Chapter 2

Basic concepts of the stochastic calculus

2.1 Wiener process

2.1.1 Definition of a Wiener process. A t-parametric system of random variables $\{w(t), t \geq 0\}$ is called a **Wiener process**, if

1. the increments $w(t + \Delta t) - w(t)$ have a normal distribution $\mathcal{N}(0, \Delta)$,
2. for any partition $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0, t]$, the increments $w(t_1) - w(t_0), w(t_2) - w(t_1), \ldots, w(t_n) - w(t_{n-1})$ are independent random variables,
3. $w(0) = 0$ almost surely.

It follows from the Kolmogorov existence theorem that a process with these properties actually exists (cf. the lectures). Moreover, it can be shown there is such a process for which, moreover, the sample paths are continuous almost surely.

In what follows, w denotes a Wiener process satisfying the continuity condition.

2.1.2 Simulating trajectories of a Wiener process. We are going to simulate various random processes so that we can observe their properties, confirm our computations, get an intuition with working with random processes, etc. The first step will be a Scilab function which allows us to generate a trajectory (a sample path) of a Wiener process. We proceed as follows:

- We will simulate an approximation - values for a discrete set of time points, which we join.
- We will simulate the values of a Wiener process at times $0, \mathrm{d}t, 2\mathrm{d}t, \ldots$, where $\mathrm{d}t$ is a sufficiently small time step.
- We know that the value at time $t = 0$ is zero.
- The increment on the time interval $[k\,\mathrm{d}t, (k+1)\,\mathrm{d}t]$ is a normally distributed random variable with the mean equal to zero and the variance equal to $\mathrm{d}t$.

Scilab has quite a general function for simulating random numbers, therefore we first define a simple function which returns a random number from $\mathcal{N}(0, 1)$ distribution.

```
function [r]=randn()
        r=rand(1,"normal");
endfunction
```

Now we can define a function which returns a trajectory of a Wiener process. The input parameters are $\mathrm{d}t$ - the time step used in the simulation and n - number of time steps.

```
function [w]=wiener(dt,n)
        w(1)=0;
        for i=1:n
                dw=sqrt(dt)*randn();
                w(i+1)=w(i)+dw;
        end;
        w=w'     // the output vector w will be a row vector
endfunction
```

Using this function we plot a trajectory:

```
dt=0.001;
n=1000;
time=(0:dt:n*dt);

figure;
plot(time,wiener(dt,n));
```

The result is shown in Figure 2.1.

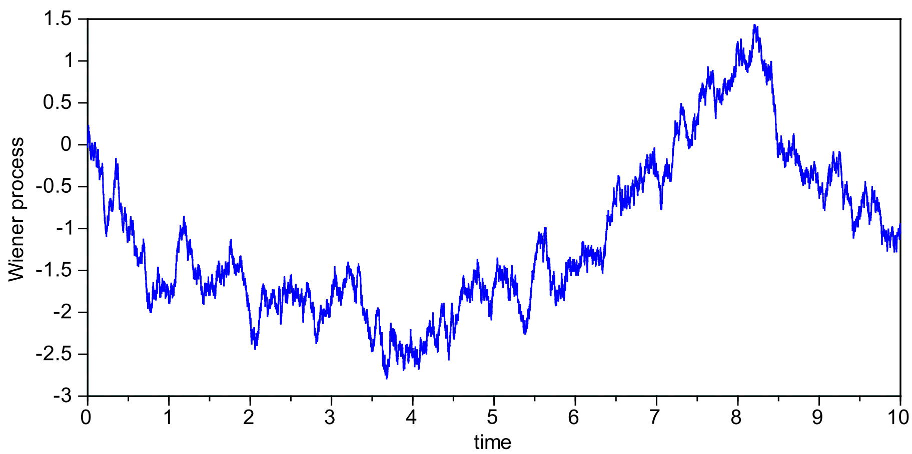

Figure 2.1: A sample path of a Wiener process.

2.1.3 Exercise: Simulating trajectories of a Wiener process. Add several sample paths of a Wiener process into one graph. Show that the probability distribution of $w(t)$ is normal for every t and compute its mean value and variance. To the figure with sample paths add a graph of the mean and 95% confidence intervals (i.e., mean +/- 2 $\times$ standard deviation). See Figure 2.2 for a sample result.

