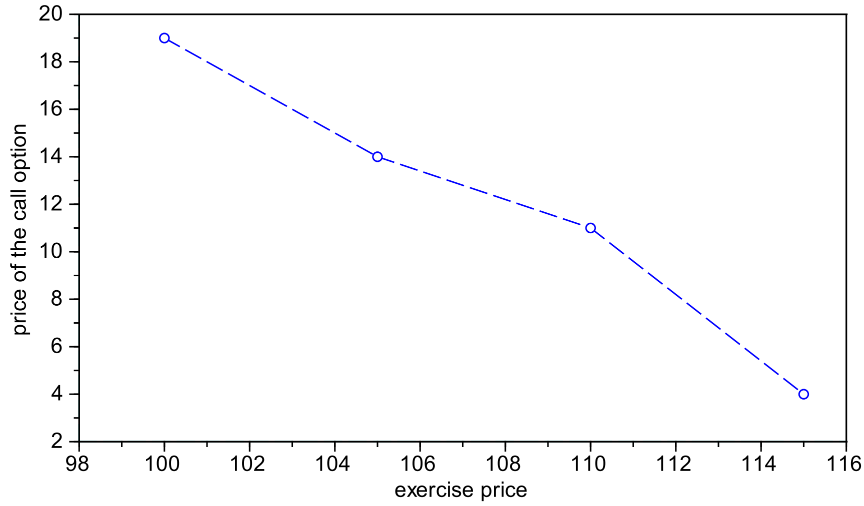


Figure 1.10: Option prices for the arbitrage finding example.

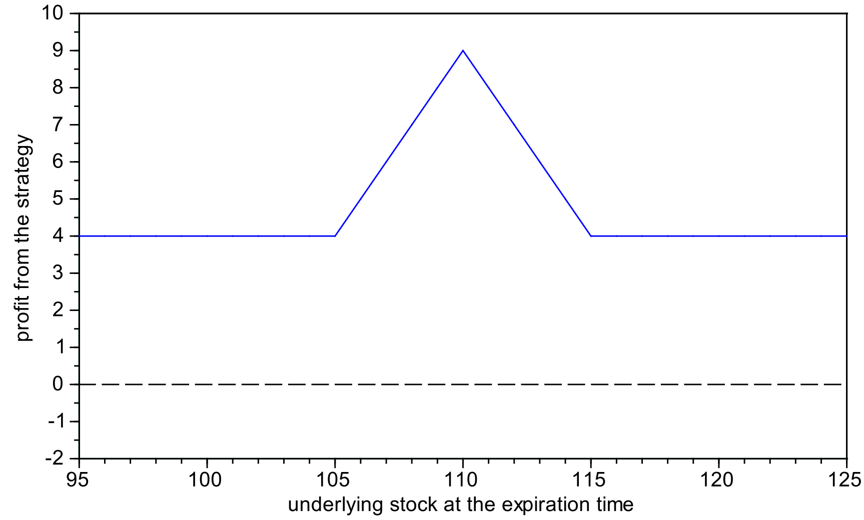


Figure 1.11: Profit diagram of the constructed arbitrage strategy.

Multiplying the amounts by 2 removes the fractions, which may make the strategy more plausible, although it is not necessary. The profit diagram of the resulting strategy is shown in Figure 1.11. It is strictly positive which confirms that the strategy is indeed an arbitrage.

1.4.5 EXERCISE. Replace the expiration price 115 in Table 1.2 by 120 and find an arbitrage in this case.

1.4.6 EXERCISE. Prove that to prevent the arbitrage possibilities, the following properties necessarily hold.

- Price of a put option is a nonincreasing function of the expiration price.
- Price of a put option is a convex function of the expiration price.
- The following inequalities hold: $S - Ee^{-r\tau} \leq c(S) \leq S$.
- If $E_1 \geq E_2$, then $c(E_2) - c(E_1) \leq (E_1 - E_2)e^{-r\tau}$.

For each property construct an example when it is not satisfied and find an arbitrage.

Chapter 2

Basic concepts of the stochastic calculus

2.1 Wiener process

2.1.1 Definition of a Wiener process. A t-parametric system of random variables $\{w(t), t \geq 0\}$ is called a **Wiener process**, if

1. the increments $w(t + \Delta t) - w(t)$ have a normal distribution $\mathcal{N}(0, \Delta)$,
2. for any partition $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0, t]$, the increments $w(t_1) - w(t_0), w(t_2) - w(t_1), \ldots, w(t_n) - w(t_{n-1})$ are independent random variables,
3. $w(0) = 0$ almost surely.

It follows from the Kolmogorov existence theorem that a process with these properties actually exists (cf. the lectures). Moreover, it can be shown there is such a process for which, moreover, the sample paths are continuous almost surely.

In what follows, w denotes a Wiener process satisfying the continuity condition.

2.1.2 Simulating trajectories of a Wiener process. We are going to simulate various random processes so that we can observe their properties, confirm our computations, get an intuition with working with random processes, etc. The first step will be a Scilab function which allows us to generate a trajectory (a sample path) of a Wiener process. We proceed as follows:

- We will simulate an approximation - values for a discrete set of time points, which we join.
- We will simulate the values of a Wiener process at times $0, \mathrm{d}t, 2\mathrm{d}t, \ldots$, where $\mathrm{d}t$ is a sufficiently small time step.
- We know that the value at time $t = 0$ is zero.
- The increment on the time interval $[k\,\mathrm{d}t, (k+1)\,\mathrm{d}t]$ is a normally distributed random variable with the mean equal to zero and the variance equal to $\mathrm{d}t$.

Scilab has quite a general function for simulating random numbers, therefore we first define a simple function which returns a random number from $\mathcal{N}(0, 1)$ distribution.

```
function [r]=randn()
        r=rand(1,"normal");
endfunction
```

Now we can define a function which returns a trajectory of a Wiener process. The input parameters are $\mathrm{d}t$ - the time step used in the simulation and n - number of time steps.

```
function [w]=wiener(dt,n)
        w(1)=0;
        for i=1:n
                dw=sqrt(dt)*randn();
                w(i+1)=w(i)+dw;
        end;
        w=w'    // the output vector w will be a row vector
endfunction
```

Using this function we plot a trajectory:

```
dt=0.001;
n=1000;
time=(0:dt:n*dt);

figure;
plot(time,wiener(dt,n));
```

The result is shown in Figure 2.1.

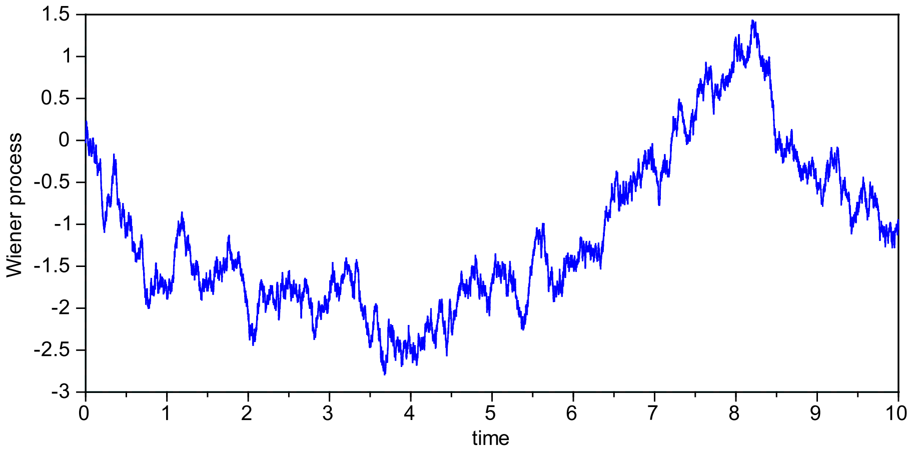


Figure 2.1: A sample path of a Wiener process.

2.1.3 Exercise: Simulating trajectories of a Wiener process. Add several sample paths of a Wiener process into one graph. Show that the probability distribution of $w(t)$ is normal for every t and compute its mean value and variance. To the figure with sample paths add a graph of the mean and 95% confidence intervals (i.e., mean +/- 2 $\times$ standard deviation). See Figure 2.2 for a sample result.

2.1.4 Exercise: Performing simulations. Simulating a process or a random variable associated with it can quickly provide an insight into its properties and behaviour. For both exercises here, the distribution can be derived analytically. This requires a certain effort and in such cases, simulations can be used to ”numerically check” our computations.

1. Denote by t_M the time, in which the sample path of the Wiener process achieved its maximum on the time interval $[0, 1]$. That is,

$$t_M = \text{argmax}\{w(t), t \in [0, 1]\},$$

see Figure 2.3. Plot a histogram[1] by simulating the realizations of the random variable t_M. A sample result can be found in Figure 2.3 as well.

[1]Scilab command for plotting a histogram is `histplot(N,data)`, where `N` is the number of bins.

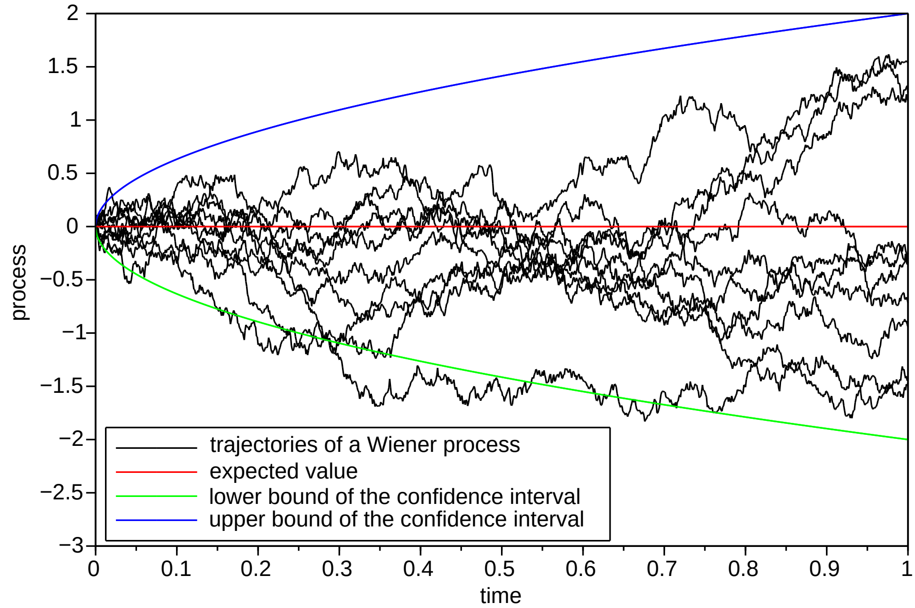


Figure 2.2: Sample paths of a Wiener process, expected value and confidence intervals.

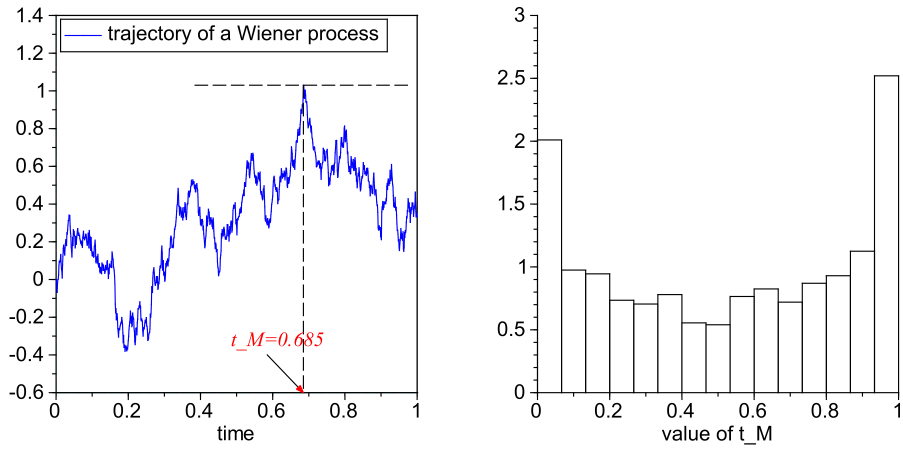


Figure 2.3: Definition of the random variable t_M (left) and histogram of its values (right).

2. Define the random process

$$M(t) = \max_{0 \le s \le t} w(s),$$

which is, at time t, the maximum value of a Wiener process on the interval $[0, t]$. Make a similar plot as shown in Figure 2.4, simulating a trajectory of a Wiener process and determining the corresponding $M(t)$. Then, plot a histogram of $M(1)$ and estimate its expected value.

Note that by taking a maximum over a discrete set of time points instead of the whole interval $[0, 1]$, we are underestimating the maximum corresponding to a given trajectory of a Wiener process. Hence, when for example estimating the expected value, in order to achieve a higher precision it is necessary to refine the time grid, in addition to increasing the number of simulations.

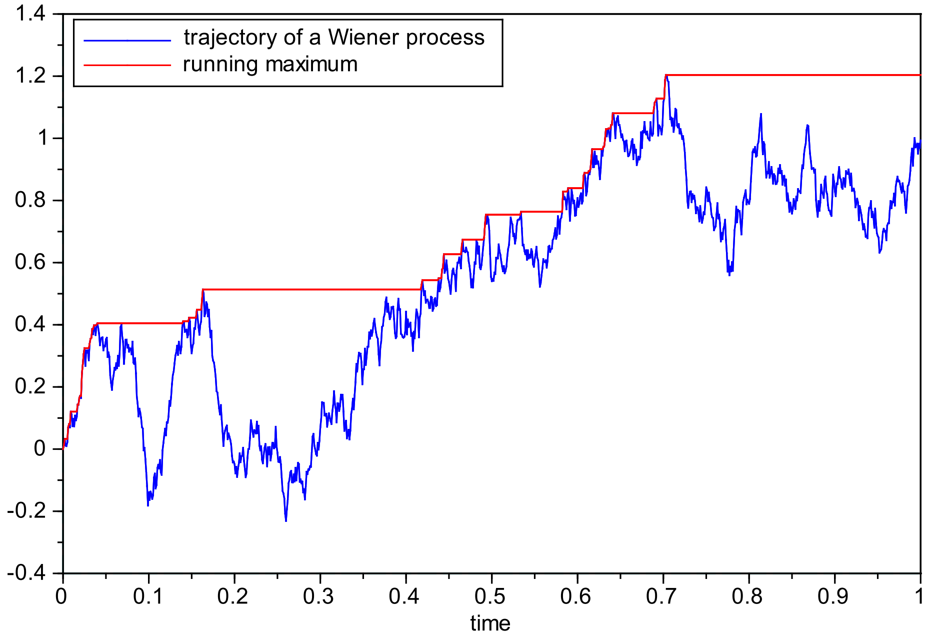


Figure 2.4: Wiener process $w(t)$ and the running maximum $M(t)$.

2.1.5 Exercise: Theoretical computations with a Wiener process.

1. Compute the probability distributions of the following random variables:
   - $x_1 = w(2) - w(1)$,
   - $x_2 = 3\,(w(1) - w(2))$,
   - $x_3 = 5\,w(2)$,
   - $x_4 = w(1) + w(2)$,
     Hint. Note that $w(1)$ and $w(2)$ are not independent. Write x_4 as $x_4 = [w(2) - w(1)] + 2w(1) = [w(2) - w(1)] + 2\,[w(1) - w(0)]$ and use the independence of increments of a Wiener process.
   - $x_5 = 2\,w(1) + 3\,w(2)$,
   - $x_6 = w(1) + w(2) + w(3)$.

2. Let w be Wiener process. Show that the following processes are also Wiener processes (i.e., check that they satisfy the properties from the definition):
   - $w_1(t) = -w(t)$,
   - $w_2(t) = c\,w(t/c^2)$, where $c > 0$ is a constant.

3. Replace the distribution of the increments $w(t + \Delta t) - w(t)$ by $\mathcal{N}(0, \sqrt{\Delta t})$, while leaving the other properties from the definition on a Wiener process unchanged.
   - Show that there is no process satisfying the new conditions.
   - Where does an attempt to apply the Kolmogorov existence theorem (used to establish the existence of a Wiener process) fail?

4. Show that the covariance between the values of the Wiener process are given by $Cov(w(t), w(s)) = \min(t, s)$. Plot the behaviour of the correlation $Cor(x(t), x(s))$ for a fixed t as a function of s.

2.1.6 Exercise. For $t \in [0, \infty)$ define the process

$$x(t) = \frac{w(t)}{1 + t}.$$

- Plot some trajectories of the process. How does the variance change in time? A sample graph showing five trajectories is shown in Figure 2.5. Plot more trajectories, so that the typical behaviour of the process can be better observed.

- Compute the mean and variance of the process analytically. At which time achieves the variance its maximum? What is its limit as time approaches infinity? Compare these results with the simulations of the trajectories.

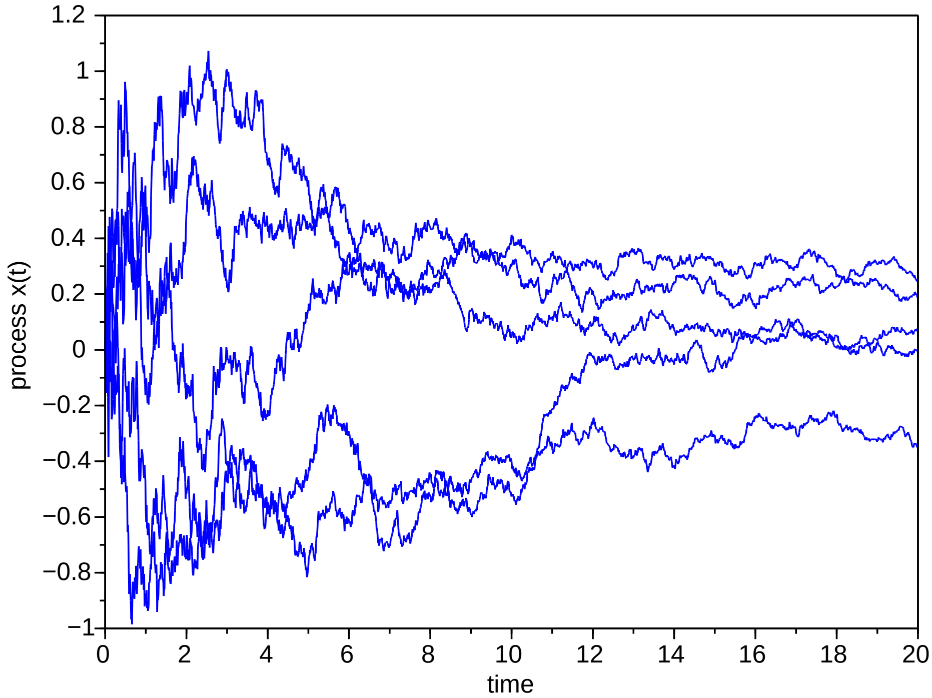


Figure 2.5: Trajectories of the process $x(t) = w(t)/(1+t)$.

2.1.7 Exercise: Brownian bridge. For $t \in [0,1]$ define the process $x(t) = w(t) - tw(1)$. This is known as **Brownian bridge**.

- Plot some trajectories of the process. Where does its name come from?

- Compute the mean and the variance of the process at each time. When is the variance minimal (why?) and when is it maximal?

- Show that the covariance is given by $Cov(x(t), x(s)) = \min(t,s) - ts$. Plot the behaviour of the correlation $Cor(x(t), x(s))$ for a fixed t as a function of s.

2.2 Brownian motion

2.2.1 Definition of a Brownian motion If w is a Wiener process, then the process

$$x(t) = \mu t + \sigma w(t),$$

where μ and σ are constants, is called a **Brownian motion**.

2.2.2 Exercise. Consider the process $x(t) = x_0 + \mu t + \sigma w(t)$.

- Plot some trajectories of the process and note how its typical behaviour depends on the parameters μ, σ, x_0
- The processes in Figure 2.6 are typical trajectories of the following processes:

  1. $x_1(t) = 2\,w(t)$,
  2. $x_2(t) = 0.5\,w(t)$,
  3. $x_3(t) = 3 + 2\,w(t)$,
  4. $x_4(t) = 3 - 2\,w(t)$,
  5. $x_5(t) = -3 + 2\,w(t)$.

  Add the process to the corresponding trajectory.

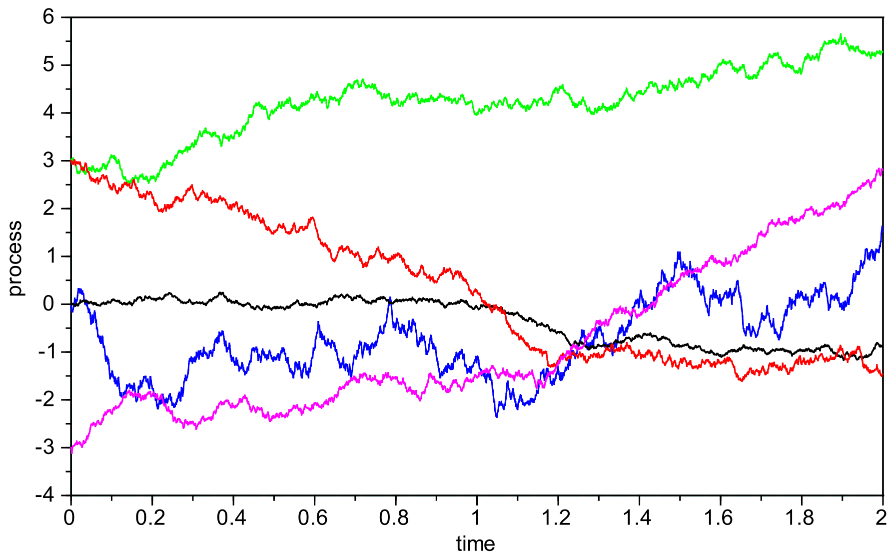


Figure 2.6: Trajectories of the processes of the kind $x(t) = x_0 + \mu t + \sigma w(t)$

2.3 Geometrical Brownian motion

2.3.1 Definition of a geometrical Brownian motion. If w is a Wiener process, then the process

$$x(t) = x_0 e^{\mu t + \sigma w(t)},$$

where μ, σ and x_0 are constants, is called a **geometrical Brownian motion**.

2.3.2 Lognormal probability distribution. Geometrical Brownian motion is closely related to a **lognormal probability distribution**. Recall that a random variable X has a lognormal distribution if $\log(X)$ (by $\log$ we denote the natural logarithm) has a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Then the probability density function of the variable X is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

The first two moments of the random variable X are given by

$$\begin{aligned} \mathbb{E}[X] &= e^{\mu t + \frac{1}{2}\sigma^2 t}, \\ \mathbb{D}[X] &= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \end{aligned}$$

2.3.3 Probability distribution of a geometrical Brownian motion. Based on the given properties of the lognormal distribution, derive the density and the first two moments of the value of a geometrical Brownian motion at time t.

Simulate trajectories of a geometrical Brownian motion with selected parameters; a sample result is shown in Figure 2.7. Add the expected value of the process to the graph.

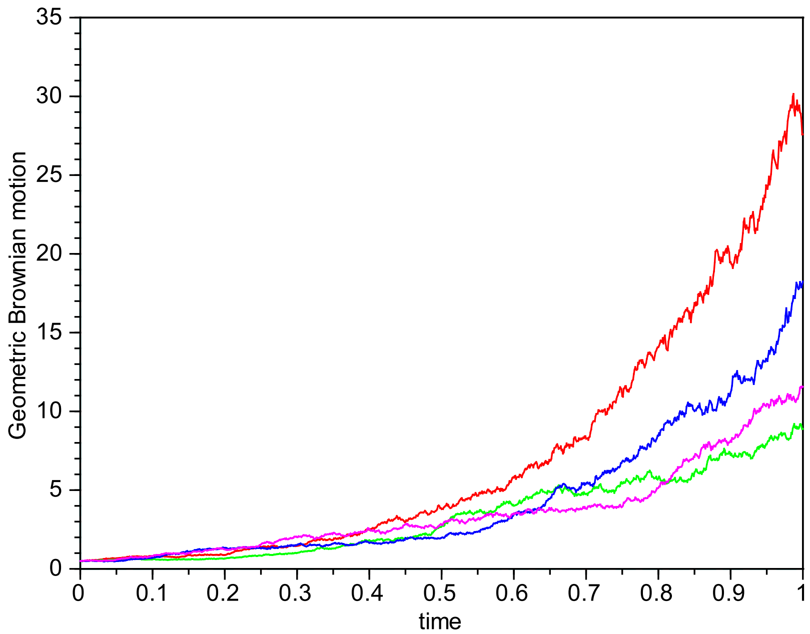


Figure 2.7: Geometrical Brownian motion.

2.3.4 Modelling stock prices with a GBM, estimating the parameters. Geometrical Brownian motion can be used as a simple model for the stock prices[2]. It means that if the initial stock price S_0 at time $t = 0$ is given, the future stock prices are modelled as

$$S(t) = S_0 e^{\mu t + \sigma^2 t}. \tag{2.1}$$

Then the (logarithmic) returns

$$\text{return}_t = \log\left(\frac{S_t}{S_{t-\Delta t}}\right) = \mu \Delta t + \sigma \Delta w$$

are independent random variables with $\mathcal{N}(\mu\Delta t, \sigma^2 \Delta t)$ distribution, which is a base for the estimation of the parameters from the stock price data.

We rewrite the estimation procedure from the lectures into Scilab, assuming that the data are given in the text file `stock.txt` which is located in the current working directory. Finish the code using the outline below:

```
s=fscanfMat("stock.txt");

dt=1/252 ; // time step, in years
           // 1/252 for the daily data
returns= ; //create the vector of returns
```

[2]It is also one of the assumptions of the Black-Scholes model for pricing derivatives, which we will study later.

```
muDelta=mean(returns);              // estimate of mu*dt
sigma2Delta=variance(returns);      // estimate of (sigma^2)*dt
mu=muDelta/dt                       // estimate of mu
sigma=sqrt(sigma2Delta/dt)          // estimate of sigma
```

Download the historical data[3] for a selected stock from `finance.yahoo.com` or `finance.google.com` (see Figure 2.8 for a snapshop of the data table at `finance.google.com`) and use them for the following tasks:

- Display the evolution of the stock prices and the returns.
- Estimate the parameters of the geometrical Brownian motion.
- Add the estimated expected value of the stock price in the future, conditioned on its last observed value.

