I. Derivatives, call and put options, boundaries for option prices, combined strategies

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Financial derivatives, winter term 2014/2015

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What are financial derivatives

• Term DERIVATIVE in a dictionary:

noun

1 something which is based on another source:

the aircraft is a derivative of the Falcon 20G

• a word derived from another or from a root in the same or another language:

‘fly-tip’ is a derivative of the phrase ‘on the fly’

• a substance that is derived chemically from a specified compound:

crack is a highly addictive cocaine derivative

2 (often derivatives) Finance an arrangement or product (such as a future, option, or warrant) whose value derives from and is dependent on the value of an underlying asset, such as a commodity, currency, or security:

[as modifier]:

the derivatives market

3 Mathematics an expression representing the rate of change of a function with respect to an independent variable.

http://oxforddictionaries.com/definition/derivative
Derivatives

- Aristotle writes about Thales of Miletus (Politics, Book I, Chapter XI):

  ... while it was yet winter, having got a little money, he gave earnest for all the oil works that were in Miletus and Chios, which he hired at a low price, there being no one to bid against him; but when the season came for making oil, many persons wanting them, he all at once let them upon what terms he pleased; and raising a large sum of money

  English translation: http://www.gutenberg.org

- Right to use the oil presses - its value depends on the crop in the given year

- Some presses may stay unused; Thales has a right, but not an obligation to use the presses
Derivatives

- A right but also an obligation to realize an arranged trade - some historical examples:
  - England, France, 12th century - arrangement of a future trade based on a sample, "lettre de faire"
  - Japan, 17th century - standardized rice trades
  - Chicago, 19th century - wheat, establishment of Chicago Board of Trade (1848)
  - 1898 - Chicago Mercantile Trading, butter and eggs, later also other agricultural commodities
  - 1978 - International Monetary Market as a part of Chicago Mercantile Trading, foreign exchange, later also (e.g.) S&P 500 derivatives
Akcie

• Mostly we will deal with derivatives in stock market
• Example: evolution of DIS (*The Walt Disney Company*) stock price

http://finance.yahoo.com
Stocks

- Evolution of a stock price consists of a trend and random fluctuations
- Example of a trend: NFLX (Netflix, Inc.), August 2012 - August 2014:

http://finance.google.com
Stocks

• Example of fluctuations: NFLX (Netflix, Inc.)

http://finance.google.com

• Next lecture: mathematical modelling of this observation trend + fluctuations
Stock options

• **European call option** is a right - but not an obligation - to buy the asset for the predetermined price $E$ (strike price, exercise price) in the predetermined time $T$ (expiration time)

• **European put option** is a right - but not an obligation - to sell the asset for the predetermined price $E$ (strike price, exercise price) in the predetermined time $T$ (expiration time)

• **American call/put options** - a right to buy/sell the stock not only at the expiration time $T$, but at any time prior to the expiration time
Stock options

- Example of real data: put options on Disney stock

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<tr>
<th>Strike</th>
<th>Symbol</th>
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<th>Bid</th>
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<td>2</td>
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</tbody>
</table>

http://finance.yahoo.com
Option price

- Option price consists of two parts:
  - **intrinsic value** - value of the option if exercised now
  - **time value** - remaining part of the price:
    - holder of the option pays this value, expecting that the option brings him profit in the future
    - risk premium for the writer of the option
Intrinsic and time value: example

- Put prices from page 9 - we will use the last realized price
- Stock price: 87.40 USD

Let us consider the put option with exercise price 70 USD which costs 0.17 USD:
  - intrinsic value: 0
  - time value: 0.17
Intrinsic and time value: example

• Questions:
  ◦ Why do all the options (page 9) zero intrinsic value?
  ◦ Which puts would have a positive intrinsic value?
  ◦ How about call options? Use data below:

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<td>0.00</td>
</tr>
</tbody>
</table>
Example

• We sell the DIS stock for the current bid price (the price a buyer is willing to pay): 87.33 USD.

http://finance.yahoo.com
Example

- Then, we sell a put option with exercise price 60 USD and expiration in October for - we find the *bid price* - 0.04 USD

<table>
<thead>
<tr>
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http://finance.yahoo.com
Example

