II. Stochastic processes

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Stochastic processes in finance: motivation

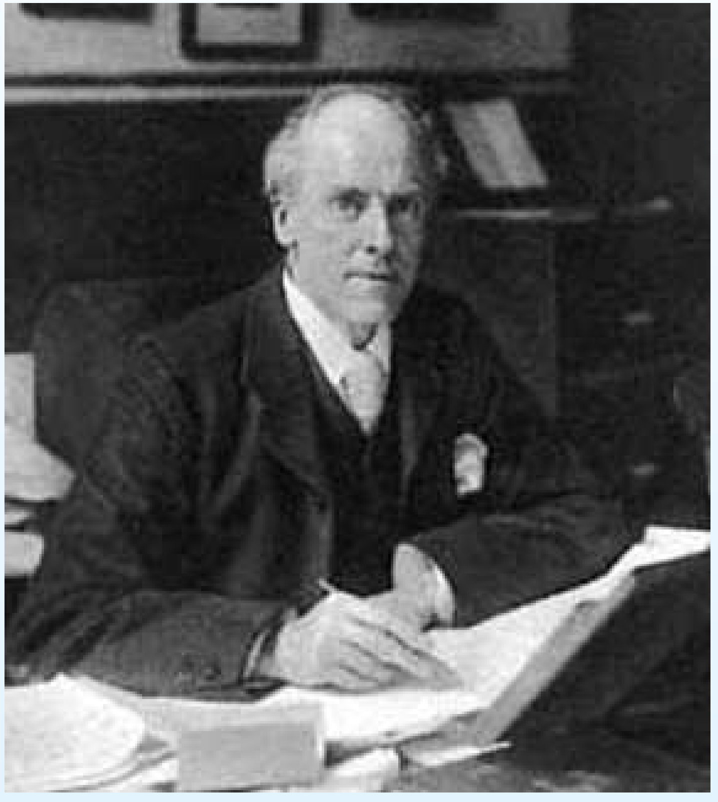
- Stock prices - it is clear that their evolution is not deterministic, for modelling we need the theory of stochastic processes:

http://finance.google.com

- Stochastic process: t-parametric system of random variables $\{X(t), t \in I\}$, I - interval or a discrete set
- This means: at each time t, the value $X(t)$ of the process is a random variable

Random walk

- **Karl Pearson**, 1905: modelling migration of mosquitoes infesting jungle regions



http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Pearson.html

Writes in *Nature*, 27 July 1905:

A man starts from a point O and walks l yards in a straight line, he then turns though any angle whatever and walks another l yards in a second straight line. He repeats the process n times. I require the probability that after these n steps he is a distance between r and $r + \delta r$ from his origin O.

Random walk

- Answered by **Lord Rayleigh** (John William Strutt) - refering to his paper from 1880 concerned with sound vibrations
- The answer, given for large n (number of steps), is an expression originating from normal distribution

http://www-history.mcs.st-andrews.ac.uk/Biographies/Rayleigh.html

Random walk

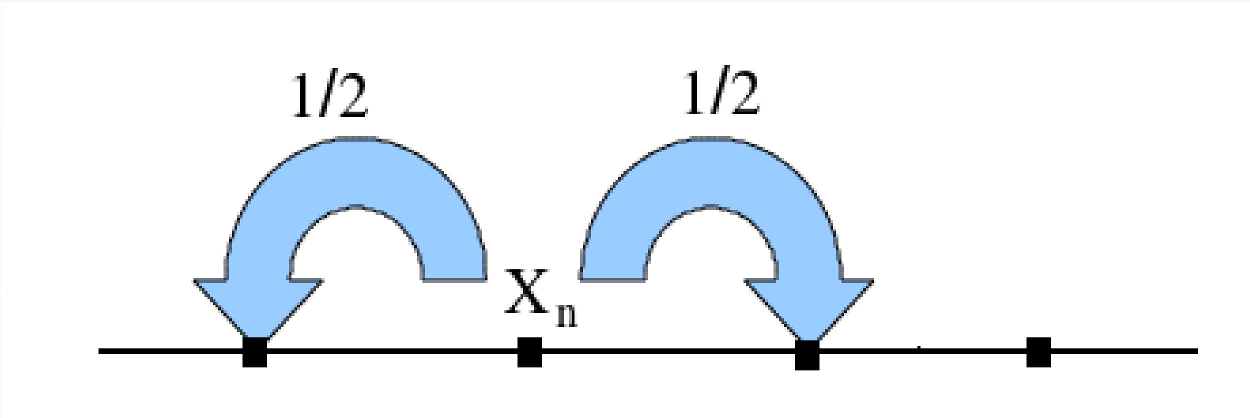
- Pearson comments in *Nature*:

  > *The lesson of Lord Rayleigh's solution is that in open country the most probable place of finding a drunken man who is at all capable of keeping on his feet is somewhere near his starting point.*

- PLAN FOR THE NEXT PART OF THE LECTURE:
  - A random walk example, known as drunken sailor problem or drunkard problem.
  - We study it in one-dimensional case
  - Then we make a limit to a continuous case ("random flight")
  - We arrive to a so called Wiener process

The drunken sailor problem

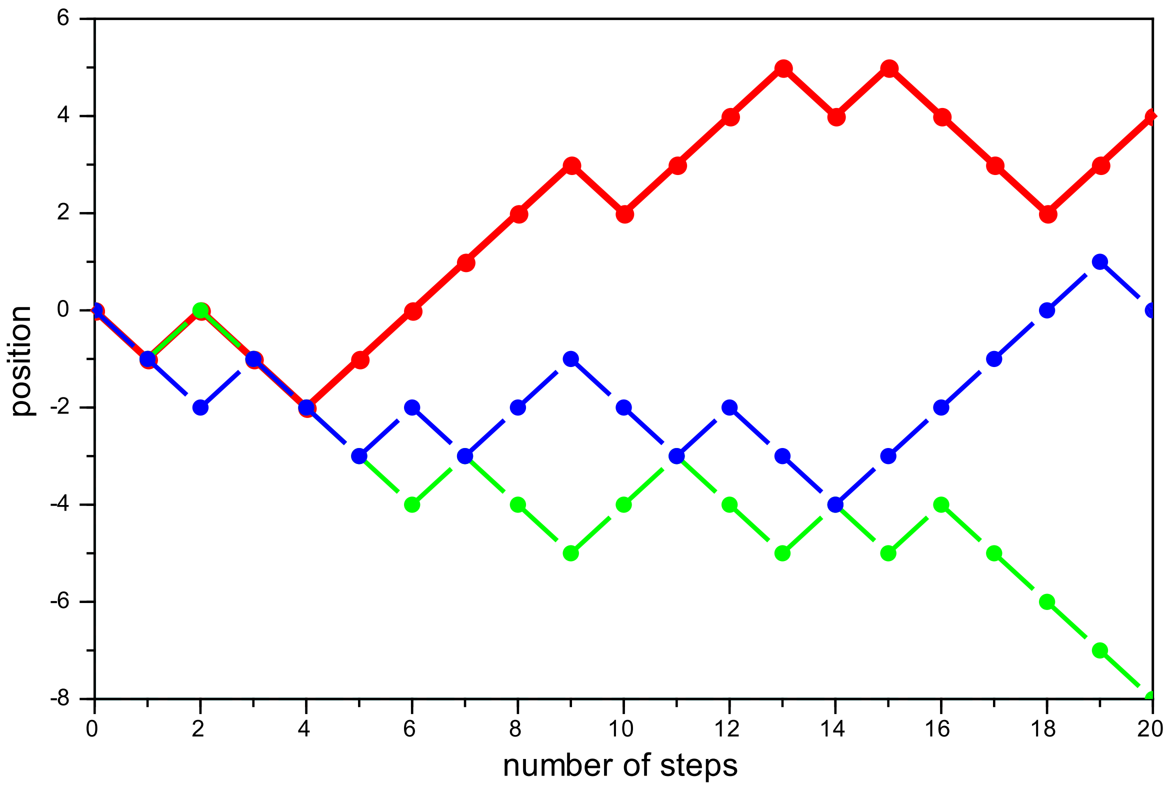
- The sailor moves on a line starting off at $x_0 = 0$
- With probability $1/2$ makes one step forward; with probability $1/2$ makes one step back
- We denote by X_n his position after n steps