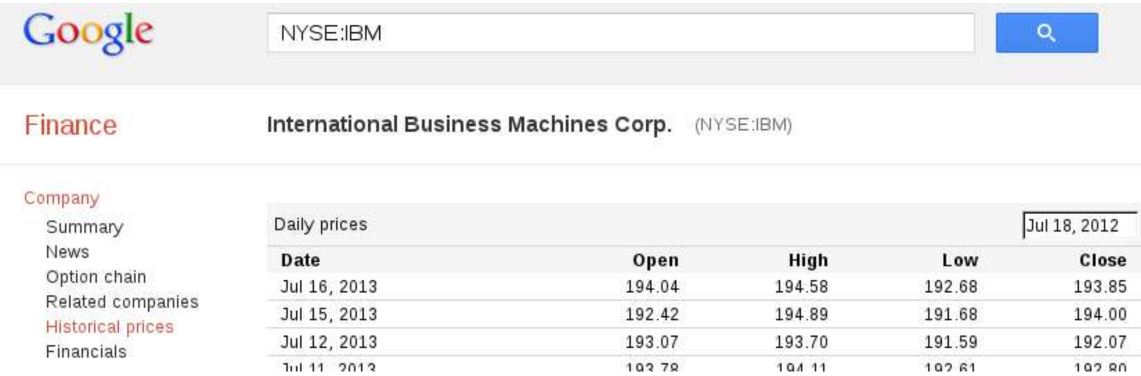


Figure 2.8: Historical stock price data at `finance.google.com`.

2.3.5 Exercise: Forecasting the stock price evolution using GBM. Suppose that the stock price follows the geometric Brownian motion (2.1) with parameters $\mu = 0.15$ and $\sigma = 0.20$. The price of the stock today is 120 USD.

- Plot the density of the stock price in one month. Perform simulations: generate 1000 values of the stock price and plot their histogram. Compare the two plots.

  Hint. Outline of the Scilab code for the density:

```
// density of S(t) for the given S(0)=s0
// when modelling S by  a geometric BM:  S(0)*exp(mu*t+sigma*w(t))
function [pdf]=densityS(s,mu,sigma,s0,t)
          muS=log(s0)+mu*t;
          sigma2S=t*sigma^2;
          pdf=exp(-(log(s)-muS).^2/(2*sigma2S))./...
                    (s*sqrt(2*%pi*sigma2S));
endfunction

// left: exact density, right: histogram from simulations
figure;
subplot(1,2,1);
s= ;   // suitable range of s (stock price)
f= ;   // densityS(...) - density values
plot(s,f);
```

[3]Note that the data are usually displayed with the most recent in to the top. It is easy to adjust the computation of the parameters, but it is better to adjust the data - in order to be able to produce graphs, etc.

and for the simulated values:

```
subplot(1,2,2);

// vector of 1000 iid N(0,1) realizations
n=rand(1,1000,"normal");
// use the vector n to simulate the stock prices
Ssim= ;

// plot a histogram: histplot(N,data), where N = number of bins
```

A sample result is shown in Figure 2.9.

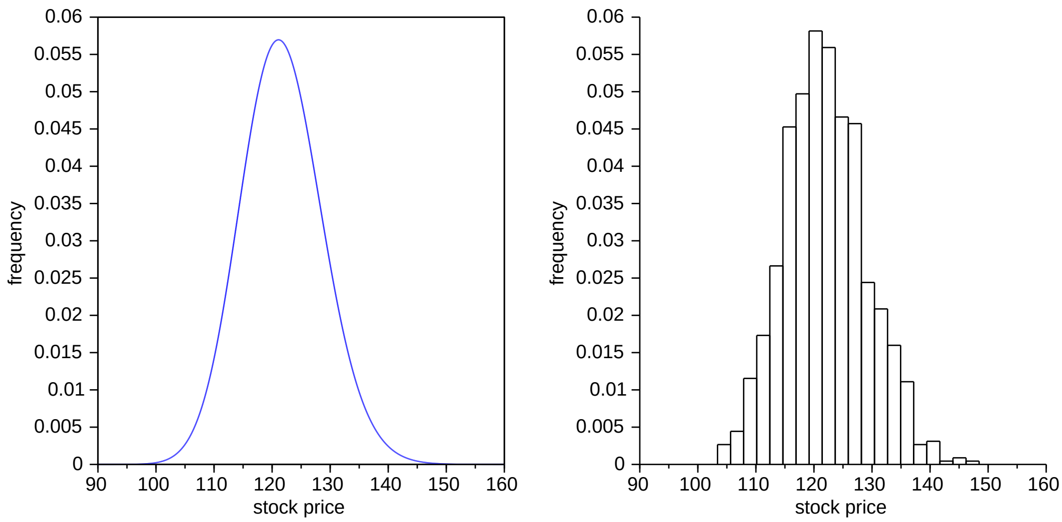


Figure 2.9: Density of the future stock price and histogram of the simulated values.

- What is the expected value of the quarterly return? What is the probability that it will be negative?

  Hint. We can simplify the original Scilab function for cumulative distribution function and define a new function `normcdf` for the $\mathcal{N}(0,1)$ distribution which has only one parameter:

```
function [cdf]=normcdf(x)
    cdf=cdfnor("PQ",x,0,1);
endfunction
```

- What is the probability that in one year the stock price exceeds 150 USD? What is the probability that it falls below 100?

2.3.6 Exercise: Estimating the parameters of a GBM and choice of the time period. Select a stock and download the historical data of its prices. Estimate the parameters of the GBM, using the data from different time period: last quarter, last year, last couple of years, etc. Plot the stock prices for each period, list the estimates and comment on the differences.

2.3.7 Exercise: Pricing options - Monte Carlo method. In this course we are going to study derivatives pricing using the partial differential equation approach. Another alternative is probabilistic characterization of the option price. This allows the computation of the derivatives prices by simulations.

In this exercise we outline the basic idea. We assume that the stock price behaves according to the geometric Brownian motion (2.1) and that the derivative has the payoff $\bar{V}(S)$ at the time of its expiration.

Firstly, we need to realize that the correct price of the derivative is not the expected value of $\bar{V}(S)$ (or its discounted value, taking interest rates into account). The derivative is not a simple "bet" in which we receive $\bar{V}(S)$ with a certain probability distribution of S. The difference from a bet is, that in our case it is possible to trade also the underlying stock.

It can be shown that the price *is* the discounted expected value of the payoff, but under another - so called *risk neutral* - probability measure. The stock price follows a geometric Brownian motion also under this risk neutral measure, but instead of the parameter μ there is $r - \frac{\sigma^2}{2}$ where r is the interest rate. The volatility σ remains the same.

This means that the price of the derivative at time t, when the stock price equals S, equals

$$V(S,t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[\bar{V}(S) \right],$$

where T is the expiration time and $\mathbb{Q}$ denotes the risk neutral measure. The expected value can be approximated by the arithmetic average of the realizations, and hence

$$V(S,t) \approx e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} \bar{V}(S_i),$$

where $S_1, \ldots, S_N$ are simulated values of the stock price at the expiration time under the risk neutral measure $\mathbb{Q}$.

Let us consider a specific call option:

```
// GBM for the stock price:
mu=0.35;
sigma=0.30;

// Current stock price
s0=150;

// Call option
E=175;     // exercise price
tau=1/2;   // time to expiration

// Interest rate
r=0.01;
```

At the time of expiration we have:

```
Z=randn();          // N(0,1), randn() has been defined earlier
wT=sqrt(tau)*Z;     // Wiener process at time T

// risk neutral measure
// => "r-0.5*sigma^2" instead of "mu"  in the GBM
sT=s0*exp((r-0.5*sigma^2)*tau+sigma*wT);  // stock at time T
vT=max(0,sT-E);                           // option at time T
```

Repeat this in a loop and after each new simulation compute the current approximation of the option price based on the simulations performed so far. The approximations converge to the options price with increasing number of simulations. A sample result is shown in Figures 2.10 (convergence of the intermediate results) and 2.11 (histogram of 1000 values attained after 10000 simulations). It is possible to price this option also analytically[4] using the Black-Scholes formula (this will be studied later), its price is 4.8572 USD.

Note that the variance of the Monte Carlo simulations performed in this direct way is quite high. There are methods, so called *variance reduction methods* whose aim is to decrease this variance[5]

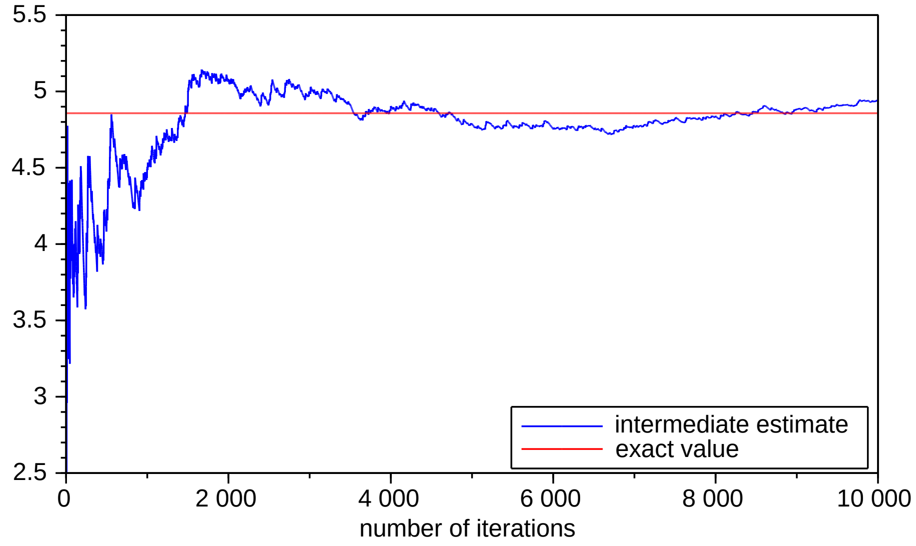


Figure 2.10: Computing the option price using Monte Carlo method.

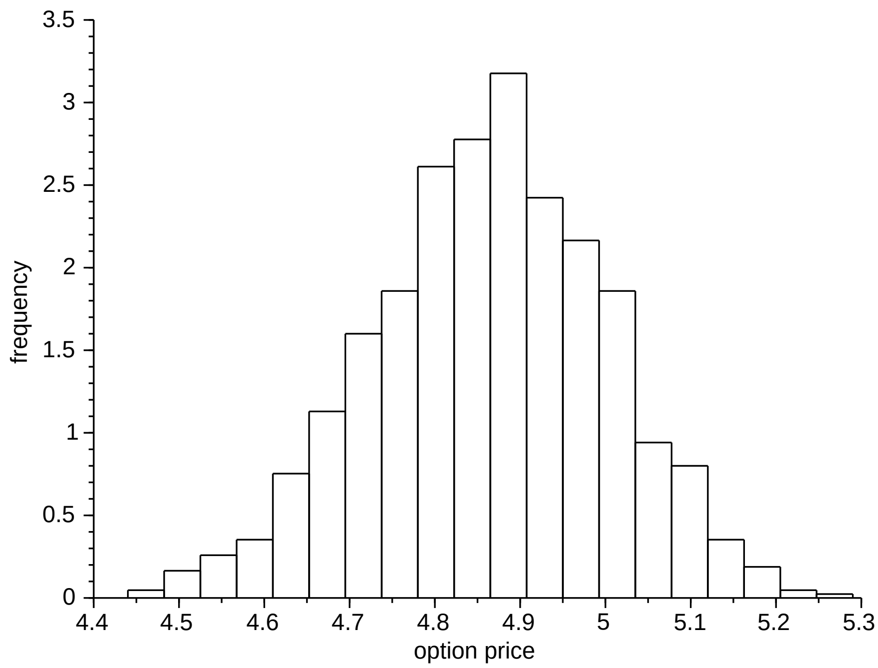


Figure 2.11: Computing the option price using Monte Carlo method - histogram of the values after 10000 simulations.

2.3.8 Exercise. Find the cover of the 6th edition of the book *Stochastic differential equations* by Berndt Oksendal. What is in the picture? Produce such a plot.

[4]This is not possible or the more complicated derivatives, hence the need for approximate methods, for example Monte Carlo simulations. Using these methods in the simple cases with analytical solution enables us to test their efficiency and precision.

[5]For a simple introduction to these techniques see the chapter 3.5.4. of the book [4].

2.4 Stochastic differential equations, Itō lemma

2.4.1 Exercise: Ornstein-Uhlenbeck process. Consider the stochastic differential equation

$$\mathrm{d}x = \kappa(\theta - x)\mathrm{d}t + \sigma \mathrm{d}w, \quad (2.2)$$

where κ, θ, σ are positive constants. In this example we will see how to visualize a process given by a stochastic differential equation by discretizing it, and its dependence on parameters.

- For the beginning disregard the stochastic part; this leaves an ordinary differential equation $\mathrm{d}x = \kappa(\theta - x)\mathrm{d}t$. How does its solution (for a given initial condition $x(0) = x_0$) look like? What is the role of the parameters κ and σ, how do they influence the behaviour of the solution?

- The stochastic parts ads fluctuations. We can get a quick idea about the process by discretizing it. The simplest way is replacing the differentials in (2.2) by differences:

$$\begin{aligned} \Delta x &= \kappa(\theta - x)\Delta t + \sigma \Delta w, \\ x_{i+1} &= x_i + \kappa(\theta - x_i)\Delta t + \sigma \Delta w, \end{aligned}$$

  where Δw in each step are independent random variables with $\mathcal{N}(0, \Delta t)$ distribution. Being a generalization of the Euler method for obtaining a numerical solution of an ordinary differential equation, this procedure is known as **Euler-Maruyama method**.

```
kappa=10; theta=1; sigma=0.25;     // parameters
x0=0.5;                            // initial value

dt=0.001; // time step
n=1000;   // number of steps

// Euler-Maruyama
x(1)=x0;
for i=1:n
        dw=sqrt(dt)*randn();
        dx=kappa*(theta-x(i))*dt+sigma*dw;
        x(i+1)=x(i)+dx;
end;

t=0:dt:n*dt; // time
figure;
plot(t,x);
```

  A sample path generated in this way is shown in Figure 2.12.

- Define a Scilab function of the following form which returns a vector with the values of an Ornstein-Uhlenbeck process:

```
function [x]=ou(dt,n,x0,kappa,theta,sigma)
// insert the body of the function
endfunction

// Then we can make plots such as:
plot(t,ou(dt,n,x0,kappa,theta,sigma));
```

  Plot the trajectories of the process with different parameters to get an intuition about the behaviour of the process.

- The trajectories in Figure 2.13 we obtained for the following sets of parameters:

  1. $\kappa = 20, \theta = 1, \sigma = 0.5$,

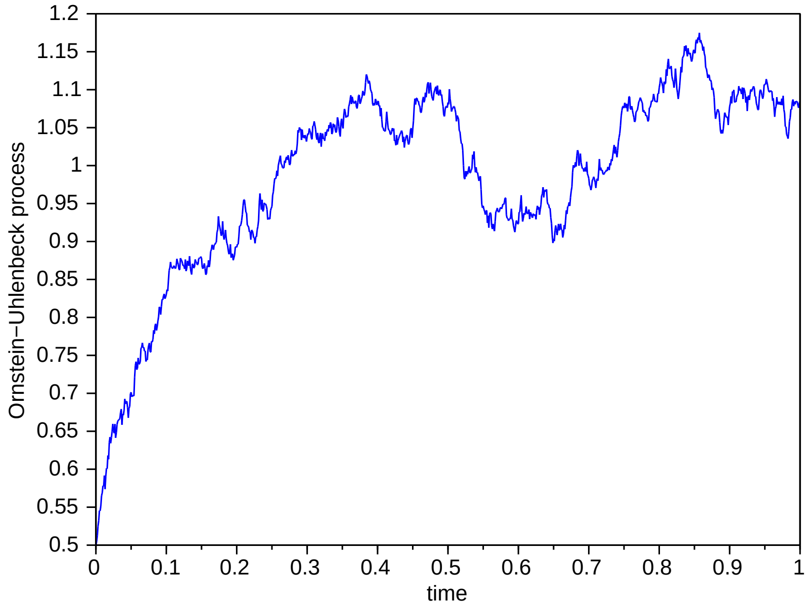


Figure 2.12: A sample path of an Ornstein-Uhlenbeck process obtained by Euler-Maruyama discretization.

2. $\kappa = 3, \theta = 1, \sigma = 0.5$,
3. $\kappa = 10, \theta = 3, \sigma = 0.2$,
4. $\kappa = 10, \theta = 3, \sigma = 0.5$.

Match the parameter set with the corresponding trajectory in Figure 2.13.

2.4.2 EXERCISE: ORNSTEIN-UHLENBECK PROCESS IN FINANCIAL MODELLING. One of the applications of the OU process in finance is modelling interest rates. So far, we have considered constant interest rates, which is a simplification of reality. We return to interest rate modelling in later chapters, now we show an example to illustrate the use of the OU process.

We refer to the paper *Further evidence on alternative continuous time models of the short-term interest rate* published in 2000[6], in which data from many countries have been analyzed. The model considered is

$$\mathrm{d}r = (\alpha + \beta r)\mathrm{d}t + \sigma r^{\gamma}\mathrm{d}w \tag{2.3}$$

Note that for $\gamma = 0$ we obtain constant volatility, as is the case of the OU process, which is known as **Vasicek process** in the context of interest rate modelling. Since both (2.2) and (2.3) have a general form of a linear drift, so they are only reparametrizations of the same function. We are going to work with the parameters which are given in Table 2.1 (we present only the estimates of the parameters and omit their standard deviations and other statistics associated with them). They were estimated from monthly data of New Zealand interest rates from April 1986 to April 1998. The table shows the estimates of the general model (2.3), Vasicek model (which we are going to use) and CIR model (which is studied later in lectures, when dealing with interest rate modelling).

- Write the Vasicek process in the parameterization with κ, θ, σ. What is the level, to which the interest rates converge according to this models?
- Plot some trajectories starting from a chosen initial value and see the speed of the convergence to the equilibrium level computed above.

[6] A. Episcopos, *Further evidence on alternative continuous time models of the short-term interest rate*, Journal of International Financial Markets, Institutions and Money 10 (2000) 199-212.

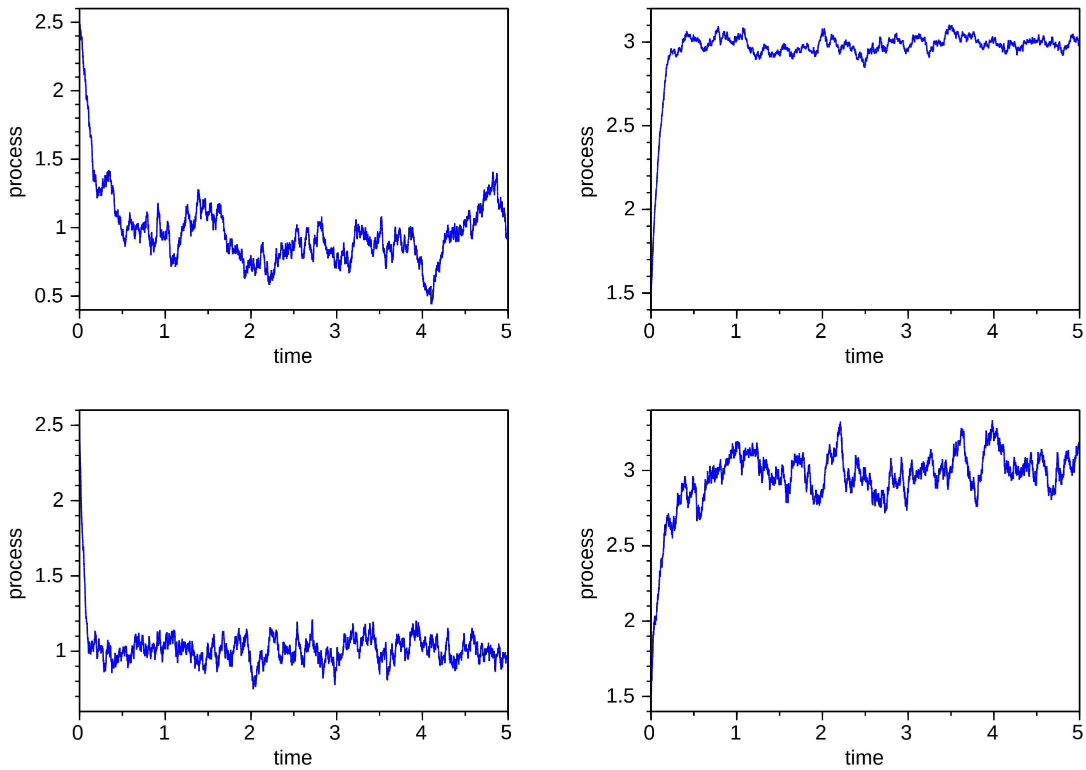


Figure 2.13: Sample path of an Ornstein-Uhlenbeck processes with different parameters.

- How would you need to adjust the parameters to observe a higher speed of convergence? Generate trajectories with the suggested value and show the difference.
- Suppose that the model is estimated with the current data. What differences in the estimates of the parameters would you expect?

Model	α	β	σ^2	γ
Unrestricted	0.0045	-0.048	0.0034	0.7815
Vasicek	0.0046	-0.0487	0.0001	0
CIR	0.0041	-0.0447	0.0010	0.5

Table 2.1: Estimated parameters for the process $\mathrm{d}r = (\alpha + \beta r)\mathrm{d}t + \sigma r^{\gamma}\mathrm{d}w$ and some of its special cases. (Episcopos, 2000)

2.4.3 Itō lemma Let x be a process satisfying the stochastic differential equation

$$\mathrm{d}x = \mu(x,t)\,\mathrm{d}t + \sigma(x,t)\,\mathrm{d}w$$

and $f = f(x,t)$ be a smooth function. Then the process $y = f(x)$ satisfies

$$\begin{aligned} \mathrm{d}y &= \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\,\mathrm{d}t \\ &= \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)\,\mathrm{d}t + \sigma(x,t)\frac{\partial f}{\partial x}\,\mathrm{d}w. \end{aligned} \tag{2.4}$$

2.4.4 Example: Application of the Itō lemma. Let x be a process satisfying the stochastic differential equation $\mathrm{d}x = 2x\,\mathrm{d}t + x\,\mathrm{d}w$. We define $y = e^{2t}x^2$ and compute $\mathrm{d}y$ using Itō lemma.

SOLUTION I. We use the formula (2.4). In this case we have $\mu(x,t) = 2x$, $\sigma(x,t) = x$ and $f(x,t) = e^{2t}x^2$. We compute the partial derivatives

$$\frac{\partial f}{\partial t} = 2e^{2t}x^2, \frac{\partial f}{\partial x} = 2e^{2t}x, \frac{\partial^2 f}{\partial x^2} = 2e^{2t}.$$

and substituting them into (2.4) we obtain

$$\begin{aligned} \mathrm{d}y &= \left(2e^{2t}x^2 + (2x)(2e^{2t}x) + \frac{1}{2}(x^2)(2e^{2t})\right)\mathrm{d}t + x(2e^{2t}x)\,\mathrm{d}w \\ &= 7y\,\mathrm{d}t + 2y\,\mathrm{d}w \end{aligned}$$

SOLUTION II. We do not need to remember the formula (2.4). Instead we can proceed as in its derivation (cf. the lectures) - compute the expansion up to the second order with respect to t and x, use the "rule" $(\mathrm{d}w)^2 = \mathrm{d}t$ and ignore the terms of higher order than $\mathrm{d}t$ and $\mathrm{d}w$:

$$\begin{aligned} \mathrm{d}y &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}x)^2 \\ &= (2e^{2t}x^2)\,\mathrm{d}t + (2e^{2t}x)(2x\,\mathrm{d}t + x\,\mathrm{d}w) + \frac{1}{2}(2e^{2t})(2x\,\mathrm{d}t + x\,\mathrm{d}w)^2 \\ &= 2y\,\mathrm{d}t + (4y\,\mathrm{d}t + 2y\,\mathrm{d}w) + \frac{1}{2}(2e^{2t})(x^2)\,\mathrm{d}t \\ &= 2y\,\mathrm{d}t + (4y\,\mathrm{d}t + 2y\,\mathrm{d}w) + y\,\mathrm{d}t = 7y\,\mathrm{d}t + 2y\,\mathrm{d}w. \end{aligned}$$

2.4.5 EXERCISES: ITŌ LEMMA.

1. Compute dy for the following processes
   - $y = e^{-t}x^3$, where $\mathrm{d}x$ is the same as in the previous problem, i.e., $\mathrm{d}x = 2x\mathrm{d}t + x\mathrm{d}w$,
   - $y = \log x$, where $\mathrm{d}x = -2x\mathrm{d}t + 3x\mathrm{d}w$,
   - $y = e^{2x}$, where $\mathrm{d}x = 3\mathrm{d}t + 4\mathrm{d}w$.
2. Compute dy for the following processes (there may be several possibilities for choosing a process x and a function $f(x,t)$):
   - $y = 3t + 2w$,
   - $y = 3e^{2w}$,
   - $y = e^t w$,
   - $y = 2e^{3t+4w}$,
   - $y = t^2 e^w$.

2.4.6 EXERCISE: STOCHASTIC DIFFERENTIAL EQUATION FOR A GEOMETRIC BROWNIAN MOTION. Let y be a geometric Brownian motion $y = y_0 e^{\mu t + \sigma w}$.