2.1.4 Exercise: Performing simulations. Simulating a process or a random variable associated with it can quickly provide an insight into its properties and behaviour. For both exercises here, the distribution can be derived analytically. This requires a certain effort and in such cases, simulations can be used to ”numerically check” our computations.

1. Denote by t_M the time, in which the sample path of the Wiener process achieved its maximum on the time interval $[0, 1]$. That is,

   $$t_M = \text{argmax}\{w(t), t \in [0, 1]\},$$

   see Figure 2.3. Plot a histogram[1] by simulating the realizations of the random variable t_M. A sample result can be found in Figure 2.3 as well.

[1] Scilab command for plotting a histogram is `histplot(N,data)`, where `N` is the number of bins.

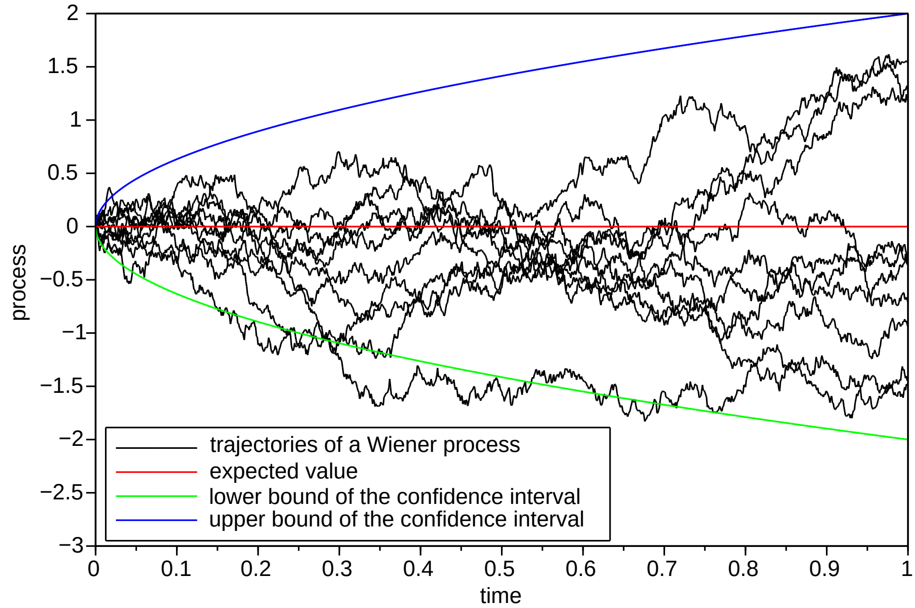

Figure 2.2: Sample paths of a Wiener process, expected value and confidence intervals.

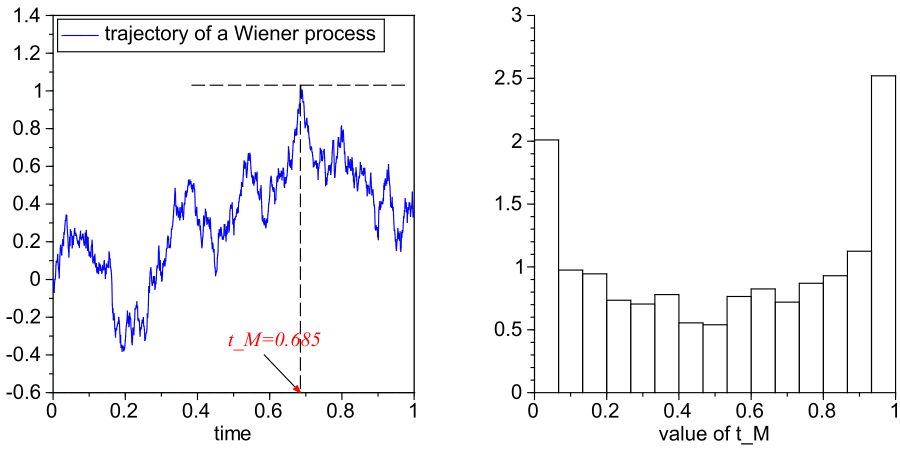

Figure 2.3: Definition of the random variable t_M (left) and histogram of its values (right).

