III. - supplement: First passage time of a stochastic process

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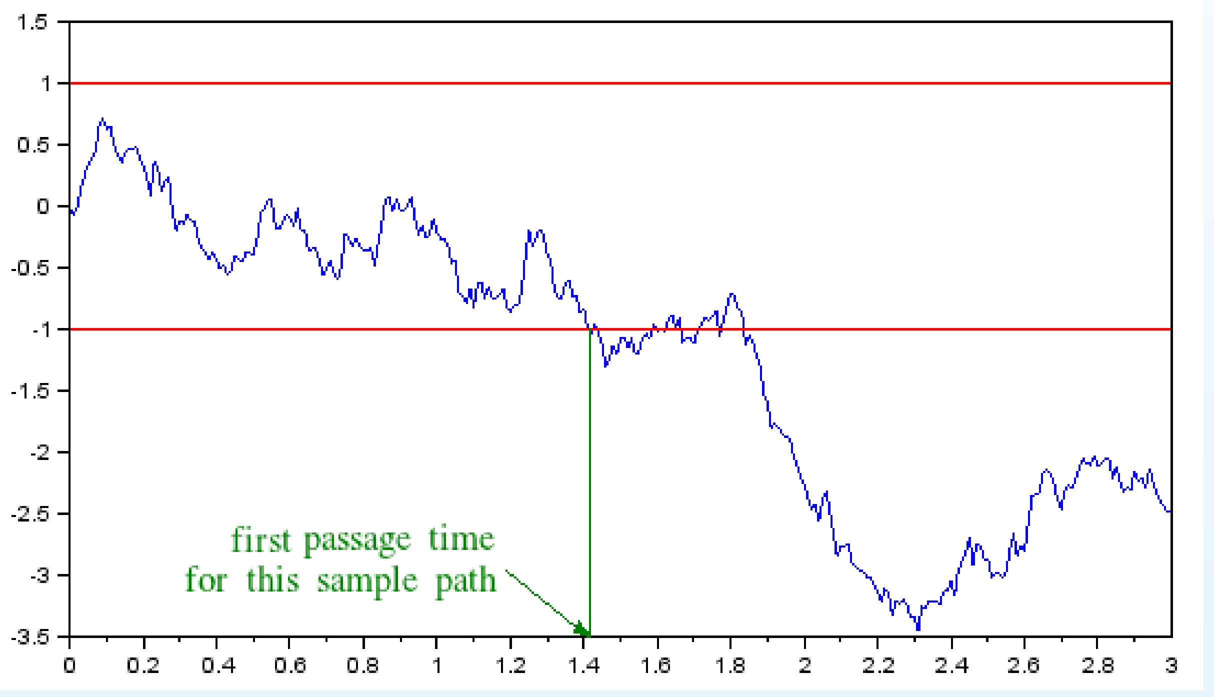
First passage time

- Consider stochastic process $x(t)$
- The first passage time T is a random variable, giving the time when the process reaches the given boundary for the first time
- The mean first passage time is the expected value of T, i.e. $\mathbb{E}[T]$

Mean first passage time: Wiener process

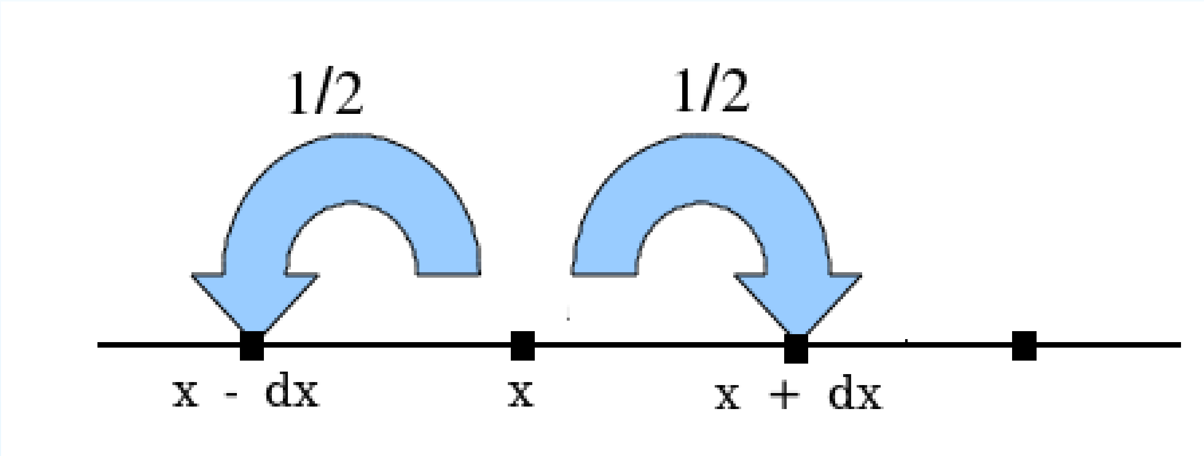
- Consider Wiener process $w(t)$
- When does it cross the boundary $[-A, B]$ (with $A, B > 0$) for the first time?