- Compute $\mathrm{d}y$.
- The assumption of the geometrical Brownian motion for the stock price S is often written in the differential form as $\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}w$. Write S in the explicit form and note that it is *not* equal to $S_0 e^{\mu t + \sigma w}$.
- Suppose that the model for the stock price is given in the differential form as $\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}w$. How do you estimate parameters μ and σ from the data; what changes in the algorithm from 2.3.4 are necessary?

2.4.7 EXERCISE: STOCHASTIC DIFFERENTIAL EQUATION. Show that the process $x = \frac{w}{1+t}$ satisfies the stochastic differential equation

$$\mathrm{d}x = -\frac{x}{1+t}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}w$$

and the initial condition $x(0) = 0$.

2.4.8 On uniqueness of a solution to a stochastic differential equations. Similarly as it is the case for ordinary differential equation, the existence and uniqueness of a solution to a stochastic differential equation does not automatically hold. There are conditions which guarantee it (see, for example, [3] or other text specialized on stochastic processes); here we present an example showing the possibility of nonuniqueness of a solution. Compare this result with the ordinary differential equation obtained by taking only the deterministic part of the equation and a question of uniqueness of a solution to this ODE.

Consider the stochastic differential equation

$$\mathrm{d}x = \frac{1}{3}x^{1/3}\mathrm{d}t + x^{2/3}\mathrm{d}w$$

with the initial condition $x(0) = x_0$.

- Show that the process $x = \left(x_0^{1/3} + \frac{1}{3}w\right)^3$ is a solution
- Find another solution in the case $x_0 = 0$.

Based on this example, write another, analogous stochastic differential equation with an initial condition, for which the solution is not uniquely determined.

2.4.9 Solving stochastic differential equations. In certain cases we are able to solve a given stochastic differential equation. The following problems show some of the methods that can be used - guessing a form of the solution (and determining the coefficients) and substitution leading to an equation which is easier to solve.

1. Solve the following stochastic differential equation, i.e., find a process which satisfies the given SDE and the initial condition:
$$\mathrm{d}x = 4\mathrm{d}t + 3\mathrm{d}w,\ x(0) = 2.$$
Hint. The solution is a shifted Brownian motion.
2. Solve the equation
$$\mathrm{d}x = 2x\mathrm{d}t + x\mathrm{d}w,\ x(0) = x_0$$
by the following two methods:
    - Note that this is a special case of a geometric Brownian motion; hence write a general form of a GBM, compute its differential and match the coefficients.
    - Alternatively, make a substitution $y = \log x$, find the stochastic differential equation for y, solve it and finally express the original process x.
3. Use the same two methods as in the previous problem to solve the equation
$$\mathrm{d}x = x\mathrm{d}w,\ x(0) = x_0.$$
4. Use a substitution $y = e^t x$ to solve the equation
$$\mathrm{d}x = -x\mathrm{d}t + e^{-t}\mathrm{d}w,\ x(0) = x_0.$$

2.4.10 Exercise. Suppose that the random process x can be written as $x = f(y, z)$ and we know the differentials $\mathrm{d}y$ and $\mathrm{d}z$. How can they be used to compute $\mathrm{d}x$? Show this procedure on computation of $\mathrm{d}x$, where

- $x = (w^3 - t^2)\, e^{w-t^4}$,
- $x = (w^3 - t^2)^2\, e^{w-t^4}$.

Compare with a direct application of the Itō lemma.

2.4.11 Exercise: Accuracy of the Euler-Maruyama discretization. In Exercises 2.1.6 and 2.4.7 we studied the process

$$x = \frac{w}{1+t}$$

and we have seen that it satisfies the stochastic differential equation

$$\mathrm{d}x = -\frac{x}{1+t}\mathrm{d}t + \frac{1}{1+t}\mathrm{d}w$$

with the initial condition $x(0) = 0$. Having an explicit solution to the stochastic differential equation enables us to check the accuracy of the Euler-Maruyama discretization scheme. We generate the sample path of the process and then we use the same trajectory of the Wiener process in the discretization. The outline of the script is the following:

```
T=5;          // time interval [0,T]
dt=0.001;     // time step
t=0:dt:T;     // vector of time

// generate the trajectory of a Wiener process
// common to both exact process and discretization
w=...

// x = exact process
// use its definition and w generated above
x=...

// xD = discretization of the SDE
// note that dw needs to be computed from w above
xD(1)=...;
for ...
        xD(i+1)=... ;
end;

// compare
plot(t,x);
plot(t,xD,'r');
```

Finish the script and run it for different time steps. Sample results are shown in Figure 2.14.

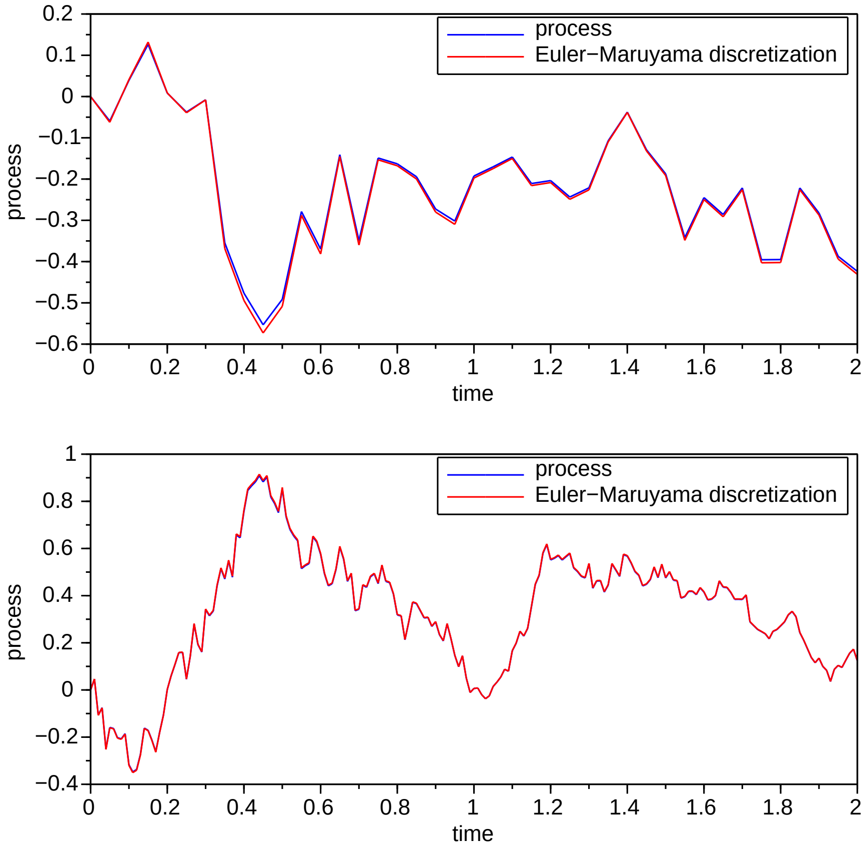


Figure 2.14: Checking the accuracy of Euler-Maruyama discretization: time step equals 0.05 (above) and 0.01 (below).

Chapter 3

Black-Scholes model

3.1 Black-Scholes partial differential equation

3.1.1 Black-Scholes partial differential equation. Under the assumptions of the Black-Scholes model (cf. the lectures), the price $V = V(S,t)$ of a European-type derivative satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3.1}$$

for the stock price $S \in (0,\infty)$ and time $t \in [0,T)$, where T is the expiration time. In the equation (3.1), r is the interest rate and σ is a volatility of a stock which is modelled by a geometric Brownian motion $\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}w$.

The value at the time of expiration is given by the payoff as $V(S,T) = \bar{V}(S)$ for $S \in (0,\infty)$.

3.1.2 Exercise: Some simple solutions to the Black-Scholes PDE. Show that the following function satisfy the Black-Scholes partial differential equation (3.1). What is their value at time $t = T$ and what is the financial interpretation of the price $V(S,t)$?

- $V(S,t) = Ke^{-r(T-t)}$,
- $V(S,T) = KS$,

where $K > 0$ is a constant

3.1.3 Exercise: Put-call parity as a consequence of linearity of the Black-Scholes PDE.

- What does it mean that a PDE is linear? Verify that the Black-Scholes PDE (3.1) is linear.
- Denote by $V^{call}(S,t)$ and $V^{put}(S,t)$ the prices of a European call and put options with the same exercise prices and define a new function $V(S,t) = V^{put}(S,t) - V^{call}(S,t)$. Derive the partial differential equation and the terminal condition at time $t = T$ for the function $V(S,t)$.
- Using the previous exercise and the linearity of the PDE find the solution $V(S,t)$. Finally, from the definition of V write the relation between $V^{call}(S,t)$ and $V^{put}(S,t)$ (put-call parity).

3.1.4 Exercise: Solution to the Black-Scholes PDE in the special form. Find the solutions to the Black-Scholes PDE (3.1) which have the form $V(S,t) = V(S)$, i.e., the price of the derivative depends only on the current price of the stock. (Note that as a special case of your general solution you should obtain also a solution KS from an earlier exercise.)

3.1.5 Exercise: Mathematical properties of the solutions and their financial interpretation. Prove the following properties using only the mathematical formulation of $V(S,t)$ as a solution to the Black-Scholes PDE (3.1). Then, give their financial interpretation.

- If $\bar{V}(S) \geq 0$ and is positive on some interval, then $V(S,t) > 0$ for all $S > 0$ and $t \in [0,T)$.

- Suppose that $\bar{V}(S)$ is a differentiable function of S and that $\frac{\partial \bar{V}(S)}{\partial S}(S) > 0$ for all S. Prove that then also $\frac{\partial V}{\partial S}(S,t) > 0$ for all S and t.
  Hint. Firstly show the following: if $V(S,t)$ is a solution to (3.1), then also $S\frac{\partial V}{\partial S}(S,t)$ is a solution.

3.1.6 Exercise: Adding continuous dividends to the Black-Scholes model. Suppose that the stock pays continuous dividends with dividend rate q. The remaining assumptions remain the same.

- Following the lectures and the derivation of the Black-Scholes model, modify the equation for the change of the portfolio value if the stock pays dividends as described above.

- Repeat the subsequent steps of the derivation and show that the partial differential equation for the price of a derivative changes to

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3.2}$$

  for the stock price $S \in (0,\infty)$ and time $t \in [0,T)$. The value at the time of expiration is again given by the payoff as $V(S,T) = \bar{V}(S)$ for $S \in (0,\infty)$.

- Show that $V(S,t) = S$ is not a solution to (3.2) if $q > 0$. Why? Find the solution to the equation (3.2) with the terminal condition $V(S,T) = S$ for $S \in (0,\infty)$. What is its financial interpretation?

- Based on the previous question, modify the put-call parity for the case of positive continuous dividends.

3.2 Pricing call and put options, and combined strategies

3.2.1 Black-Scholes formula for a call option. The solution to the equation (3.2) with the terminal condition $V(S,T) = \max(0, S-E)$, i.e., the price of a European call option can be expressed using the cumulative distribution function N of a normalized normal distribution $\mathcal{N}(0,1)$:

$$V(S,t) = Se^{-q\tau}N(d_1) - Ee^{-r\tau}N(d_2), \tag{3.3}$$

where

$$d_1 = \frac{\log(S/E) + (r+q+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, d_2 = \frac{\log(S/E) + (r-q-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \tag{3.4}$$

and $\tau = T - t$ is time remaining to expiration of the option.

At this point we suggest writing the following functions into a separate `.sci` Scilab file. These functions will be used in the various exercises and keeping them separated adds to clarity of the files and computations.

```
// we redefine our earlier function normcdf
// so that it can take also vector argument
function [cdf]=normcdf(x)
        cdfN=length(x);
        cdf=cdfnor("PQ",x,zeros(1,cdfN),ones(1,cdfN));
endfunction

function [v]=Call(S,E,r,q,sigma,tau)
        d1=(log(S/E)+(r-q+0.5*sigma^2)*tau)./(sigma*sqrt(tau));
        d2=(log(S/E)+(r-q-0.5*sigma^2)*tau)./(sigma*sqrt(tau));
        v=S*exp(-q*tau).*normcdf(d1)-E*exp(-r*tau)*normcdf(d2);
endfunction
```

Now, after loading these functions by `getd()` command[1], we can compute for example the price of a call option with exercise price of 100 USD and expiry in one month, written on a non-dividend paying stock with volatility 0.30, which has the current price 105 USD. The interest rate is one percent. We use the function `Call` defined above and write:

```
Call(105,100,0.01,0,0.3,1/12)
```

The resulting option price is 6.645 USD.

The function `Call` can also take a vector of stock prices as an argument, which allows us to easily plot for example the dependence of the option price on the current stock price. Figure 3.1 shows the option above, but for different times to maturity as a function of the stock price.

```
S=0:200;
plot(S,Call(S,100,0.01,0,0.3,1/252);
plot(S,Call(S,100,0.01,0,0.3,1/12,"r");
plot(S,Call(S,100,0.01,0,0.3,1/2,"g");
plot(S,Call(S,100,0.01,0,0.3,1,"y");
```

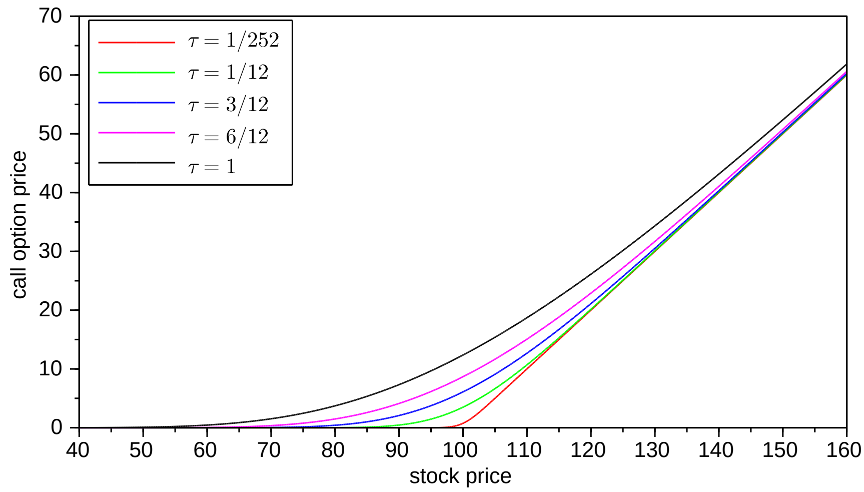


Figure 3.1: Price of a call option.

3.2.2 Black-Scholes formula for a put option. Show that the price of a put option can be expressed as

$$V(S,t) = Ee^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1), \tag{3.5}$$

where d_1, d_2 are given be (3.4) as before and $\tau = T - t$ is time remaining to expiration of the option.
Hint. Use the put-call parity and the symmetry of the cumulative distribution function of $\mathcal{N}(0,1)$ distribution: $N(-x) = 1 - N(x)$.

Write a Scilab function `Put(S,E,r,q,sigma,tau)` for computation of a put option price. Produce similar plot as in Figure 3.1, but for a put option. A sample result is given in Figure 3.2.

3.2.3 Pricing combined strategies. From the linearity of the Black-Scholes equation it follows that if the payoff of a strategy is a linear combination of payoffs of call and put options, the price of the strategy is the same linear combination of call and put options prices.

Use this property to make a graph of a dependence of a chosen combined strategy value on the underlying stock price for a couple of times to expiration. Figure 3.3 shows such a plot for a butterfly

[1] This has to be done with the working directory set to the folder which includes the `.sci` file with the functions, see also the website for more information on working in Scilab.

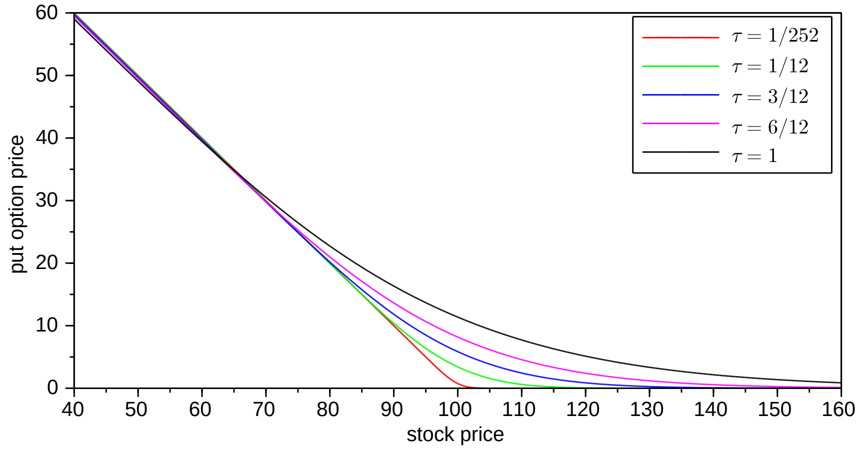


Figure 3.2: Price of a put option.

strategy, which consists of buying a call option with an exercise price of 80 USD and a call option with an exercise price of 100 USD, and selling two call options with an exercise price of 120 USD.

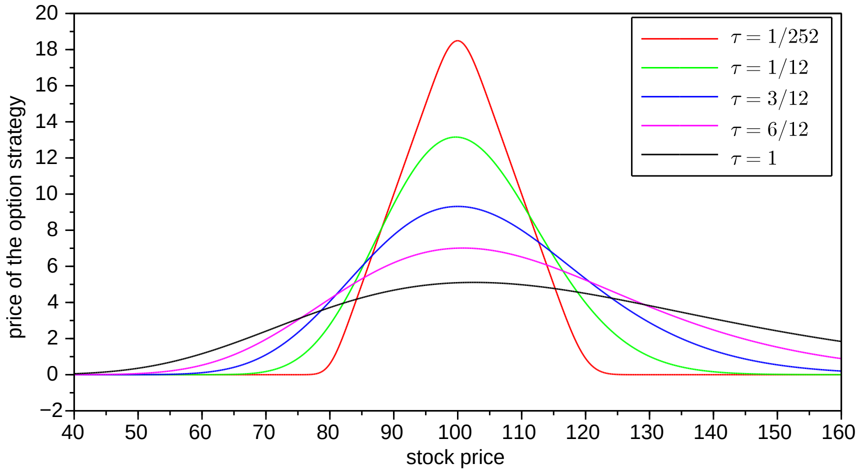


Figure 3.3: Price of a butterfly strategy.

3.2.4 Comparison of the Black-Scholes prices with market prices. Consider the stock that does not pay dividends or approximate the dividends paid by the stock by continuous dividends. Take current market prices for the call options or consider those from the first chapter. Estimate the volatility using the historical data and find the interest rate. Other parameters are given directly by the option and the stock. Compare the resulting Black-Scholes prices with the market data.

How is the answer sensitive to the choice of the interest rate? How is it sensitive to the choice of the dataset used to estimate the historical volatility? We are going to deal with dependence of the options prices on parameters in the later sections.

3.2.5 Black-Scholes prices of call and put vs. their payoff diagrams. The following properties of the call and put prices will be useful later, when dealing with so called American style of options.

- Show that if the stock does not pay dividends, than the price of a call option is above the payoff diagram for all $\tau > 0$.

- Show that if the stock does pays dividends, than the price of a call option crosses the payoff diagram for all $\tau > 0$ and is below the payoff for large S. Put more precisely: Let $\tau > 0$ be fixed. Show that there is S^* such that for $S > S^*$ the inequality $V^{call}(S, \tau) < \max(0, S - E)$ hold.
- Show that the price of a put option crosses the payoff diagram for all $\tau > 0$, regardless on whether the underlying stock pays dividends or not.

These properties are graphically presented in Figure 3.4.

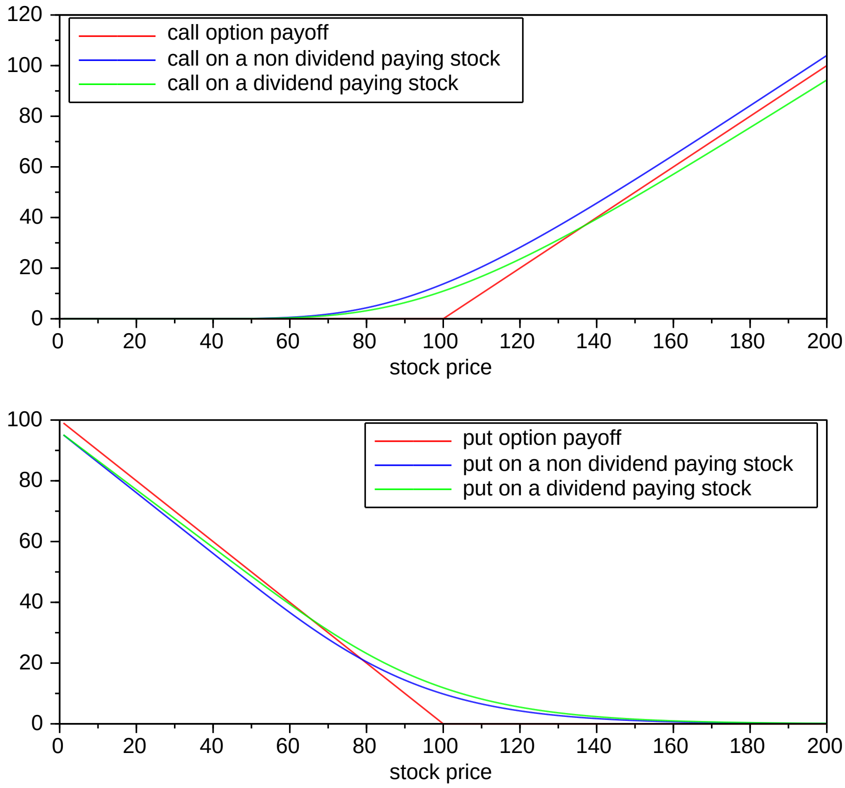


Figure 3.4: Option prices vs. payoff diagram.

3.2.6 Limit properties of call and put options prices. Let $\tau > 0$ be fixed.

- Show that the price of a put option converges to zero as S approaches infinity. Similarly, the price of a call option approaches zero as S approaches zero.
- Compute the limit of a put option as S approaches zero.
- Show that the ratio $\frac{V^{call}}{S}$ converges to unity as S approaches infinity. Show that the same hold for the ratio $\frac{V^{call}}{Se^{-q\tau} - Ee^{-r\tau}}$. Numerically compare the approximations $V^{call} \approx Se^{-q\tau} - Ee^{-r\tau}$ and $V^{call} \approx S$ for large S (relative to the expiration price E) - which one is more precise?

Why could we expect this behaviour without knowing the explicit formulae for the options prices and computing the given quantities?

3.3 Implied volatility

3.3.1 Dependence of the call option price on volatility. Recall from the lectures the following properties of the Black-Scholes price of a call option written on a stock which does not pay dividends:

- The call option price is an increasing function of the volatility σ.

- If the volatility σ approaches infinity, the option price converges to the current price of the underlying stock S.
- If the volatility σ approaches zero, the limit of the option price depends on the relation between the stock price S and the exercise price E of the option: if $S > Ee^{-r\tau}$, the limit is $S - Ee^{-r\tau}$; if $S \leq Ee^{-r\tau}$, the limit is zero. We can write this limit in a compact form as $\max(0, S - Ee^{-r\tau})$.

3.3.2 Exercise. Plot a graph showing the properties stated in the previous point, such as the one shown in Figure 3.5.

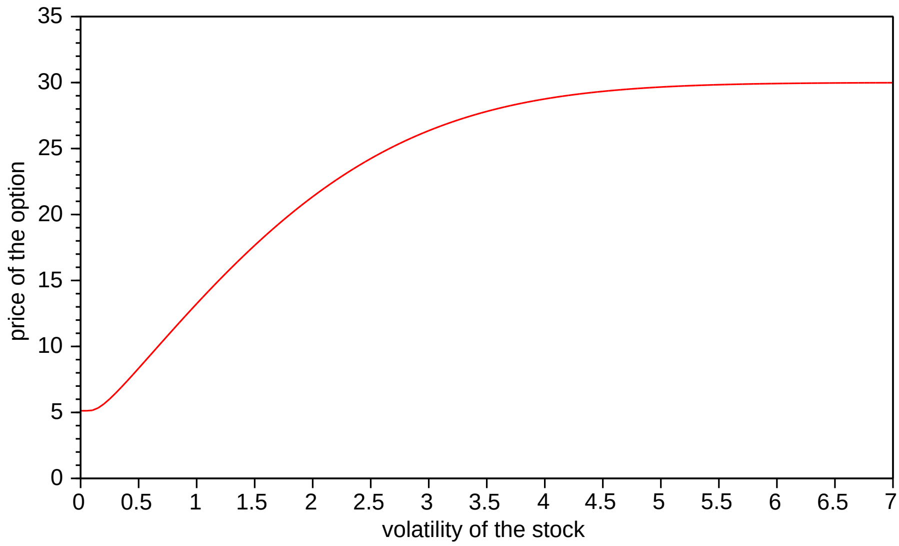


Figure 3.5: Price of a call option on a non-dividend paying stock - dependence on volatility.

3.3.3 Exercise. What would happen if the price of a call option was greater than S? What if it was less than $S - Ee^{-r\tau}$ (assuming that this quantity is positive)? Note that this does not depend on the Black-Scholes pricing, but on general principles which hold for options, recall the first chapter.

3.3.4 Exercise: call option on a dividend paying stock. Adjust the computations from the lecture for a stock which pays dividends. In particular:

- Show that the call option price is an increasing function of σ.
- Compute the limits of the option price as volatility approaches zero and infinity and show that

$$\begin{aligned} \lim_{\sigma \to 0^+} V^{call}(S, \tau; \sigma) &= \max(Se^{-q\tau} - Ee^{-r\tau}, 0), \\ \lim_{\sigma \to \infty} V^{call}(S, \tau; \sigma) &= Se^{-q\tau}. \end{aligned}$$

Demonstrate these properties graphically.