2. Define the random process

$$M(t) = \max_{0 \le s \le t} w(s),$$

which is, at time t, the maximum value of a Wiener process on the interval $[0, t]$. Make a similar plot as shown in Figure 2.4, simulating a trajectory of a Wiener process and determining the corresponding $M(t)$. Then, plot a histogram of $M(1)$ and estimate its expected value.

Note that by taking a maximum over a discrete set of time points instead of the whole interval $[0, 1]$, we are underestimating the maximum corresponding to a given trajectory of a Wiener process. Hence, when for example estimating the expected value, in order to achieve a higher precision it is necessary to refine the time grid, in addition to increasing the number of simulations.

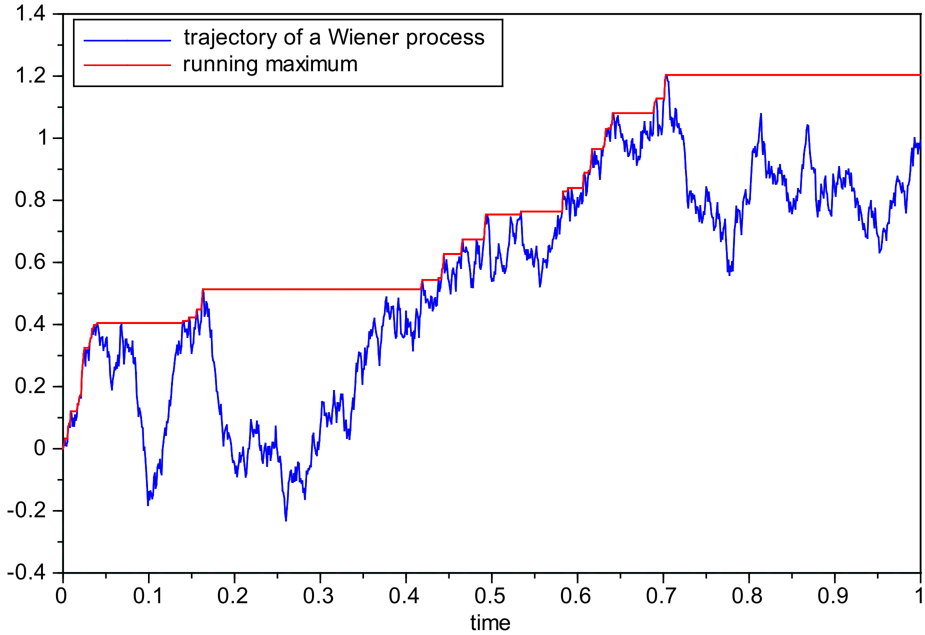

Figure 2.4: Wiener process $w(t)$ and the running maximum $M(t)$.

2.1.5 Exercise: Theoretical computations with a Wiener process.

1. Compute the probability distributions of the following random variables:

   - $x_1 = w(2) - w(1)$,
   - $x_2 = 3\,(w(1) - w(2))$,
   - $x_3 = 5\,w(2)$,
   - $x_4 = w(1) + w(2)$,
     Hint. Note that $w(1)$ and $w(2)$ are not independent. Write x_4 as $x_4 = [w(2) - w(1)] + 2w(1) = [w(2) - w(1)] + 2\,[w(1) - w(0)]$ and use the independence of increments of a Wiener process.
   - $x_5 = 2\,w(1) + 3\,w(2)$,
   - $x_6 = w(1) + w(2) + w(3)$.

2. Let w be Wiener process. Show that the following processes are also Wiener processes (i.e., check that they satisfy the properties from the definition):

   - $w_1(t) = -w(t)$,
   - $w_2(t) = c\,w(t/c^2)$, where $c > 0$ is a constant.

3. Replace the distribution of the increments $w(t + \Delta t) - w(t)$ by $\mathcal{N}(0, \sqrt{\Delta t})$, while leaving the other properties from the definition on a Wiener process unchanged.

   - Show that there is no process satisfying the new conditions.
   - Where does an attempt to apply the Kolmogorov existence theorem (used to establish the existence of a Wiener process) fail?

4. Show that the covariance between the values of the Wiener process are given by $Cov(w(t), w(s)) = \min(t, s)$. Plot the behaviour of the correlation $Cor(x(t), x(s))$ for a fixed t as a function of s.