• How much are you willing to pay for a call option with the same exercise price and the same expiration time?
• Recall the evolution of DIS stock price; the data are from August 11, 2014:

http://finance.yahoo.com
Example

- Russel Sage, New York, 19th century:
  - bought a stock and a put option, sold a call with the same exercise price and the same expiration time
  - avoided bounds on interest rates given by usury laws

- **Example - continued:**
  - We show that this strategy is - in fact - a loan (so called *synthetised loan*)
  - What interest rate did you agreed on by your accepted price of the call option?
Call-put parity

• Consider a portfolio:
  ◦ write 1 call option with exercise price \( E \)
  ◦ sell 1 put option with the same exercise price and the same expiration time
  ◦ buy 1 stock

• What will be the portfolio value at the time of expiration?

\[
\text{portfolio} = -1 \text{ call} + 1 \text{ put} + 1 \text{ stock} \\
\Rightarrow \\
\text{payoff} = -[\text{payoff of call}] + [\text{payoff of put}] + [\text{stock price}]
\]
Call-put parity

- Hence, depending on the stock price $S$ at the time of expiration:
  - if $S \leq E$:
    \[
    \text{payoff} = -[0] + [E - S] + [S] = E
    \]
  - if $S \geq E$:
    \[
    \text{payoff} = -[S - E] + [0] + [S] = E
    \]

So, without any risk we end up with $E$

- Therefore, the value of the portfolio today has to be
  \[
  -c(S, E, \tau) + p(S, E, \tau) + S = E e^{-r\tau}
  \]

- we have obtained a relation between call and put prices, known as call-put parity
Payoff diagram

- Payoff diagram of an option - value of the option at the time of expiration, as a function of the stock price at this time
- Call option: $\max(0, S - E)$, put option: $\max(E - S, 0)$
Profit diagram

- Profit diagram of an option - payoff of the option minus the value of our initial investment:
  - If $r = 0$, then
    \[
    \text{profit} = \text{payoff} - \text{costs}
    \]
  - In general:
    \[
    \text{profit} = \text{payoff} - \text{costs} \times e^{r\tau}
    \]

(to pay costs today is the same as to pay costs$ \times e^{r\tau}$ at the expiration time)
Profit diagram - example 1

- Consider a call option with exercise price 105 USD which costs 15 USD
- Profit diagram (for $r = 0$):

![Profit Diagram](image-url)
Profit diagram - example 2

- Analyze the following profit diagram of a put option (for $r = 0$):
Profit diagram - example 2

- **SIMPLE QUESTIONS:**
  - What is the exercise price of the option? How much did it cost?
  - Is the possible profit bounded? If it is bounded, when it is maximal? If it is not bounded, when it rises without bounds?
  - Is the possible loss bounded? If it is bounded, when it is maximal? If it is not bounded, when it rises without bounds?
Bounds for option prices

- We show some inequalities for prices, which have to hold - otherwise there is an arbitrage on the market
- All the options considered have the same expiration time
- We denote the riskless interest rate by $r$.
- Notation:
  - $c(S, E, \tau)$ is the market price of a call option with exercise price $E$, if the stock price today is $S$ and time remaining to expiration is $\tau$
  - $p(S, E, \tau)$ is the market price of a put option with exercise price $E$, if the stock price today is $S$ and time remaining to expiration is $\tau$
Bounds for option prices

• Outline:
  ◦ We consider two portfolios - such that at the time of expiration we have:
    \[(\text{value of portfolio I.}) \leq (\text{value of portfolio II.})\]
  ◦ To avoid a possibility of arbitrage, also today necessarily
    \[(\text{value of portfolio I.}) \leq (\text{value of portfolio II.})\];
  
  the portfolios are constructed in such a way that this is the inequality that we need to prove
Bounds for option prices - examples

**Example 1:** Clearly

\[ c(S, E, \tau) \geq 0, \quad p(S, E, \tau) \geq 0 \]

**Example 2:** Show that

\[ E_1 \geq E_2 \Rightarrow c(S, E_1, \tau) \leq c(S, E_2, \tau) \]

**Solution** Let \( E_1 \geq E_2 \) Consider the following portfolios

portfolio I.: option with exercise price \( E_1 \)

portfolio II.: option with exercise price \( E_2 \)

We compare their value at the expiration time, depending on the stock price \( S \) at this time.
Bounds for option prices - examples