3.3.5 Implied volatility of a call option. The volatility σ for which the Black-Scholes value of the option equals its market value is called **implied volatility**. From the properties stated above and the continuous dependence if the Black-Scholes option price on σ it follows that of the real market price V^{real} satisfies

$$V^{real} \in \left(\max(Se^{-q\tau} - Ee^{-r\tau}, 0), Se^{-q\tau}\right),$$

then the implied volatility exists and is uniquely determined.

As an example, consider the market data on GM option in Table 3.1. The come from the end of trading on 22nd July 2013, which makes 19 trading days until the expiration of the options on 17th

August 2013. The last traded stock price was 36.76 USD, the interest rate was 0.01 percent. The stock does not pay dividends.

exercise price	price of the option
34	3.05
35	2.25
36	1.57
37	0.97
38	0.59
39	0.32

Table 3.1: Data from 22nd July 2013 on GM options expiring on 17th August 2013; the stock price is 36.76 USD.

We define a function to compute of the implied volatility of a call option. It uses the earlier function `Call(S,E,r,q,sigma,tau)` which computes the Black-Scholes price; the initial point is chosen to be 0.2 which the usual order of the volatility (however, the algorithm is not sensitive to the choice of this starting point):

```
function [sigmaImpl]=ImplVolCall(S,E,r,q,tau,V)
   function [d]=difference(sigma)
        d=Call(S,E,r,q,sigma,tau)-V;
   endfunction
   sigmaImpl=fsolve(0.2,difference);
endfunction
```

We use it to find the implied volatility of the option with the expiration price 35 USD in Table 3.1:

```
ImplVolCall(36.76,35,0.0001,0,19/252,2.25)
```

The resulting implied volatility is 0.296. Compute all the implied volatilities and make a graph of the implied volatility as a function of the exercise price of the option, see Figure 3.6 for the result.

3.3.6 Exercise. Take the current data and compute the implied volatilities for the options. Moreover, compare the obtained implied volatilities with the historical volatility estimated from the historical stock prices.

3.3.7 Exercise: Implied volatility of a put option. Derive the necessary properties of the Black-Scholes price of a put option and state a theorem about the existence and uniqueness of the implied volatility. Write a function for its computation.

3.3.8 Exercise. Suppose that the stock price equals 130 USD, the exercise price of a call option is 140 USD and the option expires in one year. The interest rate is 0.5 %. Show that for all of the following potential option prices the implied volatility exists: 10 USD, 15 USD, 20 USD, 25 USD. Which of them results in the highest implied volatility? How can we answer the previous question without actually computing the implied volatilities?

3.3.9 Exercise. Consider the option prices in Table 3.1. Find the value of σ which minimized the sum of squared relative errors in option prices:

$$F(\sigma) = \sum_{i=1}^{n} \left(\frac{V_i^{real} - V_i^{bs}(\sigma)}{V_i^{real}} \right)^2, \tag{3.6}$$

where V_i^{real} $(i = 1, \dots, n)$ is the real price of the i-th option and $V_i^{bs}(\sigma)$ $(i = 1, \dots, n)$ is its Black-Scholes price for the given σ.

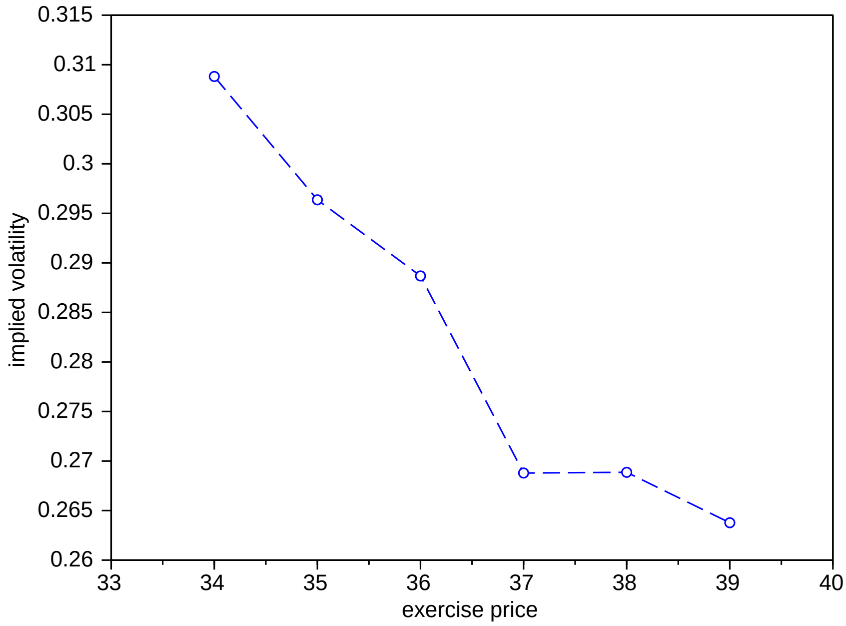


Figure 3.6: Implied volatilities for the options in Table 3.1.

3.4 Delta of a derivative, delta hedging

3.4.1 Delta of a derivative, case of a call option on non-dividend paying asset. By **delta** of a derivative we denote the partial derivative of the Black-Scholes price V with respect to the price of the underlying asset S:

$$\Delta = \frac{\partial V}{\partial S}$$

It has been shown in lectures that for the call option on a stock which does not pay the dividends, the delta equals

$$\Delta = \frac{\partial V^{call}}{\partial S} = N(d_1),$$

where N is the cumulative distribution function of $\mathcal{N}(0,1)$ distribution and d_1 is given by (3.4) with $q = 0$. Note that both d_1 and d_2 in (3.3) and (3.4) depend on S, which has to be taken into account when computing the derivative of V^{call} with respect to S. Recall that the lemma

$$S\frac{\partial N(d_1)}{\partial S} - Ee^{-r\tau}\frac{\partial N(d_2)}{\partial S} = 0 \tag{3.7}$$

has been used to eliminate the remaining terms which are obtained when differentiating the option price.

3.4.2 Exercise: Delta of a call option on a dividend paying stock. Show that if the underlying stock pays dividends, then the delta of a call option is given by

$$\Delta = e^{-q\tau}\frac{\partial V^{call}}{\partial S} = N(d_1),$$

where N is the cumulative distribution function of $\mathcal{N}(0,1)$ distribution and d_1 is given by (3.4) How does the lemma (3.7) change?

We define the function `DeltaCall(S,E,r,q,sigma,tau)` using the results above, which allows us to numerically compute the delta of a call option (it uses our earlier function `normcdf`) and add it to the `.sci` file with Black-Scholes related functions:

```
function [delta]=DeltaCall(S,E,r,q,sigma,tau)
  d1=(log(S/E)+(r-q+0.5*sigma^2)*tau)./(sigma*sqrt(tau));
  v=exp(-q*tau)*normcdf(d1);
endfunction
```

3.4.3 Interpretation of the delta and the delta hedging. Recall from the derivation of the Black-Scholes model that $\Delta = \frac{\partial V}{\partial S}$ denotes the number of the underlying stocks which are present at a given moment in the riskless portfolio with one sold derivative.

More generally, denote the amounts of the derivatives and the stocks by Q_V and Q_S. Then the portfolio is riskless if the ratio $\frac{Q_S}{Q_V}$ equals to minus Δ. Maintaining this ratio by selling and buying stocks is known as **delta hedging**.

It should be noticed that the hedge is valid only for a short moment, since the delta changes both with time and with movements of the stock price. The portfolio has to be rehedged; in theory, the hedging is continuous.

In what follows, by hedging we mean delta hedging.

3.4.4 Exercise: Delta hedging. Suppose that we have sold 1000 options with the exercise price 25 USD which expires in one month. The volatility of the underlying stock is 0.25, it does not pay dividends and the interest rate is 0.5%. The current price of the stock is 23 USD. How many stocks do we need in our portfolio, to hedge the options which we have sold?

Solution. We have $Q_V = -1000$, hence

$$\Delta = -\frac{Q_S}{Q_V} \Rightarrow Q_S = -\Delta Q_v = 1000\Delta,$$

which we obtain by the command

```
1000*DeltaCall(23,25,0.005,0,0.25,1/12)
```

that yields a numerical answer 132.744. We therefore buy 133 stocks.

3.4.5 Exercise. Suppose that the volatility of the underlying stock is 0.2, it pays dividends with the dividend rate 2 percent and its current price of is 92.50 USD. The interest rate is one percent. Determine the number of stocks that we need to hedge the options in the following cases:

- We have sold 100 call options with the exercise price of 100 USD which expire in two months.
- We have bought 100 call options with the exercise price of 100 USD which expire in two months.

Determine the sign of the answer (i.e., whether we will be buying the stocks or going to short positions) in advance, before computing the numerical result.

3.4.6 Exercise: Dependence of delta on the parameters. To motivate the following considerations, let us firstly consider the situation from Exercise 3.4.4, changing the current value of the stock to 26 USD. Since we have sold options which allow the holder to buy the stock for 25 USD in one month, there is a higher chance now that these options will be exercised. It is therefore natural to expect that more stocks will be needed to hedge this risk. Indeed, the command

```
1000*DeltaCall(26,25,0.005,0,0.25,1/12)
```

produces the result of 720.832, and hence in this modified situation we buy 721 stocks.

- Consider the delta of a call option as a function of the stock price S, while keeping the remaining parameters fixed, i.e., $\Delta = \Delta(S)$. Show that it is an increasing function of S and compute its limits as $S \to 0^+$ and $S \to \infty$. Compare the results with Figure 3.7. Give a financial interpretation of these results.

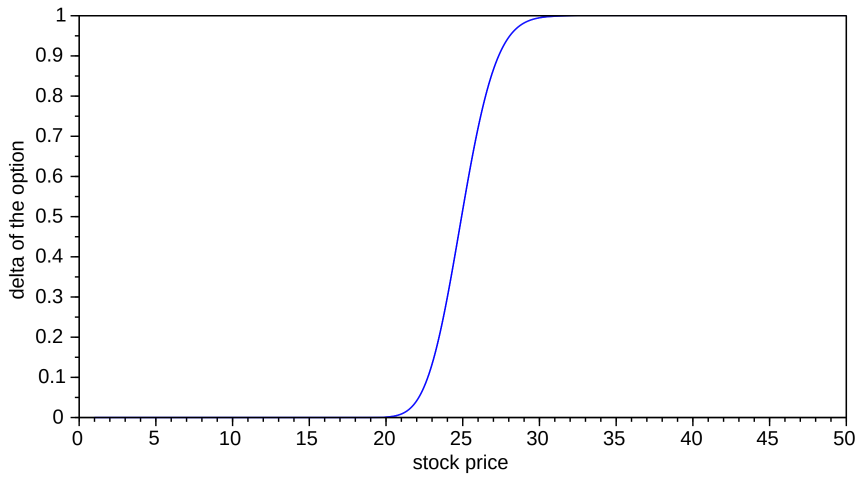


Figure 3.7: Delta of the option.

- What is the range of possible values of Δ? Explain the consequences for delta hedging: how many stocks can there possibly be in the portfolio?
- Consider $\Delta = \Delta(\tau)$ and in Figure 3.8 see the plot of deltas for the options with different expiration times, which have the remaining parameters the same Compute the limit of the delta as $\tau \to 0^+$, i.e., the time approaching the exercise time of the option. Why is it logical that the result depends of the relation between the stock price and the exercise price?
- Plot deltas of options written on stocks with different dividend rates. Explain the dependence of Δ on q when the other parameters are fixed, find a financial interpretation and prove your statements by performing the necessary computations.

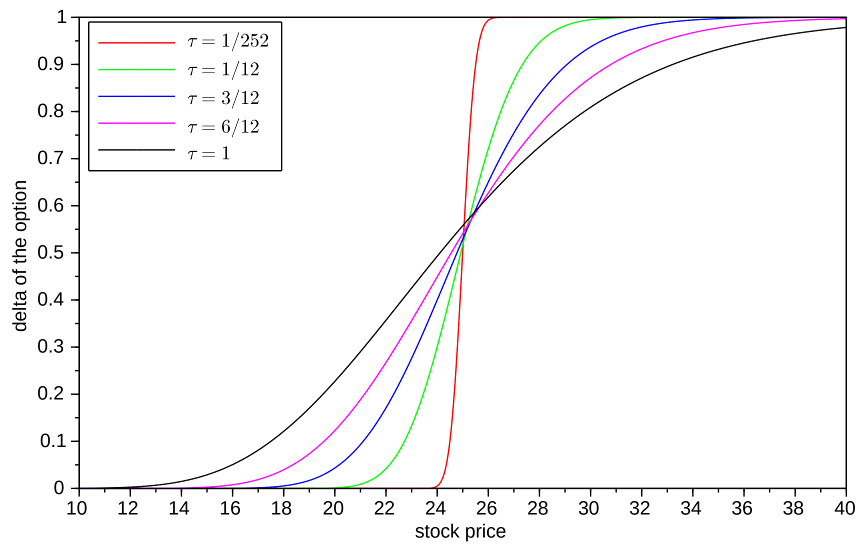


Figure 3.8: Deltas of options with different expiration times.

3.4.7 Exercise. Figure 3.9 shows deltas of three call options: an ITM option, an ATM option and an

OTM option. Which graph corresponds to each of the options?

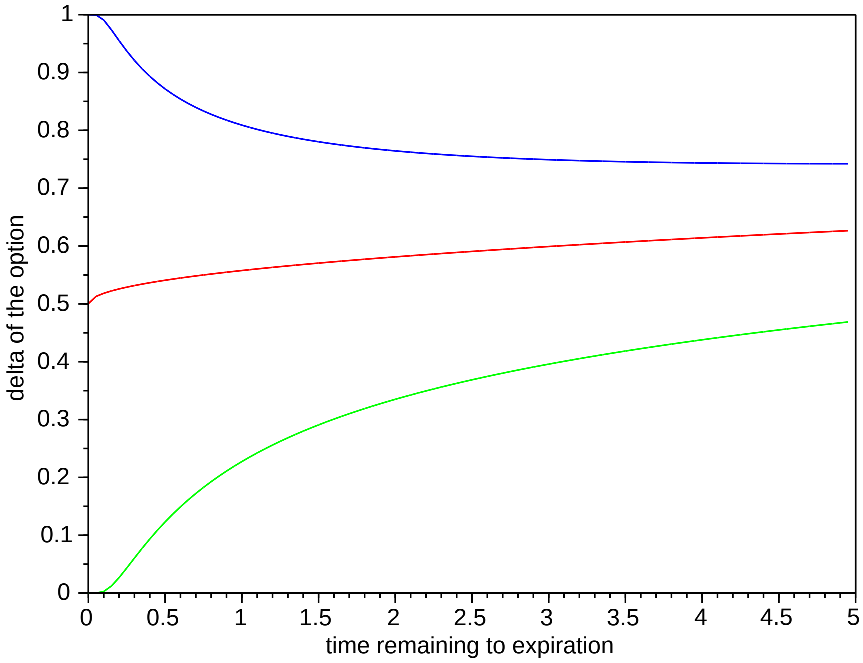


Figure 3.9: Delta of the options

3.4.8 Exercise: Delta of a put option. Use the put-call parity to derive that

$$\Delta^{put} = \Delta^{call} - e^{-q\tau}. \tag{3.8}$$

Suppose that the volatility of the underlying stock is 0.2, it pays dividends with the dividend rate of 2 percent and its current price of is 92.50 USD. The interest rate is one percent. Determine the number of stocks that we need to hedge the options in the following cases:

- We have sold 100 put options with the exercise price of 100 USD which expire in two months.
- We have bought 100 put options with the exercise price of 100 USD which expire in two months.

Determine the sign of the answer (i.e., whether we will be buying or short selling the stocks) in advance, before computing the numerical result - both by a financial intuition and using the formula (3.8)

3.4.9 Exercise: Delta of a put option - properties. Perform the required computations and give financial interpretation of the results. Plot graphs illustrating the given properties.

- Show that delta of a put option is always negative
- Consider the delta of a put option as a function of the stock price S, while keeping the remaining parameters fixed, i.e., $\Delta = \Delta(S)$. Show that it is an increasing function of S and compute its limits as $S \to 0^+$ and $S \to \infty$.
- Compute the limit of the delta as $\tau \to 0^+$, i.e., the time approaching the expiration time of the option.
- How does the delta depend on the dividend rate q?

3.4.10 Exercise. Figure 3.10 shows delta of the following options, written on the same stock, as functions of the stock price S:

1. call option with expiration price 25 USD on a non-dividend paying stock, which expires in one year,
2. call option with expiration price 25 USD on a stock paying continuous dividends with 3 percent rate, which expires in one year,
3. call option with expiration price 55 USD on a non-dividend paying stock, which expires in one year,
4. put option with expiration price 55 USD on a non-dividend paying stock, which expires in one year,
5. put option with expiration price 55 USD on a non-dividend paying stock, which expires in one month.

Determine which graph corresponds to each of the options.

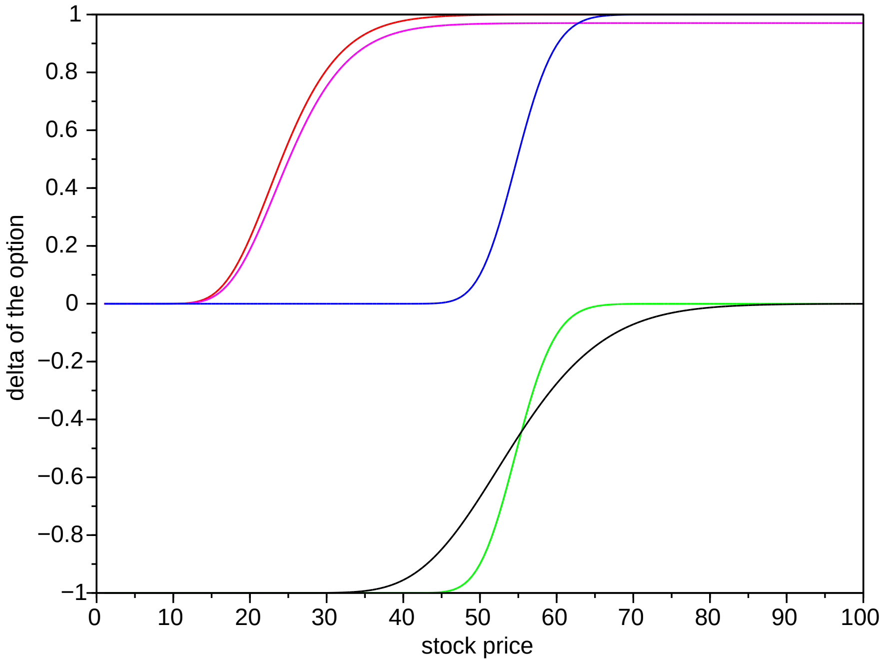


Figure 3.10: Deltas of the options.

3.4.11 Delta of a portfolio. From the linearity of the Black-Scholes equation it follows that delta of the portfolio can be computed by adding deltas of its components. This can be also seen from the assumptions of the model, since hedging a portfolio is equivalent to hedging each of its components independently (this is not the case, for example, in the presence of transaction costs).

3.5 Greeks: sensitivities of the option to the parameters

3.5.1 Greeks: definitions and values. Greeks are the partial derivatives of the option price on the parameters. They give sensitivities of the option price to these parameters. Table 3.2 summarizes their definitions and values. The name *greeks* comes from the fact they most of them are denoted by letters from the Greek alphabet.

name	notation	definition	value for a call	value for a put
delta	Δ	$\frac{\partial V}{\partial S}$	$e^{-q\tau}N(d_1)$	$-e^{-q\tau}N(-d_1)$
gamma	Γ	$\frac{\partial^2 V}{\partial S^2}$	$e^{-q\tau}\frac{e^{-d_1^2/2}}{\sqrt{2\pi\tau}S\sigma}$	$e^{-q\tau}\frac{e^{-d_1^2/2}}{\sqrt{2\pi\tau}S\sigma}$
rho	ρ	$\frac{\partial V}{\partial r}$	$E\tau e^{-r\tau}N(d_2)$	$-E\tau e^{-r\tau}N(-d_2)$
theta	Θ	$\frac{\partial V}{\partial t}=-\frac{\partial V}{\partial \tau}$	$-\frac{E\sigma}{2\sqrt{\tau}}e^{-r\tau}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}$ $-rEe^{-r\tau}N(d_2)$ $+qSe^{-q\tau}N(d_1)$	$-\frac{E\sigma}{2\sqrt{\tau}}e^{-r\tau}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}$ $+rEe^{-r\tau}N(-d_2)$ $-qSe^{-q\tau}N(-d_1)$
vega	Υ	$\frac{\partial V}{\partial \sigma}$	$Ee^{-r\tau}\sqrt{\tau}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}$	$Ee^{-r\tau}\sqrt{\tau}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}$

Table 3.2: Options greeks.

Note that Θ is sometimes defined as a partial derivative with respect to τ, instead of t. In this text we use the definition $\Theta = \partial V/\partial t$, as given in Table 3.2.

3.5.2. Exercise: Signs of the greeks. With one exception, the greeks presented in Table 3.2 do not change their sign. Determine the signs of the greeks - except for the theta of a put option - and explain why these signs could have been expected.

Theta of a put option can be both positive and negative. Recall that it is a derivative of a put price with respect to time, i.e., it measures the change of the put option price as time passes, if the other parameters (including stock price) are fixed. We know that as the time approaches maturity, the option price approaches the payoff. Therefore, according to the plot of the option price in Figure 3.11, we can expect the negative theta for the price S_1 and positive theta for the price S_2. Find the concrete values of the parameters, for which the theta of a put option is positive and for which it is negative.

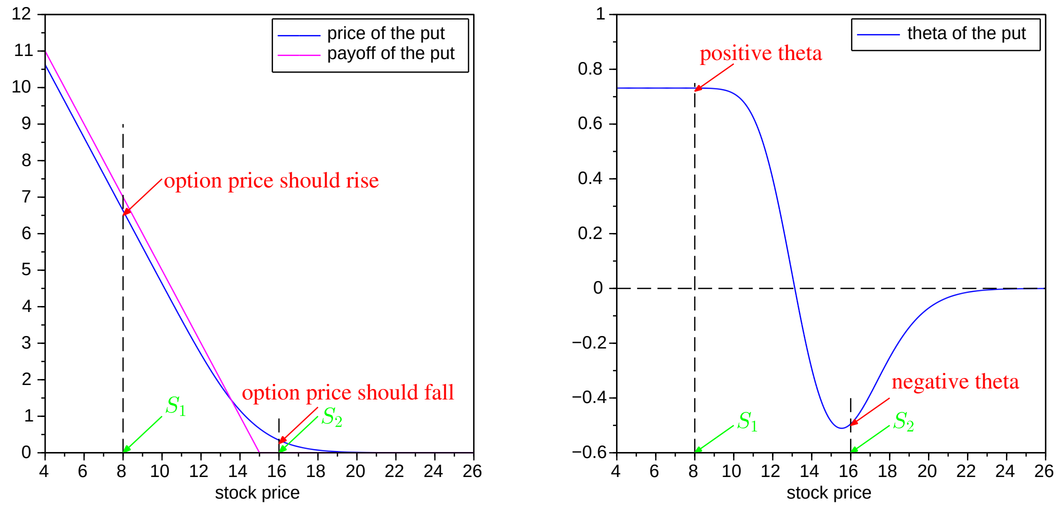


Figure 3.11: Put option: price (left) and theta (right).

3.5.3 Exercise: Gamma of an option and its relation to option price and delta. Gamma, the second derivative of the option price with respect to the stock price S, can be interpreted in the following ways: as a curvature of the graph of $V(S)$ and as a measure of how much the delta changes with a change of the stock price. Explain these interpretations using Figure 3.12 and make similar graphs for a put option.

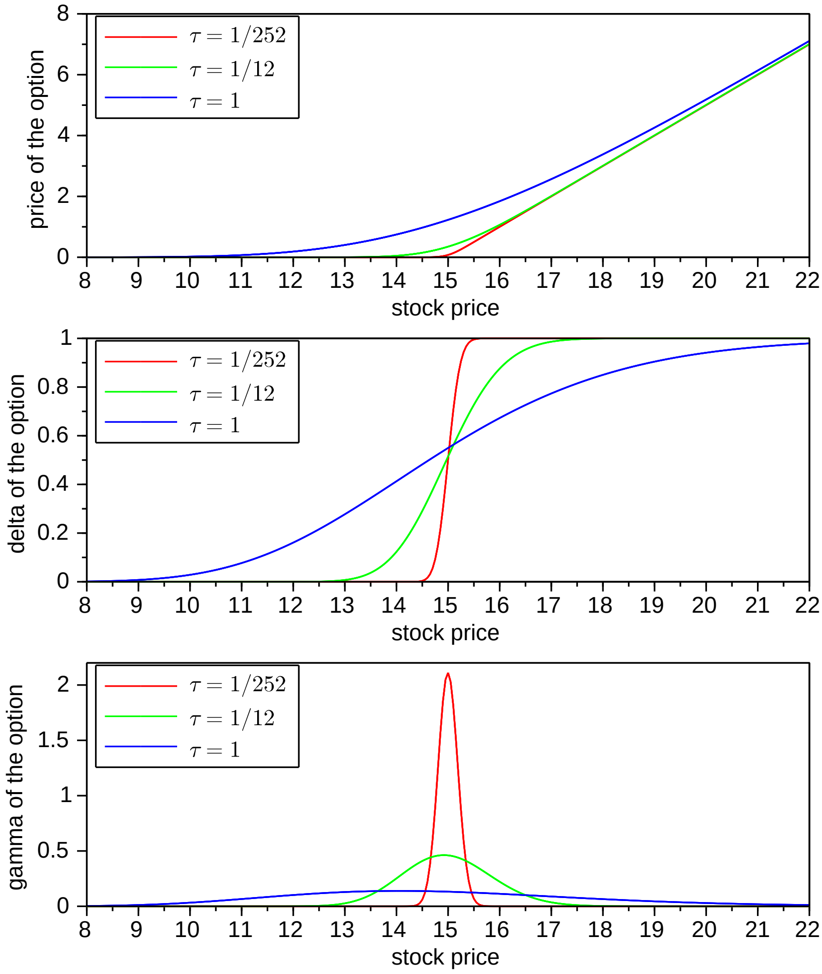


Figure 3.12: Option price, delta and gamma of a call option.

