IX. Leland model: European call and put options

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Leland PDE

- Recall the Leland PDE for the price of a derivative:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign \left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- The PDE holds for $S > 0, t \in [0, T]$, we add the terminal condition $V(S, T)$ depending on the derivative, e.g., $V(S, T) = \max(0, S - E)$ for $S > 0$ in the case of a call option
- Nonlinear PDE because of the term containing *signum*
- Recall the for the Black-Scholes prices of call and put options we have $\frac{\partial^2 V}{\partial S^2} > 0$ (positive gamma) $\Rightarrow$ $sign\left(\frac{\partial^2 V}{\partial S^2}\right) = 1$

Leland PDE - call and put

- What happens if inserting Black-Scholes price of a call/put with adjusted volatility $V(S,t;\tilde{\sigma})$:

$$\tilde{\sigma}^2 = \sigma^2 \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \right]$$

into the Leland PDE:

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign\left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV &= \\
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 - \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} \right] + \frac{\partial V}{\partial S} S - rV & \\
\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} S - rV &= 0
\end{aligned}$$

Leland PDE - call and put

- It means that Black-Scholes price of a call/put with adjusted volatility $V(S,t;\tilde{\sigma})$:

$$\tilde{\sigma}^2 = \sigma^2 \left[1 - \frac{c}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}} \right] = \sigma^2 \left[1 - Le \right]$$

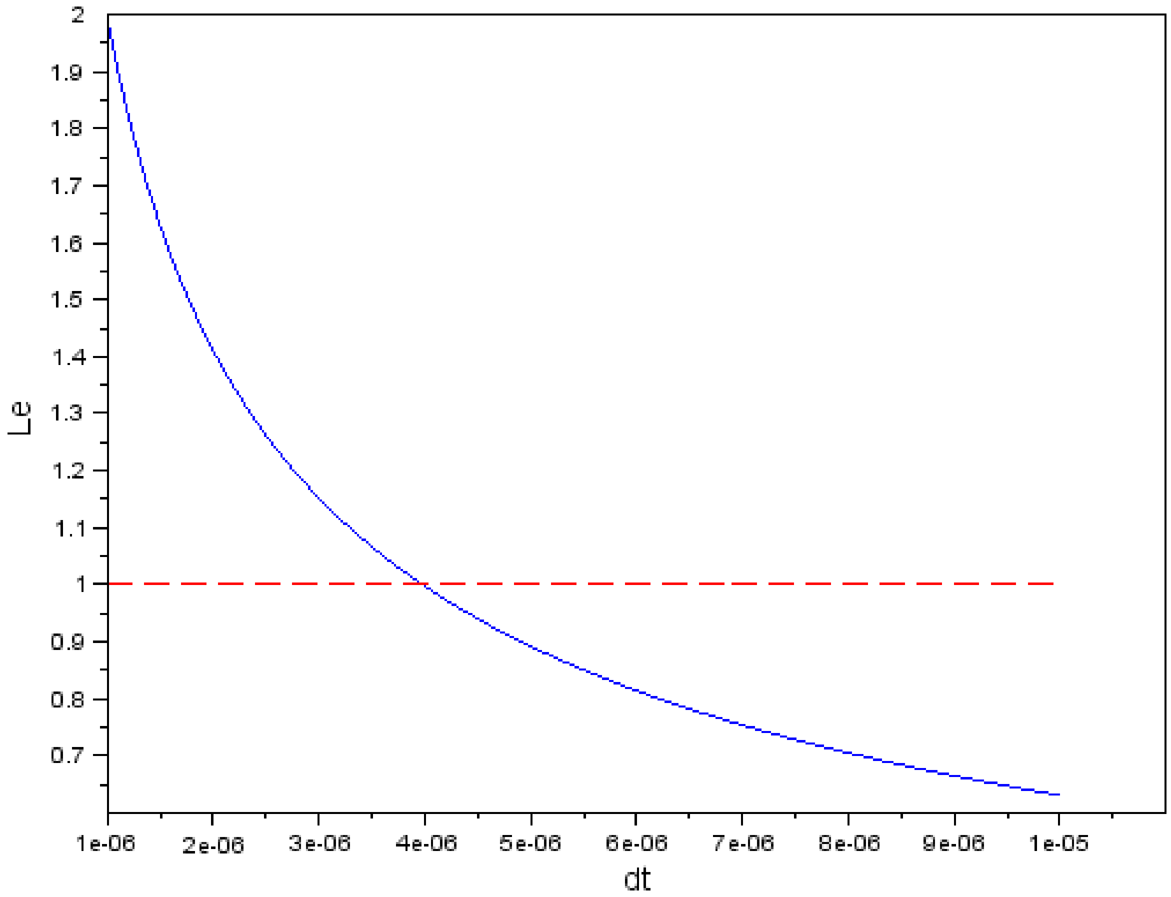
is a solution to the Leland PDE for a European call/put.

