the rate of change of the value of the derivative security with respect to the interest rate. The rhos of the European vanilla call and put values are given, respectively, by

$$\rho_c = \frac{\partial c}{\partial r} = SN'(d_1)\,\frac{\partial d_1}{\partial r} + \tau X e^{-r\tau} N(d_2) - X e^{-r\tau} N'(d_2)\,\frac{\partial d_2}{\partial r} \tag{23a}$$
$$= \tau X e^{-r\tau} N(d_2) > 0 \quad \text{for } \tau > 0 \text{ and } X > 0,$$

and

$$\rho_p = \frac{\partial p}{\partial r} = \frac{\partial c}{\partial r} - \tau X e^{-r\tau} \qquad \text{(from the put-call parity relation)} \tag{23b}$$
$$= -\tau X e^{-r\tau} N(-d_2) < 0 \quad \text{for } \tau > 0 \text{ and } X > 0.$$

The signs of the rhos of the option prices confirm the above claims on the effects of changing interest rate on call and put prices.

2.1.4 Calculation of implied volatility

The option prices obtained from the Black-Scholes pricing framework are functions of five parameters: asset price S, strike price X, interest rate r, time to expiry τ and volatility σ. Except for the volatility parameter, the other four parameters are observable quantities. The difficulties of setting volatility value in the valuation formulas lie in the fact that the model should be the forecast value over the remaining life of the option rather than an estimate value from the past market data of the asset price. The estimate of the volatility value from past data is commonly called the *historical volatility*. Here, we propose the following inverse problem: instead of computing the option price given the volatility value using the Black-Scholes formula, we solve for the volatility value from the observed market option price. The volatility value implied by an observed option price is called the *implied volatility*, which indicates a consensual view about the volatility level determined by the market. In particular, several implied volatility values obtained simultaneously from different options on the same underlying asset provide an extensive market viewpoint about the volatility of the stochastic movement of that asset. Such information may be useful for a trader to set the volatility value for the underlying asset of an option that he is interested in. In financial markets, it becomes a common practice for traders to quote an option's market price in terms of implied volatility, σ_{imp}. In essence, σ_{imp} becomes a means of quoting prices.

Since σ cannot be solved explicitly in terms of S, X, r, τ and option price V from the pricing formulas, the determination of the implied volatility must be accomplished by an iterative algorithm as commonly performed for the root-finding procedure for a non-linear equation. Since the option price is known to be an increasing function of the volatility value (see Eqs. (22a, 22b)), the iterative algorithms become more simplified. We propose two simple algorithms, namely, the *bisection search method* and the *Newton-Raphson method* for the calculation of the implied volatility.

Bisection search method
The bisection search method takes advantage of the monotonicity property of the option price function $V(\sigma)$. As a preliminary step, we find two estimated volatility values σ_{low} and σ_{high} by trial and error such that $V(\sigma_{low}) < V_{market}$ and $V(\sigma_{high}) > V_{market}$, where V_{market} is the market value of the option price. Let σ_{imp} be the implied volatility such that $V(\sigma_{imp}) = V_{market}$. By the monotonicity property of $V(\sigma)$, σ_{imp} is guaranteed to lie within $(\sigma_{low}, \sigma_{high})$. We take $\sigma_{mid} = \frac{\sigma_{low}+\sigma_{high}}{2}$ and bisect the interval $(\sigma_{low}, \sigma_{high})$ into two equal half intervals $(\sigma_{low}, \sigma_{mid})$ and $(\sigma_{mid}, \sigma_{high})$. In the following procedure, we delete either one of the half intervals while ensuring that the remaining half interval contains σ_{imp}:

> If $Q = [V_{market} - V(\sigma_{mid})]\,[V_{market} - V(\sigma_{high})] < 0$, then delete $(\sigma_{low}, \sigma_{mid})$ and take $\sigma_{low} = \sigma_{mid}$; otherwise, if $Q > 0$, then delete $(\sigma_{mid}, \sigma_{high})$ and take $\sigma_{high} = \sigma_{mid}$.