- i-th step: $s_i = \pm 1$, each with probability $1/2$; steps are independent
- Position after n-th step: $X_n = \sum_{i=1}^{n} s_i$

The drunken sailor problem

- Sample trajectories:



The drunken sailor problem

- We derive the distribution of X_n
- We have:
  - i-th step: $s_i = \pm 1$, each with probability $1/2 \Rightarrow$ $\mathbb{E}[s_i] = 0$, $\mathbb{D}[s_i] = 1$
  - independent steps
- So for X_n we obtain
  - the expected value:

$$\mathbb{E}[X_n] = \mathbb{E}\left[\sum_{i=1}^n s_i\right] = \sum_{i=1}^n \mathbb{E}[s_i] = 0$$

  - and - using independence of the steps - the dispersion:

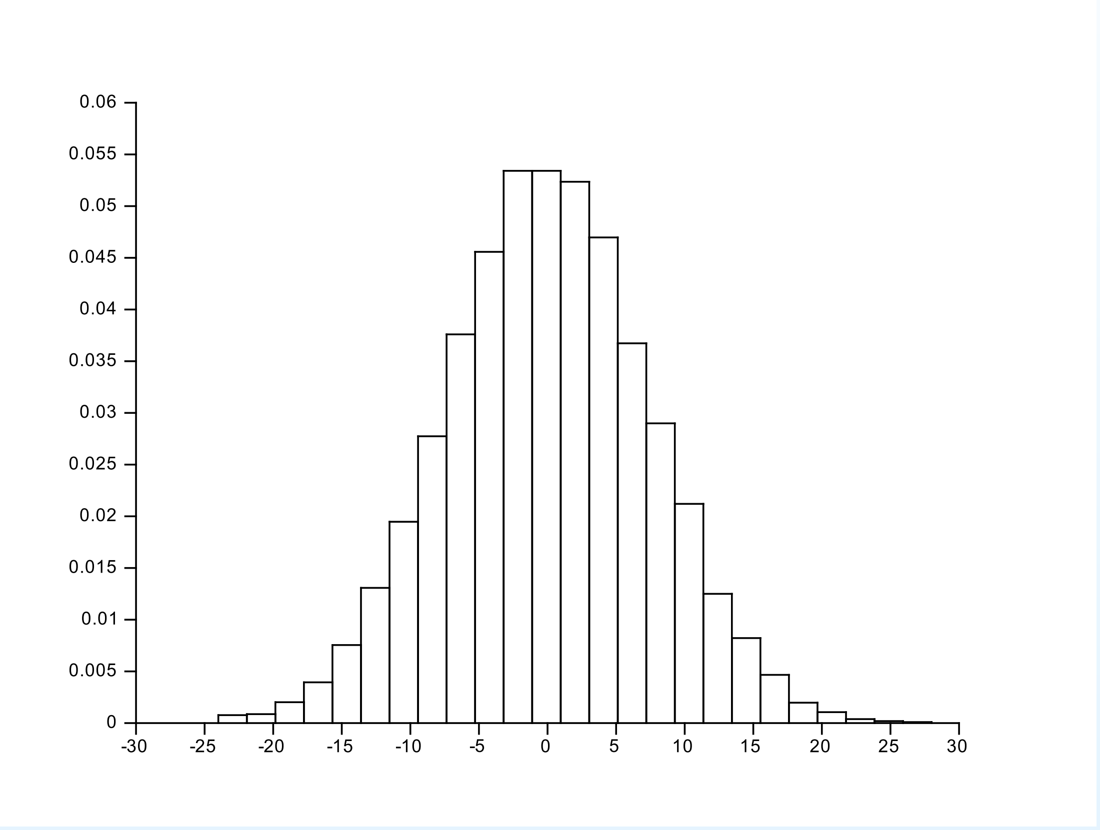
$$\mathbb{D}[X_n] = \mathbb{D}\left[\sum_{i=1}^n s_i\right] = \sum_{i=1}^n \mathbb{D}[s_i] = n$$

- From central limit theorem: for large n, the distribution is close to normal

The drunken sailor problem

EXAMPLE:

- Position after 50 steps - a sample histogram:



- Our earlier compuations: the distribution should be close to $N(0, 50)$ - agrees with the simulations

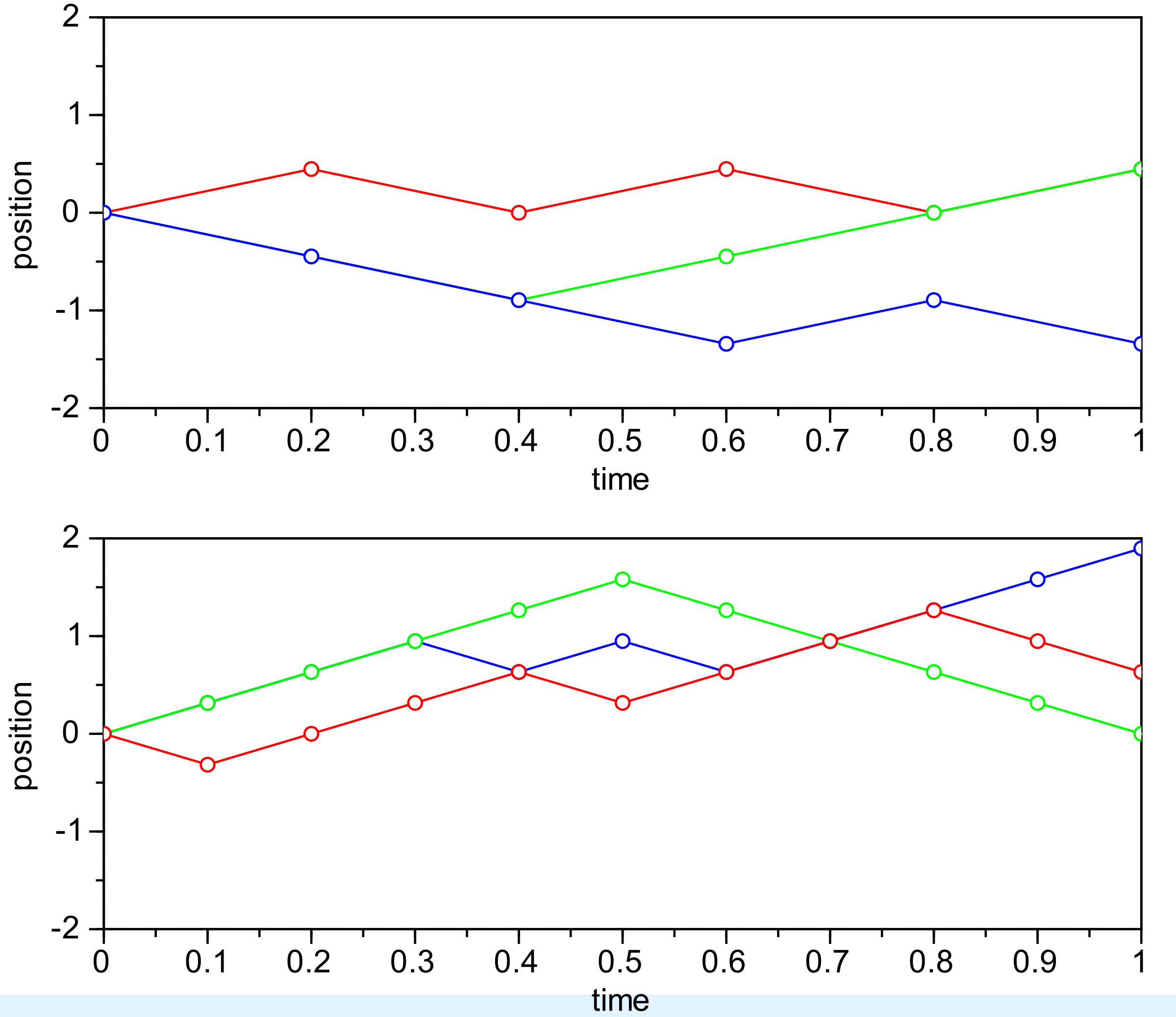
The drunken sailor problem

INTUITIVE APPROACH TO LIMIT:

- Shorter time interval between steps:
  - now, it is a unit of time
  - let us make it $1/k$ (i.e., k steps during a unit of time)
- We need to scale the step length to keep the distribution:
  - step of length $s_i = \pm\ell$, each with probability $1/2$
    $\Rightarrow$ the expected value is 0 and the dispersion is ℓ^2
  - at time n, the sailor's position is $\sum_{i=1}^{nk} s_i$
    $\Rightarrow$ the expected value is 0 and the dispersion is $nk\ell^2$
  - to keep variance at time n equal to n, we need to take step length $\ell = 1/\sqrt{k}$
- We would like to make a limit as $k \to \infty$

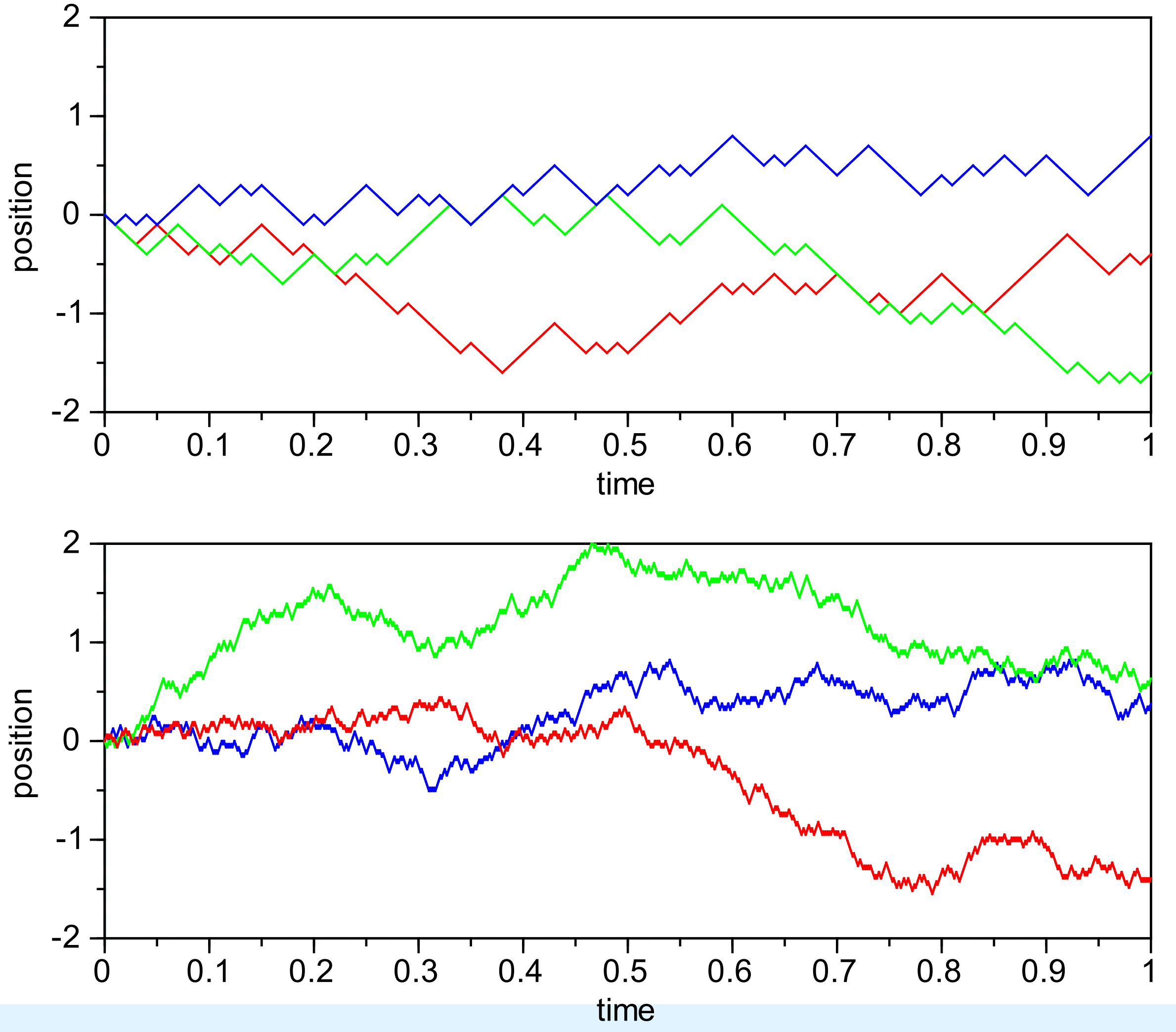
The drunken sailor problem

- EXAMPLE: 5 and 10 steps during a unit of time:



The drunken sailor problem

- EXAMPLE: 100 and 1000 steps during a unit of time:



The drunken sailor problem

INTUITIVE EXPECTATIONS ABOUT THE LIMIT:

- Distribution of the increment on interval $[t, t+\Delta t]$ is the limit of the following setting:

  we make $k\Delta t$ steps and each step has length $\ell = 1/\sqrt{k}$

  where the increment has
  - expected value equal to zero
  - dispersion equal to $k\Delta t\, \mathbb{D}[\ell] = \Delta t$
  - approximately normal distribution
- In the limit, we expect exact normal distribution $\mathcal{N}(0, \Delta t)$
- Increments on non-overlapping time intervals are independent

Wiener process: definition

- Wiener process $\{w(t), t \geq 0\}$ is a random process which satisfies:

  i) $w(0) = 0$

  ii) increments $w(t + \Delta t) - w(t)$ have a normal distribution with expected value 0 and dispersion Δt

  iii) for every partition $t_0 = 0 < t_1 < t_2 < t_3 < ... < t_n$ the increments

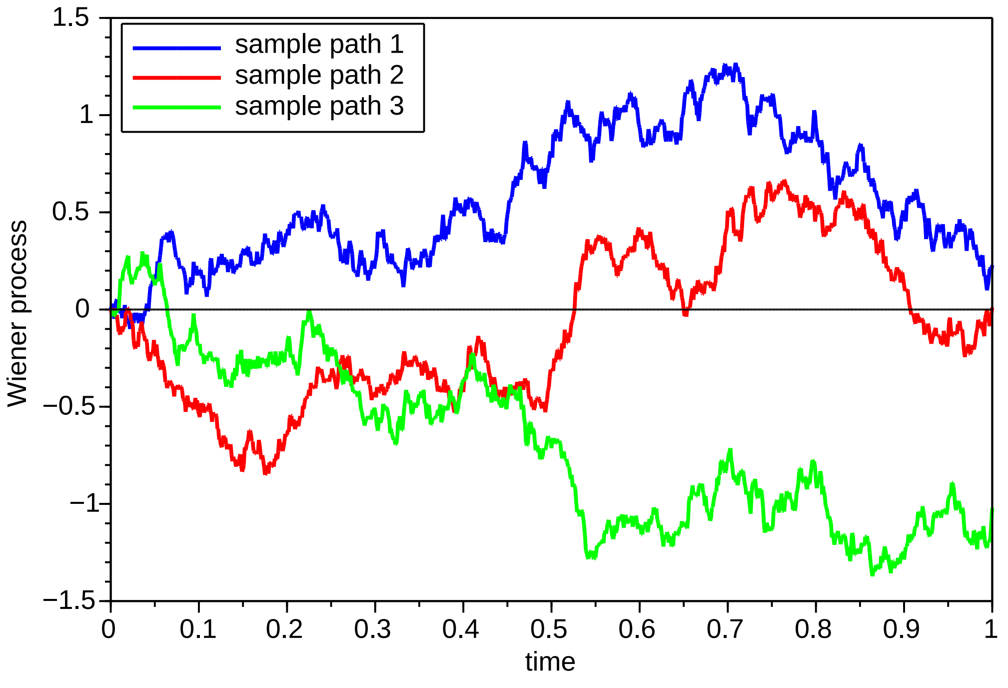
  $$w(t_1) - w(t_0), w(t_2) - w(t_1), \ldots, w(t_n) - w(t_{n-1})$$