- We compute the mean first passage time $u(x_0)$ for the process $w(t) + x_0$

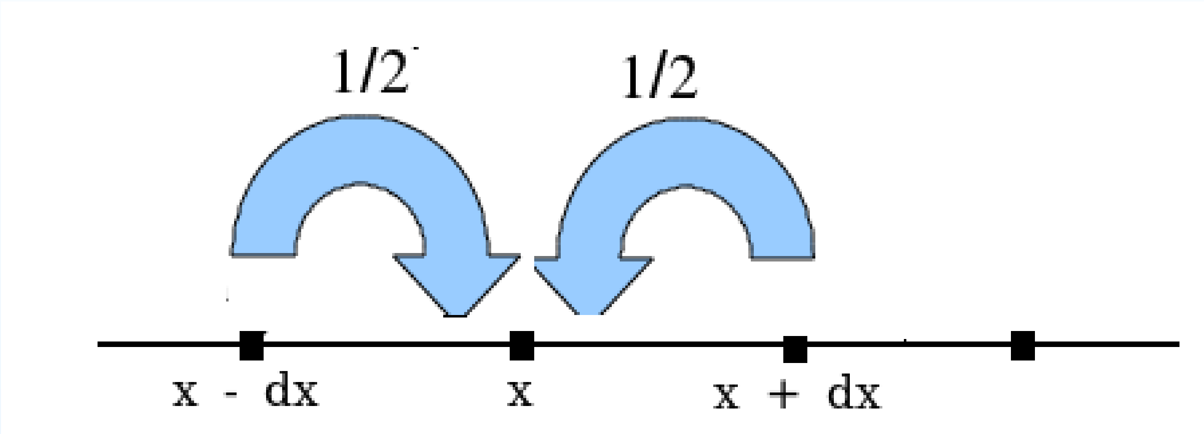
Mean first passage time: Wiener process

- Recall that a Wiener process can be seen as a limit of a random walk with time step Δt and space steps $\pm\Delta$ with $\Delta x = \sqrt{\Delta t}$ (each with the same probability $1/2$)



Mean first passage time: Wiener process

- Opposite way: how can be get to the point x?



Mean first passage time: Wiener process

- Recall: $u(x)$ = expected time of staying inside $(-A, B)$ for a particle which is at point x
- Hence:

$$u(x) = \Delta t + \frac{1}{2}u(x - \Delta x) + \frac{1}{2}u(x + \Delta x)$$

since:

- the "trip" from $x - \Delta x$ or $x + \Delta x$ lasted a time interval Δt
- each of these trips has the same probability $1/2$
- so:

[expected time when at point x] = [length of trip] + [probability of coming from $x - \Delta x$] $\times$ [expected time when at point $x - \Delta x$] + [probability of coming from $x + \Delta x$] $\times$ [expected time when at point $x + \Delta x$]

Mean first passage time: Wiener process

- Taylor expansion:

$$
\begin{aligned}
u(x) &= \Delta t + \frac{1}{2}\left[u(x) - u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 + o((\Delta x)^2)\right] \\
&\quad + \frac{1}{2}\left[u(x) + u'(x)\Delta x + \frac{1}{2}u''(x)(\Delta x)^2 + o((\Delta x)^2)\right]
\end{aligned}
$$

- Substituting $\Delta t = (\Delta x)^2$ and taking limit $\Delta x \to 0$:

$$\frac{1}{2}u''(x) = -1$$

- Boundary conditions (when we are at the boundary, no more time is spent in the interval of interest): $u(-A) = 0, u(B) = 0$

Mean first passage time: Wiener process

- General solution:

$$u(x) = -x^2 + c_1 x + c_2$$

  compute the constants from the boundary conditions
- Example for the interval $(-1, 1)$:
  - solution to the ODE is $u(x) = -x^2 + 1$
  - Wiener process starts at $x = 0$
  - so the mean first passage time is $u(0) = 1$

Mean first passage time: general case

- Consider the process

$$dx = \mu(x)dt + \sigma(x)dw$$

- The ODE for the mean first passage time is

$$\mu(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x) = -1$$

with boundary conditions

$$u(-A) = 0, u(B) = 0$$

- Can be proved *using Fokker-Planck PDE*, see e.g. http://www3.ul.ie/gleesonj/SDEs_Finance/first_passage_times.pdf