3.5.4 Maximum of gamma. Find the stock price S, for which the option gamma attains its maximum and compare the answer with previous graphs. Alternatively, for a given stock price S, which exercise price of an option leads to the highest gamma?

3.5.5 Exercise: Maximal gamma and the illusions of risk. The title of this exercise is taken from the section title in the paper *Know Your Weapon: Part 1* by E. G. Haug, published in the *Wilmott Magazine*. The author writes:

> One day in the trading room of a former employer of mine, one of the BSD traders suddenly got worried over his gamma. He had a long dated deep-out-of-the money call. The stock price had been falling, and the further the out-of-themoney the option went the lower the gamma he expected. As with many option traders he believed the gamma was largest approximately at-the-money-forward. Looking at his Bloomberg screen, however, the further out of the money the call went the higher his gamma got. Another BSD was coming over, and they both tried to come up with an explanation for this. Was there something wrong with Bloomberg?[2]

This can indeed occur, as it can be seen by plotting 3-dimensional graph of gamma as a function of stock price S and time remaining to expiration τ, see Figure 3.13. The OTM options really have high gammas, when there is a long time remaining to maturity.

[2]E. G. Haug, *Know your weapon, Part 1,* Wilmott Magazine 5 (2003), p. 53

To make a 3-dimensional plot, we can use the function `meshgrid`, as shown below. In the following code we assume that we have already defined a function `bsGamma` which computes the Black-Scholes gamma of a call or a put option, as well as the vectors `s` and `tau` for stock prices and expiration times, which will be used in the plot.[3].

```
[s0,tau0]=meshgrid(s,tau);
g=bsGamma(s0,50,0.01,0,0.2,tau0);

plot3d(s,tau,g);
```

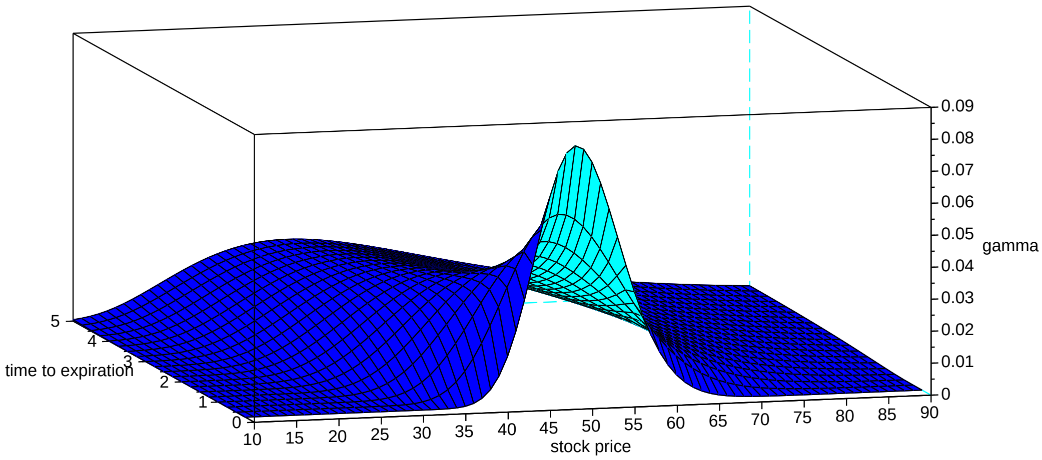


Figure 3.13: 3-dimensional graph of gamma.

Recall that gamma measures the approximate change in delta if the stock price changes by one dollar. However, if the stock price is close to zero, a change of its value by one dollar is actually a big change (for example, a change of a price from 1 USD to 3 USD is a 200 percent rise). It is not probable that the price will change from 1 USD to 3 USD suddenly, contrary to a change such as going from 500 USD to 503 USD which may likely happen in quite a short period of time.

Let us therefore define a "modified gamma" as a percentage change of delta for a percentage change of the underlying stock price, denoted by Γ_P in the cited paper.

- Explain, why $\Gamma_P = S\Gamma/100$.
- Plot a figure similar to that in Figure 3.13, but for Γ_P and show that the feature of a surprisingly high gamma for OTM options with long time remaining to their exercise disappears.

3.5.6 Exercise: One more experience by E. Haug. Consider the second derivative of the option price V with respect to volatility σ and the stock price S, i.e., $\frac{\partial^2 V}{\partial S \partial \sigma}$, which is known as **vanna**. This can be interpreted as the derivative of vega with respect to stock price, thus showing how the vega is sensitive to the change of the stock price. Alternatively, it can be seen as the derivative of delta with respect to sigma, and thus it tells how sensitive is delta to the stock volatility.

- Derive a closed form expression for vanna.
- In the paper cited in the previous exercise, we can also read:

[3] To obtain the plot shown in Figure 3.13, we have afterwards selected `Edit` and `Axes properties` from the menu, where we have checked the checkbox `Cube scaling` in the tab `Aspect` and chosen a suitable point in the `Viewpoint` tab. See the website for more detailed information on plotting graphs in Scilab.

> One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures - I was supposed to do ”arbitrage trading”. That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their DdeltaDvol was very high. So I immediately asked what volatility the risk management used to calculate their delta. As expected, the volatility in the risk-management-system was considerable below the market and again was leading to a very low delta for the options. This example is just to illustrate how a feeling of your DdeltaDvol can be useful. If you have a high DdeltaDvol the volatility you use to compute your deltas becomes very important. [4]

Explain the assertions made in the quote above: Why do those options have high DdeltaDvol[5]? Why is the choice of volatility important in this situation and why does using too low volatility lead to too low deltas?

3.5.7 Exercise: Delta for simple ”derivatives”.

- Recall that $V(S,t) = S$ and $V(S,t) = Ke^{-r(T-t)}$ are solutions to the Black-Scholes PDE with $q = 0$. What is their delta? Give an interpretation of this result.
- Recall the solution to the general Black-Scholes equation (general $q \geq 0$) with the terminal condition $V(S,t) = S$. What is its delta? Explain the effect of dividends.

3.5.8 Exercise: Greeks for a cash-or-nothing option. Recall the binary options from the lectures and for simplicity assume $q = 0$. Consider cash-or-nothing binary option with payoff

$$V(S,T) = \begin{cases} 0 & \text{for } S \leq E, \\ 1 & \text{for } S > E. \end{cases}$$

We know that its price is $V(S,t) = e^{-q\tau}N(d_2)$, where d_2 is the expression (3.4) from the Black-Scholes formula, and it is shown in Figure 3.14 as a function of the stock price.

- Compute its delta and plot its dependence on the stock price S for different times remaining to expiration. See Figure 3.15 for a sample result. By relating it to hedging such an option, explain, why such a behaviour could have been expected.
- Explain the following quote on a cash-or-nothing option:

  > What starts off as a placid instrument turns into an unmanageable monster over the last few hours of its life as the at-the-money delta becomes so high that the option becomes unhedgeable.[6]

- Based on the graph of delta, sketch the graph of gamma.
- How does the option price depend on volatility? Plot the dependence of vega on the stock price for different times remaining to expiration and explain its behaviour. See Figure 3.16 for a sample result.
- How does the option price depend on time? Plot the dependence of theta on the stock price for different times remaining to expiration and explain its behaviour. See Figure 3.17 for a sample result.

[4] Ibid., p. 51
[5] This notation is used for derivative of the delta with respect to the volatility.
[6] http://www.binaryoptions.com/binary-call-options-delta/

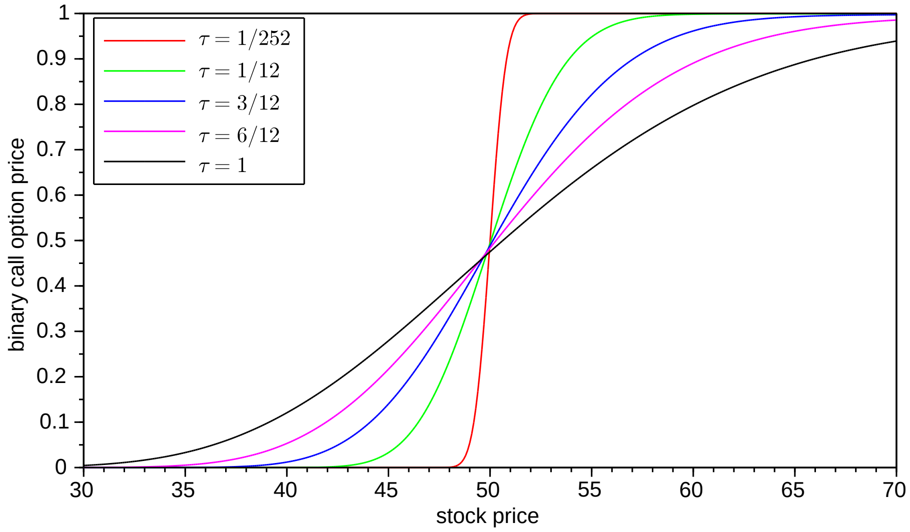


Figure 3.14: Price of a cash-or-nothing option.

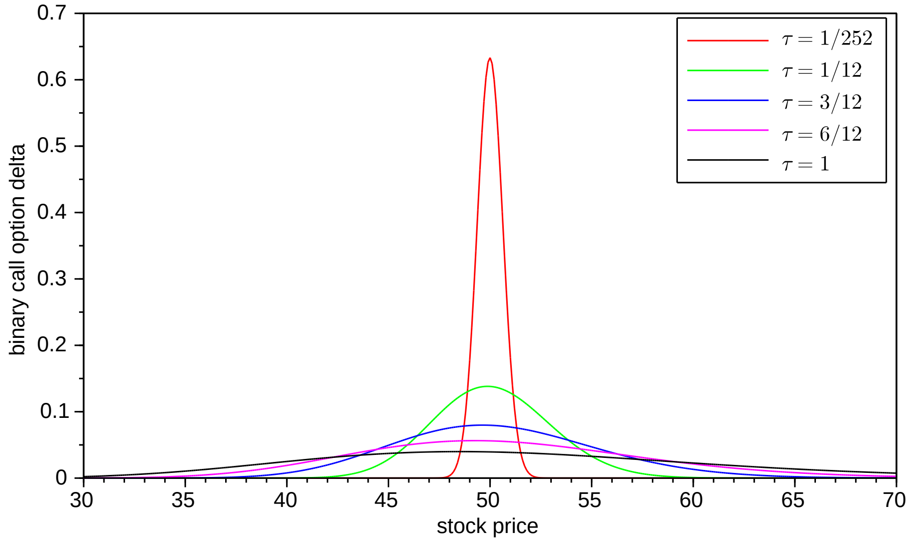


Figure 3.15: Delta of a cash-or-nothing option.

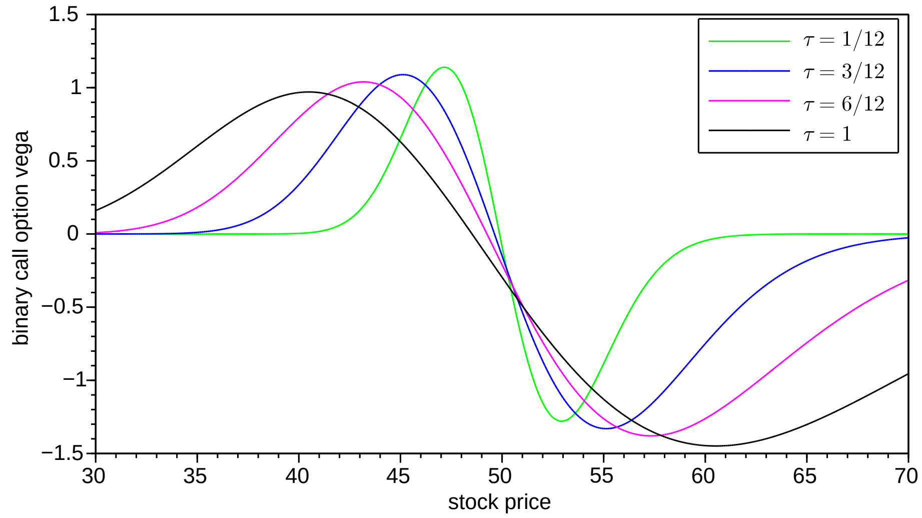


Figure 3.16: Vega of a cash-or-nothing option.

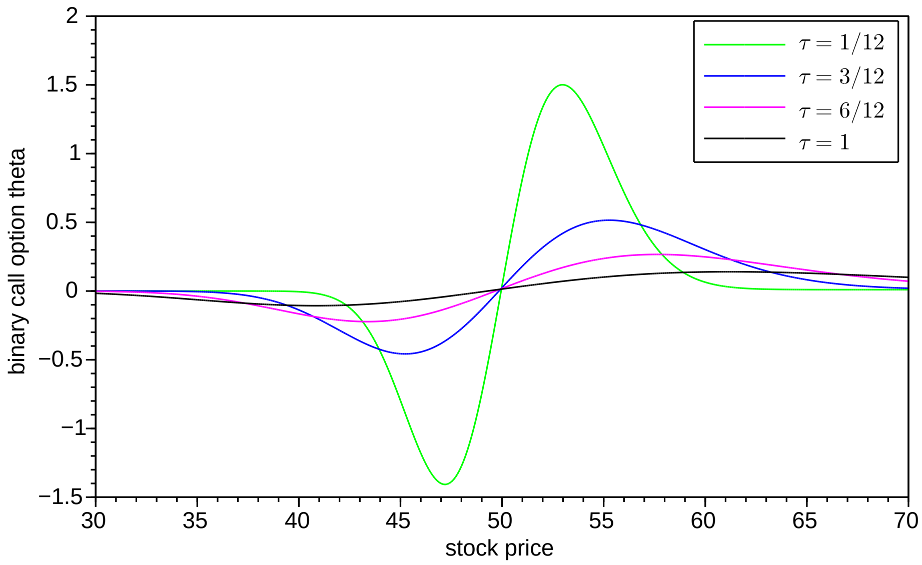


Figure 3.17: Theta of a cash-or-nothing option.

Chapter 4

Leland model: modelling transaction costs

4.1 Definition of the model and the PDE for the price of a derivative

4.1.1 Transaction costs in Leland model. Recall from the lectures that the transaction costs in Leland model come from different bid (S_{bid}) and ask (S_{ask}) prices for the stock, while the quantity entering the model and being modelled by a geometric Brownian motion $\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}w$ is their average

$$S = \frac{S_{ask} + S_{bid}}{2}. \tag{4.1}$$

A nondimensional parameter characterizing the transaction costs is defined by

$$c = \frac{S_{ask} - S_{bid}}{S}. \tag{4.2}$$

Then, transaction costs arising from buying or selling one asset are equal to $cS/2$.

4.1.2 Exercise: Computing the parameter c defining transaction costs. Figure 4.1 shows bid and ask prices of AMZN stock (prices in the **Bid** and **Ask** rows) as presented at `finance.yahoo.com`. Compute the parameter c based on these values. Repeat with the stock prices given in Table 4.1.

Amazon.com Inc. (AMZN) - NasdaqGS

289.78 ↑3.90 (1.36%) 10:18AM EDT - Nasdaq Real Time Price

Prev Close:	285.88	Day's Range:	286.18 - 290.79
Open:	286.38	52wk Range:	212.61 - 290.79
Bid:	289.83 x 100	Volume:	1,009,269
Ask:	290.07 x 300	Avg Vol (3m):	3,046,600
1y Target Est:	314.73	Market Cap:	131.92B
Beta:	0.64	P/E (ttm):	N/A
Next Earnings Date:	22-Jul-13	EPS (ttm):	-0.19
		Div & Yield:	N/A (N/A)

Figure 4.1: Bid and ask prices of AMZN stock at `finance.yahoo.com` from the beginning of trading on 8th July 2013.

stock	bid price	ask price
IBM	195.50	195.55
MSFT	34.50	34.51
VOW.DE	153.75	153.95
YHOO	25.79	25.80

Table 4.1: Selected stock prices from the beginning of trading on 8th July 2013.

4.1.3 PDE FOR THE OPTION PRICE IN THE LELAND MODEL. Recall from the lectures that the PDE satisfied by the option price $V = V(S,t)$ reads as follows:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \left(1 + \sqrt{\frac{2}{\pi}} \frac{c}{\sigma\sqrt{\Delta t}} \operatorname{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\right) \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = 0 \tag{4.3}$$

for $S \in (0,\infty)$ and $t \in [0,T)$, where Δt is the time between two adjustments of the portfolio, since a continuous hedging (which it was the case in Black-Scholes setting) is not possible in the presence of transaction costs. The terminal condition $V(S,T) = \bar{V}(S)$ for $S \in (0,\infty)$ is determined by the option type.

4.2 Call and put prices

4.2.1 BID PRICE OF A CALL AND PUT OPTION. Bid price is a price suggested by a potential buyer of the option. Hence its price comes from hedging a portfolio with one call option.

The main outline of the derivation of the price is as follows (see lectures for more details): For the Black-Scholes price of both call and put option, the second derivative with respect to S is always positive. It follows that, if the quantity

$$\tilde{\sigma}^2 := \left(1 - \sqrt{\frac{2}{\pi}} \frac{c}{\sigma\sqrt{\Delta t}}\right) \sigma^2 \tag{4.4}$$

is positive, we can compute the Leland price as the Black-Scholes price, by replacing σ^2 in the Black-Scholes formula by $\tilde{\sigma}^2$ given by (4.4).

4.2.2 EXERCISE. Give an example of a payoff, for which the second derivative of the Black-Scholes price with respect to S changes the sign. For such a derivative, the reasoning above does not apply and the Leland price cannot be computed simply as the Black-Scholes price with adjusted volatility.

4.2.3 EXERCISE: NONLINEARITY OF THE LELAND PDE. Note that the PDE (4.3) is nonlinear, which means that a linear combination of solutions is not necessarily a solution again. Give an example of such solutions and their linear combination. How does this property relate to hedging such a portfolio?

4.2.4 LELAND NUMBER. Define the **Leland number** as

$$\mathrm{Le} = \sqrt{\frac{2}{\pi}} \frac{c}{\sigma\sqrt{\Delta t}}. \tag{4.5}$$

Then the adjustment of the volatility (4.4) can be written as $\tilde{\sigma}^2 := (1 - \mathrm{Le})\sigma^2$. The constraint on the adjusted volatility then translates into the condition

$$\mathrm{Le} < 1. \tag{4.6}$$

Since the other parameters are given by the market, we are only able to choose Δt, the time between two adjustments of the portfolio. Hence the condition (4.6) is, in fact, a restriction on Δt.

4.2.5 EXERCISE: ADMISSIBLE Δt AND DETERMINING THE BID PRICES OF OPTIONS. Consider one of the stocks for which we have computed the transaction costs c.

- Use the historical prices of the stock to estimate the volatility S (in Leland model it is the volatility of the average of bid and ask prices, use the closing prices as a proxy.
- Plot the dependence of the Leland number on Δt. What are the admissible values of Δt, for which the Leland number satisfies the condition (4.6)? For an illustrative graph see Figure 4.2. Note that Δt is measured in years. Write the condition on Δt in units which are more easily interpreted in this context (days, hours[1], or minutes).
- Find the interest rate and select the parameters of the option which you are going to use for the following questions.
- Choose a concrete admissible value of Δt and compute the corresponding option price.
- Plot the dependence of the option price on Δt, over the admissible values of Δt. Is this dependence increasing or decreasing? Why?

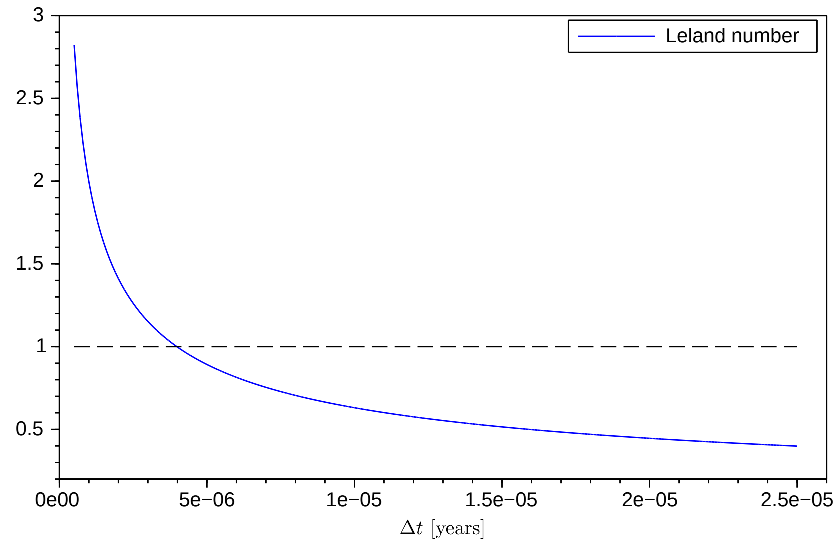


Figure 4.2: Leland number as a function of the time interval between two adjustments of the portfolio.

4.2.6 Exercise. How does the range of admissible values for Δt depend on transaction costs measure c?

4.2.7 Exercise. Suppose that the difference between the bid and ask price of a stock equals to 0.4 percent of their average value. We would like to hedge a call option every 5 minutes. For what values of the stock volatility would it be an admissible time?

4.2.8 Exercise: Ask price of the option. Show that the ask price of the call and put option can be computed by the Black-Scholes formula, where instead of σ we use the adjusted volatility given by

$$\tilde{\sigma}^2 = (1 + \mathrm{Le})\sigma^2, \tag{4.7}$$

where Le is given by (4.5). Use both the approaches below:

- Go through the derivation of the PDE for the bid price from the lectures, but instead of having one option in the portfolio consider minus one option.

[1] Similarly as with considering only trading days (e.g., the convention of 252 trading days a year), we should also consider only hours during which the stock market is open for trading.

- Consider plus one option (hence you are able to use the results above), but with the payoffs $-\max(0, S-E)$ and $-\max(0, E-S)$.
  Hint. How does the opposite sign of a payoff change the second partial derivative with respect to S?

4.2.9 Exercise: Comparison with the real data. Consider the AMZN stock and the parameter c estimated in Exercise 4.1.2. Use historical prices to estimate the volatility σ and find the interest rate. Select an option from those listed in Table 4.2

- What are the admissible values of Δt?
- Plot the bid and ask prices as functions of Δt and compare them with market prices.

option code	bid price	ask price
AMZN130817C00280000	18.25	18.45
AMZN130817C00285000	15.20	15.40
AMZN130817C00290000	12.30	12.45
AMZN130817C00295000	10.15	10.35

Table 4.2: Selected options prices from the beginning of trading on 8th July 2013.

4.2.10 Exercise: Show that the following inequality between bid and ask prices (V_{bid}, V_{ask}) from the Leland model and the Black-Scholes price (V_{bs}) holds for both call and put options:

$$V_{bid}(S,t) < V_{bs}(S,t) < V_{ask}(S,t)$$

for all $S > 0$ and $t < T$. This property is graphically illustrated in Figure 4.3.

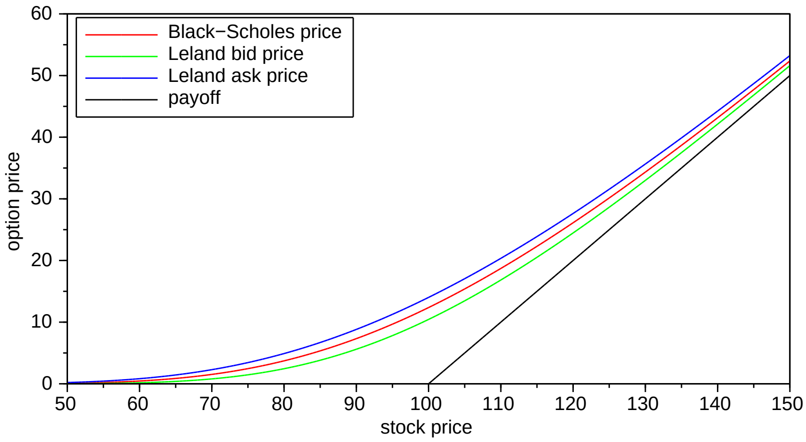


Figure 4.3: Bid and ask prices from the Leland model compared with the Black-Scholes price.

4.2.11 Exercise: Implied parameters. Consider the option priced in the previous exercise and the remaining parameters except for the historical volatility. We are going to compute the implied volatility σ and implied time Δt using the following steps (cf. also the lectures):

- Compute the Black-Scholes implied volatility σ_{ask} using the ask price of the option (note that the value of the stock price is given by S, not S_{ask}).
- Compute the Black-Scholes implied volatility σ_{bid} using the bid price of the option (again, note that the value of the stock price is given by S, not S_{bid}).

- From the formulae for computations of Leland prices of call and put options it follows that

$$\sigma_{ask} = (1 + \text{Le})\sigma, \ \ \sigma_{bid} = (1 - \text{Le})\sigma.$$

  Use these two equations to determine the implied values of the volatility σ and the Leland number Le.

- Using the volatility σ and the Leland number Le from the previous point (and the other necessary parameters), compute the implied time Δt and express it in suitable units. Compare the implied volatility with its value estimated from the historical data of the stock prices.