  are independent

  iv) the sample paths (trajectories) are continuous

Wiener process - example

- Sample trajectories of a Wiener process



- Distribution of a Wiener process at time t :

$$w(t) = w(t) - w(0) \sim N(0, t)$$

Wiener process - remark on dispersion

QUESTION:
Does the dispersion of the increment $w(t+\Delta t) - w(t)$ have to be Δt , or could it be also for example $\Delta^2, \sqrt{\Delta}, \dots$?

- We arrived to dispersion Δt by considering (intuitively) a limit of a certain random walk
- Is it possible to arrive to a different dispersion via another discrete process?
- Can we simply define another random process by setting another dispersion in point ii) of the definition?

Wiener process - remark on dispersion

- Let $0 = t_0 < t_1 < ... < t_n = t$ be a partition of $[0, t]$. Then:

(1) $$w(t) - w(0) = \sum_{i=1}^{n} w(t_i) - w(t_{i-1}).$$

- From the independence of increments (dispersion of a sum of random variables equals the sum of their dispersions):

$$\mathbb{D}\left[\sum_{i=1}^{n} w(t_i) - w(t_{i-1})\right] = \sum_{i=1}^{n} \mathbb{D}[w(t_i) - w(t_{i-1})]$$

- Dispersions of the left- and right-hand side of (1) have to be equal:

$$\mathbb{D}\left[w(t) - w(0)\right] = \sum_{i=1}^{n} \mathbb{D}[w(t_i) - w(t_{i-1})]$$

- this holds of $\mathbb{D}[w(t + \Delta) - w(t)]$ is a multiple of Δ, we norm it to Δ

Wiener process - remark on dispersion

- Denote $f(x) = \mathbb{D}[w(t+x) - w(t)]$; then we have the condition

$$f(x_1 + x_2 + \dots x_n) = f(x_1) + f(x_2) + \dots f(x_n) \tag{2}$$

- Furthemore, if we require the function f to be continuous or increasing (both are natural in this context), then

$$f(x) = cx,$$

where c is a constant, is the only solution

- HOMEWORK:
Prove the proposition above.

Wiener process - remark on dispersion

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique,
Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. ANALYSE ALGÉBRIQUE.

DE L'IMPRIMERIE ROYALE.

Chez Debure frères, Libraires du Roi et de la Bibliothèque du Roi,
rue Serpente, n.° 7.

1821.

https://archive.org/stream/coursdanalysede00caucgoog

(see page 104 for the equation)

Remark

- The equation (2) is known as Cauchy's functional equation
- Appears in the (equivalent) form $f(x+y) = f(x) + f(y)$ with a condition of continuity in his *Cours d'analyse de l'École royale polytechnique* from 1821.
- Nontrivial solutions if no additional condition required

Wiener process - proof of existence

HOMEWORK:

- State the theorem and show its application for this case, which is used to prove the existence of a process with the properties from p. 14, cf. *Financial mathematics* lectures

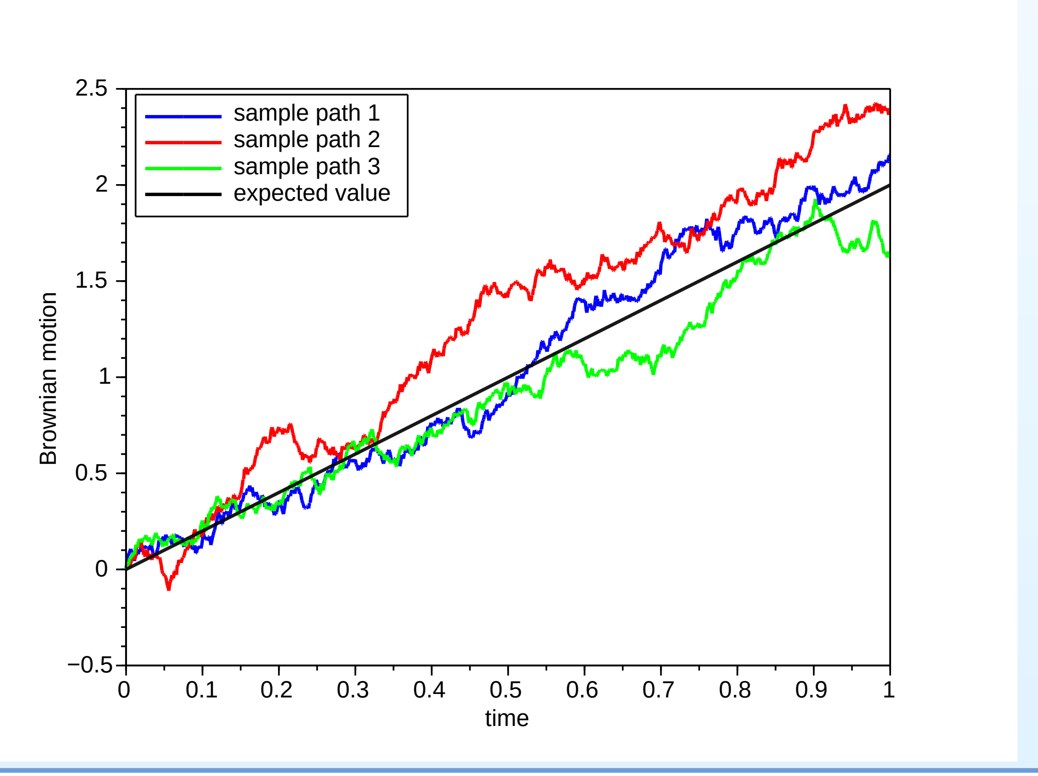
  So far we only used intuitive limit (recall different kinds of convergence for random variables; this was not specified here) and did not arrive to a contradiction when considering dispersion
- Where does it fail for (e.g.) $\mathbb{D}[w(t+\Delta t) - w(t)] = (\Delta t)^2$?

Brownian motion

- Brownian motion :

$$x(t) = \mu t + \sigma w(t),$$

  where $w(t)$ is a Wiener process
- Probability distribution $x(t) \sim N(\mu t, \sigma^2 t)$
- Sample trajectories together with the expected value:



Brown, Wiener *et al.*: Historical remarks

Henry William Pickersgill: *Portrait of Robert Brown*, 1864

http://nla.gov.au/nla.pic-an11278663

- **Robert Brown** (1773 - 1858), a Scottish botanist
- Paper entitled *A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies*

Brown, Wiener *et al.*: Historical remarks

http://en.wikipedia.org/wiki/File:Clarkia_pulchella.jpg

http://commons.wikimedia.org/wiki/File:Clarkia_pulchella_flower.jpg

- Observing pollen of the plant *Clarkia pulchella* (photos in the left) immersed in water with his microscope
- Small particles from pollen grains in vivid motion
- Motion exhibited also by particles obtained from inorganic material
- Today we can explain the phenomenon by collisions of molecules

Brown, Wiener *et al.*: Historical remarks

- **Albert Einstein** (1905), **Marian Smoluchowski** (1906): explanation of the Brownian motion
- **Jean Baptiste Perrin**: a series of experiments (1900-1912), determinig a value of Avogadro's number; awarded the Nobel Prize in Physics 1926
  Perrin's Nobel prize lecture:
  http://www.nobelprize.org/nobel_prizes/physics/laureates/1926/perrin-lecture.html
- **Norbert Wiener** (1923): the first complete mathematical construction of a Brownian motion as a continuous stochastic process
- **Norbert Wiener, Paul Lévy, ...**: mathematical properties

Applications in finance: Historical remarks

Louis Bachelier (1900): defends his thesis *Théorie de la spéculation* at Sorbonne University

- Beginning of the thesis: description of the products available in the French stock market of that time
- Model: Brownian motion as a model for a stock price
- Results:
  - he derives probability distribution, relates the density function to the heat equation
  - calculates the probability that a Brownian motion does not exceed a fixed level
  - finds the distribution of the supremum of the Brownian motion

Applications in finance: Historical remarks

THÉORIE

DE

LA SPÉCULATION,

PAR M. L. BACHELIER.