- Le is called Leland number
- Term $\tilde{\sigma}^2$ has to be positive $\Rightarrow$ this gives a bound on feasible times Δt - i.e. possible times between two changes of the portfolio (parameters σ, c are given):

$$\Delta t > \frac{2}{\pi}\frac{c^2}{\sigma^2}$$

Feasible values of Δt

GRAPHICALLY: dependence of Le on Δt for $c = 5 \times 10^{-4}$, $\sigma = 0.2$



Feasible values of Δt

NUMERICALLY: what is the borderline of feasible Δt:

```
-->function [f]=f(dt)
-->    c=5*10^(-4);
-->    sigma=0.2;
-->    le=sqrt(2/%pi)*c/(sigma*sqrt(dt));
-->    f=le-1;
-->endfunction

-->t=fsolve(4*10^(-6),f);

-->mprintf('%e\n', t)
3.978874e-006

--> t*252*7*60
 ans  =

    0.4211240
```

Assume 252 trading days in a year and the market opened 7 hours a day $\Rightarrow$ Δt has to be more than approximately 0.42 min.

Computation of the option price I.

- Let us take $\Delta t =$ 5 minutes, i.e., $\Delta t = 5/(60 * 7 * 252)$
- Leland number is then feasible (less than 1) if:

```
-->dt=5/(60*7*252);

-->le(dt)
 ans  =

    0.2902151
```

- Adjusted volatility, to be used in the Black-Scholes formula:

```
-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
 sigmaTC  =

    0.1684975
```

Computation of the option price I.

- We compute the price of a call option with exercise price $E = 110$ which expires in $\tau = 1$ year, if the interest rate equals $r = 1\%$ and the underlying stock price is $S = 100$
- For a comparison - price in the absence of transaction costs

```
-->Call(100,110,0.01,sigmaTC,0.5)
 ans  =

    1.6108991

-->Call(100,110,0.01,sigma,0.5)
 ans  =

    2.3394205
```

Computation of the option price I.

- The same option if $\Delta t = 1/252$, i.e., 1 day:

```
-->dt=1/(252);

-->le(dt)
 ans  =

    0.0316651

-->sigmaTC=sqrt((1-le(dt))*(sigma^2))
 sigmaTC  =

    0.1968080

-->Call(100,110,0.01,sigmaTC,0.5)
 ans  =

    2.2630352
```

Bid a ask prices of options in Leland model

- When deriving the Leland PDE, we considered the portfolio: 1 option, δ stocks $\Rightarrow$ the resulting price is *bid* price
- Let us consider the portfolio minus 1 option, δ stocks $\Rightarrow$ the resulting price will be *ask* price
- In the same way we obtain that the ask price satisfies

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \left[1 + \frac{c}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}} sign\left(\frac{\partial^2 V}{\partial S^2} \right) \right] + \frac{\partial V}{\partial S} S - rV = 0$$

- Call a put options: Black-Scholes price with adjusted volatility $\sigma_{TC}^2 = (1 + Le)\sigma^2$

Implied parameters

- If we have bid and ask prices of the stock and the option, we can compute:
  - implied volatility
  - implied time between two changes of the portfolio

  (i.e., the values, for which the theoretical and market bid and ask option prices will coincide)

INPUTS:

- Stock - bid and ask prices S_{bid}, S_{ask}
- Option - bid and ask prices V_{bid}, V_{ask}, exercise price E, time τ remaining to expiration of the option
- Other market parameters: interest rate r

Implied parameters

PROCEDURE:

- Using bid and ask prices of the stock we compute $S = (S_{ask} + S_{bid})/2$ and $c = (S_{ask} - S_{bid})/S$
- Using S, E, r, τ and
  - V_{bid} we compute the Black-Scholes implied volatility, then $\sqrt{(1 - Le)\sigma^2} := \sigma_{bid}$
  - V_{ask} we compute the Black-Scholes implied volatility, then $\sqrt{(1 + Le)\sigma^2} := \sigma_{ask}$
- By solving the system of equations $(1 - Le)\sigma^2 = \sigma_{bid}^2$, $(1 + Le)\sigma^2 = \sigma_{ask}^2$ we compute the implied volatility σ and Leland number Le
- From the definition of the Leland number we compute the implied time Δt between two changes of the hedging portfolio

Implied parameters - example

EXAMPLE:

- Data from 8.3.2014 morning
- Stock:

General Motors Company (GM) - NYSE ★ Follow

37.65 ↑0.11 (0.29%) 9:44AM EST - Nasdaq Real Time Price

Prev Close:	**37.54**	Day's Range:	**37.54 - 38.01**
Open:	**N/A**	52wk Range:	**27.11 - 41.85**
Bid:	**37.90 x 1000**	Volume:	**594,013**
Ask:	**37.94 x 400**	Avg Vol (3m):	**26,332,800**

Implied parameters - example

- Call option:

GM Mar 2014 37.000 call (GM140322C00037000) - OPR

1.00 ↑0.10(11.11%) Mar 6

Prev Close:	0.90	Day's Range:	1.00 - 1.24
Open:	1.24	Contract Range:	N/A - N/A
Bid:	1.20	Volume:	434
Ask:	1.27	Open Interest:	64,168
Strike:	37.00		
Expire Date:	22-Mar-14		

Implied parameters - example

- Interest rates:

US Treasury Bonds Rates

Maturity	Yield	Yesterday	Last Week	Last Month
3 Month	0.04	0.04	0.04	0.04
6 Month	0.07	0.07	0.08	0.04
2 Year	0.37	0.35	0.31	0.32
3 Year	0.77	0.71	0.66	0.65
5 Year	1.64	1.57	1.47	1.49
10 Year	2.81	2.73	2.65	2.67
30 Year	3.74	3.69	3.58	3.65

Implied parameters - example

- Hence we have:

```
Sbid=37.90; Sask=37.94;
Vbid=1.20; Vask=1.27;
E=37;
r=0.04/100;

tau=11/252;


S=(Sask+Sbid)/2;
c=(Sask-Sbid)/S;
```

Implied parameters - example

- We compute the implied volatilities

```
->sigmaBid=ImplVolCall(S,E,r,tau,Vbid)
sigmaBid  =


   0.2039042

->sigmaAsk=ImplVolCall(S,E,r,tau,Vask)
sigmaAsk  =


   0.2298972
```

- Remarks:
  - S is common (not S_{bid}, S_{ask})
  - implied volatilities are from Black-Scholes model

Implied parameters - example

- From the system of equations

$$(1 - Le)\sigma^2 = \sigma^2_{bid}, \; (1 + Le)\sigma^2 = \sigma^2_{ask}$$

we compute Leland number Le and implied volatility σ :

```
-->Le=(sigmaAsk^2-sigmaBid^2)/(sigmaAsk^2+sigmaBid^2);
```

```
->sigma=sigmaAsk/sqrt(1+Le)
sigma  =

   0.2172898
```

Implied parameters - example

- From the definition of the Leland number we compute the implied time Δt:

```
-->dt=(2/%pi)*(c/(sigma*Le))^2;

-->dt*252
ans  =

    0.2651599    dt in days
```

SUMMARY:

- implied volatility $\sigma_{impl} = 0.217$
- implied time between two changes of the portfolio Δt_{impl} is approximately $1/4$ days