4.2.12 Exercise: Show that the bid-ask spread for a call and a put option with the same parameters is the same.

4.2.13 Exercise: Modelling bid-ask spreads.

- For the selected parameters, plot the difference between ask and bid price of a selected option, as a function of the stock price, with other parameters fixed. Figure 4.4 shows a sample result. Numerically find the value for which the difference is the highest.
- Find a general formula for the stock price, for which the bid-ask spread is highest.

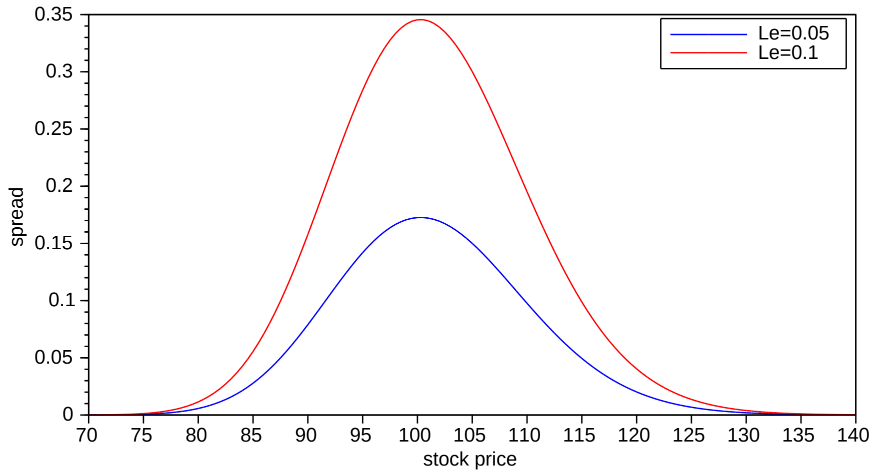


Figure 4.4: Bid-ask spreads in Leland model.

Chapter 5

Numerical methods for the Black-Scholes equation

5.1 Transformation to heat equation

5.1.1 Motivation. The Black-Scholes equation for pricing European call and put option can be solved explicitly but we have two reasons for solving it numerically:

- testing the numerical schemes by comparing the obtained numerical results with the exact solution,
- using a modification of the method to price American-style option, where no closed form solution is available.

5.1.2 Transformation to s heat equation on the line. The transformation consists of the following steps:

1. Transformation of the time $\tau = T - t$ - it leads to having an initial condition instead of a terminal condition.
2. Logarithmic transformation of the stock price $x = \log(S/E)$ - the new variable x is defined on the whole line $x \in \mathbb{R}$; note that a simple logarithm $\log(S)$ would suffice, but this transformation is suitable for generating the mesh, see the following exercise.
3. The function $Z(x, \tau)$ defined by $Z(x, \tau) = V(Ee^x, T - t)$ satisfies a parabolic partial differential equation with constant coefficients. This can be transformed to a heat equation by transformation $u(x, \tau) = e^{\alpha x + \beta \tau} Z(x, \tau)$, where the coefficients α and β are chosen so that the terms u and $\partial u/\partial x$ diminish and we obtain a heat equation. This leads to the choice of

$$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}, \beta = \frac{r+q}{2} + \frac{\sigma^2}{8} + \frac{(r-q)^2}{2\sigma^2}. \tag{5.1}$$

The resulting PDE for the function $u(x, \tau)$ then reads as

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \quad \text{for } x \in \mathbb{R}, \tau \in (0, T] \tag{5.2}$$

and the initial condition is transformed into

$$u(x, 0) = e^{\alpha x} \bar{V}(Ee^x) \quad \text{for } x \in \mathbb{R}.$$

5.1.3 Exercise: Logarithmic transformation. Consider the logarithmic transformation in the second step described in the previous point.

- In the variable x, a numerical solution will be computed on a finite interval symmetric around zero, which we denote by $[-L, L]$. What stock price does the value $x = 0$ correspond to?
- What interval of the stock prices does the interval $x \in [-L, L]$ correspond to? Give numerical answer for different expiration prices, when $L = 2$, $L = 3$, or some other choices of L.
- When numerically solving the heat equation, we are going to need boundary conditions for $x = -L$ and $x = L$. Based on the answer to the previous question - what stock prices does this correspond to? We are able to deduce the option price was very low stock price, close to zero, as well for the very high price. Show that our transformation produces reasonable boundary points, which can be made common for different expiration prices.
- How do the answers to the previous questions change if we consider the logarithmic transformation $x = \log(S)$ which also leads to a heat equation?

5.1.4 Exercise: Boundary conditions. Explain, why we should expect that the options prices satisfy

$$V^{call}(0,t) \approx 0,\ V^{call}(S,t) \approx Se^{-r\tau} - Ee^{-r\tau} \text{ for large } S$$

for the call option and

$$V^{put}(0,t) \approx Ee^{-r\tau},\ V^{put}(S,t) \approx 0 \text{ for large } S$$

for the put option.

Earlier (cf. 3.2.6), we have seen that these approximations hold for the closed-form solution to the Black-Scholes equation. We are, however, interested in financial reasoning. In other models, where a closed-form solution is not available and we need to impose boundary conditions, this has to be done by analyzing what is happening at these boundary points.

5.1.5 Mesh for numerical solution. Numerical solution will be defined on the mesh with the time step k and the space step h. In what follows, the index i corresponds to space and index j corresponds to time. The grid points are denoted by x_i (for $i = -n, -n+1, \dots, -1, 0, 1, \dots, n-1, n$) and τ_j (for $j = 0, 1, \dots, m$).

5.1.6 Practical implementation: initial steps. We start the practical implementation in Scilab. These steps are be common for both schemes which we will consider.

- At the beginning, we set the parameters of the call option which we are going to price:

```
// parameters of the call option
E=50;
q=0.12;
sigma=0.4;
T=1;
// interest rate
r=0.04;
```

- We choose the parameter L defining the space interval $[-L, L]$, on which the solution will be computed:

```
L=2;     // x in [-L,L]
```

the parameters of the time and space discretization:

```
// discretization in space
n=20;    // x_i  for i=-n,-n+1,..,-1,0,1,...,n-1,n
h=L/n;   // space step

// discretization in time
m=12;    // tau_j for j=0,1,...,m
k=T/m;   // time step
```

Now we can define the mesh:

```
x = -L:h:L;
tau = 0:k:T;
```

- When defining the initial and boundary conditions, we will need the constants used in the transformation of the Black-Scholes equation to the heat equation (since the initial and boundary conditions are originally defined for the option price and we need to define them for the heat equation):

```
alpha=(r-q)/(sigma^2)-0.5;
beta=(r+q)/2+(sigma^2)/8+((r-q)^2)/(2*sigma^2);
```

  Also, these constants will be needed at the end, when expressing the option prices from the solution of the heat equation.

- The boundary and initial conditions will be defined as functions. Derive these expressions from the boundary conditions for call option prices.

```
// x=-L, i.e., price close to zero
function [phi]=phi(tau)
        phi=0;
endfunction

// x=L, i.e., price close to infinity
function [psi]=psi(tau)
        psi=E*exp(alpha*L+beta*tau).*(exp(L-q*tau)-exp(-r*tau));
endfunction

// initial condition
function [u0]=u0(x)
        u0=E*exp(alpha*x).*max(0, exp(x)-1);
endfunction
```

  This allows an easy change of the code if we want to price other derivative than a call option which we consider now.

- We define the matrix, into which we insert the computed numerical values of the function u:

```
sol=zeros(2*n+1,m+1);
```

5.1.7 Exercise: Boundary and initial conditions. Using the function `phi`, `psi` and `u0` defined above (we stress again that in this way, when pricing another derivate, only these functions will be changed, not the code written now), insert the initial and boundary conditions into the matrix `sol`.

5.2 Explicit and implicit numerical schemes

5.2.1 Discretization of the heat equation. Recall the following two dicretizations of the heat equation (5.2):

- the **explicit scheme** given by

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{\sigma^2}{2} \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}, \tag{5.3}$$

- the **implicit scheme** given by

$$\frac{u_i^j - u_i^{j-1}}{k} = \frac{\sigma^2}{2}\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}. \tag{5.4}$$

In both cases u_i^j denotes the approximation of the function u for $x = x_i$ and $\tau = \tau_j$, i.e.,

$$u_i^j \approx u(x_i, \tau_j).$$

Recall that k is the time step and h is the space step.

5.2.2 Exercise: Implementing the explicit numerical scheme, CFL condition. Compute the numerical solution on the next time layers using the explicit numerical scheme given by (5.3). Note that practically this means going through the columns of the solution matrix (each column represents one time layer) and adding the values one after another using an explicit formula (which uses only the values from the previous time layer)). Transform the numerical solution of the function u to the option prices. Compute the exact option prices using the Black-Scholes formula and compare the obtained numerical solution with the exact prices. Repeat the same procedure for different time and space steps.

There is a condition on the time and space step known as **Courant-Friedrichs-Lewy** condition which has to be satisfied to ensure the stability of the explicit numerical scheme for heat equations. If the heat equation with a diffusion coefficient a^2 (using the usual notation for a heat equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$) is being solved with time step k and space step h, the CFL condition reads as

$$a^2 \frac{k}{h^2} \leq \frac{1}{2}.$$

What does it mean in our case? Check this condition for the time and space steps used and plot the comparison of the numerically obtained values and the exact option price in a case the CFL condition is satisfied and in a case the condition is not satisfied. A sample result is shown in Figure 5.1.

For the given space step h, find the condition on the time step k, so that the CFL condition is satisfied. Suppose that we want to increase the number of grid points in space and set $n = 40$. Determine the condition on the time step, so that the CFL condition holds.

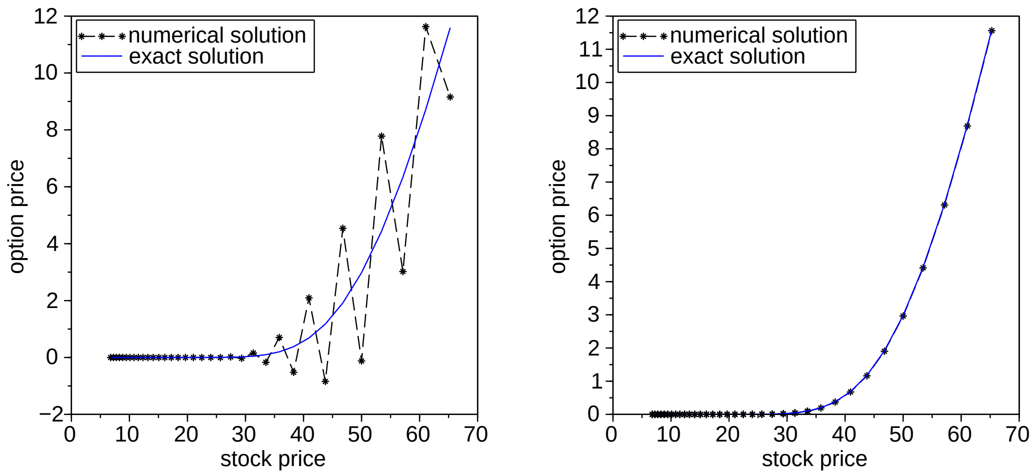


Figure 5.1: Numerical solution of the Black-Scholes equation. The heat equation is solved by explicit scheme while not satisfying the CFL condition (left) and satisfying the CFL condition (right).

5.2.3 Exercise: Trinomial tree. The explicit numerical scheme can be seen as a trinomial tree: The value u_i^j is computed as a weighted sum of u_{i-1}^{j-1}, u_i^{j-1}, u_{i+1}^{j-1} with the weights w_1, w_2, w_3 . What is the relation between the positivity of the weights w_1, w_2, w_3 and the CFL condition[1].

[1]These weights are related to so called risk neutral pricing and risk neutral probabilities.

5.2.4 Exercise: Explicit scheme in the matrix form. Write the explicit numerical scheme in the matrix form

$$u^{j+1} = \mathbf{A}u^j + b,$$

where u^j and u^{j+1} are vectors of numerical values on j-th and $(j+1)$-th time layer, $\mathbf{A}$ is a suitable matrix and b is a suitable vector. Determine $\mathbf{A}$ and b. What does the CFL condition say about the properties of the matrix $\mathbf{A}$?

5.2.5 Exercise: Binomial tree. In the special case, when $h = \sigma\sqrt{k}$, each value is computed as the average of two values from the previous time layer and hence it can be seen as a binomial tree. Show that in this case the CFL condition is satisfied. Choose the parameters of the numerical scheme which lead to a binomial tree and plot the comparison of the numerical values with the exact prices. Suppose that we want to increase the number of grid points, what is the corresponding time step?

5.2.6 Exercise: Implicit scheme in the matrix form. Show that the implicit numerical scheme (5.4) can be written as a system of linear equations which has to be solved to obtain the numerical values of the j-th time layer. Write it in the form

$$\mathbf{A}u^j = b, \tag{5.5}$$

where u^j is vector of numerical values on j-th time layer, $\mathbf{A}$ is a suitable matrix and b is a suitable vector. Determine $\mathbf{A}$ and b.

5.2.7 Exercise: Implementing the implicit scheme. Define the matrix $\mathbf{A}$ from (5.5); this will be common for the computation of all the time layers. In a cycle, for each time layer define the right hand side b and solve the system (5.5).

In Scilab, the backslash operator can be used to compute the solution of a system of linear equations[2]:

```
A\b
```

Compute the solution on the system on each time layer and compare the obtained numerical prices with the exact ones.

Note that in this case CFL condition is not needed, see Figure 5.2.

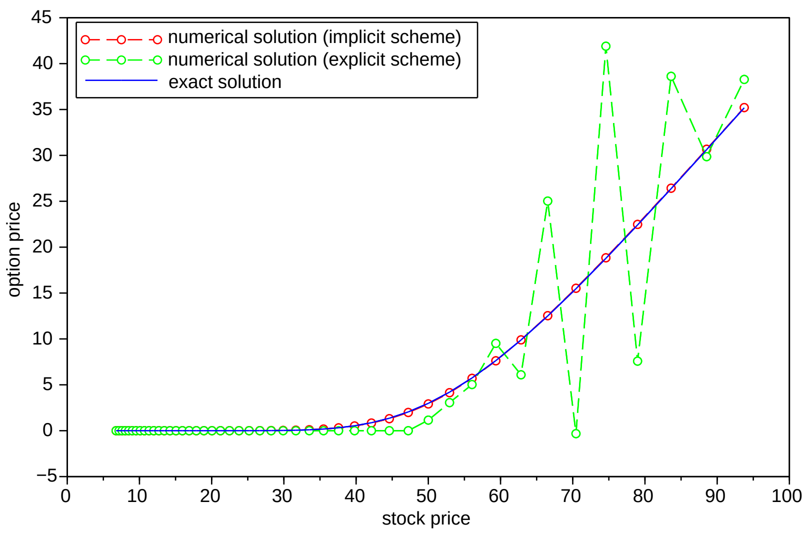


Figure 5.2: Numerical solutions obtained by explicit and implicit methods with the same time and space steps.

[2]See `http://help.scilab.org/docs/5.4.1/en_US/backslash.html` for information on what methods are used.

5.2.8 EXERCISE. Find the parameters of the numerical scheme, for which the numerical prices obtained by implicit method are accurate to four decimal places.

5.3 The SOR method for solving a system of linear equations

5.3.1 MOTIVATION. We are able to solve the system of linear equations arising from the implicit scheme by a simple command, but nevertheless, we are going to study methods for solving such a system, and **successive overrelaxation (SOR) method** in particular. Why?

- To achieve a reasonable precision in option prices, we need to use suitable space interval and step, which leads to a higher dimension of the system. The speed of the computation can be increased by choosing a suitable method. SOR method, in particular, always converges for our system and the speed of the convergence can be increased by a suitable choice of the parameter of the method.
- The SOR method will be a base for the algorithm pricing American-style options, where the numerical problem will no longer be a system of linear equations.

5.3.2 JACOBI AND GAUSS-SEIDEL METHOD. Recall the motivation behind the formulae for these two schemes: Let us write a regular system $\mathbf{A}x = b$ with $a_{ii} \neq 0$ in the form

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i \quad \text{for } i = 1, \ldots, n,$$

where n is the number of equations. The assumption $a_{ii} \neq 0$ enables us to write these equations in the following equivalent form, where we have put x_{ii} to the left hand side of the i-th equation:

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j \right) \quad \text{for } i = 1, \ldots, n.$$

Thus, if we define the iterations $x^{(k)}$ (where k denotes the number of the iteration), starting from some vector $x^{(0)}$, by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k-1)} \right) \quad \text{for } i = 1, \ldots, n. \tag{5.6}$$

for $k = 1, 2, \ldots$ and these iterations converge to a vector x, then the vector x is the solution to the system $Ax = b$. This is known as **Jacobi method**.

Note that when computing $x_i^{(k)}$ using (5.6), we already have $x_j^{(k)}$ for $j = 1, \ldots, i$. Therefore, alternatively, we may use them in (5.6) instead of the values $x_j^{(k-1)}$ ($j = 1, \ldots, i$) from the previous iteration. This leads to the scheme

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k)} - \sum_{j>i} a_{ij} x_j^{(k-1)} \right) \quad \text{for } i = 1, \ldots, n \tag{5.7}$$

for $k = 1, 2, \ldots$, which is known as **Gauss-Seidel method**.

5.3.3 REVISION: ITERATIVE SCHEMES. Recall the general iterative scheme

$$x^{(k+1)} = \mathbf{T}x^{(k)} + c \tag{5.8}$$

(with a certain initial approximation $x^(0)$ for solving a regular system of linear equations $Ax = b$.

- What is the condition for convergence of the iterative scheme (5.8)?
- Write the Jacobi and Gauss-Seidel schemes (5.6) and (5.7) in the form (5.8), using the decomposition $\mathbf{A} = -\mathbf{L} + \mathbf{D} - \mathbf{U}$.
- Write a condition ensuring convergence of the Gauss-Seidel method in terms of the matrix $\mathbf{A}$ (i.e., in terms of the matrix $\mathbf{A}$ of the linear system, not in terms of the iteration matrix $\mathbf{T}$).

- Show that the Gauss-Seidel method converges for any starting point, when applied to the system from the implicit scheme for solving the heat equation.

5.3.4 Exercise: Exploring the Gauss-Seidel method on a small system. Let us take the example from the lecture: the system $\mathbf{A}x = b$ with

```
A=[8,6;6,7];
b=[14;13];
```

which has the exact solution $(x_1, x_2)' = (1, 1)'$. We will do the similar analysis in Scilab, as was presented at lecture on a spreadsheet.

- Show that the Gauss-Seidel method converges for any starting point.
- Define the initial approximation of the solution:

  ```
  x1=[0];
  x2=[0];
  ```

  and the Gauss-Seidel iterations:

  ```
  N= ; // here choose the number of iterations

  for i=1:N
          x1new=(b(1)-A(1,2)*x2(i))/A(1,1);
          x1=[x1,x1new];
          x2new=(b(2)-A(2,1)*x1(i+1))/A(2,2);
          x2=[x2,x2new];
  end
  ```

  Plot the iterations and the exact solution:

  ```
  plot(x1,x2,'o--');
  ```

  A sample result is shown in Figure 5.3.

- Using the results from the previous point, make a table such as the one started in Table 5.1. How many Gauss-Seidel iterations are needed to achieve the precision $\|x^{(k)} - x^*\| < 10^{-5}$?
- Repeat the same analysis with another matrix (and possibly another right hand side, so that the exact solution is easily computed), while leaving is symmetric and positive definite[3].

k	$x_1^{(k)}$	$x_2^{(k)}$	$\|x^{(k)} - x^*\|$
0	0	0	1.4142136
1	1.75	0.3571429	0.9878083
2	1.4821429	0.5867347	0.6350196
3	1.309949	0.7343294	0.4082269
...	...	...	...
10	1.0140634	0.9879456	0.0185226

Table 5.1: Convergence of the Gauss-Seidel method. Here, x^* denotes the exact solution of the system and for $\|.\|$ we use the L^2-norm.

[3]The reason for these conditions will be seen later, when considering the SOR method.

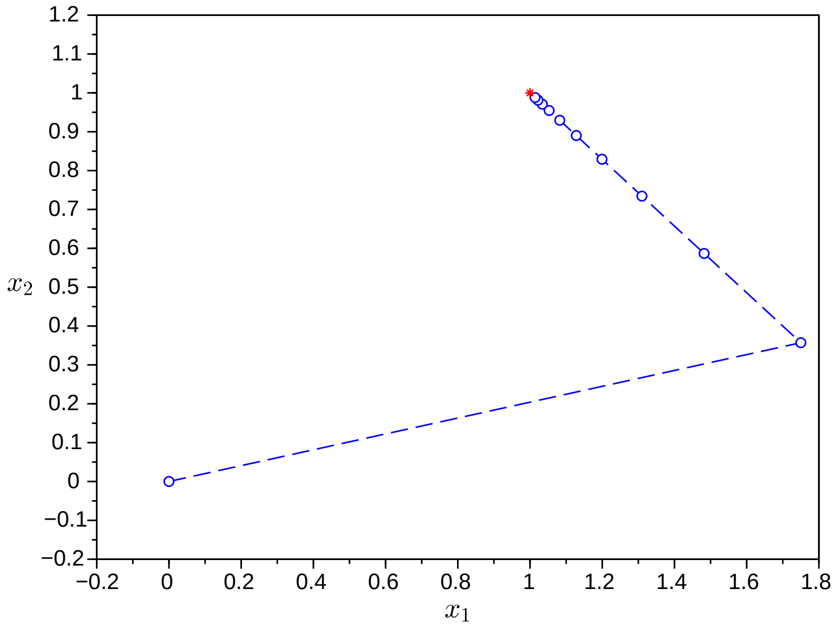


Figure 5.3: Convergence of Gauss-Seidel method; the red point is the exact solution.

5.3.5 Successive overrelaxation (SOR) method. Recall from the lectures that the Gauss-Seidel iteration can be written as

$$x_i^{(k)} = x_i^{(k-1)} + \frac{1}{a_{ii}} r_{ii}^{(k)},$$

where $r_i^{(k)}$ is the residue at the time of the computation of $x_i^{(k)}$ and $r_{ii}^{(k)}$ is its i-th element. Motived by the numerical experiments in the previous point, we define the new method by

$$x_i^{(k)} = x_i^{(k-1)} + \omega \frac{1}{a_{ii}} r_{ii}^{(k)}, \tag{5.9}$$

where ω is a parameter. For $\omega > 1$ we have an **overrelaxation** (this agrees with our original motivation) and for $0 < \omega < 1$ we have an **underrelaxation** (as we will see, the method may converge also in this case). For $\omega = 1$, the method coincides with Gauss-Seidel.

5.3.6 Exercise: Implementing and testing the SOR method, Exercise 5.3.4 continued. The SOR method (5.9) can be written as

$$x_i^{(k)} = \frac{\omega}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k)} - \sum_{j>i} a_{ij} x_j^{(k-1)} \right) + (1-\omega) x_i^{(k-1)}. \tag{5.10}$$

Use this expression to modify the Gauss-Seidel iterations in the Scilab script and produce the iterations of the SOR method.

- Add these iterations to the plot of the Gauss-Seidel iterations. A sample result is in Figure 5.4.
- How many SOR iterations are needed to achieve the precision $\|x^{(k)} - x^*\| < 10^{-5}$? Add columns to Table 5.1, indicating the same quantities for SOR method, as suggested in Table 5.2. Choose two values of ω - one for which the SOR method converges more quickly and one for which the SOR method converges but more slowly than Gauss-Seidel.
- Find an example of a value of ω, for which the SOR method diverges.

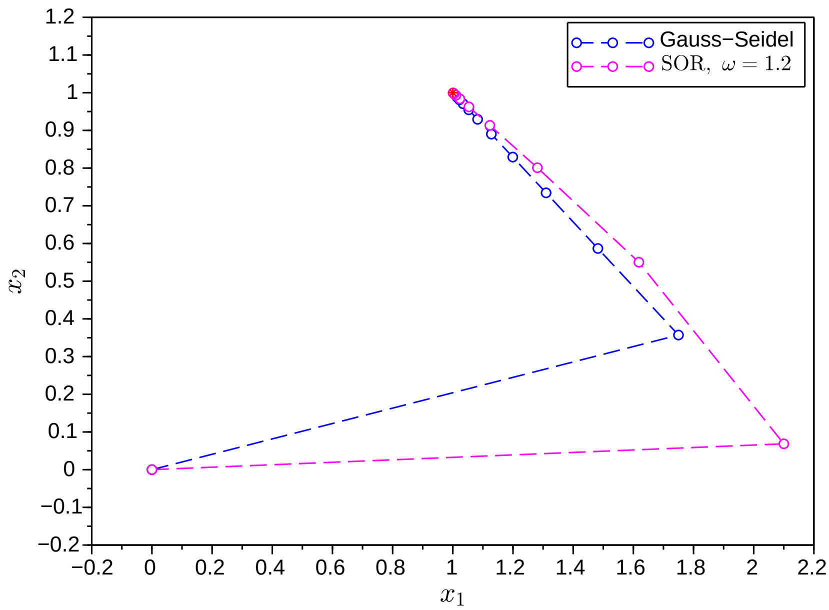


Figure 5.4: Convergence of Gauss-Seidel and SOR methods; the red point is the exact solution.

	Gauss-Seidel			SOR, $\omega = 1.2$		
k	$x_1^{(k)}$	$x_2^{(k)}$	$\lVert x^{(k)} - x^* \rVert$	$x_1^{(k)}$	$x_2^{(k)}$	$\lVert x^{(k)} - x^* \rVert$
0	0	0	1.4142136	0	0	1.4142136
1	1.75	0.3571429	0.9878083	2.1	0.0685714	1.4413741
2	1.4821429	0.5867347	0.6350196	1.6182857	0.5503347	0.7645104
3	1.309949	0.7343294	0.4082269	1.2810416	0.8008617	0.3444423
...	...	...	...	...	...	...
10	1.0140634	0.9879456	0.0185226	1.0008177	0.9994245	0.0009999

Table 5.2: Convergence of Gauss-Seidel method and SOR methods. Here, x^* denotes the exact solution of the system and for $\|.\|$ we use the L^2-norm.