After one iterative step as outlined above, the width of the bracketing interval $(\sigma_{low}, \sigma_{high})$ is halved. Suppose the above procedure is performed a sufficient number of times, the interval is eventually reduced to a width less than 2δ, where δ is some chosen small positive tolerance value. The iteration may then be terminated and σ_{mid} taken as an estimate to σ_{imp}. The error in the volatility estimation is guaranteed to be less than δ.

Newton-Raphson method
Another common method used for the root-finding procedure is the Newton-Raphson method. Applied to the present problem, the Newton-Raphson iterative scheme is given by

$$\sigma_{n+1} = \sigma_n - \frac{V(\sigma_n) - V_{market}}{V'(\sigma_n)}, \tag{24}$$

where σ_n denotes the nth iterative for σ_{imp}. Provided that the first iterate σ_1 is properly chosen, the limit of the sequence $\{\sigma_n\}$ converges to the unique solution σ_{imp}. The Newton-Raphson method enjoys its popularity due to its quadratic rate of convergence property, that is, $\sigma_{n+1} - \sigma_{imp} = K(\sigma_n - \sigma_{imp})^2$ for some K independent of n. Equation (24) may be rewritten in the following form

$$\frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} = 1 - \frac{V(\sigma_n) - V(\sigma_{imp})}{\sigma_n - \sigma_{imp}} \frac{1}{V'(\sigma_n)} = 1 - \frac{V'(\sigma_n^*)}{V'(\sigma_n)}, \tag{25}$$

where σ_n^* lies between σ_n and σ_{imp}, by virtue of the Mean Value Theorem in calculus. Manaster and Koehler (1982) proposed to choose the first iterate σ_1 such that $V'(\sigma)$ is maximized by $\sigma = \sigma_1$. Recall from Eq. (22a) that

$$V'(\sigma) = \frac{S\sqrt{\tau}\, e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0 \quad \text{for all } \sigma, \tag{26a}$$

and so

$$V''(\sigma) = \frac{S\sqrt{\tau} d_1 d_2 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}\sigma} = \frac{V'(\sigma) d_1 d_2}{\sigma}, \tag{26b}$$

where d_1 and d_2 are defined in Eq. (10b). Therefore, the critical points of the function $V'(\sigma)$ are given by $d_1 = 0$ and $d_2 = 0$, which lead respectively to

$$\sigma^2 = -2\frac{\ln\frac{S}{X} + r\tau}{\tau} \quad \text{and} \quad \sigma^2 = 2\frac{\ln\frac{S}{X} + r\tau}{\tau}. \tag{27}$$

The above two values of σ^2 both give $V'''(\sigma) < 0$. Hence, we can choose the first iterate σ_1 to be

$$\sigma_1 = \sqrt{\left|\frac{2}{\tau}\left(\ln\frac{S}{X} + r\tau\right)\right|}, \tag{28}$$

which satisfies the requirement of maximizing $V'(\sigma)$. Setting $n = 1$ in Eq. (25), we then have

$$0 < \frac{\sigma_2 - \sigma_{imp}}{\sigma_1 - \sigma_{imp}} < 1. \tag{29a}$$

In general, suppose we can establish (see Problem 8)

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1, \qquad n > 1, \tag{29b}$$

then the sequence $\{\sigma_n\}$ is monotonic and bounded and so $\{\sigma_n\}$ converges to the unique solution σ_{imp}. In conclusion, if we start with the first iterate σ_1 given by Eq. (28), then the sequence $\{\sigma_n\}$ generated by Eq. (24) will converge to σ_{imp} monotonically with a quadratic rate of convergence.

2.1.5 Pricing biases of the Black-Scholes model

The Black-Scholes model assumes the lognormal probability distribution of the asset price at any future time. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility. If the model were perfect, the implied volatility would be the same for all option market prices. However, many empirical studies have revealed that the implied volatilities strongly depend on the strike price and the maturity of European options. The dependence of the volatility on maturity can be resolved relatively easily by allowing time dependence in volatility. The relevant model with time dependent parameters will be discussed in the next section. It is more interesting to examine the dependence of volatility on strike price. If we plot implied volatilities of exchange-traded