2.1.6 Exercise. For $t \in [0, \infty)$ define the process

$$x(t) = \frac{w(t)}{1 + t}.$$

- Plot some trajectories of the process. How does the variance change in time? A sample graph showing five trajectories is shown in Figure 2.5. Plot more trajectories, so that the typical behaviour of the process can be better observed.
- Compute the mean and variance of the process analytically. At which time achieves the variance its maximum? What is its limit as time approaches infinity? Compare these results with the simulations of the trajectories.

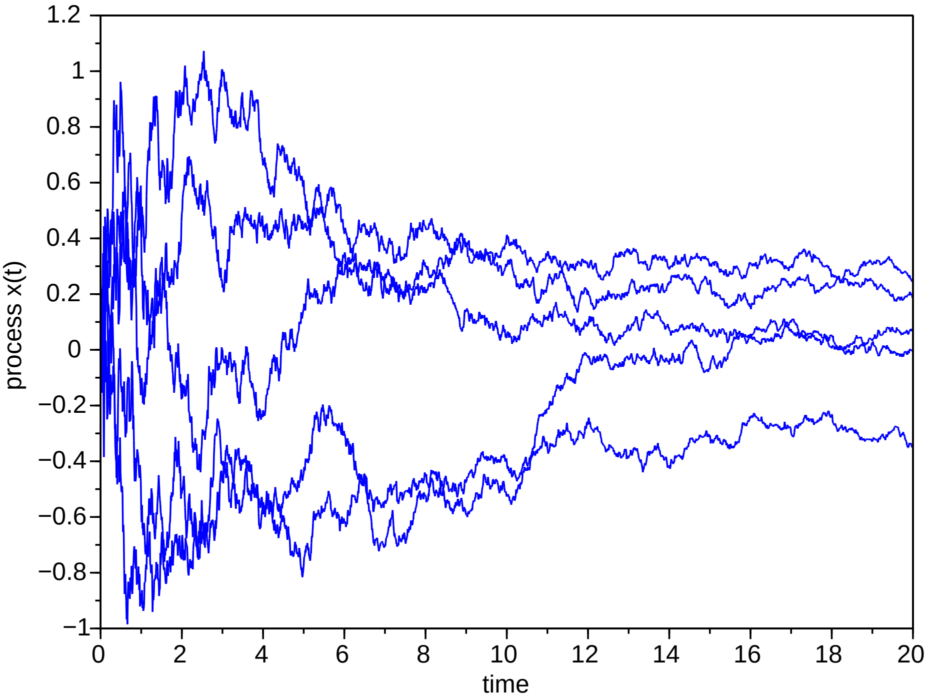

Figure 2.5: Trajectories of the process $x(t) = w(t)/(1 + t)$.

2.1.7 Exercise: Brownian bridge. For $t \in [0, 1]$ define the process $x(t) = w(t) - tw(1)$. This is known as **Brownian bridge**.

- Plot some trajectories of the process. Where does its name come from?
- Compute the mean and the variance of the process at each time. When is the variance minimal (why?) and when is it maximal?
- Show that the covariance is given by $Cov(x(t), x(s)) = \min(t, s) - ts$. Plot the behaviour of the correlation $Cor(x(t), x(s))$ for a fixed t as a function of s.

2.2 Brownian motion

2.2.1 Definition of a Brownian motion If w is a Wiener process, then the process

$$x(t) = \mu t + \sigma w(t),$$

where μ and σ are constants, is called a **Brownian motion**.

2.2.2 Exercise. Consider the process $x(t) = x_0 + \mu t + \sigma w(t)$.

- Plot some trajectories of the process and note how its typical behaviour depends on the parameters μ, σ, x_0
- The processes in Figure 2.6 are typical trajectories of the following processes:

  1. $x_1(t) = 2\,w(t)$,
  2. $x_2(t) = 0.5\,w(t)$,
  3. $x_3(t) = 3 + 2\,w(t)$,
  4. $x_4(t) = 3 - 2\,w(t)$,
  5. $x_5(t) = -3 + 2\,w(t)$.

  Add the process to the corresponding trajectory.