INTRODUCTION.

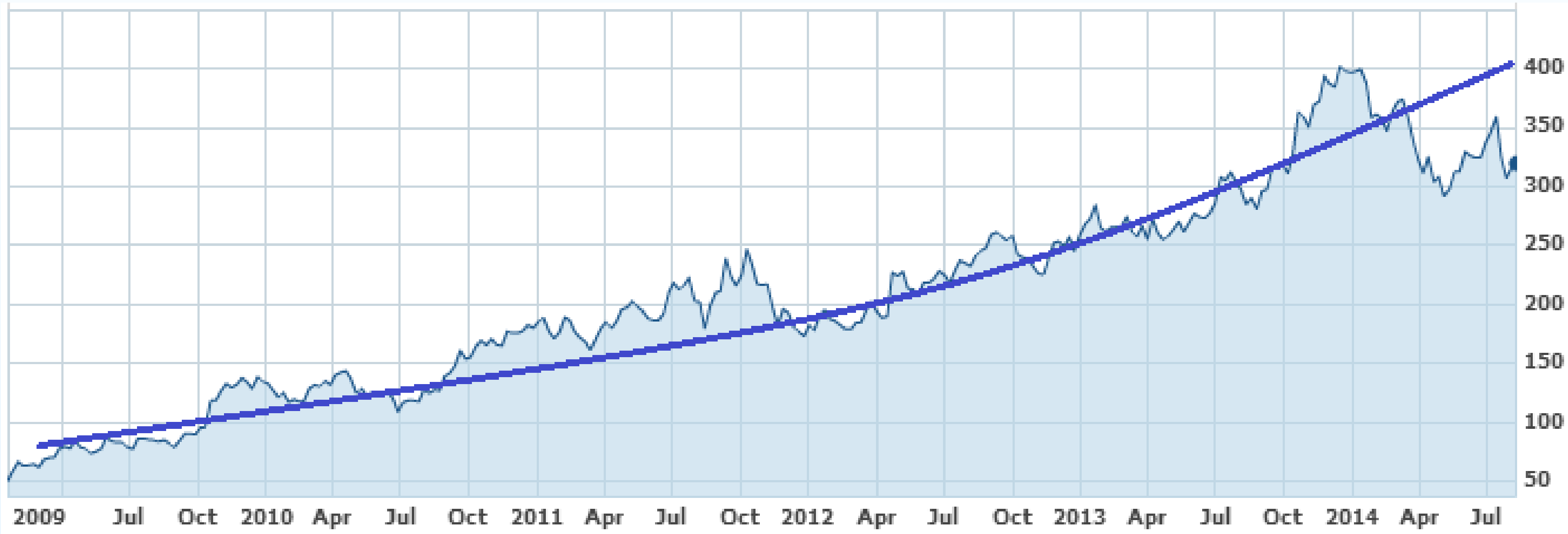
Les influences qui déterminent les mouvements de la Bourse sont

Full text: http://www.numdam.org/item?id=ASENS_1900_3_17__21_0 (in French)

CURRENT MODELLING: Brownian motion - not a model for the stock price, but its logarithm

Geometric Brownian motion: motivation

- Motivation - real stock prices (AMZN) and their trend:



http://finance.yahoo.com

- Continuous risk-free investment grows exponentially $x(t) = x_0 e^{rt}$
- We add random component: $x(t) = x_0 e^{\mu t + \sigma w(t)}$

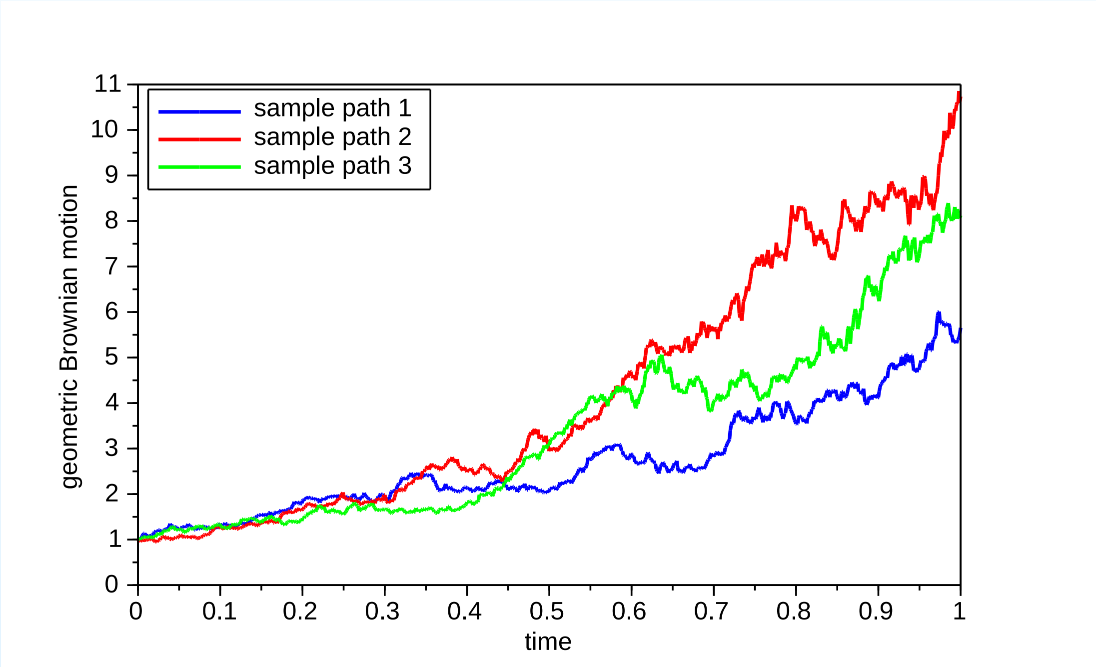
Geometric Brownian motion

- Geometric Brownian motion:

$$x(t) = x_0 \exp(\mu t + \sigma w(t)),$$

  where $w(t)$ is a Wiener process

- Sample trajectories:



- We derive the expected value and the dispersion of a GBM.

