III. Short rate models: Evolution of the short rate

Beáta Stehlíková

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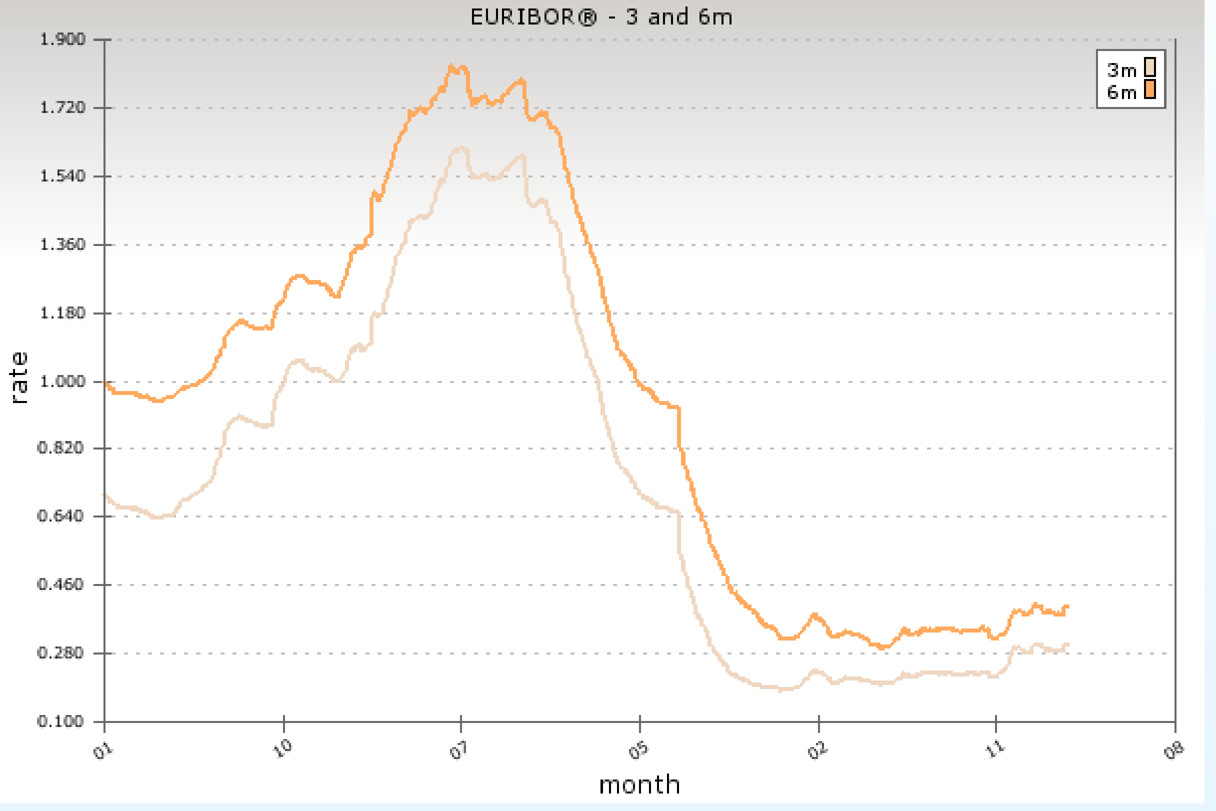
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava

Interest rate

- Assumptions in several models: the interest rate is constant , for example when pricing a riskless portfolio in the derivation of the Black-Scholes model ($dP = rPdt$, cf. *partial differential equations* lectures)
- Reasonable in some cases, but not - for example - if the derivative directly depends on the interest rate (bond, swap, ...)
- What we need to model:
  - the interest rate is not constant
  - there are interest rates with different maturities

Interest rates

- Example: Euribor (European Inter-Bank Offered Rate):



http://www.euribor-ebf.eu/

Short rate models

- Short rate - it is the instantaneous interest rate - interest rate for an infinitesimally small time interval
- Theoretical variable, in practice we use a proxy (1 month, 3 months)
- Short rate models:
  - Short rate r is modelled by a stochastic differential equation

$$dr = \mu(r,t)dt + \sigma(r,t)dw$$

  Terminology: $\mu(r,t)$ - drift, $\sigma(r,t)$ - volatility
  - Other interest rates and derivatives - solving a partial differential equation

Mean-reversion models

- Mean-reversion - reverting to some long-term equilibrium level
- This property in short rate models: the drift is taken to be

$$\mu(r,t) = \kappa(\theta - r)dt,$$

where $\kappa, \theta > 0$ are constants

- ODE for the expected value $\mathbb{E}[r]$ (for a given r_0):