<table>
<thead>
<tr>
<th></th>
<th>$0 \leq S \leq E_2$</th>
<th>$E_2 \leq S \leq E_1$</th>
<th>$E_1 \leq S$</th>
</tr>
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<tbody>
<tr>
<td>portfolio I.</td>
<td>0</td>
<td>0</td>
<td>$S - E_1$</td>
</tr>
<tr>
<td>portfolio II.</td>
<td>0</td>
<td>$S - E_2$</td>
<td>$S - E_2$</td>
</tr>
<tr>
<td>comparison</td>
<td>$0 = 0$</td>
<td>$0 \leq S - E_2$</td>
<td>$S - E_1 \leq S - E_2$</td>
</tr>
</tbody>
</table>

At the expiration time:

(value of portfolio I.) $\leq$ (value of portfolio II.)

$\Rightarrow$ also today:

(value of portfolio I.) $\leq$ (value of portfolio II.),

i.e.,

$c(S, E_1, \tau) \leq c(S, E_2, \tau)$, QED
EXAMPLE 3:
Assume zero interest rate and the following call option prices:

<table>
<thead>
<tr>
<th>exercise price</th>
<th>option price</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
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<tr>
<td>15</td>
<td>26</td>
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<tr>
<td>20</td>
<td>27</td>
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<tr>
<td>25</td>
<td>23</td>
</tr>
<tr>
<td>30</td>
<td>19</td>
</tr>
</tbody>
</table>

Find an arbitrage.

SOLUTION: We plot the dependence of the call option price on the exercise price - its decreasing character, proved in the previous example, is not satisfied.
We should have $c(S, 15, \tau) \geq c(S, 20, \tau)$; however, here $c(S, 15, \tau) < c(S, 20, \tau)$. Therefore:

- **we buy the option, which costs less than it is supposed to,** in this case the option with exercise price $E = 15$,

- **we sell the option, which costs more than it is supposed to,** in this case the option with exercise price $E = 20$. 
Bounds for option prices - examples

• We check the result of our strategy:

```matlab
S = 0:30;
payoff = max(S - 15, 0) - max(S - 20, 0);
price = 26:27; // price of the strategy = initial investment
profit = payoff - price;

figure(1);
plot(S, profit);
xlabel("stock at expiration"); ylabel("profit");
```

• What do we expect as a result? How does a profit diagram of an arbitrage look like?
Bounds for option prices - examples

- Resulting profit diagram:

⇒ this strategy is indeed an arbitrage

- EXERCISES SESSION NEXT WEEK: More practice with proving bounds for option prices and finding arbitrage opportunities
Combined strategies

• In the previous (theoretical) example we combined the options to construct an arbitrage

• This idea of buying and selling several options can be used also with real option prices - based on our expectations about future behaviours of the stock price
Combined strategies

**EXAMPLE:**

- Consider the MCD (Mac Donald’s Corp.) stock prices

http://finance.yahoo.com

and suppose (for this exercise) that we expect the stock price to be falling
Combined strategies

- The stock price is 93.72 USD and some of the available options are (data from August 11):

<table>
<thead>
<tr>
<th>Strike</th>
<th>Symbol</th>
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http://finance.yahoo.com

- We expect the stock to fall ⇒ we buy a put option, for example one with exercise price 90 USD
- However, we don’t expect it to fall too low ⇒ we sell a put option with a lower expiration price, for example 85 USD
- We expect that the latter will not be exercised, but by writing it, we lower the initial investment
Combined strategies

• Our strategy:
  *we buy a put with $E = 16$ and sell a put with $E = 14*$

• Recall bid and ask prices:
  - bid price (the lower one) - the price a buyer is willing to pay → I can sell the option for bid
  - ask price (the higher one) - the price a seller is willing to accept → I can see the option for ask

• Therefore our initial investment is **0.49**, since:
  - we buy the put with $E = 16$ for **0.66**
  - we sell the put with $E = 13$ for **0.17**
Combined strategies

- Profit diagram:
Combined strategies

- Comparison - with only buying the put with $E = 90$:

![Graph showing profit vs. stock at expiration for different options](image-url)
**Combined strategies**

- **EXTRA CREDIT:**
  Construction of a combined strategy using the real data, aiming to achieve the highest profit

- **Overview of combined strategies:**
  - Ševčovič, Stehlíková, Mikula: *Analytické a numerické metódy oceňovania finančných derivátov*. STU 2009. (In Slovak) - chapter 2.3.3.