GBP - expected value, dispersion

- Recall from the probability:
  If X is a random variable with density f, then the expected value of $g(X)$ equals $\int_{-\infty}^{\infty} g(x) f(x) dx$.
- Hence, for example: $\mathbb{E}[e^X] = \int_{-\infty}^{\infty} e^x f(x) dx$.
- In our case we need:

$$\mathbb{E}\left[x_0 \exp(\mu t + \sigma w(t))\right] = x_0 \mathbb{E}\left[\exp(\mu t + \sigma w(t))\right],$$

while we know that

$$\mu t + \sigma w(t) \sim N(\mu t, \sigma^2 t);$$

therefore $\mu t + \sigma w(t)$ has the density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

GBP - expected value, dispersion

- Hence:

$$\mathbb{E}\left[e^{\mu t+\sigma w(t)}\right] = \int_{-\infty}^{\infty} e^{x}\,\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\,dx$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\left[\frac{(x-\mu t)^2}{2\sigma^2 t}-x\right]} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{[x-(\mu t+\sigma^2 t)]^2-2\mu\sigma^2 t^2-\sigma^4 t^2}{2\sigma^2 t}} dx \\
&= e^{\mu t+\frac{\sigma^2 t}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{[x-(\mu t+\sigma^2 t)]^2}{2\sigma^2 t}} dx \\
&= e^{\mu t+\frac{\sigma^2 t}{2}}
\end{aligned}$$

GBP - expected value, dispersion

- Dispersion:
$$\mathbb{D}\left[x_0 \exp(\mu t + \sigma w(t))\right] = x_0^2\, \mathbb{D}\left[\exp(\mu t + \sigma w(t))\right]$$
- Recall that the dispersion of a random variable X can be expressed as $\mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ - we therefore need $\mathbb{E}\left[\left(e^{\mu t+\sigma w(t)}\right)^2\right]$
- Similarly as in the computation of the expected value:

$$\begin{aligned} \mathbb{E}\left[\left(e^{\mu t+\sigma w(t)}\right)^2\right] &= \int_{-\infty}^{\infty} (e^x)^2 \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} dx \\ &= e^{2\mu t+2\sigma^2 t} \end{aligned}$$

GBP - expected value, dispersion

- Summary:

$$\mathbb{E}\left[x_0 \exp(\mu t + \sigma w(t))\right] = x_0\, e^{\mu t + \frac{\sigma^2 t}{2}}$$

$$\mathbb{D}\left[x_0 \exp(\mu t + \sigma w(t))\right] = x_0^2\, e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right)$$

- Another derivation:
  - we derive the density of a GBM (lognormal distribution - we will work with it on exercises session) and use it to compute the expected value and dispersion → textbook [Ševčovič, Stehlíková, Mikula] - HOMEWORK
  - using Itō lemma → later in this lecture

Model for stock prices

- Model: stock price $S(t)$ follows geometric Brownian motion

$$S(t) = S_0 \exp(\mu t + \sigma w(t)),$$

- We use logarithmic returns:

$$return_t = \log\left(\frac{S_t}{S_{t-\Delta t}}\right)$$

REMARK ON RETURNS:

- computation I: $\frac{S_t - S_{t-\Delta t}}{S_{t-\Delta t}}$ - discrete compounding:

$$\frac{S_t - S_{t-\Delta t}}{S_{t-\Delta t}} = r \Rightarrow S_t = (1+r)S_{t-\Delta t}$$

- computation II: $\log\left(\frac{S_t}{S_{t-\Delta t}}\right)$ - continuous compounding:

$$\log\left(\frac{S_t}{S_{t-\Delta t}}\right) = r \Rightarrow S_t = e^r S_{t-\Delta t}$$

Model for stock prices

- Then, we have

$$
\begin{aligned}
returns_t &= \log\left(\frac{S_t}{S_{t-\Delta t}}\right) \\
&= \log\left(\frac{S_0 e^{\mu t+\sigma w(t)}}{S_0 e^{\mu(t-\Delta t)+\sigma w(t-\Delta t)}}\right) \\
&= \log\left(e^{\mu\Delta t+\sigma(w(t)-w(t-\Delta t))}\right) \\
&= \mu\Delta t+\sigma(w(t)-w(t-\Delta t)) \sim N(\mu\Delta t, \sigma^2 t)
\end{aligned}
$$

and these returns are independent

Stock prices - estimating parameters of a GBP

We proceed as follows:

1. Denote $\Delta t =$ the legth of the time interval between two observations [in years]
2. Compute the returns - according to the model, they are IID $\mathcal{N}(\mu\Delta t, \sigma^2\Delta t)$ random variables
3. We estimate their expected value and dispersion:
   - $m =$ arithmetic mean of the returns $\rightarrow$ estimate of the expected valule $\mu\Delta t$
   - $s^2 =$ sample dispersion of the returns $\rightarrow$ estimate of the dispersion $\sigma^2\Delta t$
4. We estimate the parameters of the GBP:

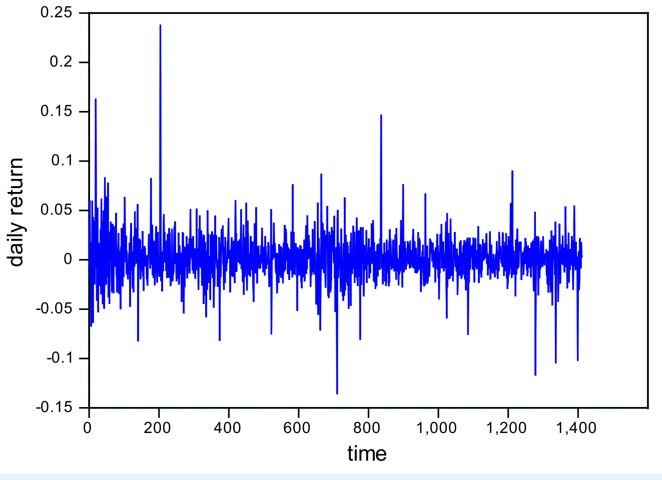
$$\mu = \frac{m}{\Delta t}, \quad \sigma = \sqrt{\frac{s^2}{\Delta t}}$$

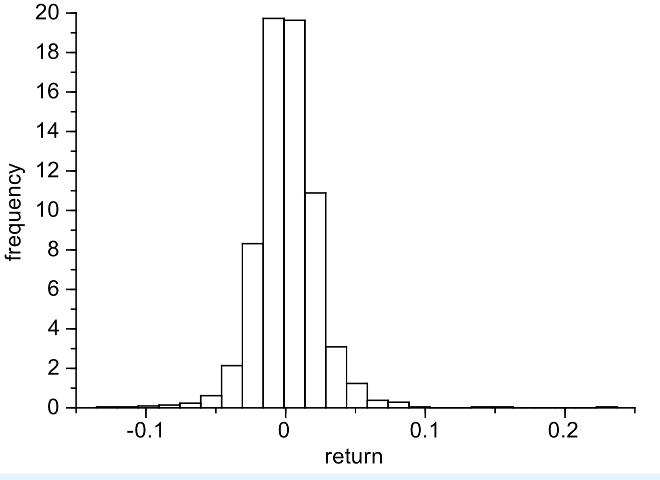
Estimating parameters - example

- We use the daily data of AMZN stock prices (page 26)
- We compute returns:

```
n=length(s);
returns=log(s(2:n)./s(1:(n-1)));
```

- The returns - time serie and histogram:





Estimating parameters - example

- Computing the estimates:

```
muDelta=mean(returns);
sigma2Delta=variance(returns);

mu=muDelta/dt;
sigma=sqrt(sigma2Delta/dt);
```