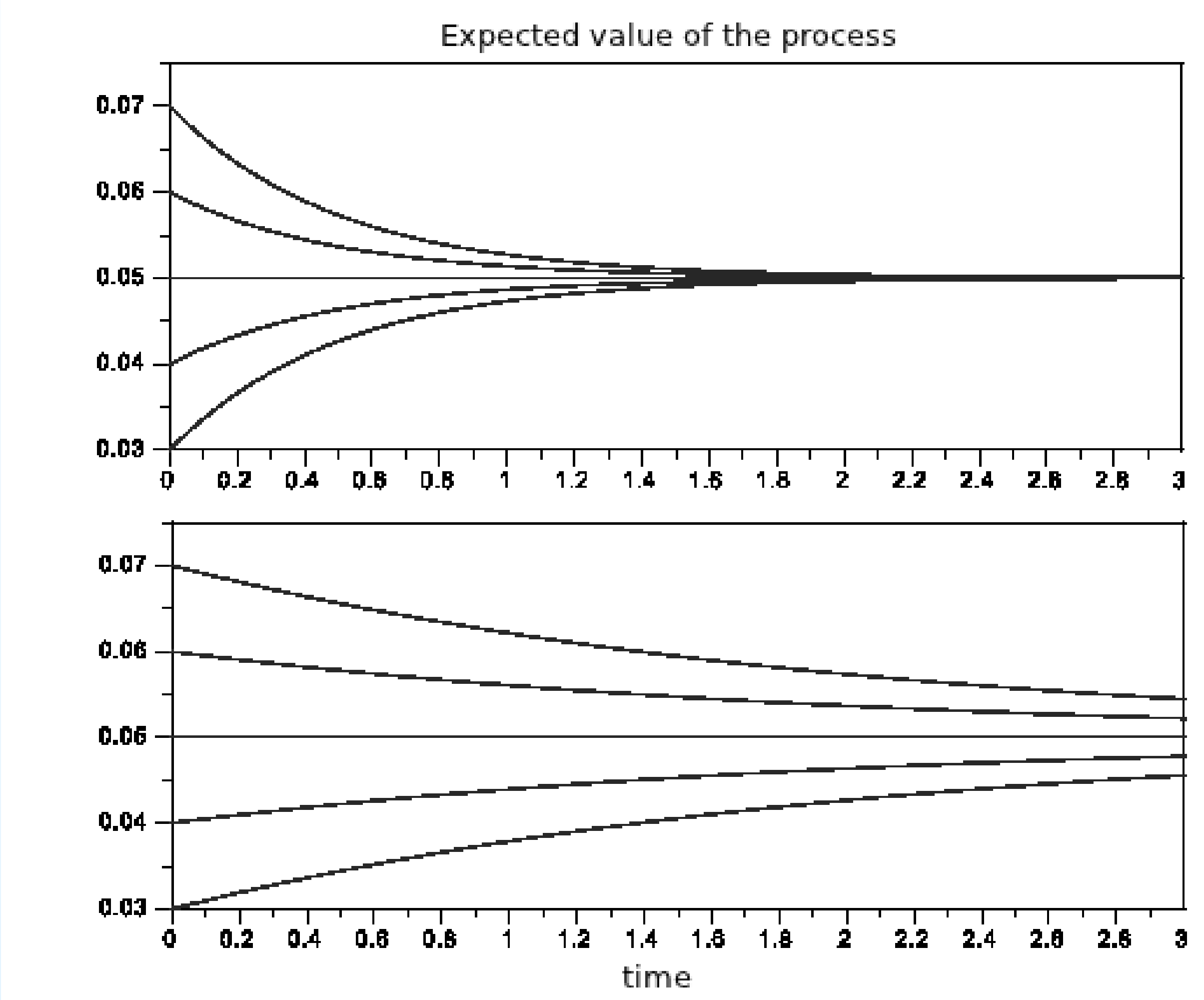
$$d\mathbb{E}[r] = \kappa(\theta - \mathbb{E}[r])dt + \mathbb{E}[\sigma(r,t)dw] = \kappa(\theta - \mathbb{E}[r])dt,$$

its solution is: $\mathbb{E}[r_t] = r_0 e^{-\kappa t} + (1 - e^{-\kappa t})\theta$

- Therefore $\mathbb{E}[r] \to \theta$ as $t \to \infty$

Mean-reversion models

- Sample solutions for selected r_0:



What is θ? What parameter is different in these two cases?