5.3.7 Ostrowski-Reich convergence theorem for SOR method. This theorem states that for a system with a positive definite matrix, the SOR method converges for any $0 < \omega < 2$ and any initial approximation.

5.3.8 Exercise. Compare your observations from Exercise 5.3.6 with Ostrowski-Reich theorem.

5.3.9 Error estimate and spectral radius. For the iterative scheme (5.8), the following estimate holds[4]

$$\|x^{(k)} - x^*\| = \|\mathbf{T}^k\,[x^{(0)} - x^*]\| \leq \|\mathbf{T}^k\|\;\|x^{(0)} - x^*\|,$$

where $x^{(k)}$ is the approximation from k-th iteration and x^* is the exact solution. This motivates finding an estimate on $\|\mathbf{T}^k\|$. This estimate follows from the following property of the spectral radius (the maximum absolute value of the eigenvalues of the matrix), known as **Gelfand theorem**:

$$\lim_{k\to\infty} \|\mathbf{T}^k\|^{1/k} = \rho(\mathbf{T}), \tag{5.11}$$

where $\rho(\mathbf{T})$ is the spectral radius of the matrix $\mathbf{T}$. Thus, $\|\mathbf{T}^k\| \approx \rho(\mathbf{T})^\mathbf{k}$ and so we have an estimate

$$\|x^{(k)} - x^*\| \leq \|\mathbf{T}^k\|\;\|x^{(0)} - x^*\| \approx \rho(\mathbf{T})^\mathbf{k}\;\|\mathbf{x}^{(\mathbf{0})} - \mathbf{x}^*\|. \tag{5.12}$$

[4]For the last inequality we need the vector and matrix norms to be compatible (cf.notes on the website).

5.3.10 Numerical examples illustrating Gelfand theorem. We are going to numerically "check" the relation (5.11).

- By typing

```
help norm
```

  get an information on the vector and matrix norms in Scilab.

- Firstly, we repeat the computation presented in lectures: Generate a random matrix, for example:

```
A=rand(3,3,"normal");
```

  and compute its spectral radius[5] and the values of $\|\mathbf{A}^k\|^{1/k}$. Plot a graph showing their convergence to spectral radius. A sample result is shown in Figure 5.5.

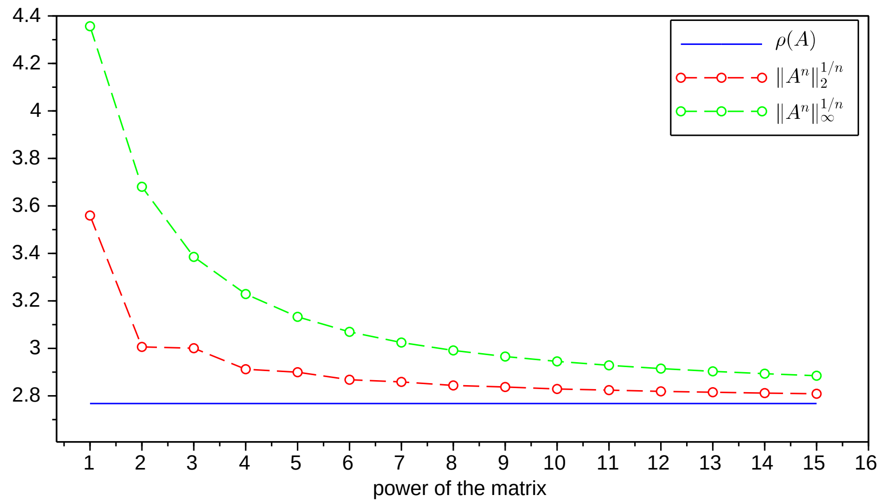


Figure 5.5: Convergence of $\|A^n\|^{1/n}$ to the spectral radius of the matrix A.

- What we actually need for our analysis of the iterative schemes for solving systems of linear equations, is the approximation $\|\mathbf{A}^k\| \approx \rho(\mathbf{A})^{\mathbf{k}}$ for a matrix $\mathbf{A}$ with spectral radius less than unity[6]. Suggest a method that generates a random matrix with spectral radius less than unity and compare the quantities $\|\mathbf{A}^k\|$ and $\rho(\mathbf{A})^{\mathbf{k}}$.

5.3.11 Estimate on the spectral radius of the SOR iteration matrix (Kahan theorem). If the diagonal elements of the matrix of the system are nonzero, then we have the following estimate on the spectral radius of the SOR iteration matrix[7] $\mathbf{T}_\omega$:

$$\rho(\mathbf{T}_\omega) \geq |\omega - \mathbf{1}|. \tag{5.13}$$

5.3.12 Exercise: Derivation of a necessary condition for the convergence of the SOR method. Use the estimate (5.13) to show that the SOR method can converge only for $\omega \in (0, 2)$.

In general, this not a sufficient condition, but we have seen a class of matrices for which this constitutes also a sufficient condition (positive definite matrices, according to the Ostrowski-Reich theorem, see point 5.3.7). However, if ω is outside of the interval $(0, 2)$, the method does not converge.

[5] Scilab command `spec` computes the eigenvalues of a matrix, i.e., its spectrum.
[6] Recall that this condition on the spectral radius of the iteration matrix is needed for convergence of the iteration scheme.
[7] The subscript denotes the dependence of the iteration matrix $\mathbf{T}$ on the parameter ω.

Again, compare this result with the numerical experiments and finding the value of ω, for which the SOR method does not converge.

5.3.13 Solving a system with positive definite tridiagonal matrix. For the SOR method applied to a system with positive definite tridiagonal matrix, the following propositions hold:

1. There is a relation between the spectral radii of the Jacobi and Gauss-Seidel iteration matrices:
$$\rho(\mathbf{T}_{gs}) = \rho(\mathbf{T}_j)^2$$
2. The optimal choice of the parameter ω for the SOR method (i.e., which leads to the smallest spectral radius of the iteration matrix) is given by
$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(\mathbf{T}_j)]^2}}. \tag{5.14}$$
With this choice of ω we have $\rho(\mathbf{T}_\omega) = \omega - 1$

Above, we have denoted $\mathbf{T}_j, \mathbf{T}_{gs}, \mathbf{T}_\omega$ the iteration matrices of Jacobi, Gauss-Seidel and SOR (with parameter ω given by the subscript) methods.

5.3.14 Exercise. Consider the system with positive tridiagonal matrix. Denote $\mathbf{T}_{\omega^*}$ the iteration matrix of the SOR method with optimal ω chosen according to (5.14). Show that then
$$\rho(\mathbf{T}_j) < \rho(\mathbf{T}_{gs}) < \rho(\mathbf{T}_{\omega^*}),$$
where $\mathbf{T}_j, \mathbf{T}_{gs}$ are the iteration matrices of Jacobi and Gauss-Seidel methods. What does this inequality say about the speed of convergence of these methods?

5.3.15 Exercise. Return to the numerical experiments with our small 2×2 system from 5.3.4 an 5.3.6.

- The SOR method can be written in the matrix form as
$$x^{(k+1)} = (\mathbf{D} - \omega\mathbf{L})^{-1}[(1-\omega)\mathbf{D} + \omega\mathbf{U}]x^{(k)} + \omega(\mathbf{D} - \omega\mathbf{L})^{-1}b \tag{5.15}$$
Make a plot of the spectral radius of the iteration matrix as a function of ω. A sample result is given in Figure 5.6. Then, show the dependence of the number of iterations necessary to achieve a given precision (for example, so that the norm of the difference between the approximation and the exact solution is less than 10^{-5}) as a function of ω. Compare the two graphs.
- Compute the optimal value of ω according to (5.14). Check the relation with the graphs from the previous question.
- Verify numerically the statement that for the spectral radius of the iteration matrix for the optimal ω equals $\omega - 1$.
- Find the range of ω, for which the iteration matrix corresponding to SOR method has a smaller spectral radius than the matrix corresponding to Gauss-Seidel method.

5.4 Application of the SOR method to the Black-Scholes equation

5.4.1 Exercise: Matrix from the numerical solution of the Black-Scholes equation Show that the matrix from the implicit scheme applied to solving the Black-Scholes equation is tridiagonal and positive definite. Which properties from the previous section therefore hold?

5.4.2 Exercise: Auxiliary function for Gauss-Seidel method applied to a symmetric tridiagonal system. Firstly, we consider the Gauss-Seidel method. Finish the code below an save it as `gs.sci`:

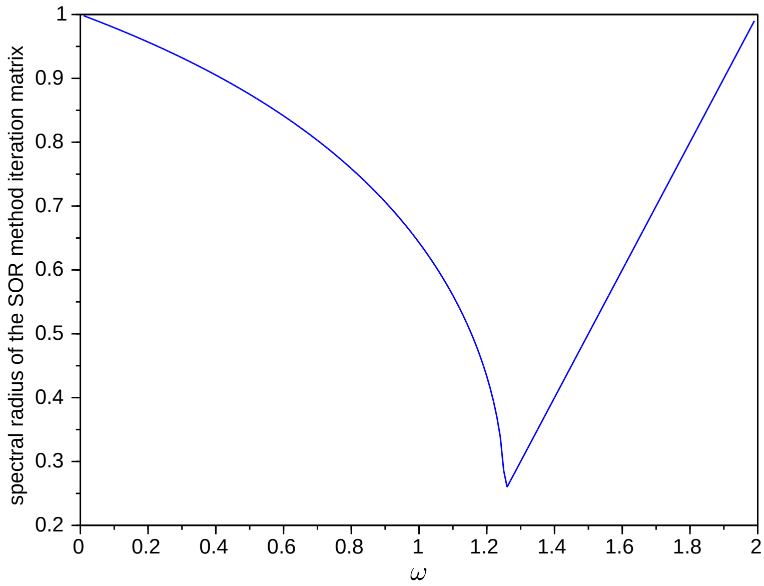


Figure 5.6: Spectral radius of the SOR method iteration matrix for different values of ω.

```
function [v] = gs(a,b,rhs,v0,epsilon)
// Gauss-Seidel method for a symmetric tridiagonal system A*v=rhs
//
// inputs:
// a,b: the matrix A has: a above and under the diagonal
//                        b on the diagonal
// rhs: right hand side of the system
// v0: the initial approximation
// epsilon: iterations finish when norm(A*v-b)<=epsilon
//
// output:
// v: approximation of the solution

function [v]=gs(a,b,rhs,v0,epsilon)

N=length(v0);

// here write the subfunction which computes te error
// for the given approximation w
//
// function [err]=err(w)
// ...
// endfunction

// loop: iterations continue while the error is still large
v=v0;
while err(v)>epsilon
        v(1)=(rhs(1)-a*v(2))/b;
        for i=2:N-1
                v(i)=(rhs(i)-a*(v(i-1)+v(i+1))/b;
```

```
        end;
        v(N)=(rhs(N)-a*v(N-1))/b;
end;

endfunction;
```

Explain this form for a symmetric tridiagonal matrix (what do the sums appearing in the formula simplify to?).

5.4.3 Exercise: Application of Gauss-Seidel method to Black-Scholes PDE. Use the function defined in the previous point to solve the system arising from the implicit scheme for solving the heat equation, as a part of the numerical solution to the Black-Scholes equation.
Hint. The only difference is in the line where the system of linear equation is being solved. Instead of using

```
A\b
```

call the function

```
gs(a,b,rhs,v0,eps)
```

with suitable parameters. Although the method converges for any initial approximation, it is a good idea to provide it close to the expected solution to speed up the computation. A good candidate is the solution from the previous time layer. Use the parameters from Exercise 5.2.8 and find suitable value of precision parameter `epsilon`, so that the option prices retain the precision. How many iterations are needed to compute one time layer? Compare with using the Gauss-Seidel method.

5.4.4 Exercise: An alternative criterion for stopping iterations. When pricing American options in the next chapter, the problem to be solved will not be a system of linear equations, and hence we will not be able to use a criterion based on the error. An alternative criterion suggesting that the convergence has been achieved is, that the two successive iterations are already close to each other. If we denote the approximation from the k-th iteration as $x^{(k)}$, then the criterion takes the form $\|x^{(k)} - x^{(k-1)}\| \leq \varepsilon$.

Implement this criterion to solving the system of linear equations using Gauss-Seidel method and find a suitable ε.

We return to this in the next chapter in the context of pricing American derivatives. In this chapter, we continue using the original criterion based on the norm of the residue.

5.4.5 Exercise: Auxiliary function for the SOR method. Define the function

```
sor(omega,a,b,rhs,v0,eps)
```

analogous to `gs(a,b,rhs,v0,eps)` by adjusting the `while` loop.

5.4.6 Exercise: Application of SOR method to Black-Scholes PDE. Use the function defined in the previous point to numerically solve the heat equation, as a part of finding a numerical solution to the Black-Scholes equation. Based on the earlier propositions and analyses, what values of ω produce a convergent scheme?

5.4.7 Exercise: Choice of parameter ω for the SOR method applied to Black-Scholes PDE: parameters of the scheme. Firstly, we should note that numerical computation of the spectral radii of the iteration matrices for different values of ω is computationally demanding and practically it is not a good way of choosing ω for a concrete numerical scheme. The aim of this exercise is to get an intuition about behaviour of the optimal ω. This intuition is also useful when studying the analytical optimality results for this problem, presented in the lectures: the resulting - otherwise complicated looking - formulae ”make sense”.

Choose different set of parameters of the numerical scheme and see, how refining the mesh (which should lead to a more precise numerical solution) affects the performance of the SOR method. For a given set of parameters compute the optimal ω using the formula (5.14) and display them graphically. How does it depend on the parameters of the numerical scheme? A sample result is shown in Figure 5.7.

Check the accuracy of the numerical solution for a given set of parameters. If a more precise solution is needed, how could it be achieved? What do you expect about the corresponding optimal value of ω?

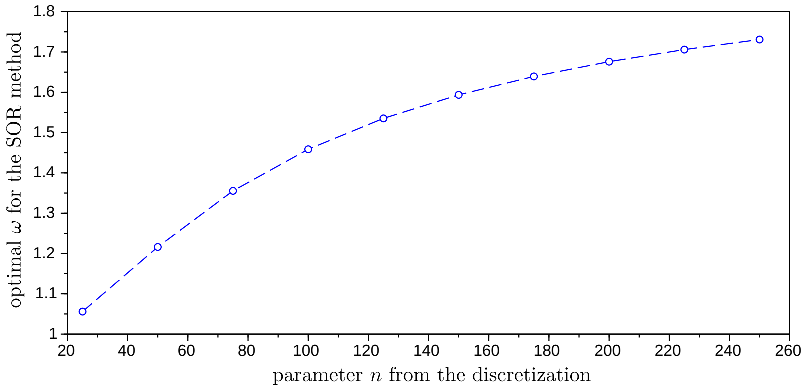


Figure 5.7: Optimal ω for the numerical schemes.

5.4.8 Exercise: Pricing the put option. We are going to price a put option. Choose L and compute the boundary points for the stock price, in which the boundary conditions are used - do you consider these prices to be sufficiently low and high, so that the boundary conditions can be applied? Based on the previous exercises, choose parameters of the scheme which you consider reliable to provide an approximation with high accuracy and the parameter ω for the SOR method, which you consider suitable with respect to speed of the convergence.

Use these values to price the selected put option. Then compare your numerical results with the exact solution. Was your computation successfull?

Chapter 6

American derivatives: properties and numerical methods

6.1 American derivatives: the basic principles and properties

6.1.1 American derivatives. The feature, which distinguishes American derivatives from their European counterparts, is the possibility of their early exercise. European derivatives can be exercised only at the predetermined exact time (time of expiration), while American derivatives can be exercised at any time prior to the time of their expiration.

6.1.2 Exercise: Price of an American derivative vs. the payoff. The price of an American derivative cannot fall under the payoff, as a consequence of a possibility of its immediate exercise.

Consider an American call option with exercise price of 100 USD. The price of the underlying stock is 120 USD. Show what arbitrage would be present, if the option price was 115 USD.

6.1.3 Exercise: Natural inequality comparing American and European option prices. In what follows, we denote by $C^a(S,t)$, $P^a(S,t)$ the prices of American call and put options, and by $C^e(S,t), P^e(S,t)$ their European counterparts. Parameter S denotes the current price of the underlying stock and t is the time. All the options considered are written on the same underlying asset and have the same exercise price. Unless stated otherwise, they have the same expiration time.

Since American options provide their holder more opportunities (exercise at any time until the expiration time) than European ones (exercise only at the expiration time), the obvious inequalities have to hold:

$$C^e(S,t) \leq C^a(S,t), P^e(S,t) \leq P^a(S,t).$$

What would happen otherwise? Explicitly describe the arbitrage strategy in the following situation: The stock price today equals 100 USD. There are European and American options with expiration price 125 USD, which expire in two months. The price of the European option is 30 USD and the price of the American option is 28 USD.

6.1.4 Exercise: American options with different expiration times. Using the same idea as above, comparing the possibilities given by two derivatives, we can deduce that the price of the American option which expires later cannot be smaller than the price of the American option with earlier expiration if they have the same exercise price.

Construct the arbitrage strategy in the following situation: The stock price today equals 100 USD. There is are two American options with expiration price 125 USD: the first one expires in two months and costs 28 USD, the second one expires in one month and costs 31 USD.

6.1.5 American call option on a non-dividend paying stock. In general, prices of the American type options are higher than those of European type. There is, however, one important difference: The price of the American call option on a stock which does not pay dividends, equals the price of the European option. We assume that $r > 0$.

This proposition says that the extra possibility of an earlier exercise, which the American call option gives, has in fact a zero value. To see why, suppose that we have the option and exercise it prior to the expiration, at time $t < T$. We receive the stock for the price E, so our profit is $S - E$, where S is the stock price at the time we decided to use the option. If instead of exercising the option, we decide to sell it, we get $C^a(S,t)$. Using the bounds on the call option price (see Exercise 1.3.6) we obtain the estimate

$$C^a(S,t) \geq C^e(S,t) \geq S - Ee^{-r(T-t)} > S - E, \tag{6.1}$$

since both r and $T - t$ are positive. Hence it is never optimal to exercise the option prior the expiration, it is always better to sell it instead. Hence the possibility to exercise the option early has no value.

6.1.6 Exercise: American call option on a non-dividend paying stock, an alternative proof. Give an alternative proof of the statement $C^a = C^e$ using an arbitrage argument. Suppose that $C^a > C^e$ at a certain moment and consider the following strategy:

- We buy the American call and sell the European call.
- If the American call is exercised at the expiration time or it is not exercised at all, we do the same with our European call.
- If the American call is exercised at time $t < T$, we go to short position in stock to fulfill our obligation and put the received premium E to the bank account. At time T we use our European option to get the stock for E and close the short position.

Show that the initial cash flow is positive and the payoff at time T is $E(e^{r(T-t)} - 1) > 0$ if the American option is exercised prior to expiration at time $t < T$ and zero otherwise.

6.1.7 American call option on a non-dividend paying stock: analysis of the proof. We have shown that the Black-Scholes price of a call option on a stock which pays dividends is below the payoff for large values of the stock price, see Exercise 3.2.5. Hence the prices of an American and an European option cannot coincide in this case. Where we have used in 6.1.5 and 6.1.6 that the stock does not pay dividends?

6.1.8 Exercise: Bounds on the difference between prices of American put and call options on a non-dividend paying stock. Recall that in the case of European option, the difference between put and call option is related to the stock price and exercise price by a put-call parity $P^e - C^e + S = Ee^{-r\tau}$, see 1.3.1.

- Why a similar equality does not hold for American options? Which step from the derivation of the put-call parity for European options cannot be done?
- Derive the following bounds on American options prices, which are written on a stock that does not pay dividends:
$$Ee^{-r(T-t)} \leq S + P^a - C^a \leq E. \tag{6.2}$$
- Using the real data of call option prices and the bounds (6.2), form an interval for possible put option prices. Compare with the actual price of the put option.

6.2 American derivatives in the Black-Scholes setting

6.2.1 Exercise: American put options. Show that regardless on whether the underlying stock does not pay dividends or not, the price of the American put option cannot be equal to the price of its European counterpart if $r > 0$.

6.2.2 Summary: what needs to be done. We are going to consider:

- call options on the non-dividend paying assets,
- put options on both dividend paying and non-dividend paying assets.

6.2.3 Free boundary problem for the American call option price. Recall the behaviour of the price from the lectures, see also Figure 6.1.

- if $S < S_f(t)$: the price satisfies Black-Scholes PDE, we keep the option (we do not exercise it)
- if $S > S_f(t)$: the price equals payoff, we exercise the option
- if $S = S_f(t)$: at this point, the price has the same value (because of the requirement of the continuity) and derivative (because of the requirement of the smoothness) as the payoff

The function $S_f(t)$ represents the **early exercise boundary**; from the mathematical point of view it is a **free boundary**.

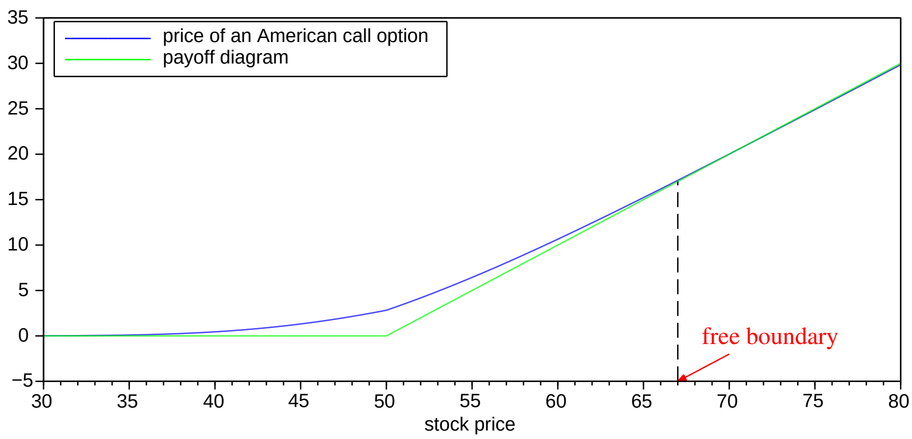


Figure 6.1: Pricing American call option.

The mathematical formulation is therefore the following:

- Function $V(S,t)$ is a solution to the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S} - rV = 0$$

  on the time-dependent domain $0 \le t < T, 0 < S < S_f(t)$.

- Terminal condition is given by

$$V(S,T) = \max(S-E, 0).$$

  for all S.

- Conditions on the boundary $S = 0$ and $S = S_f(t)$ for $0 \le t < T$ are the following:

$$V(0,t) = 0, \quad V(S_f(t),t) = S_f(t) - E, \quad \frac{\partial V}{\partial S}(S_f(t),t) = 1.$$

6.2.4 Linear complementarity problem for the American call option price. The free boundary formulation from 6.2.3 can be transformed into the following linear complementarity problem:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S} - rV \le 0,$$

$$V(S,t) \ge \bar{V}(S),$$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q) S \frac{\partial V}{\partial S} - rV \right) \left(V(S,t) - \bar{V}(S) \right) = 0$$

for all $S \in (0,\infty), 0 < \tau \leq T$.

Recall that in this transformation we have used the bound of the free boundary position

$$S_f(t) \geq E\max(1, r/q),$$

which we had derived. It is used to establish the inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV \leq 0.$$

6.2.5 Exercise: Transformation of the linear complementarity problem for the call. Use the same series of the transformations $V(S,t) \to Z(x,\tau) \to u(x,\tau)$ as in the case of pricing European options to obtain the following linear complementarity problem:

$$\begin{aligned} \left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) &= 0, \\ \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \geq 0, \quad u(x,\tau) - g(x,\tau) &\geq 0 \end{aligned}$$

for $x \in \mathbb{R}, 0 < \tau \leq T$, with transformed payoff

$$g(x,\tau) = Ee^{\alpha x + \beta\tau}\max(0, e^x - 1),$$

where α, β are given by (5.1). This is supplemented with initial condition $u(x,0), x \in \mathbb{R}$.

Using the computations done for the European option, writing the most part of the transformed problem is straightforward; we are mostly concerned in establishing whether $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \geq 0$ or $\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \leq 0$.

6.2.6 Exercise: Free boundary problem for the American put option price. Write down the free boundary formulation of the American put option pricing problem and sketch the solution (option price as a function of the current price of the underlying stock). What are the differences from the American call option pricing problem?

6.2.7 Exercise: Location of the free boundary. Consider the option prices, computed for an American put with exercise price of 10 USD, given in Table 6.1. Determine the interval in which the free boundary is located.

stock price	price of the put option
0	10.0000
2	8.0000
4	6.0000
6	4.0000
8	2.0200
10	0.6913
12	0.1711
14	0.0332
16	0.0055

Table 6.1: American put option prices.

6.2.8 Exercise: Linear complementarity problem for the American put option price. The aim is to transform also the American put option pricing problem written in its free boundary formulation as a linear complementarity problem. We follows the steps from the transformation in the case of a call.

- Recall the property of the free boundary

$$\frac{\partial V^{call}}{\partial t}(S_f(t), t) = 0,$$

derived in lectures for the price of an American call. Using the same method, derive this property of the price of an American put option.

- Recall from the lectures the derivation the the estimate on the free boundary location for the call option, which reads as

$$S_f(t) \geq \max\left(\frac{r}{q}E, E\right).$$

Derive an analogous estimate for the free boundary in the case of pricing a put option. Use the inequality to show that

$$\frac{\partial V^{put}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^{put}}{\partial S^2} + (r-q)S\frac{\partial V^{put}}{\partial S} - rV^{put} \leq 0.$$

- Write down the linear complementarity problem for the American put option price.