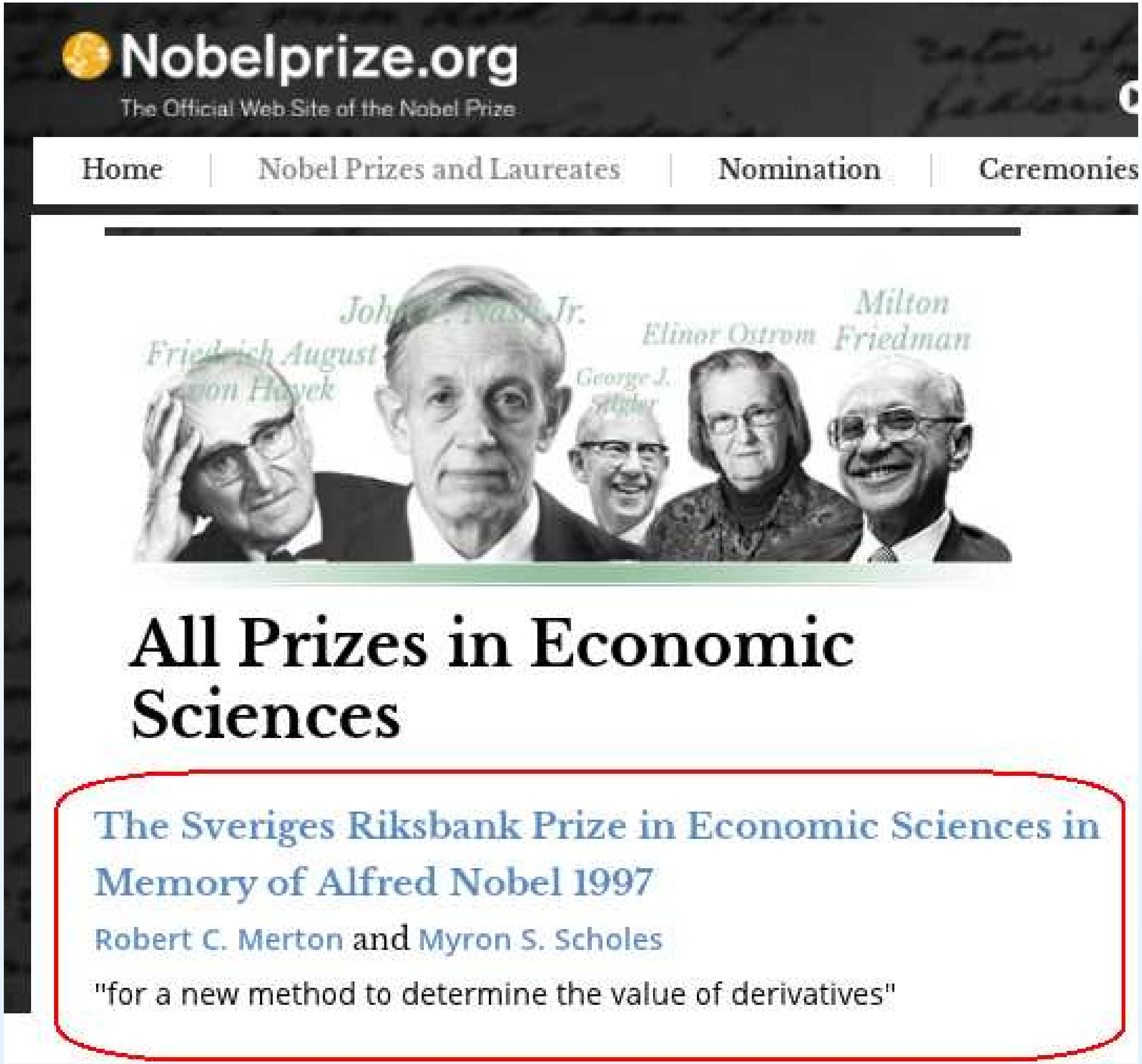
- Results:

```
-->mu
 mu  =

    0.3162172

-->sigma
 sigma  =

    0.3672489
```

Estimating parameters - example

- Applications $\rightarrow$ ON THE EXERCISES SESSION:
  - simulation of the future prices, expected value, comparison with reality
  - probability distribution of stock prices and returns
  - computation of probabilities of different events

Itō lemma


Nobelprize.org
The Official Web Site of the Nobel Prize
Home
Nobel Prizes and Laureates
Nomination
Ceremonies
Milton Friedman
Elinor Ostrom
Friedrich August von Hayek
All Prizes in Economic Sciences
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997
Robert C. Merton and Myron S. Scholes
"for a new method to determine the value of derivatives"

Itō lemma

Guillermo Ferreyra: **The Mathematics Behind the 1997 Nobel Prize in Economics**

http://www.ams.org/samplings/feature-column/fcarc-black-scholes-ito

- *Feature Column* at the American Mathematical Society website:

  *Black and Scholes determined explicitly a stochastic process model for the evolution of the price s of certain assets, and, further, developed a formula for the value $V = V(s,t)$ of the European call option. This is a function of the random variable s and of the time variable t. Once again, the work of K. Itô supplies the needed **chain rule to differentiate V as a function of these two variables**.*

Itō lemma

- Continued:

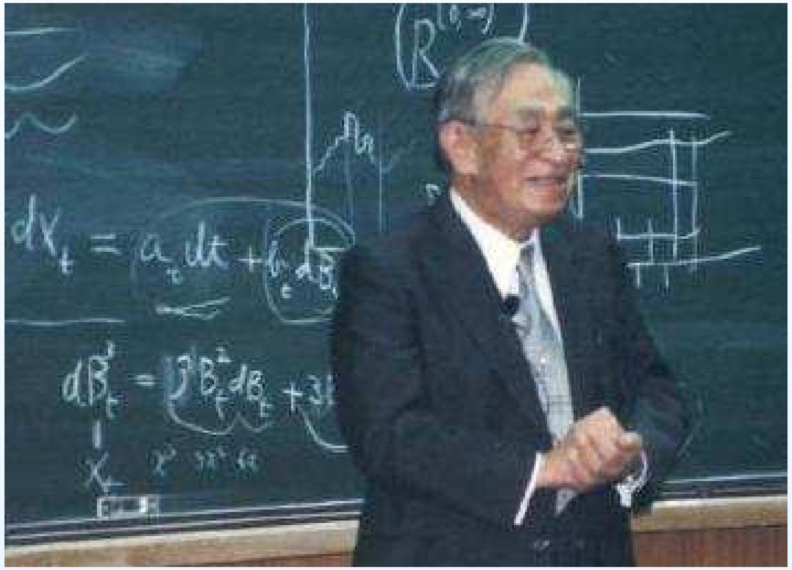
  Differentiation of composition of real valued functions is studied in freshman calculus. Itô's rule is similar in spirit, but it is the differentiation rule for functions of random processes when the random processes are solutions of SDE's. Moreover, the rule looks like the ordinary chain rule studied in calculus, but with ***an additional summand called Itô's correction term.*** *Using this rule Black and Scholes were able to deduce their formula for* V

- Later in this course we explain also:

  As a byproduct of their deduction, they obtained a ratio for the mix between options and assets so that the resulting mix is hedged against fluctuations in the market price of the asset.

Itō lemma

- Itō lemma - how to compute differentials of random functions:



Eightieth Birthday Lecture, Kyoto University, 1995

http://www.kurims.kyoto-u.ac.jp/˜kenkyubu/past-director/ito/ito-kiyosi.html

- In the photo:
  What is dx, if $x = w^3$, where w is a Wiener process?
- We see: $d(w^3) = 3w^2\,dw + \ldots$ - what is the additional term?

Itō lemma: special simpler case

- Let us consider $f = f(t, w)$ where w is a Wiener process
- Taylor expansion up to the second order:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial w} dw + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(dw)^2 + 2\frac{\partial^2 f}{\partial w \partial t} dw\, dt + \frac{\partial^2 f}{\partial t^2}(dt)^2\right) + \ldots$$

- From $dw = \Phi\sqrt{dt}$, where $\Phi \approx N(0,1)$, it follows that

$$(dw)^2 \approx dt$$

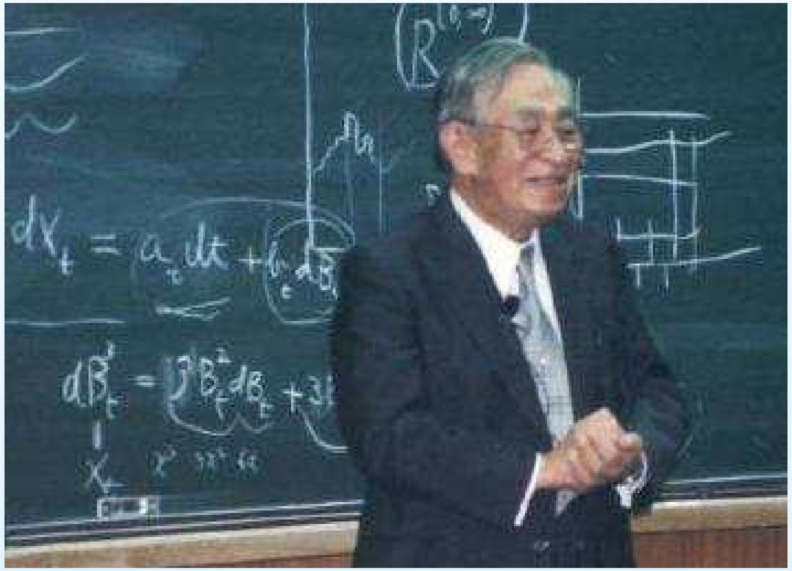
- Similarly:$dw\, dt = O((dt)^{3/2})$
- Terms of order dt, dw in the expansion of df therefore are:

$$df = \frac{\partial f}{\partial w} dw + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\right) dt$$