Mean-reversion models

- Example: $dr = \kappa(\theta - r)dt + \sigma dw$ Ornstein-Uhlenbeck process, in finance known as Vasicek model

- Oldřich Alfons Vašíček (born 1942) - a Czech mathematician
- Emigrated to the USA in 1968
- 1969: employed in the management science department of Wells Fargo Bank.

Photo: http://www.risk.net/risk-magazine/feature/1506410/presenting-risk-awards-2002

About Vasicek: http://www.risk.net/risk-magazine/feature/1506624/2002-winner-lifetime-achievement-award-oldrich-alfons-vasicek

Examples of one-factor models

- Already mentioned Vasicek model
  - Short rate: $dr = \kappa(\theta - r)dt + \sigma dw$
  - Drawback: allows negative interest rates (intuitively for now: also for r very close to zero, the volatility is always the same
- Cox-Ingersoll-Ross:
  - J. C. Cox., J. E. Ingersoll Jr, S. A. Ross, **A theory of the term structure of interest rates**, Econometrica (1985) 385-407.
  - Short rate: $dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$
  - Solves the previous problem: does not allow negative interest rates (intuitively: for $r = 0$, the volatility is zero and the drift is positive
  - It can be shown that if $2\kappa > \sigma^2$, then $r = 0$ has a zero probability (intuition: SDE for $y = \ln(r)$ and analysis of the drift)

Examples of one-factor models

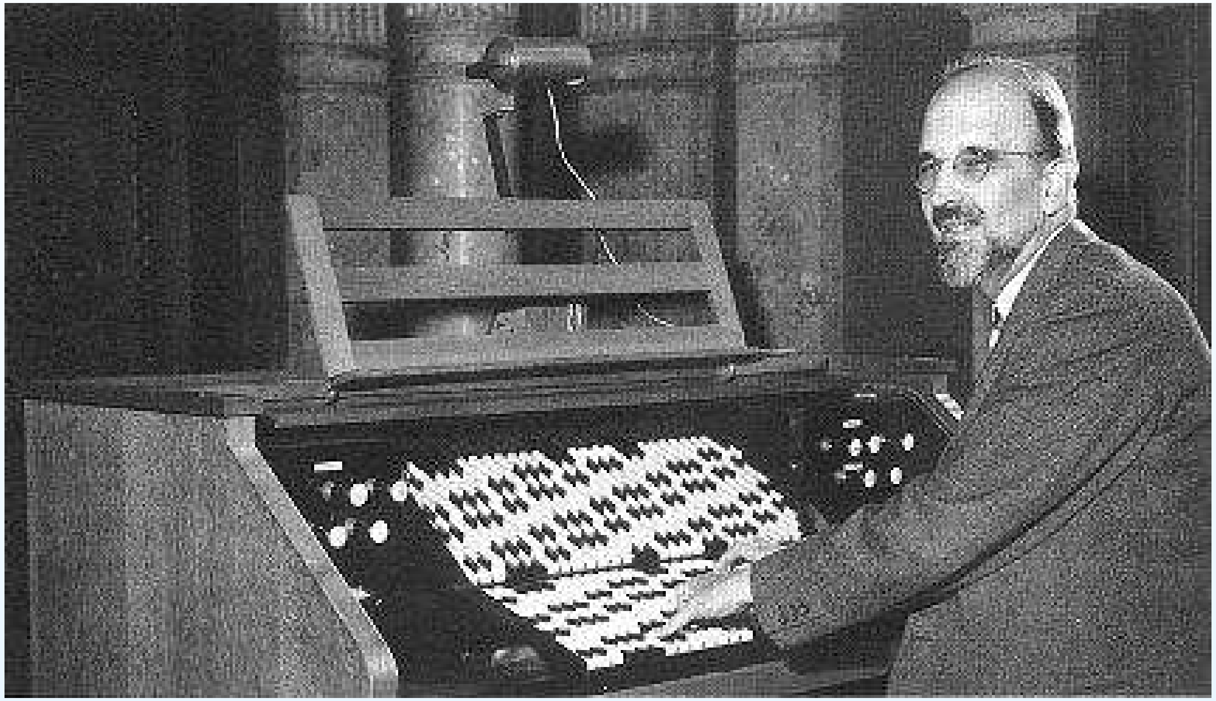
- Chan-Karolyi-Longstaff-Sanders:
  - C. K. Chan, G. A. Karolyi, F. A. Longstaff, A. B. Sanders, **An empirical comparison of alternative models of the short-term interest rate**, The Journal of Finance 47 (1992) 1209-1227.
  - Short rate: $dr = \kappa(\theta - r)dt + \sigma r^\gamma$
  - Vašíček a CIR are special cases ($\gamma = 0, \gamma = 1/2$)
  - They estimated a general model (optimal γ turned out to be 1.5) and tested $\gamma = 0, \gamma = 1/2$ as restrictions on parameters $\rightarrow$ they were rejected
  - Later many other studies of this kind (different data sets, different statistical methods)