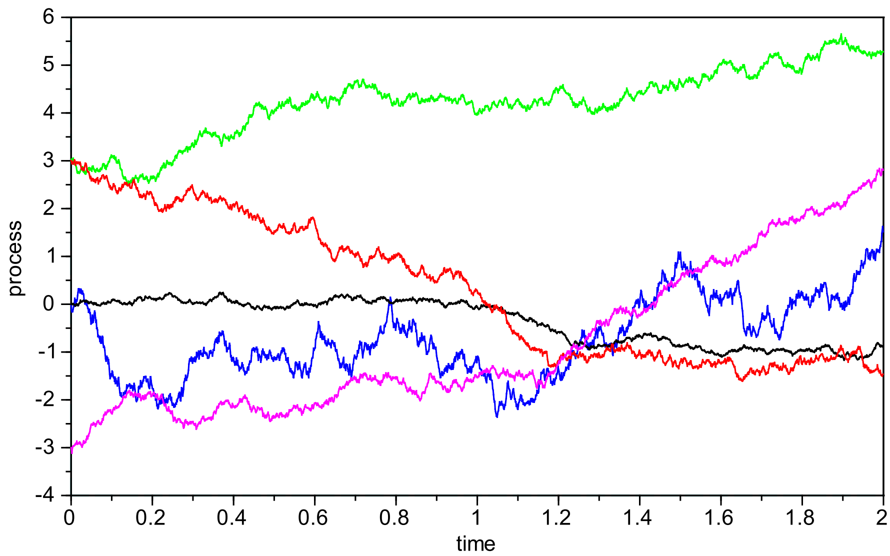

Figure 2.6: Trajectories of the processes of the kind $x(t) = x_0 + \mu t + \sigma w(t)$

2.3 Geometrical Brownian motion

2.3.1 Definition of a geometrical Brownian motion. If w is a Wiener process, then the process

$$x(t) = x_0 e^{\mu t + \sigma w(t)},$$

where μ, σ and x_0 are constants, is called a **geometrical Brownian motion**.

2.3.2 Lognormal probability distribution. Geometrical Brownian motion is closely related to a **lognormal probability distribution**. Recall that a random variable X has a lognormal distribution if $\log(X)$ (by $\log$ we denote the natural logarithm) has a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Then the probability density function of the variable X is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

The first two moments of the random variable X are given by

$$\begin{aligned} \mathbb{E}[X] &= e^{\mu t + \frac{1}{2}\sigma^2 t}, \\ \mathbb{D}[X] &= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \end{aligned}$$

2.3.3 Probability distribution of a geometrical Brownian motion. Based on the given properties of the lognormal distribution, derive the density and the first two moments of the value of a geometrical Brownian motion at time t.

Simulate trajectories of a geometrical Brownian motion with selected parameters; a sample result is shown in Figure 2.7. Add the expected value of the process to the graph.

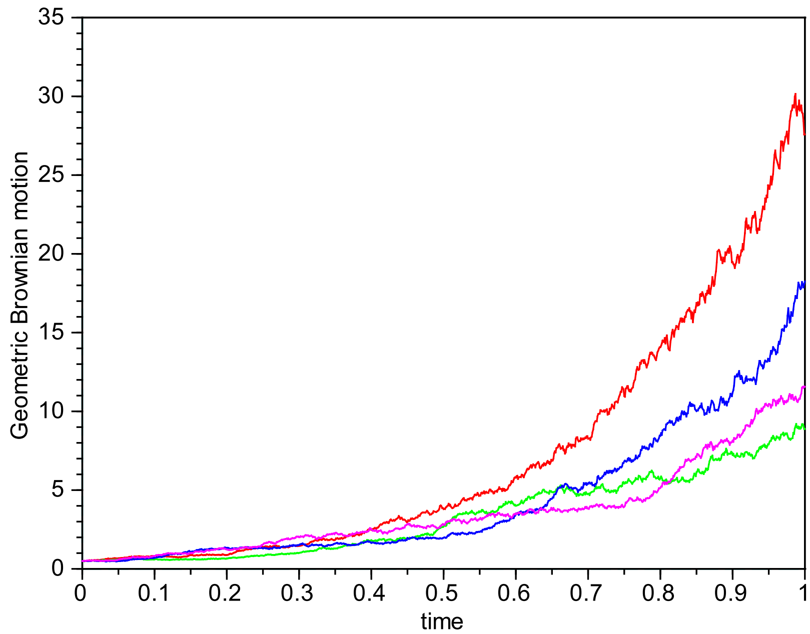

Figure 2.7: Geometrical Brownian motion.