6.3 Numerical algorithm for pricing American call and put options

6.3.1 Parameters of the test example. We are going to price the put option on a stock with volatility $\sigma = 0.40$ which does not pay dividends. The exercise price is 10 USD and the option expires in 3 months. The interest rate is assumed to be equal to 10 percent *per annum*. Thus we have the following:

```
// parameters of the option
E=10;
q=0;
sigma=0.4;
T=1/4;
// interest rate
r=0.1;
```

The prices, accurate to four decimal places are given in Table 6.1, which we can use for checking the accuracy of our computations.

6.3.2 Exercise: Transformed problem. Based on the computations from the previous section, write down the transformed linear complementarity problem and its discretization in the form

$$\mathbf{A}u \geq b, \quad u \geq g, (\mathbf{A}u - b)_i(u_i - g_i) = 0 \quad \text{for all } i, \tag{6.3}$$

which we need to solve on each time layer.

6.3.3 Exercise: Analysis of the algorithm for European options. The beginning of the code can be preserved: definition of the mesh, definition of the initial condition, inserting the initial condition into the matrix. We need to take care about the boundary conditions: at one boundary, the original boundary condition is preserved; on the other boundary, the value is set to be equal to the payoff.

What boundary condition remains the same and which one equals the payoff in our case of a put option? How is it in the case of a call option? Change the definitions of the transformed boundary conditions in the code:

```
// x=-L, i.e., price close to zero
function [phi]=phiUS(tau)
        phi= ... ;
endfunction

// x=L, i.e., price close to infinity
function [psi]=psiUS(tau)
        psi=...;
endfunction
```

Insert the values into the matrix with solution.

6.3.4 Numerical method for solving the discrete linear complementarity problem. At each time layer we are solving a problem which has the form (6.3). The **PSOR (projected successive overrelaxation)** method for solving this system is given by

$$u_i^{(p+1)} = \max\left[\frac{\omega}{A_{ii}}\left(b_i - \sum_{j<i} A_{ij} u_j^{(p+1)} - \sum_{j>i} A_{ij} u_j^{(p)}\right) + (1-\omega) u_i^{(p)},\ g_i\right]$$

and some initial approximation u^0. It means that in each step we compare the SOR iteration with the transformed payoff. If the SOR iteration falls below the transformed payoff (which is not permitted for the solution), it is replaced by the transformed payoff. Otherwise it is kept. The convergence of the algorithm has been proved in lectures.

6.3.5 Implementation of the PSOR algorithm to American option pricing. Implement the PSOR method on each time layer to solve the discrete linear complementarity problem. Transform the obtained values into the option prices and compare them with the values given in Table 6.1. If the precision is not satisfactory, adjust the parameters of the numerical scheme.

6.3.6 Numerically finding the free boundary. Using the developed algorithm, find the numerical approximation of the free boundary. Plot it and give the implications of the result for trading.

Chapter 7

Vasicek model of interest rates

7.1 Modelling the short rate

7.1.1 Stochastic differential equation for the short rate. The short rate (the instantaneous interest rate) r is modelled by an Ornstein-Uhlenbeck process

$$dr = \kappa(\theta - r)dt + \sigma dw, \tag{7.1}$$

where κ, θ, σ are positive parameters. We have already studied this process and mentioned its application to interest rate modelling, see Exercises 2.4.1 and 2.4.2. Recall the effect of the parameters to the typical evolution of the short rate.

7.1.2 Itō integral. In lectures, the conditional distribution of the short rate, given its value at an earlier time, was derived using the Fokker-Planck equation. Here, we present an alternative derivation, which uses the notion of **Itō integral**.

The Itō integral of a nonrandom function $f = f(t)$ is defined by[1]

$$\int_a^b f(\tau)dw(\tau) = \lim_{\nu \to 0} \sum_{i=0}^{n} f(\tau_i)[w(\tau_{i+1}) - w(\tau_i)], \tag{7.2}$$

where $a = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = b$ is a partition of the interval $[a, b]$ and ν is the norm of this partition. Note the similarity of (7.2) with the definitions of Riemann and Riemann-Stieltjes integrals.

7.1.3 Exercise: Properties of Itō integral. We see that the Itō integral (7.2) depends on the Wiener process, so it is a random variable. The aim of the following questions is to explore this concept numerically by computer simulations and to compute its probability distribution.

Part I: Numerical simulations.

- In Exercise 2.1.2 we have simulated trajectories of a Wiener process in the following way:

```
function [r]=randn()
        r=rand(1,"normal");
endfunction

function [w]=wiener(dt,n)
        w(1)=0;
        for i=1:n
                dw=sqrt(dt)*randn();
                w(i+1)=w(i)+dw;
```

[1] We stay at this intuitive level; the details on the class of integrable functions, existence of the limit, etc. in a text specialized on stochastic calculus, for example [3].

```
        end;
        w=w';
endfunction
```

We define a function which we are going to integrate:

```
function [f]=f(t)
        f=t.^2;
endfunction
```

Now, we have everything necessary to numerically simulate Itō integral.

- Create a trajectory of a Wiener process:

```
dt=0.01;
n=100;
t=(0:dt:n*dt);

w=wiener(dt,n);
```

Compute the approximation of the Itō integral corresponding to this trajectory by evaluating the sum at the right hand side of (7.2).

- Simulate more trajectories of the Wiener process and compute the corresponding values of the Itō integral. Plot the histogram of all the values generated.
- Repeat for the simulation with a smaller time step.

PART II: COMPUTATION OF THE PROBABILITY DISTRIBUTION.

- Show that the probability distribution of the sum on the right hand side of (7.2) is normal and compute its expected value and variance.
- What is the limit of the expected value and variance, as the norm of the partition approaches zero?
- Deduce the probability distribution of the Itō integral and compare with the results of the simulations.

7.1.4 EXERCISE: CONDITIONAL DISTRIBUTION OF THE SHORT RATE. The solution to the stochastic differential equation (7.1) with initial condition $r(t_0) = r_0$ can be written in a closed form, using the Itō integral:

$$r(t) = e^{-\kappa(t-t_0)}\, r_0 + (1 - e^{-\kappa(t-t_0)})\, \theta + \sigma \int_{t_0}^{t} e^{-\kappa(t-s)} \mathrm{d}s. \tag{7.3}$$

- By differentiating (7.3) verify that it is indeed a solution to the stochastic differential equation (7.1) and check that it also satisfies the initial condition $r(t_0) = r_0$.
- Use the properties of the Itō integral to show that the conditional distribution of $r(t)$, given that $r(t_0) = r_0$, is a normal distribution and its moments are given by

$$\begin{aligned} \mathbb{E}[r(t)|r(t_0) = r_0] &= e^{-\kappa(t-t_0)}\, r_0 + (1 - e^{-\kappa(t-t_0)})\, \theta, \\ \mathbb{D}[r(t)|r(t_0) = r_0] &= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-t_0)}). \end{aligned}$$

- Find the limiting distribution as $t \to \infty$.

7.1.5 Exercise. Consider the conditional distribution of $r(t)$, given that $r(0) = r_0$. How does the expected value and dispersion depend on t? Sketch their graphs. How do they depend on the parameters of the process?

Match the following parameter values with evolutions of expected value and variance of the future interest rates in Figure 7.1; in all the cases, θ is equal to 0.05 and the initial value of the short rate is 0.07:

1. $\kappa_0 = 0.75, \sigma_0 = 0.01$,
2. $\kappa_1 = 5 \times \kappa_0, \sigma_1 = 2 \times \sigma_0$,
3. $\kappa_1 = 10 \times \kappa_0, \sigma_1 = \sigma_0$.

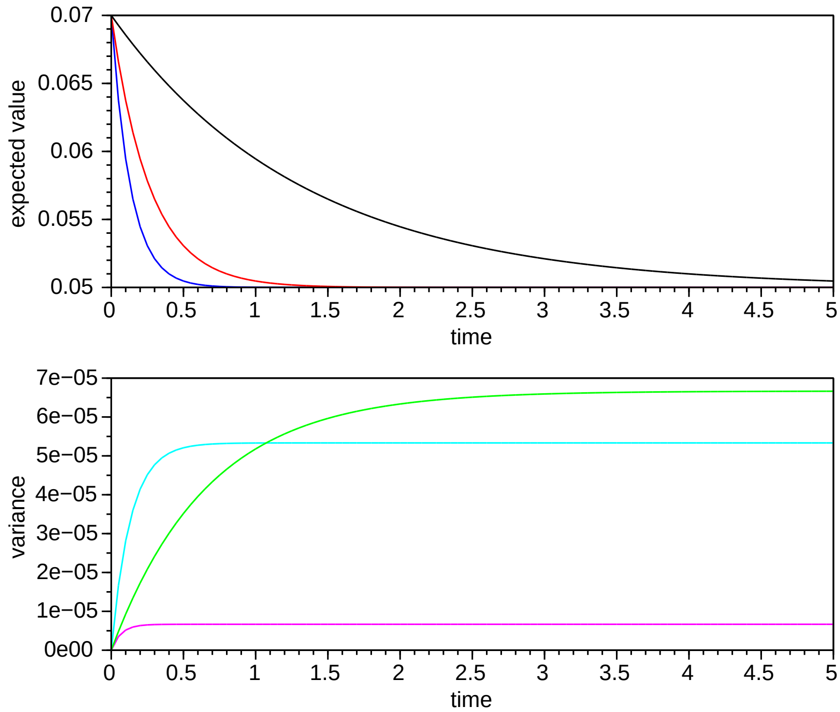


Figure 7.1: Evolution of the expected value and variance of the future interest rates.

7.1.6 Exercise. Consider the parameters values of the Vasicek model from Table 2.1 and choose the initial value of the interest rate. Plot the probability density functions of the interest rate at future times. Show how they converge to the limiting distribution. A sample output is presented in Figure 7.2.

What parameters does the speed of convergence depend on? Adjust the parameters so that the convergence to the limiting distribution is faster and plot similar graphs.

7.1.7 Exercise: Accuracy of the Euler-Maruyama discretization for Vasicek model. In Exercise 2.4.1 we used the Euler-Maruyama discretization to simulate trajectories of the Vasicek model. Now, knowing the exact distribution, we can check the accuracy of the the Euler-Maruyama discretization by comparing distribution of the interest rates, when simulated from the exact distribution and when using the approximation. We are going to compare the densities.

Firstly, derive the conditional distribution of the interest rate simulated using the Euler-Maruyama discretization with the time step Δt.

Now, consider the parameters of Vasicek model from Table 2.1 and choose the initial value of the interest rate. Suppose that we simulate daily data. Plot the density functions of the interest rate generated from exact distribution and using the discretization.

How does the comparison change when we increase the time step (for example, if we simulate monthly data)?

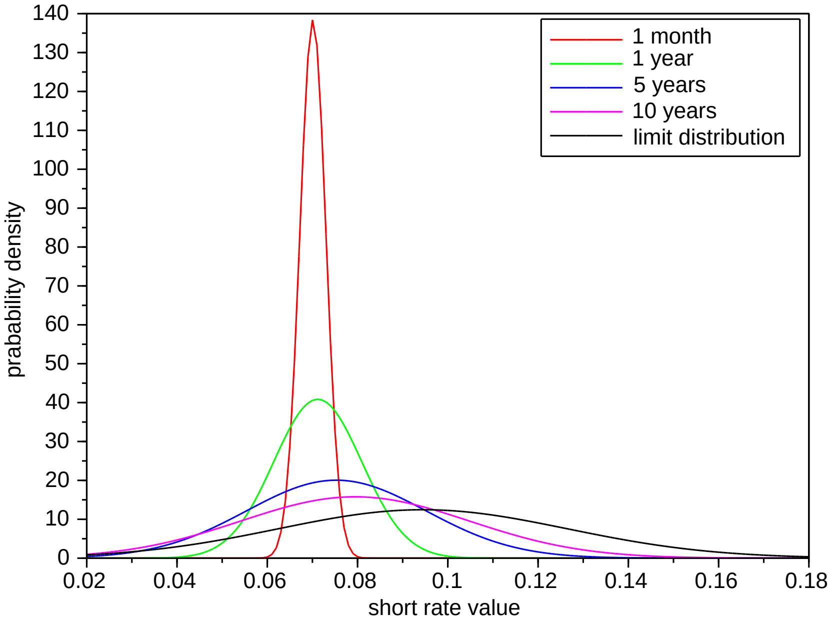


Figure 7.2: Densities of the interest rates in the future.

7.1.8 Exercise. Consider again the parameters of Vasicek model from Table 2.1. Assuming the the current interest rate is 6 percent, compute:

- the expected value of the interest rate in 1 week, 1 month, 1 year,
- the 95 % confidence interval for the interest rate in 1 week, 1 month, 1 year,
- probability than in 1 year the interest rate will be lower than 4 percent,
- probability than in 1 year the interest rate will be higher than today.

7.1.9 Exercise: Probability of the negative interest rates. One of the problems arising in the Vasicek model is a positive probability of negative interest rates, which is a consequence of normal distribution. The seriousness of this problem depends on the parameters, which affect this probability.

Assume the parameters from Table 2.1. Compute the probability that

- the interest rate in one year[2] is negative, if its value today is 5 percent,
- the interest rate in one year is negative, if its value today is 2 percent,
- the interest rate under the limiting distribution is negative (note that this does not depend on the initial interest rate).

How do these probabilities depend on the initial level of the short rate?

Find a set of parameters for which these probabilities are not negligible (i.e., what combinations of parameters cause problems?).

7.1.10 Maximum likelihood estimation of the parameters. Knowledge of the conditional distribution of the interest rates enables us to form the likelihood function for the given time series of interest

[2]Note that this refers to the interest rate at one precise time, not to a possibility that the interest rate is negative at a certain time during the year. The latter question is related to so called first passage times, see lectures and the additional notes at the website.

rates. Since this distribution is normal, the likelihood function is sufficiently simple to allow the explicit computation of the maximum likelihood estimates.

Suppose that we have the equidistant data $r_1, r_2, \ldots, r_n$ and the length of the time interval between two observations is dt. Define

$$A = e^{-\kappa \,\mathrm{d}t}, \quad V^2 = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa \,\mathrm{d}t});$$

note that V^2 is exactly the conditional variance and the parameter κ appears in the conditional expected value through the new parameter A - hence it is natural to expect that the new parameters A and V^2 will make a suitable reparametrization for the maximum likelihood estimation. Then the maximum likelihood estimates of the parameters θ, A, V^2 can be computed using the following code:

```
// r - vector with the data

n=length(r);

A=((n-1)*sum(r(2:n).*r(1:n-1))-sum(r(2:n))*...
         sum(r(1:n-1)))/((n-1)*sum(r(1:n-1).^2)-(sum(r(1:n-1)))^2);
theta=sum(r(2:n)-A*r(1:n-1))/((n-1)*(1-A));
v2=sum((r(2:n)-A*r(1:n-1)-theta*(1-A)*ones(n-1,1)).^2)/(n-1);
```

7.1.11 Exercise: Maximum likelihood estimation. Transform the estimates of the parameters A, θ, V^2 from the previous section into the original parameters κ, θ, σ.

Download data[3], choosing some interest rate as a proxy for the short rate. Model these data by the Vasicek model and use the maximum likelihood method to estimate the parameters. Plot the time series of the data and add the predicted expected value of the interest rate in the future.

7.2 Bond prices and term structure of interest rates

7.2.1 Bond, interest rate, term structure of interest rates. A zero-coupon bond is a security which pays its holder a specified amount of money at a specified time (called **maturity** of the bond). If it pays a unit amount of money, it is called a discount bond. In what follows, by a **bond** we mean a discount zero-coupon bond.

The bond prices define **interest rates**. If we denote the price at time t of a bond with maturity T as $P(t,T)$ and the corresponding interest rate by $R(t,T)$, they are connected through the formulae

$$P(t,T) = e^{-R(t,T)(T-t)} \Rightarrow R(t,T) = -\frac{\log P(t,T)}{T-t}.$$

Plotting the interest rates at the given time with different maturities produces **term structure of interest rates**. Figure 7.3 shows an example of market data.

7.2.2 Bond prices in general one-factor model. If the short rate follows a stochastic differential equation

$$\mathrm{d}r = \mu(t,r)\mathrm{d}t + \sigma(t,r)\mathrm{d}w,$$

then the bond price $P = P(t,r)$ satisfies the partial differential equation

$$\frac{\partial P}{\partial t} + (\mu(r,t) - \lambda(r,t)\sigma(r,t))\frac{\partial P}{\partial r} + \frac{\sigma^2(r,t)}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0 \qquad (7.4)$$

for all r and all $t \in [0,T)$.

[3] Interest rates data are available for example at the following websites:
http://www.euribor-ebf.eu/
http://www.federalreserve.gov/releases/h15/data.htm
http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/

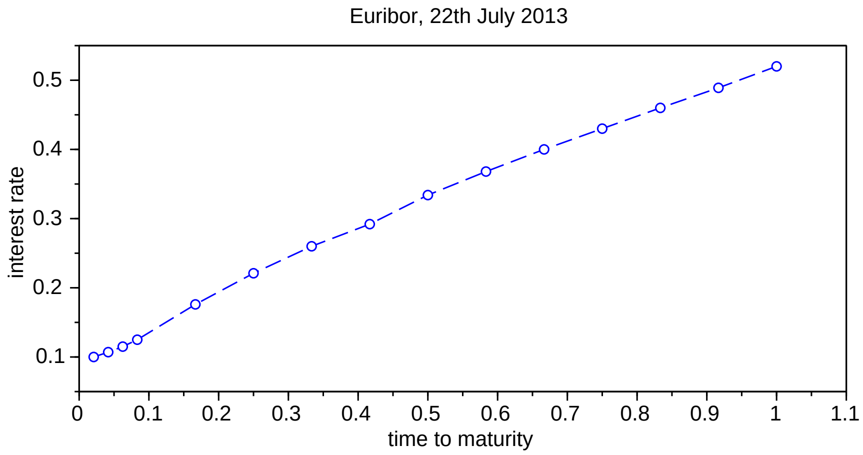


Figure 7.3: Term structure of interest rates, Euribor rate on 22nd July 2013.

The function $\lambda(r,t)$ is so called **market price of risk** and it denotes the expected increase of the return associated with a unit increase in risk:

$$\lambda(r,t) = \frac{\mu_B(r,t,T) - rP(r,t,T)}{\sigma_B(r,t,T)}, \tag{7.5}$$

where μ_B and σ_B are drift and volatility of the price of a bond which has maturity at time T. Note that λ does not depend on T, it is a function of t and r only. It is an additional input that has to be provided to price bonds, in addition to the stochastic differential equation for the short rate.

The terminal condition at time T is given by the payoff of the bond, i.e., $P(T,r) = 1$ for all r.

7.2.3 Bond prices and interest rates in the Vasicek model. In Vasicek model, it is customary to take a constant market price of risk (7.5), i.e., $\lambda(r,t) = \lambda$. Then, the bond pricing equation (7.4) becomes

$$\frac{\partial P}{\partial t} + (\kappa(\theta - r) - \lambda\sigma)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0. \tag{7.6}$$

Writing its solution in the form

$$P(\tau, r) = A(\tau)e^{-B(\tau)r}, \tag{7.7}$$

where $\tau = T - t$ is time remaining to maturity, transforms the partial differential equation (7.6) into a system of two ordinary differential equations for functions A and B. The system has a closed-form solution:

$$\begin{aligned} B(\tau) &= \frac{1 - e^{-\kappa\tau}}{\kappa}, \\ \log A(\tau) &= \left[\frac{1}{\kappa}(1 - e^{-\kappa\tau}) - \tau\right] R_\infty - \frac{\sigma^2}{4\kappa^3}(1 - e^{-\kappa\tau})^2, \end{aligned}$$

where

$$R_\infty = \theta - \frac{\lambda\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}. \tag{7.8}$$

From (7.7) it follows that the interest rates are computed as

$$R(\tau, r) = -\frac{\log P(\tau, r)}{\tau} = -\frac{A(\tau)}{\tau} + \frac{B(\tau)}{\tau} r. \tag{7.9}$$

Sample bond prices and term structures of interest rates, corresponding to parameters $\kappa = 0.5, \theta = 0.05, \sigma = 0.01, \lambda = -0.1$ and several values of the short rate r, are shown in Figure 7.4.

7.2.4 Exercise: Limit of the term structures. Show that

$$\lim_{\tau \to \infty} R(\tau, r) = R_\infty,$$

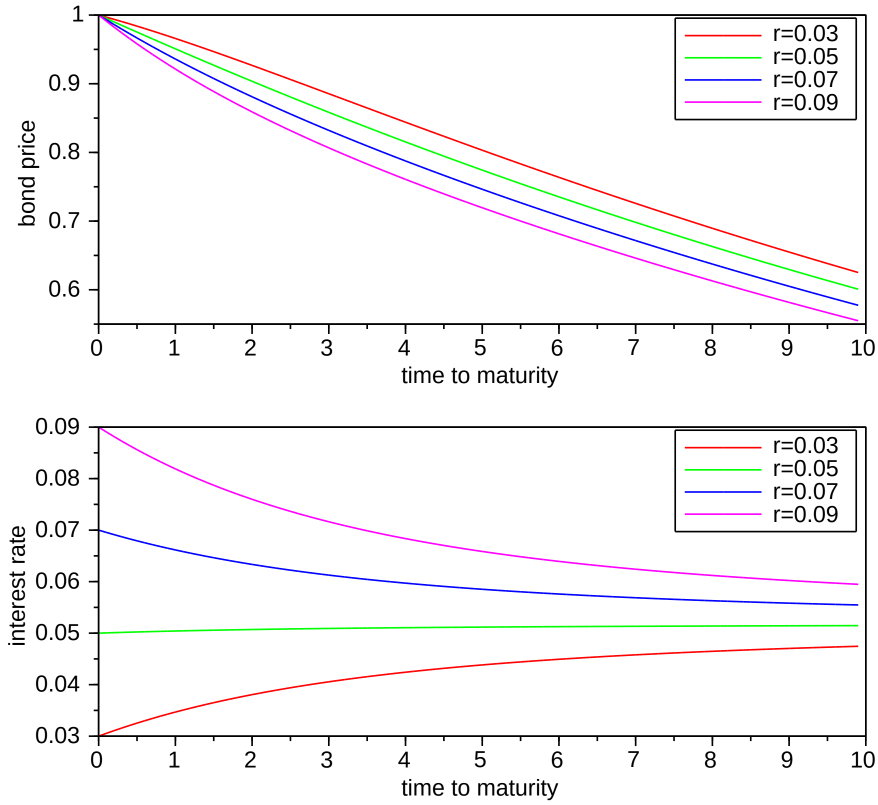


Figure 7.4: Bond prices (top) and term structures of interest rates (bottom) from the Vasicek model.

i.e., the term (7.8) represents the limit of term structures as time to maturity approaches infinity. Moreover, it does not depend on the short rate level (which is the beginning of the term structure). Compare this results with numerical examples of the term structures in Figure 7.4.

7.2.5 Exercise. Download data of interest rates with different maturities. Use another interest rate with shorter maturity as a proxy for the short rate and estimate the parameters κ, θ, σ of the Vasicek model by maximum likelihood. Plot term structures for these parameters and different values of the market price of risk λ. How do they differ? Compare them with the real term structures. What values of λ seem most plausible?

7.2.6 Exercise: Forecasting term structures. In the previous section we considered the distribution of the future values of the short rate r, given its value today. Having the distribution of r, the formula (7.9) enables us to predict also the term structures.

- Given the value of the short rate r today, at time t_0, compute the distribution, expected value and variance of the interest rate at time $t > t_0$ with maturity $t + \tau$.
- Plot a numerical example, showing the expected term structure of interest rates and 95% confidence intervals for the interest rates.
- How does the width of the confidence interval for a given maturity τ depend on the prediction time t? How does it depend on the maturity τ for a fixed prediction time t? Find a general answer and compare with the numerical example from the previous point. Give an interpretation of these results.

7.2.7 Exercise: Shapes of term structures. There are three possibilities for a shape of the term structure in the Vasicek model:

- the term structure is monotonically increasing if

$$r \leq R_\infty - \frac{\sigma^2}{4\kappa^2},$$

- the term structure is humped (initially increasing and then decreasing) if

$$R_\infty - \frac{\sigma^2}{4\kappa^2} < r < R_\infty + \frac{\sigma^2}{2\kappa^2},$$

- the term structure is monotonically decreasing if

$$r \geq R_\infty + \frac{\sigma^2}{2\kappa^2}.$$

Demonstrate these properties graphically.

7.2.8 Exercise. Consider the parameters of the Vasicek model given in Table 2.1.

- Compute the market price of risk, if the limit of term structures is 10 percent.
- Compute the market price of risk, if the limit of term structures equals 1.5 multiple of the long term limit of the short rate.
- The term structure corresponding to the short rate equal to 8 percent is humped. What values of market price of risk are consistent with this?

7.2.9 Exercise: Problem from an internet forum. This question related to the Vasicek model appeared on an internet forum[4]:

> My attempts to solve this problem are no where near correct. Could someone please help me understand how to do it?
>
> Let P(r, t, T) denote the price at time t of $1 to be paid with certainty at time T, t<=T, if the short rate at time t is equal to r.
>
> For the Vasicek model, you are given:
>
> P(0.07, 3, 5) = 0.8654
> P(0.06, 1, 3) = 0.9152
> P(r*, 2, 4) = 0.8337
>
> Calculate r*.

Can you solve this problem?

[4] The whole post and the discussion can be found at `http://www.actuarialoutpost.com/actuarial_discussion_forum/showthread.php?t=134262`

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