Fokker-Planck PDE

- Fokker-Planck PDE - partial differential equation for the density of probability distribution of the value of a stochastic process
- Out of curiosity:
    - Max Karl Ernst Ludwig Planck (1858-1947) was singing, playing the piano, organ and cello, composed songs and opera, ... but he decided to study physics
    - Adriaan Daniël Fokker (1887-1972) was interested in microtonal music, proposed a 31-tonal organ which was exhibited in *Teylers Museum* v Haarleme (the oldest museum in the Netherlands, Fokker was a curator of the physical cabinet)

Fokker-Planck PDE

A. D. Fokker and his organ:



This and other photos: http://www.huygens-fokker.org/instruments/fokkerorgan.html

Fokker-Planck PDE

- Consider the following process

$$dx = \mu(x,t)dt + \sigma(x,t)dw$$

  and define $g(x,t)$ as a conditional density of the value of the process at time t if the value x_0 at time $t = 0$ is given

- THEOREM:
  Then the function $g(x,t)$ is a solution to the Fokker-Planck PDE

$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right)$$

  with initial condition $g(x,0) = \delta(x - x_0)$.

Fokker-Planck PDE

Remark on function δ from the initial condition - it is so called Dirac function:

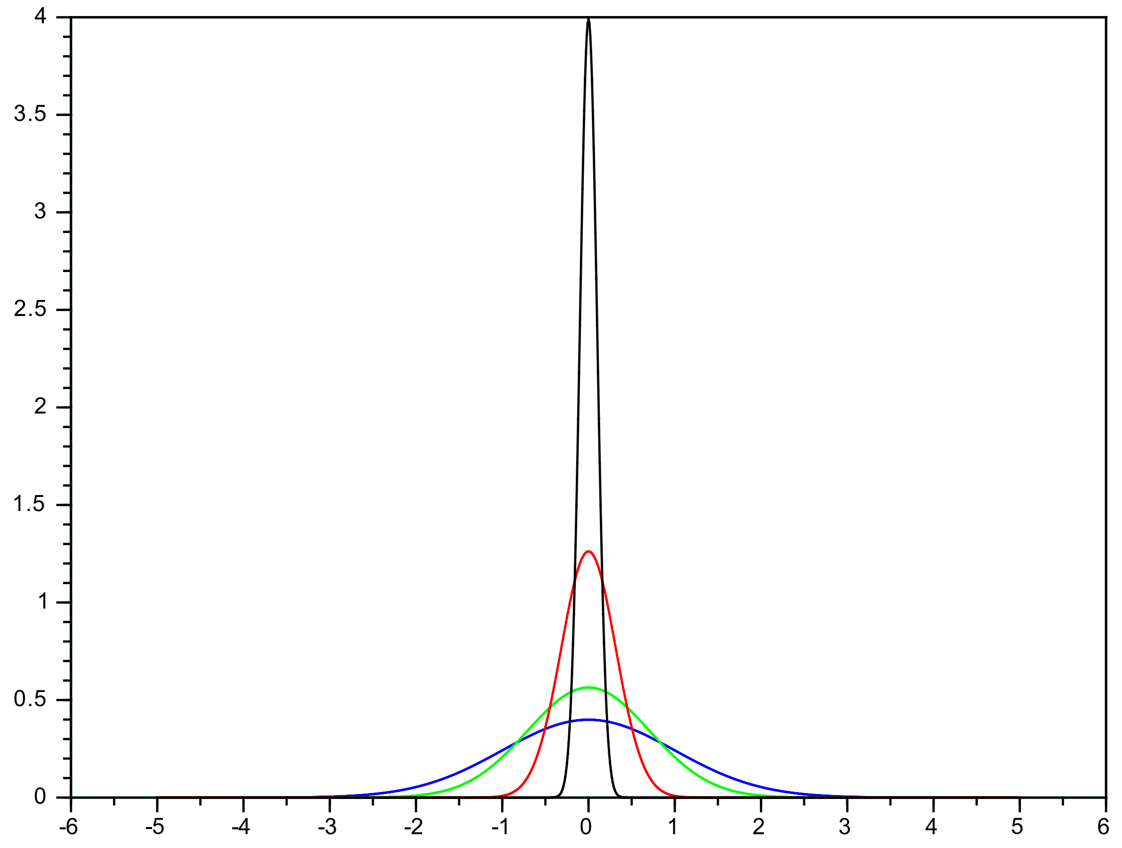
- Not a function is the classical sense
- Intuition:
  - function satisfying

$$\delta(x - x_0) = \begin{cases} 0 & \text{for } x \neq x_0 \\ +\infty & \text{for } x = x_0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(x - x_0)dx = 1.$$