Itō lemma - example

- EXAMPLE:
  Problem from the photo (p. 41); we have $f(t,w) = w^3$:

$$d(w^3) = 3w^2 dw + 3w dt$$



Itō lemma - formulation

Let $f(x,t)$ be a C^2 smooth function of x,t variables and let a process $\{x(t), t \geq 0\}$ satisfy a stochastic differential equation

$$dx = \mu(x,t)dt + \sigma(x,t)dw,$$

Then

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt \\ &= \left(\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial f}{\partial x}dw \end{aligned}$$

Itō lemma - intuition behind the proof

- As before: expansion up to the second order:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dx\,dt + \frac{\partial^2 f}{\partial t^2}(dt)^2\right) + \ldots$$

- From $dw = \Phi\sqrt{dt}$, where $\Phi \approx N(0,1)$, it follows that

$$\begin{aligned} (dx)^2 &= \sigma^2(dw)^2 + 2\mu\sigma dw\,dt + \mu^2(dt)^2 \\ &\approx \sigma^2 dt + O((dt)^{3/2}) + O((dt)^2) \end{aligned}$$

- Similarly: $dx\,dt = O((dt)^{3/2}) + O((dt)^2)$
- Terms of order dt, dw in the expansion of df therefore are:

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt$$

Itō lemma - example

- EXAMPLE: Stock price following a geometric Brownian motion $S(t) = S_0 e^{\mu t + \sigma w(t)}$:

$$dS = \left(\mu + \frac{\sigma^2}{2}\right) S dt + \sigma S dw$$

- EXAMPLE: Sometimes the model is written as

$$dS = \mu S dt + \sigma S dw$$

- also a GBM, but with different parameters:

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma w(t)}$$

- More practice on exercises session

Moments of a GBP: derivation using Itō lemma

- Let $Y(t) = x_0 \exp(X(t))$, where $X(t) = \mu t + \sigma w(t)$
- We set $f(t,x) = x_0 e^x \;\Rightarrow$

$$dY = \left(\mu + \frac{\sigma^2}{2}\right) Y\,dt + \sigma Y\,dw$$

- Then:

$$\begin{aligned} d\mathbb{E}[Y] &= \left(\mu + \frac{\sigma^2}{2}\right) \mathbb{E}[Y]dt + \sigma \mathbb{E}[Y\,dw] \\ &= \left(\mu + \frac{\sigma^2}{2}\right) \mathbb{E}[Y]dt \end{aligned}$$

  which is an ODE for $\mathbb{E}Y(t)$
- Initial condition: $\mathbb{E}Y(0) = x_0$
- Solution to this ODE with the initial condition: $\mathbb{E}[Y] = x_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)t}$

Moments of a GBP: derivation using Itō lemma

To compute $\mathbb{D}[Y]$ we need $\mathbb{E}[Y^2] = \mathbb{E}[x_0^2 \exp(X(t))^2]$:

- We set $f(t,x) = x_0^2(e^x)^2 = x_0^2 e^{2x} \Rightarrow$

$$d(Y(t)^2) = df = 2(\mu + \sigma^2)Y(t)^2 dt + 2\sigma Y(t)^2 dw$$

- Then

$$d\mathbb{E}[(Y(t)^2)] = 2(\mu + \sigma^2)\mathbb{E}[(Y(t)^2]dt$$

  which is an ODE for $\mathbb{E}[Y^2]$
- Initial condition: $\mathbb{E}[Y(0)^2] = x_0^2$
- HOMEWORK: Finish this computation.

Itō lemma in pricing financial derivatives

- First step in the derivation of the Black-Scholes model for pricing financial derivatives:
  - Suppose that stock price follows the stochastic differential equation $dS = \mu S dt + \sigma S dw$
  - Price of a derivative V (e.g., an option) depends on time t and on the underlying stock price S, hence $V = V(S, t)$
  - Stochastic differential equation for the price of a derivative is therefore obtained using the Itō lemma

Multidimensional Itō lemma - motivation

- A derivative may depend on several underlying assets
- Example: spread option - for example and option with a payoff

$$V(S_1, S_2, T) = \max(0, S_1 - S_2)$$

- We need dV for $V = V(S_1, S_2, t)$

Multidimensional Itō lemma: general case

- Random processes (for $i = 1, \ldots, n$)

$$dx_i = \mu_i(\vec{x}, t)dt + \sum_{k=1}^{n} \sigma_{ik}(\vec{x}, t)dw_k,$$

where $\vec{w} = (w_1, w_2, ..., w_n)'$ is a vector of Wiener processes with correlation matrix $(\rho_{ij})_{i,j=1}^{n}$

$$\mathbb{E}[dw_i \, dw_j] = \rho_{ij} dt$$

- Smooth function

$$f = f(\vec{x}, t) = f(x_1, x_2, ..., x_n, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}$$

- We compute df

Multidimensional Itō lemma: general case

- Again Taylor expansion:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t} dt + \nabla_x f . d\vec{x} \\
&\quad + \frac{1}{2}\left((d\vec{x})^T \nabla_x^2 f \, d\vec{x} + 2\frac{\partial f}{\partial t} . \nabla_x f \, d\vec{x} \, dt + \frac{\partial^2 f}{\partial t^2} (dt)^2 \right) + \ldots
\end{aligned}
$$

- Terms $dt\, dx_i$, $(dt)^2$ are of higher order than dt
- Terms $dx_i\, dx_j$:

$$
\begin{aligned}
dx_i\, dx_j &= \sum_{k,l=1}^{n} \sigma_{ik}\sigma_{jl} dw_k \, dw_l + O((dt)^{3/2}) + O((dt)^2) \\
&\approx \sum_{k,l=1}^{n} \sigma_{ik}\sigma_{jl} \rho_{kl} dt + O((dt)^{3/2}) + O((dt)^2)
\end{aligned}
$$

Multidimensional Itō lemma: concrete example

We demonstrate this on a concrete example:

- EXAMPLE:
  Compute dV, if $V = V(t, S_1, S_2)$ where S_1 and S_2 are geometric Brownian motions

$$dS_i = \mu_i S_i dt + \sigma_i S_i dw_i \quad (i = 1, 2)$$

and $Cor(w_1, w_2) = \rho$.

Words of Kiyoshi Itō (1915-2008)

In precisely built mathematical structures, mathematicians find the same sort of beauty others find in enchanting pieces of music, or in magnificent architecture.

...

Without numerical formulae, I could never communicate the sweet melody played in my heart. Stochastic differential equations, called "Ito Formula", are currently in wide use for describing phenomena of random fluctuations over time. When I first set forth stochastic differential equations, however, my paper did not attract attention. It was over ten years after my paper that other mathematicians began reading my "musical scores" and playing my "music" with their "instruments".

K. Ito, Abstract of the commemorative lecture

My Sixty Years along the Path of Probability Theory (1998).

The whole lecture: http://www.inamori-f.or.jp/laureates/k14_b_kiyoshi/img/lct_e.pdf