2.3.4 Modelling stock prices with a GBM, estimating the parameters. Geometrical Brownian motion can be used as a simple model for the stock prices[2]. It means that if the initial stock price S_0 at time $t = 0$ is given, the future stock prices are modelled as

$$S(t) = S_0 e^{\mu t + \sigma^2 t}. \tag{2.1}$$

Then the (logarithmic) returns

$$\text{return}_t = \log\left(\frac{S_t}{S_{t-\Delta t}}\right) = \mu\Delta t + \sigma\Delta w$$

are independent random variables with $\mathcal{N}(\mu\Delta t, \sigma^2\Delta t)$ distribution, which is a base for the estimation of the parameters from the stock price data.

We rewrite the estimation procedure from the lectures into Scilab, assuming that the data are given in the text file `stock.txt` which is located in the current working directory. Finish the code using the outline below:

```
s=fscanfMat("stock.txt");

dt=1/252 ; // time step, in years
           // 1/252 for the daily data
returns= ; //create the vector of returns
```

[2]It is also one of the assumptions of the Black-Scholes model for pricing derivatives, which we will study later.

```
muDelta=mean(returns);              // estimate of mu*dt
sigma2Delta=variance(returns);      // estimate of (sigma^2)*dt
mu=muDelta/dt                       // estimate of mu
sigma=sqrt(sigma2Delta/dt)          // estimate of sigma
```

Download the historical data[3] for a selected stock from `finance.yahoo.com` or `finance.google.com` (see Figure 2.8 for a snapshop of the data table at `finance.google.com`) and use them for the following tasks:

- Display the evolution of the stock prices and the returns.
- Estimate the parameters of the geometrical Brownian motion.
- Add the estimated expected value of the stock price in the future, conditioned on its last observed value.

Figure 2.8: Historical stock price data at `finance.google.com`.

2.3.5 Exercise: Forecasting the stock price evolution using GBM. Suppose that the stock price follows the geometric Brownian motion (2.1) with parameters $\mu = 0.15$ and $\sigma = 0.20$. The price of the stock today is 120 USD.

- Plot the density of the stock price in one month. Perform simulations: generate 1000 values of the stock price and plot their histogram. Compare the two plots.

  Hint. Outline of the Scilab code for the density:

```
// density of S(t) for the given S(0)=s0
// when modelling S by  a geometric BM:  S(0)*exp(mu*t+sigma*w(t))
function [pdf]=densityS(s,mu,sigma,s0,t)
        muS=log(s0)+mu*t;
        sigma2S=t*sigma^2;
        pdf=exp(-(log(s)-muS).^2/(2*sigma2S))./...
                (s*sqrt(2*%pi*sigma2S));
endfunction

// left: exact density, right: histogram from simulations
figure;
subplot(1,2,1);
s= ;   // suitable range of s (stock price)
f= ;   // densityS(...) - density values
plot(s,f);
```

[3]Note that the data are usually displayed with the most recent in to the top. It is easy to adjust the computation of the parameters, but it is better to adjust the data - in order to be able to produce graphs, etc.

and for the simulated values:

```
subplot(1,2,2);

// vector of 1000 iid N(0,1) realizations
n=rand(1,1000,"normal");
// use the vector n to simulate the stock prices
Ssim= ;

// plot a histogram: histplot(N,data), where N = number of bins
```

A sample result is shown in Figure 2.9.