  - "density" of a random variable which takes the value x_0 with probability 1
- We have: $\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0)$
- It can be defined in a mathematically precise way (we will not do this)

Fokker-Planck PDE

Intuitively - functions "converging" to a Dirac function:



Fokker-Planck PDE: proof

- Let $V = V(x,t)$ be an arbitrary function with compact support, i.e., $V \in C_0^\infty(\mathbb{R} \times (0,T))$
- From Itō lemma:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) dt + \sigma \frac{\partial V}{\partial x} dW.$$

- Let E_t is the expected value with respect to the distribution given by the density $g(x,t)$
- Then

$$dE_t(V) = E_t(dV) = E_t \left[\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) dt \right].$$

Fokker-Planck PDE: proof

- We have $V(x,0) = V(x,T) = 0$ and $V(x,t) = 0$ for $|x| > R$ where $R > 0$ is sufficiently large
- Integration *per partes*:

$$
\begin{aligned}
0 &= \int_0^T \frac{d}{dt} E_t(V) dt = \int_0^T E_t\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) dt \\
&= \int_0^T \int_{\mathbb{R}} \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) g(x,t)\, dx\, dt \\
&= \int_0^T \int_{\mathbb{R}} V(x,t) \left(-\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right) \right) dx\, dt.
\end{aligned}
$$

- Since $V \in C_0^\infty(\mathbb{R} \times (0,T))$ was arbitrary, for the density $g = g(x,t)$ we obtain the Fokker-Planck equation

Fokker-Planck PDE for the Vasicek model

Let x_t be an Ornestein-Uhlenbeck/Vasicek process

- Constant at $dw \rightarrow$ we can expect normal distribution
- We have already computed the expected values
- We derive an equation for the variance:

```
(%i1) mi(t):=x0*exp(-kappa*t)+theta*(1-exp(-kappa*t)) $

(%i2) g(x,t):=exp(-(x-mi(t))^2/(2*s2(t)))/sqrt(2*%pi*s2(t));
```

(%o2) $g(x,t):=\dfrac{\exp\left(\dfrac{-(x-\mathrm{mi}(t))^2}{2\,\mathrm{s2}(t)}\right)}{\sqrt{2\pi\,\mathrm{s2}(t)}}$

```
(%i3) -diff(g(x,t),t)+
      (sigma^2/2)*diff(g(x,t),x,2)+
      kappa*g(x,t)-
      kappa*(theta-x)*diff(g(x,t),x) $
```

(Computations in the `wxMaxima` software: http://wxmaxima.sourceforge.net)

Fokker-Planck PDE for the Vasicek model

- Factoring the expression:

```
factor(%);
```

(%o4) $$-\left(\left(\frac{d}{d\,t}\,s2(t)+2\,\kappa\,s2(t)-\sigma^2\right)\left(x0^2-2\,\%e^{\kappa\,t}\,x\,x0+2\,\%e^{\kappa\,t}\,\theta\,x0-2\,\theta\,x0+\%e^{2\,\kappa\,t}\,x^2-2\,\%e^{2\,\kappa\,t}\,\theta\,x+2\,\%e^{\kappa\,t}\,\theta\,x+\%e^{2\,\kappa\,t}\,\theta^2-2\,\%e^{\kappa\,t}\,\theta^2+\theta^2-\%e^{2\,\kappa\,t}\,s2(t)\right)\%e^{-\frac{\%e^{-2\,\kappa\,t}\,x0^2}{2\,s2(t)}+\frac{\%e^{-\kappa\,t}\,x\,x0}{s2(t)}-\frac{\%e^{-\kappa\,t}\,\theta\,x0}{s2(t)}+\frac{\%e^{-2\,\kappa\,t}\,\theta\,x0}{s2(t)}-\frac{x^2}{2\,s2(t)}-\frac{\%e^{-\kappa\,t}\,\theta\,x}{s2(t)}+\frac{\theta\,x}{s2(t)}+\frac{\%e^{-\kappa\,t}\,\theta^2}{s2(t)}-\frac{\%e^{-2\,\kappa\,t}\,\theta^2}{2\,s2(t)}-\frac{\theta^2}{2\,s2(t)}-2\,\kappa\,t}\right)/\left(2^{3/2}\sqrt{\pi}\,s2(t)^{5/2}\right)$$

