Financial derivatives 2014/2015
Homework: revision of numerical methods

- Homework can be solved independently or in pairs.
- Solutions should be e-mailed as a **pdf file** (typed or scanned) to **beata.ulohy@gmail.com** with subject **derivatives 2014 - numerics - name/names**
- Deadline: 2 working days before the exam

1. **Numerical methods for heat equation.** Consider a heat equation

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \;\; x \in [-L, L], t \in (0, T]$$

with initial condition $u(x,0) = u_0(x)$ for $x \in [-L, L]$ and boundary conditions $u(-L,t) = \phi(t), u(L,t) = \psi(t)$ for $t \in (0,T]$.

Consider its discretization with time step k and space step h.

(a) Write down the notation which you are going to use in your derivation of numerical schemes.

(b) Derive the explicit and implicit scheme for the heat equation.

(c) When using the explicit scheme, we cannot choose k, h arbitrarily. What condition has to be satified and why (shortly - what happens if it is not satisfied)?

(d) Write the implicit scheme in a matrix form

$$Au^{(m)} = b,$$

where $u^{(m)}$ is a vector with approximation of the solution on m-th time level (except for boundary points, where the values are determined by boundary conditions - so we do not compute them in our numerical scheme), A is a matrix (write its elements explicitly) and b is the right hand side vector.

2. **Solving a system of linear equations**

(a) Define a vector norm (i.e. the properties of a general norm of a vector from $\mathbb{R}^n$).

(b) Define a matrix norm (i.e. the properties of a general norm of an $n \times n$ matrix). Give an example of a matrix norm, example of a matrix (small one, so that the computation can be done by hand) and compute the given norm for the given matrix.

(c) In general (for any pair of matrix and vector norms) the inequality $\|Ax\| \leq \|A\| \, \|x\|$ does not necessarily hold. Give an example of a pair of matrix and vector norms, for which this inequality holds (with a reference to a book or website). Where - in the analysis of numerical methods for solving linear systems - have we used this property?

(d) Consider the iterative scheme for solving the system $Ax = b$ in the form $x^{(k+1)} = Tx^{(k)} + g$, where T is iteration matrix and g is a vector. What has to be satisfied by the matrix T, so that the iterative scheme is convergent?

(e) Is it possible that a sequence of vectors in $\mathbb{R}^n$ (e.g. a sequence of iteration from the previous point) converges in some norm, but does not converge in another norm? Explain.

(f) Derive the formulation of the Gauss-Seidel method

$$u_i^{p+1} = \frac{1}{A_{ii}} \left(b_i - \sum_{j<i} A_{ij} u_j^{p+1} - \sum_{j>i} A_{ij} u_j^p \right)$$

in the matrix form $x^{(k+1)} = Tx^{(k)} + \omega$. Explain your notation (in case you introduce any new matrices related to A).

(g) Prove that the Gauss-Seidel method, applied to the system from implicit scheme for heat equation, is convergent. State the convergence criterion (without a proof, a reference is sufficient) and show that it is satisfied by our matrix.