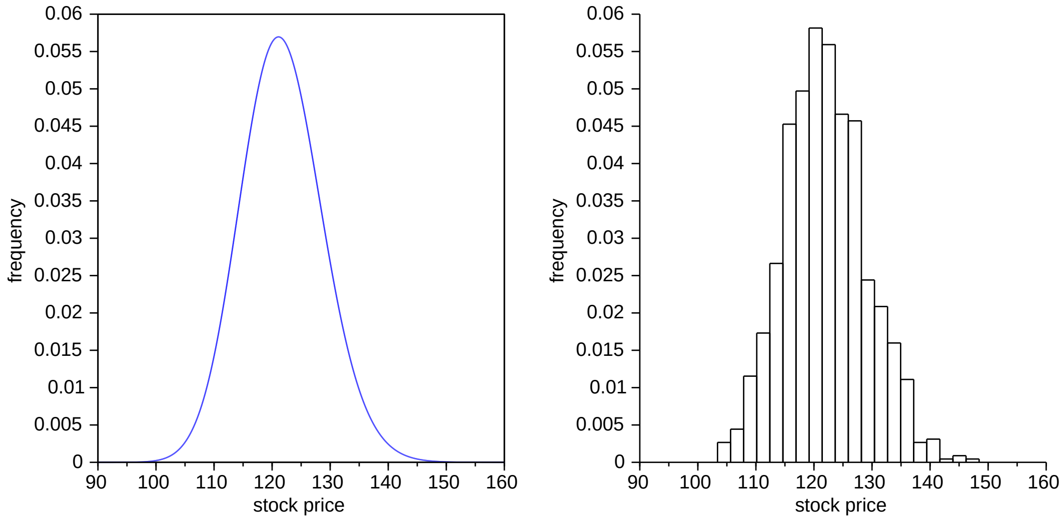

Figure 2.9: Density of the future stock price and histogram of the simulated values.

- What is the expected value of the quarterly return? What is the probability that it will be negative?

  Hint. We can simplify the original Scilab function for cumulative distribution function and define a new function `normcdf` for the $\mathcal{N}(0,1)$ distribution which has only one parameter:

```
function [cdf]=normcdf(x)
    cdf=cdfnor("PQ",x,0,1);
endfunction
```

- What is the probability that in one year the stock price exceeds 150 USD? What is the probability that it falls below 100?

2.3.6 Exercise: Estimating the parameters of a GBM and choice of the time period. Select a stock and download the historical data of its prices. Estimate the parameters of the GBM, using the data from different time period: last quarter, last year, last couple of years, etc. Plot the stock prices for each period, list the estimates and comment on the differences.

2.3.7 Exercise: Pricing options - Monte Carlo method. In this course we are going to study derivatives pricing using the partial differential equation approach. Another alternative is probabilistic characterization of the option price. This allows the computation of the derivatives prices by simulations.

In this exercise we outline the basic idea. We assume that the stock price behaves according to the geometric Brownian motion (2.1) and that the derivative has the payoff $\bar{V}(S)$ at the time of its expiration.

Firstly, we need to realize that the correct price of the derivative is not the expected value of $\bar{V}(S)$ (or its discounted value, taking interest rates into account). The derivative is not a simple "bet" in which we receive $\bar{V}(S)$ with a certain probability distribution of S. The difference from a bet is, that in our case it is possible to trade also the underlying stock.

It can be shown that the price *is* the discounted expected value of the payoff, but under another - so called *risk neutral* - probability measure. The stock price follows a geometric Brownian motion also under this risk neutral measure, but instead of the parameter μ there is $r - \frac{\sigma^2}{2}$ where r is the interest rate. The volatility σ remains the same.

This means that the price of the derivative at time t, when the stock price equals S, equals

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\bar{V}(S)\right],$$

where T is the expiration time and $\mathbb{Q}$ denotes the risk neutral measure. The expected value can be approximated by the arithmetic average of the realizations, and hence

$$V(S,t) \approx e^{-r(T-t)}\frac{1}{N}\sum_{i=1}^{N}\bar{V}(S_i),$$

where $S_1,\ldots,S_N$ are simulated values of the stock price at the expiration time under the risk neutral measure $\mathbb{Q}$.

Let us consider a specific call option:

```
// GBM for the stock price:
mu=0.35;
sigma=0.30;

// Current stock price
s0=150;

// Call option
E=175;     // exercise price
tau=1/2;   // time to expiration

// Interest rate
r=0.01;
```

At the time of expiration we have:

```
Z=randn();          // N(0,1), randn() has been defined earlier
wT=sqrt(tau)*Z;     // Wiener process at time T

// risk neutral measure
// => "r-0.5*sigma^2" instead of "mu"  in the GBM
sT=s0*exp((r-0.5*sigma^2)*tau+sigma*wT);  // stock at time T
vT=max(0,sT-E);                           // option at time T
```

Repeat this in a loop and after each new simulation compute the current approximation of the option price based on the simulations performed so far. The approximations converge to the options price with increasing number of simulations. A sample result is shown in Figures 2.10 (convergence of the intermediate results) and 2.11 (histogram of 1000 values attained after 10000 simulations). It is possible to price this option also analytically[4] using the Black-Scholes formula (this will be studied later), its price is 4.8572 USD.