- More clearly:


```
factor(%);
```

(%o4) $$-\left(\left(\frac{d}{d\,t}\,s2(t)+2\,\kappa\,s2(t)-\sigma^2\right)\left(x0^2-2\,\%e^{\kappa\,t}\,x\,x0+2\,\%e^{\kappa\,t}\,\theta\,x0-2\,\theta\,x0+\%e^{2\,\kappa\,t}\,x^2-2\,\%e^{2\,\kappa\,t}\,\theta\,x+2\,\%e^{\kappa\,t}\,\theta\,x+\%e^{2\,\kappa\,t}\,\theta^2-2\,\%e^{\kappa\,t}\,\theta^2+\theta^2-\%e^{2\,\kappa\,t}\,s2(t)\right)\%e^{-\frac{\%e^{-2\,\kappa\,t}\,x0^2}{2\,s2(t)}+\frac{\%e^{-\kappa\,t}\,x\,x0}{s2(t)}-\frac{\%e^{-\kappa\,t}\,\theta\,x0}{s2(t)}+\frac{\%e^{-2\,\kappa\,t}\,\theta\,x0}{s2(t)}-\frac{x^2}{2\,s2(t)}-\frac{\%e^{-\kappa\,t}\,\theta\,x}{s2(t)}+\frac{\theta\,x}{s2(t)}+\frac{\%e^{-\kappa\,t}\,\theta^2}{s2(t)}-\frac{\%e^{-2\,\kappa\,t}\,\theta^2}{2\,s2(t)}-\frac{\theta^2}{2\,s2(t)}-2\,\kappa\,t}\right)/\left(2^{3/2}\sqrt{\pi}\,s2(t)^{5/2}\right)$$

- This has to equal to zero

Fokker-Planck PDE for the Vasicek model

- This expression has to be equal to zero ⇒ this holds if

$$s2'(t) + 2\kappa\, s2(t) - \sigma^2 = 0$$

- Variance at time $t = 0$ is zero ⇒ initial condition $s2(0) = 0$
- Solution:

$$s2(t) = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

- CONCLUSION:
  Distribution of an Ornstein-Uhlenbeck process is a normal distribution with expected value $r_0 e^{-\kappa t} + (1 - e^{-\kappa t})\theta$ and variance $s2(t) = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$

Vasicek model: estimating parameters

- We have a time serie of interest rate (proxy for the short rate) $\rightarrow$ we want to estimate the parameters of the Vasicek model
- Knowledge of the conditional distribution allows us to construct the likelihood function for the given values of the interest rate $r_1, r_2, \ldots, r_n$ observed in the market:

$$L = \prod_{i=1}^{n-1} f(r_{i+1}|r_i)$$

- Maximizing L (equivalently, its logarithm) yields estimates of the parameters
- Vasicek model: functions f are normal distribution densities; it is possible to find closed form expressions for the estimates; we will use then on exercises session

CKLS model: estimating parameters

- Recall that $dr = \kappa(\theta - r)dt + \sigma r^\gamma$
- Conditional distribution known only for $\gamma = 1/2$; even that quite complicated $\rightarrow$ approximation of the likelihood:
  - The volatility σr^γ on time interval $[t, t + \Delta t)$ between two observations is approximated by its value in time t
  - In this approximation, volatility on $[t, t + \Delta t)$ is constant $\rightarrow$ normal distribution
  - Known as Nowman's Gaussian estimates (since based on Gaussian approximation
- Maximum likelihood estimates $\rightarrow$ testing hyptheses using likelihood ratio test

CKLS model: estimating parameters - example

Athanasios Episcopos: **Further evidence on alternative continuous time models of the shortterm interest rate**, Journal of International FinancialMarkets, Institutions andMoney Volume 10, Issue 2, June 2000, pp. 199-212

- Estimates for 10 countries (general CKLS model and several restrictions given by existing models)
- Next page: example for the USA ((data: 1/1986 - 4/1998, 148 observations)
- We go through the procedure of testing Vasicek and CIR model as restrictions of the CKLS model (computation of the test statistics and corresponding p-value)

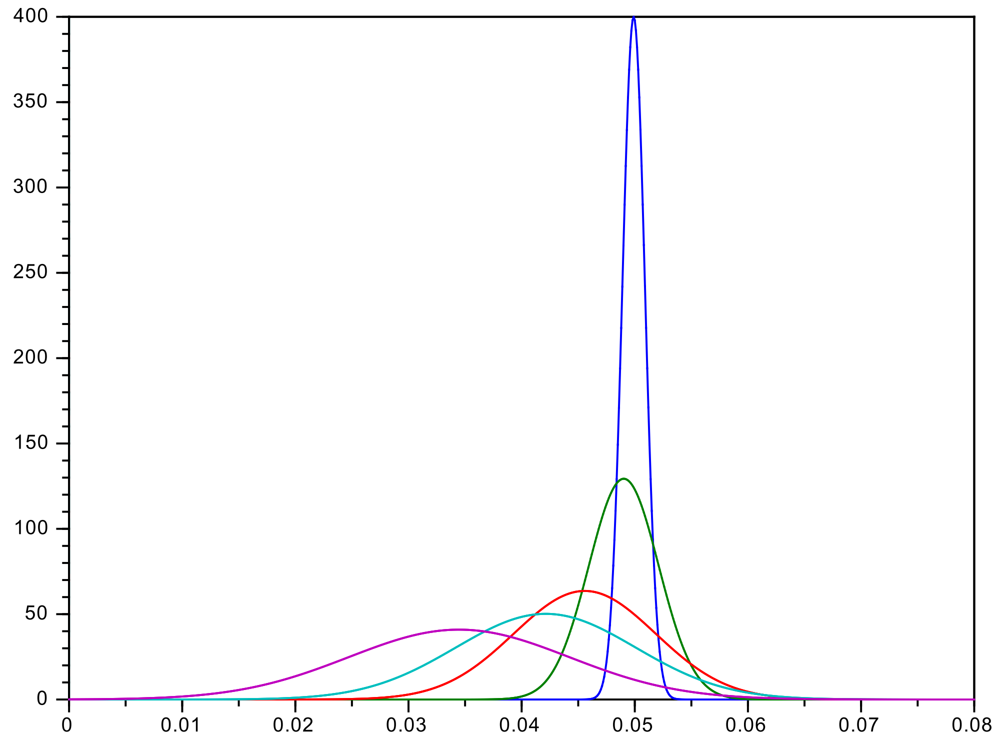
CKLS model: estimating parameters - example

Results for the USA:

Model[b]	α	β	σ^2	γ	Avg. Log L	χ^2-test[c]	df
Unrestricted	0.0013	−0.0234	0.0001	0.4239	5.1768		
	(1.4696)	(−1.5710)	(1.0309)	(2.5099)			
Vasicek	0.0013	−0.0235	0.0000	0	5.1569	5.8655	1
	(4.3499)	(−5.6112)	(16.1077)			(0.0154)	
CIR SR	0.0013	−0.0241	0.0002	0.5	5.1761	0.201	1
	(4.6916)	(−5.8893)	(13.0558)			(0.6539)	
BR-SC	0.0014	−0.0255	0.0038	1	5.1365	11.8748	1
	(6.1214)	(−6.1221)	(14.9738)			(0.0006)	
CIR VR	0	0	0.0794	1.5	5.0220	45.529	3
			(21.0933)			(0.0000)	
CEV	0	−0.003	0.0001	0.4063	5.1705	1.8477	1
		(−0.6401)	(24.7981)	(31.0657)		(0.1740)	

Fokker-Planck PDE: limiting distribution

- EXAMPLE: Ornestein-Uhlenbeck process - densities for a given x_0 and a couple of times t:



Fokker-Planck PDE: limiting distribution

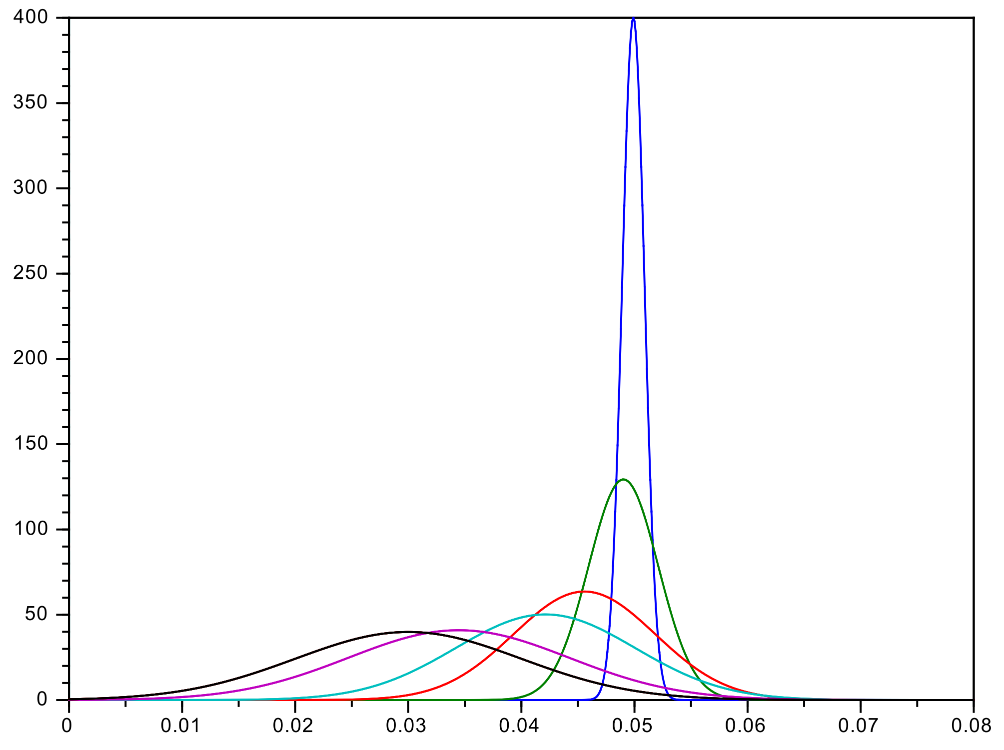
- Densities converge to a certain limiting distribution
- As $t \to \infty$, we have:

$$\mathbb{E}[r_t|r_0] = r_0 e^{-\kappa t} + (1 - e^{-\kappa t})\theta \to \theta$$

$$\mathbb{D}[r_t|r_0] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \to \frac{\sigma^2}{2\kappa}$$

Fokker-Planck PDE: limiting distribution

- EXAMPLE: Ornestein-Uhlenbeck process - densities for a given x_0 and a couple of times t (from the previous plot) and the limiting density (black line):



Fokker-Planck PDE: limiting distribution

- We do not need the conditional distributions to compute the limining distribution (this is often complicated)
- Direct computation from the Fokker-Planck PDE:
  - we know that the density $g(x,t)$ for time t satisfies the PDE

$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right)$$

  - consider limit $f(x) := \lim_{t\to\infty} g(x,t)$
  - this limit then satisfies the stationary Fokker-Planck equation:

$$0 = \frac{1}{2}\frac{d^2}{dx^2}\left(\sigma^2 f\right) - \frac{d}{dx}\left(\mu f\right)$$

with a normalization condition (it is a density function) $\int_{-\infty}^{\infty} f(x)dx = 1$

Fokker-Planck PDE: limiting distribution

EXAMPLE: CIR model for interest rates

- Recall the stochastic differential equation for the short rate $dx = \kappa(\theta - x)dt + \sigma\sqrt{x}dw$
- Let $2\kappa > \sigma^2$ (then the zero value cannot be achieved)
- Density $g(x,t)$ at time t satisfies the PDE

$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\kappa(\theta - x)g\right)$$

with initial condition

$$g(x,0) = \delta(x - x_0)$$

- There is an explicit expression for $g(x,t)$,but it is quite complicated (noncentral chi-squared distribution, modified Bessel function)

Fokker-Planck PDE: limiting distribution

EXAMPLE - CONTINUED:

- Explicit solution (without a proof):

$$g(x,t) = c\, e^{-u-v} \left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}),$$

(for $x > 0$, otherwise $g(x,t) = 0$), where I_q is a modified Bessel function of the first kind of order q and

$$\begin{aligned} c &= \frac{2\kappa}{\sigma^2(1-e^{-\kappa t})} \\ u &= c\, x_0\, e^{-\kappa t} \\ v &= c\, x \end{aligned}$$

- Complicated, but limiting distribution can be found also without a knowlenge of this conditional distribution

Fokker-Planck PDE: limiting distribution

EXAMPLE - CONTINUED:

- Limiting density $f(x) := \lim_{t\to\infty} g(x,t)$ satisfies stationary Fokker-Planck equation: (for $x > 0$, otherwise it is zero since the process never has negative values)

$$0 = \frac{1}{2}\frac{d^2}{dx^2}\left(\sigma^2 x f\right) - \frac{d}{dx}\left(\kappa(\theta - x)f\right)$$

- HOMEWORK: Integrating gives

$$f(x) = K\, x^{\frac{2\kappa\theta}{\sigma^2}-1}\, e^{-\frac{2\kappa}{\sigma^2}x}$$

- Constant K is computed from the condition $\int_{-\infty}^{\infty} f(x)dx = 1$
- Note that this is a density of a gamma distribution