Note that the variance of the Monte Carlo simulations performed in this direct way is quite high. There are methods, so called *variance reduction methods* whose aim is to decrease this variance[5]

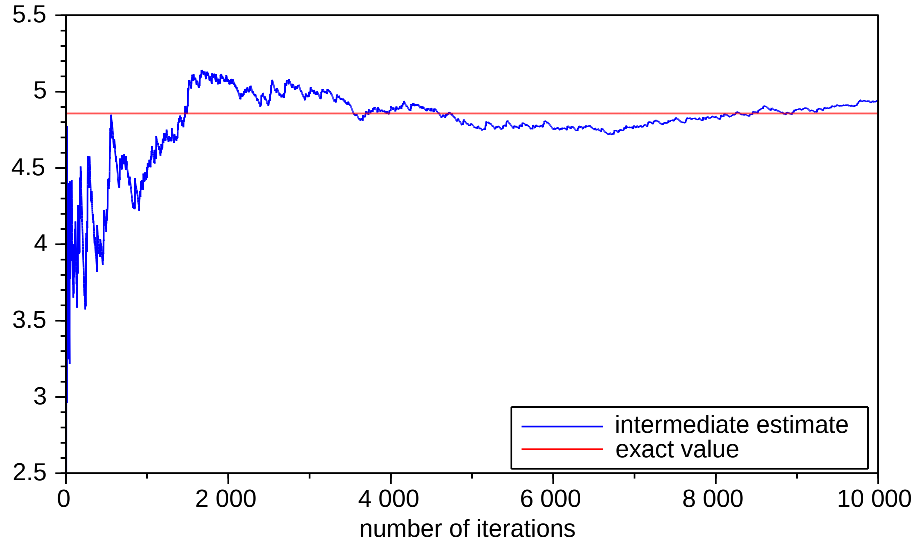

Figure 2.10: Computing the option price using Monte Carlo method.

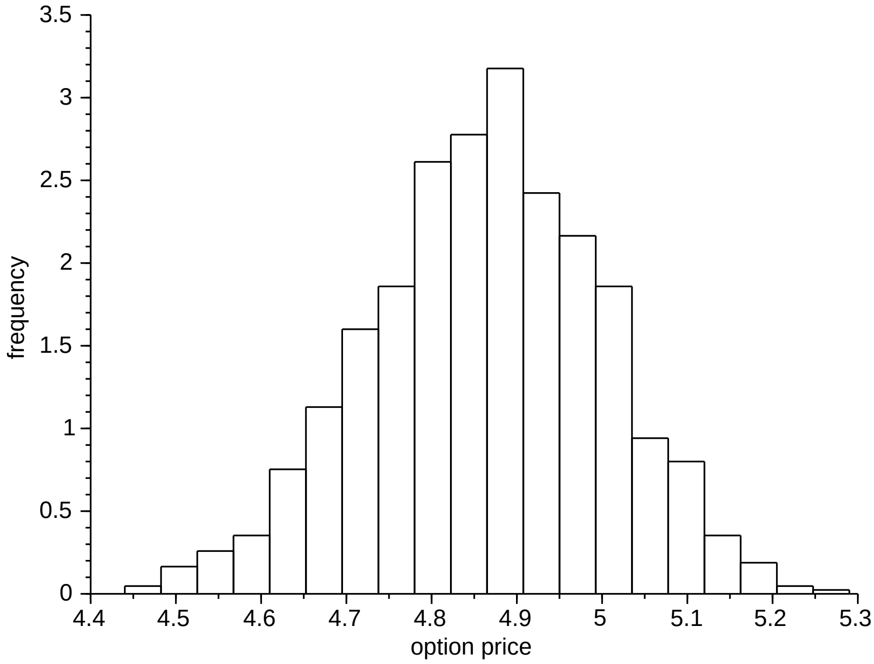

Figure 2.11: Computing the option price using Monte Carlo method - histogram of the values after 10000 simulations.

2.3.8 Exercise. Find the cover of the 6th edition of the book *Stochastic differential equations* by Berndt Oksendal. What is in the picture? Produce such a plot.

[4]This is not possible or the more complicated derivatives, hence the need for approximate methods, for example Monte Carlo simulations. Using these methods in the simple cases with analytical solution enables us to test their efficiency and precision.

[5]For a simple introduction to these techniques see the chapter 3.5.4. of the book [4].